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Robust Linear Quadratic Designs with Real Parameter Uncertainty

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Abstract—**This note derives a linear quadratic regulator which is robust to real parametric uncertainty, by using the overbounding method of Petersen and Hollot. The resulting controller is determined from the solution of a single modified Riccati equation. This controller has the same guaranteed robustness properties as standard linear quadratic designs for known systems. It is proven that when applied to a structural system, the controller achieves its robustness by minimizing the potential energy of uncertain stiffness elements, and minimizing the rate of dissipation of energy by the uncertain damping elements.**

I. Introduction

In this note, we will examine a Linear Quadratic Regulator (LQR) based control design which is robust to parametric uncertainties. We shall refer to this class of controllers as "Robust LQR (RLQR)." We will focus on structured uncertainty in the open loop "A" matrix. This is representative of a structural system where mode frequencies and damping values, which appear in the "A" matrix, are unknown.

Although there are some inherent robustness properties in the classical LQR design (specifically, we are guaranteed an infinite upwards gain margin and a downwards gain margin of .5, or a phase margin of $\pm 60°$, in each control channel independently and simultaneously [12]), the linear quadratic regulator is not robust to parametric uncertainty. In fact, "blindly" designing an LQR controller on some nominal system does not guarantee the stability of the actual system, even if the actual system is guaranteed to be open-loop stable. An example of this is shown in [3].

We would like to adapt LQR so that we have robustness to parametric uncertainty. Additionally, we would like to retain the inherent robustness properties (e.g., the MIMO gain and phase margins) of LQR designs, so that we will have limited robustness to unstructured uncertainty. The RLQR design achieves this robustness.

Our control design is based upon Petersen's Riccati equation approach [10], [11]. We have reinterpreted this approach as a method of doing LQR design on an uncertain system. Several interesting properties arise in the design, which help direct us when designing controllers for uncertain systems. Similar approaches in this framework include [1], [2], [6], and [13].

Being an LQR based approach, we assume full feedback of all state variables. Though this may not be valid in realistic applications, understanding the underlying fundamentals in this RLQR framework will help direct us when we assume knowledge of only the output variables.

II. Derivation of RLQR

We assume we have an uncertain linear system of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1}$$

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and a standard quadratic cost functional

$$J = \int_0^\infty [x^T(t)Q_0x(t) + \rho u^T(t)u(t)]\,dt. \tag{2.2}$$

We will study the case when there are uncertain, but constant, parameters. We will further assume that all the uncertainty is in the "A" matrix. This is typical of space structures, where stiffness and damping coefficients which appear in the "A" matrix are quite uncertain, while mass values, which also influence the "B" matrix, are known with a greater degree of accuracy.

We model the uncertain A matrix in the form

$$A = A_0 + \sum_{i=1}^{p} q_i E_i \qquad |q_i| \le 1 \tag{2.3}$$

where A_0 represents the "nominal" system, and each real uncertain constant parameter is known to be in a bounded interval; we assume we have p uncertain parameters. The E_i matrices represent the structure of the uncertainty, and are scaled so that the magnitudes of the scalars q_i are less than 1, $|q_i| \le 1$.

We want to derive a multi-input/multi-output (MIMO) LQR controller which is robust to this model of parametric uncertainty. One way to do this would be to look at Nyquist plots of the uncertain system, and see if we can "bound" the uncertainty in the complex plane. It turns out that this is a difficult thing to do. What is possible, however, is to get an expression for the return difference function, which is the key to the Nyquist plot, in terms of the LQR design parameters. This will help guide us in "robustifying" the LQR design.

A. Frequency Domain Equality (FDE)

We begin by stating the LQR Riccati equation for the nominal system A_0, and the cost (2.2) [7]:

$$-PA_0 - A_0^TP - Q_0 + \frac{1}{\rho}PBB^TP = 0. \tag{2.4}$$

To account for the uncertainty, add and subtract $PA + A^TP$, where A is the unknown (but constant) matrix. Also add and subtract sIP (where s is the frequency domain variable) and rearrange to get

$$P(sI - A) - (sI + A^T)P + P(A - A_0) + (A^T - A_0^T)P - Q_0 + \frac{1}{\rho}PBB^TP = 0. \tag{2.5}$$

Next postmultiply both sides of the equation by $(sI - A)^{-1}B$, and premultiply by $B^T(-sI - A^T)^{-1}$. For more compact notation, let

$$\Phi(s) = (sI - A)^{-1}, \qquad G = \frac{1}{\rho}B^TP. \tag{2.6}$$

Simple algebraic manipulations produce the equation

$$[I + G\Phi(-s)B]^T[I + G\Phi(s)B] = I + \frac{1}{\rho}B^T\Phi^T(-s)[P(A_0 - A) + (A_0^T - A^T)P + Q_0] \cdot \Phi(s)B. \tag{2.7}$$

This is called the Robust FDE (RFDE), and is the first main result. It is a frequency domain relation for the actual return difference transfer function matrix $I + G\Phi(s)B$.

B. Robustness Implications

From the RFDE (2.7), it is clear that if

$$P(A_0 - A) + (A_0^T - A^T)P + Q_0 \geq 0 \tag{2.8}$$

then

$$P(A_0 - A) + (A_0^T - A^T)P + Q_0 = F^T F \tag{2.9}$$

for some matrix F. If we define $G_{RLQR} = G\Phi(s)B$, then it is clear that

$$\sigma_i[I + G_{RLQR}(s)] = \sqrt{1 + \frac{1}{\rho}\sigma_i^2[F\Phi(s)B]}. \tag{2.10}$$

Equation (2.10) guarantees the same robustness as LQR designs on the class of uncertain systems described in the introduction in terms of MIMO gain and phase margins [8], [12]. In the complex plane for SISO systems, the expression states that the Nyquist plot of the uncertain system remains outside the unit disk centered at the critical point. Thus, we will acquire a certain level of robustness to unstructured uncertainty as well as stability and performance robustness to the parametric uncertainty.

C. The RLQR Robust Riccati Equation

Having given the motivation and philosophy behind the robust controller, we will now derive a Riccati equation which guarantees (2.8). We will use a method due to Petersen and Hollot [11]. The resulting controller will be called the "Robust LQR," or "RLQR" design.

We start by substituting the standard Riccati equation for the nominal system into (2.8). We want to find a value for P which guarantees the bound, now given by:

$$-PA - A^T P + \frac{1}{\rho}PBB^T P \geq 0. \tag{2.11}$$

Substitute in (2.11) the actual value of the A matrix (2.3) to get

$$-PA_0 - A_0^T P - \sum_{i=1}^{p} q_i P E_i - \sum_{i=1}^{p} q_i E_i^T P + \frac{1}{\rho}PBB^T P \geq 0. \tag{2.12}$$

Factor each E_i in minimal rank decompositions and define the matrices L and N as follows:

$$E_i \triangleq l_i n_i^T \qquad L = [l_1 l_2 l_3 \cdots] \qquad N = [n_1 n_2 n_3 \cdots]. \tag{2.13}$$

Recall that Q_0 is the state weighting matrix we would use on the nominal system if there were no uncertainty. Using the Petersen–Hollot bounds [10], [11] a sufficient condition for (2.12) is

$$PA_0 + A_0^T P + (Q_0 + \gamma NN^T) - P\left(\frac{1}{\rho}BB^T - \frac{1}{\gamma}LL^T\right)P = 0. \tag{2.14}$$

Thus, to design a controller to guarantee stability and robustness, we need to find the positive definite solution P, if it exists, to this modified Robust Riccati equation (2.14), and apply the feedback:

$$u(t) = -Gx(t) \qquad G = \frac{1}{\rho}B^T P. \tag{2.15}$$

The ability to find a solution may depend on the choice of factorization (2.13).

Similar Riccati equations to (2.14) have appeared in the literature; for example, see [10]. This reference discusses sufficient conditions for this type of Riccati equation to have a solution. Note that if we find a solution $P = P^T > 0$ in the Robust Riccati equation (2.14), we could define a modified state weighting matrix Q by

$$Q \triangleq -PA_0 - A_0^T P + \frac{1}{\rho}PBB^T P = Q_0 + \gamma NN^T + \frac{1}{\gamma}PLL^T P. \tag{2.16}$$

Then the RLQR controller is the optimal controller when we are minimizing the cost functional

$$J = \int_0^\infty [x^T(t)Qx(t) + \rho u^T(t)u(t)]\, dt \tag{2.17}$$

rather than that defined in (2.2). Thus the RLQR can be interpreted as an LQR design, with a suitably modified state weighting matrix Q. Thus, we are guaranteed the same robustness as in LQR designs because it is an LQR design itself [8], [12]. A similar result was found using a different approach by Petersen and McFarlane in [9].

We can interpret our results as adding guaranteed stability robustness to structured uncertainty and robustness guarantees by adding terms to the nominal LQR cost functional. The term $(1/\gamma)PLL^T P$ is equivalent to an $\mathcal{H}_\infty$ term [5]. Thus, through this term we are finding the worst possible disturbance coming in the direction defined by the L matrix, which depends on which parameters are uncertain. This "equivalent" disturbance arises from the mismatched dynamics. The γNN^T term has a physical interpretation as will be shown in Section III. In general, it modifies the original state weighting Q_0 in the direction defined by the N matrix.

The relative importance of these two terms in the cost functional is determined by the scalar γ. Since γ affects the bandwidth of the closed-loop system, an intermediate value is desired (very high or very low γ typically results in a high bandwidth [3], [4]). It is not surprising to find the bandwidth of the system with the RLQR design higher than that with the mismatched LQR design, since we are desensitizing the system to parameter variations. The parameter γ can help tune the bandwidth to an acceptable level. Note that a higher bandwidth implies less robustness to high-frequency unstructured uncertainty. This is one of the prices we pay for improved robustness to parametric uncertainty.

D. Guaranteed Performance

When we guaranteed $\sigma_i[I + G_{RLQR}(s)] \geq 1$ in Section II-B, we guaranteed the singular values of the sensitivity function are less than unity, i.e., $\sigma_i[I + G_{LQR}(s)]^{-1} \leq 1$, and therefore we have guaranteed performance robustness. This is a guaranteed property of LQR designs in the absence of uncertain parameters [8] (though note that it is not guaranteed to hold when we design an LQR controller for one system and apply it to another system with a different "A" matrix).

We will now state a theorem which shows that we have better performance robustness in the RLQR design than in a design using the nominal LQR parameters applied to the uncertain system. Note that in this nominal design case, the dynamics are mismatched to the design parameters, and thus it is called the mismatched LQR design.

Theorem 2.1: The maximum singular value of the sensitivity function of the actual plant with the RLQR design is always less than or equal to the maximum singular value of the sensitivity of the same plant with the mismatched LQR design at any given frequency.

The basic idea of the proof is as follows. The mismatched LQR is designed by solving the Riccati equation (2.4) and has the associated FDE (where G_0 is the nominal LQ feedback gain matrix)

$$[I + G_0\Phi(-s)]^T[I + G_0\Phi(s)B] = I + \frac{1}{\rho}B^T\Phi^T(-s)Q_0\Phi(s)B \tag{2.18}$$

where this FDE is derived in the same manner as the robust FDE. The RLQR design is similarly characterized by (2.7) and (2.14). By subtracting (2.7) from (2.18), and substituting $A = A_0 + \sum_{i=1}^{p} q_i E_i$ and the Riccati equations (2.14) and (2.4), we can show

$$[I + G_0\Phi(-s)B]^T[I + G_0\Phi(s)B] \leq [I + G\Phi(-s)B]^T[I + G\Phi(s)B]. \tag{2.19}$$

This implies

$$\sigma_{\max}\{(I+G_0\Phi(s)B])^{-1}\} \geq \sigma_{\max}\{(I+G\Phi(s)B])^{-1}\}. \quad (2.20)$$

Equation (2.20) is the desired result, since it states that the maximum singular value of the sensitivity function of the mismatched LQR design is greater than that of the RLQR design at every frequency. This quantifies the improvement of performance robustness in the RLQR design from a sensitivity transfer function perspective.

III. Interpretations for Structural Systems

In this section, we wish to examine the role of the term γNN^T in (2.14) and see how it increases the robustness to parameter uncertainty. To do so, we consider a structural system to gain physical insight into how this term impacts robustness. We remark that stiffness uncertainty and damping uncertainty constitute the major parametric uncertainties in this class of problems.

A. Uncertain Stiffness

Let us assume that we deal with structural dynamic systems, which can be represented as

$$M\ddot{\nu}(t)+D\dot{\nu}(t)+(K+\tilde{K})\nu(t)=f(t) \quad (3.1)$$

where $\nu(t)$ is a generalized position vector, $f(t)$ is a force vector, $M=M^T>0$ is a mass matrix, $D=D^T\geq 0$ is a damping matrix, $K=K^T\geq 0$ is a stiffness matrix consisting of elements whose stiffness values are known, and $\tilde{K}=\tilde{K}^T\geq 0$ is a stiffness matrix consisting of uncertain elements. We can rewrite the system (3.1) as

$$\begin{bmatrix}\dot{\nu}(t)\\ \ddot{\nu}(t)\end{bmatrix}=\begin{bmatrix}0 & I\\ -M^{-1}K-M^{-1}\tilde{K} & -M^{-1}D\end{bmatrix}\begin{bmatrix}\nu(t)\\ \dot{\nu}(t)\end{bmatrix}+\begin{bmatrix}0\\ M^{-1}\end{bmatrix}f(t). \quad (3.2)$$

Let us assume there are p uncertain stiffness parameters. Then we will write

$$\tilde{K}=\sum_{i=1}^{p}\tilde{r}_i\tilde{K}_i \quad (3.3)$$

where $\tilde{K}_i=\tilde{K}_i^T\geq 0$ is a known matrix which represents the structure of how the ith uncertain stiffness element affects the system, and $\tilde{r}_i>0$ is a scalar which represents the uncertain value of the unknown stiffness element.

Given uncertainty intervals for each uncertain stiffness element, we can scale each $\tilde{K}_i$ so as to write

$$\tilde{r}_i=\tilde{r}_{i0}+q_i \qquad |q_i|\leq 1 \quad (3.4)$$

where $\tilde{r}_{i0}$ is the "nominal" value of the ith uncertain element, and is chosen at the midpoint of the interval, and q_i represents the uncertain value.

The potential energy for the system (3.1) is equal to

$$\tfrac{1}{2}\nu^T(t)(K+\tilde{K})\nu(t) \quad (3.5)$$

and we can therefore see that the uncertain potential energy in the system is

$$\frac{1}{2}\nu^T(t)\tilde{K}\nu(t)=\frac{1}{2}\sum_{i=1}^{p}(\tilde{r}_{i0}+q_i)\nu^T(t)\tilde{K}_i\nu(t). \quad (3.6)$$

Hence, the potential energy in the ith uncertain stiffness element is

$$\tfrac{1}{2}(\tilde{r}_{i0}+q_i)\nu^T(t)\tilde{K}_i\nu(t). \quad (3.7)$$

Since each $\tilde{K}_i$ is symmetric, positive-semidefinite, we can write

$$\tilde{K}_i=\left(\frac{1}{\gamma_i}\eta_i\right)(\gamma_i\eta_i^T) \quad (3.8)$$

where γ_i is a scalar scaling factor which represents how we factored the matrix $\tilde{K}_i$. We can now write the total uncertainty in the RLQR setup of (2.3) as

$$\sum_{i=1}^{p}q_iE_i=\sum_{i=1}^{p}\begin{bmatrix}0 & 0\\ -M^{-1}q_i\tilde{K}_i & 0\end{bmatrix}=\sum_{i=1}^{p}q_i\begin{bmatrix}0\\ -\frac{1}{\gamma_i}M^{-1}\eta_i\end{bmatrix}[\gamma_i\eta_i^T \quad 0]. \quad (3.9)$$

Note that the midpoint matrix is grouped with the nominal matrix in the RLQR framework, and thus the term $\tilde{r}_{i0}\tilde{K}$ is not in the uncertainty matrix. Also, q_i in the RLQR framework is exactly the same q_i as in (3.4), and explains our choice of notation.

From (3.9), we see that the N matrix is

$$N=\left[\begin{bmatrix}\gamma_1\eta_1\\ 0\end{bmatrix}\begin{bmatrix}\gamma_2\eta_2\\ 0\end{bmatrix}\begin{bmatrix}\gamma_3\eta_3\\ 0\end{bmatrix}\cdots\right] \quad (3.10)$$

so that

$$x^T(t)NN^Tx(t)=\sum_{i=1}^{p}\gamma_i^2\nu^T(t)\eta_i\eta_i^T\nu(t). \quad (3.11)$$

Comparing (3.11) and (3.7) we see that the term $x^T(t)NN^Tx(t)$ in this general setup is proportional to a weighted sum of the energies in each of the uncertain stiffness elements. The weightings depend upon the factorizations (3.8).

Thus, for all structural systems of the form (3.1), the term NN^T is a weighting of a sum of the uncertain potential energies of the uncertain stiffness parameters. Thus, in the RLQR formation one tries to minimize the effects of uncertain potential energy in addition to the nominal state penalty.

B. Uncertain Dampers

Next we interpret the term NN^T in the case when there is uncertainty in the "D" matrix in the structural system of the form

$$M\ddot{\nu}(t)+(D+\tilde{D})\dot{\nu}(t)+K\nu(t)=f(t) \quad (3.12)$$

where $\nu(t)$ is the generalized position vector, $f(t)$ is the force vector, $M=M^T>0$ is a mass matrix, $K=K^T\geq 0$ is a stiffness matrix (which in this case is known), $D=D^T\geq 0$ is a damping matrix consisting of elements whose values are known, and $\tilde{D}=\tilde{D}^T\geq 0$ is a damping matrix consisting of uncertain elements.

Following similar steps as with uncertain stiffness elements, we find

$$x^T(t)NN^Tx(t)=\sum_{i=1}^{p}\gamma_i^2\dot{\nu}^T(t)\eta_i\eta_i^T\dot{\nu}(t). \quad (3.13)$$

To interpret this term, consider the energy in the system (3.12):

$$PE=\tfrac{1}{2}\nu^T(t)K\nu(t) \qquad KE=\tfrac{1}{2}\dot{\nu}^T(t)M\dot{\nu}(t)$$
$$TE=PE+KE \quad (3.14)$$

where PE is potential energy, KE is kinetic energy, and TE is the total energy.

The rate of change of total energy in the system is

$$\frac{d}{dt}(TE)=\frac{1}{2}(\ddot{\nu}^T(t)M\dot{\nu}(t)+\dot{\nu}^T(t)M\ddot{\nu}(t)+\dot{\nu}^T(t)K\nu(t)+\nu^T(t)K\dot{\nu}(t)) \quad (3.15)$$
$$=\dot{\nu}^T(t)(-(D+\tilde{D})\dot{\nu}(t)+f(t)). \quad (3.16)$$

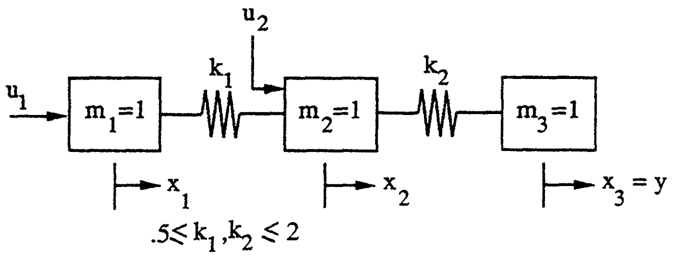

Fig. 1. Two spring example.

The term $\dot{\nu}^T(t)f(t)$ is the rate of change of energy due to the force vector, and $\dot{\nu}^T(t)(D+\tilde{D})\dot{\nu}(t)$ is the rate of dissipation of energy due to damping (the negative sign signifies that energy is dissipated).

So the rate of change of energy (dissipated power) through the ith uncertain damper is

$$\dot{\nu}^T(t)(\bar{r}_{i0}+q_i)\tilde{D}_i\dot{\nu}(t) = (\bar{r}_{i0}+q_i)\dot{\nu}^T(t)\eta_i\eta_i^T\dot{\nu}(t). \quad (3.17)$$

Now we can clearly see that $x^T(t)NN^Tx(t) = \sum_{i=1}^p \gamma_i^2\dot{\nu}^T(t)\eta_i\eta_i^T\dot{\nu}(t)$ is a weighted sum of energy dissipation rates through the uncertain dampers in the system. In this case, the RLQR design is robust to parametric uncertainty by weighting the effect of the uncertain energy dissipation.

Of course when there is uncertainty in both K and D, it is clear that $x^T(t)NN^Tx(t)$ represents a weighted sum of uncertain potential energies and uncertain energy dissipation rates. The weights evolve from the choice of the factorization of E_i into l_i and n_i. Thus, we can exploit this knowledge to intelligently choose the factorization. We put larger relative weights on those uncertain elements whose dynamics degrade our performance to a greater degree. For example, in a system with many uncertain springs, to further reduce the uncertain potential energy of the ith uncertain spring, we would change the factorization from $E_i = l_in_i^T$ to $E_i = ((1/\gamma_i)l_i)(\gamma_in_i^T)$, with $\gamma_i > 1$.

C. A Mass-Spring MIMO Example

Consider, as shown in Fig. 1, three unit masses coupled by two springs with uncertain stiffness values $k_1, k_2 \in [.5, 2]$. We wish to control the position $y(t)$ of the third mass by exerting control forces $u_1(t)$ and $u_2(t)$ on the first two masses.

Following the procedure for structural systems, we can write our uncertain system as $\dot{x} = Ax + Bu$. For this example, the L and N matrices were

$$L = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ .866 & 0 \\ -.866 & .866 \\ 0 & -.866 \end{bmatrix} \qquad N = \begin{bmatrix} -.866 & 0 \\ .866 & -.866 \\ 0 & .866 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (3.18)$$

As a basis for comparison, we designed a standard LQR control for the nominal system, characterized by the midpoint stiffness values $k_1 = k_2 = 1.25$, and applied the control to the system with different values of the spring constants. The nominal design parameters used were

$$\rho = .01 \qquad Q_0 = \mathrm{diag}(0,0,1,0,0,0). \quad (3.19)$$

The selection of Q_0 implies the nominal goal of regulating the position $y(t) = x_3(t)$ of the third mass. The resulting LQR control gains are:

$$G_0 = \begin{bmatrix} 0.518 & -0.423 & -0.206 & 1.001 & 0.185 & -0.341 \\ 1.270 & 6.375 & 2.355 & 0.185 & 3.566 & 8.951 \end{bmatrix}. \quad (3.20)$$

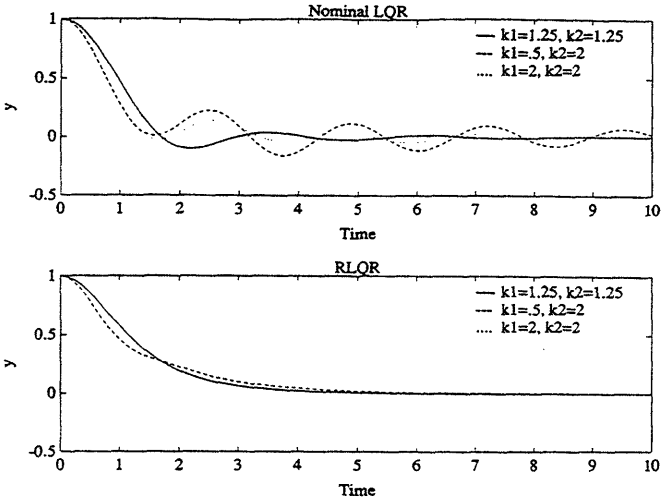

Fig. 2. Output response for two spring example. The upper plot contains mismatched LQR transients (designed with $k_1 = k_2 = 1.25$) using the gains of (3.20), while the lower plot contains RLQR output responses using the gain matrix (3.21). The performance robustness of RLQR designs is self-evident.

To design the RLQR controller, we used the values of ρ and Q_0 in (3.19), L and N of (3.18), and $\gamma = 1$. The resulting RLQR control gains are:

$$G = \begin{bmatrix} 8.224 & 3.701 & -2.327 & 4.156 & 0.609 & 12.725 \\ 1.171 & 27.673 & -1.925 & 0.609 & 7.855 & 36.912 \end{bmatrix}. \quad (3.21)$$

Typical output transients are shown in Fig. 2, where the upper plot shows the mismatched LQR design (based upon $k_1 = k_2 = 1.25$), and the lower plot shows the corresponding RLQR design. For the case when $k_1 = k_2 = 1.25$, the LQR design is matched to the system, and optimal with respect to the standard cost functional implied by (3.19).

Note from Fig. 2 that the transient response of the mismatched LQR controller can vary widely depending on the actual value of the spring stiffness parameters. The "differences" in the shape of the transient responses are an indication of the "performance unrobustness" in this numerical example and are the consequences of the wide variation of potential energy among the mismatched LQR designs. In this example, the system always remains stable, although this is by no means guaranteed in mismatched classical LQR designs.

In this example (and others [3], [4]), the RLQR controller yields similar transients for all values of the stiffness elements. It is apparent from this figure that we achieved a certain level of performance robustness with the RLQR.

Additionally, the RLQR control responded, in all cases, so as to first move the masses so that the springs were at their equilibrium length, $(x_1 - x_2) \approx 0$ and $(x_2 - x_3) \approx 0$, (in which case there is small uncertainty in the stored potential energy), and then the RLQR control moved the three masses in unison slowly back to the desired equilibrium position. This behavior was quite different than that of the classical LQR mismatched designs in which the controls moved the masses towards their zero position and then reduced the spring lengths to equilibrium. Thus, the RLQR design acted as if it "knew" that the uncertainty was in the spring constants, and it worked to keep the uncertainty in the stored potential energy from adversely affecting the dynamics of the motion. This was accomplished by the two additional terms γNN^T and $(1/\gamma)PLL^TP$ in the Robust Riccati equation (2.14). A similar effect was noted for other systems [3], [4].

For the example of Fig. 1, we calculated the maximum singular value of the sensitivity function for the system with $k_1 = k_2 = 2$,

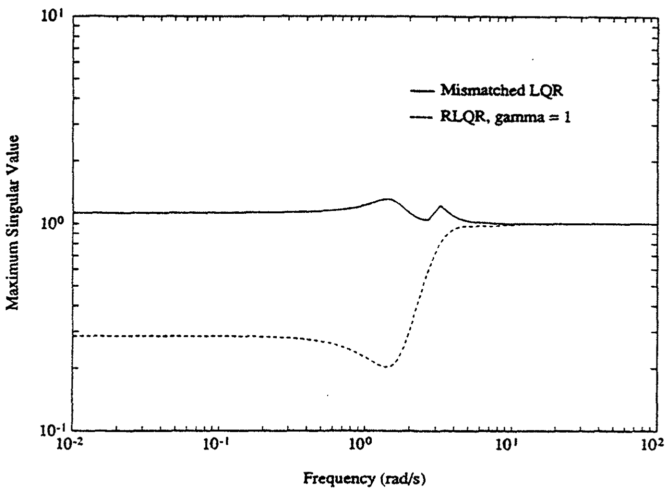

Fig. 3. Typical maximum singular value sensitivity plots of the three mass example with $k_1 = k_2 = 2$.

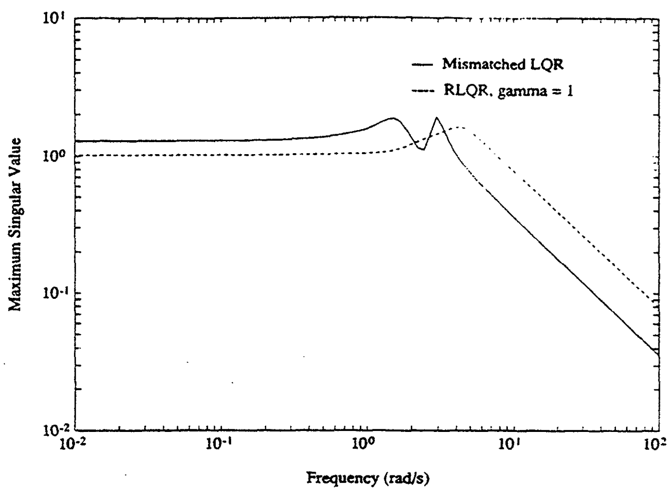

Fig. 4. Typical maximum singular value complementary sensitivity plots of the three mass example with $k_1 = k_2 = 2$

and with the gains listed above. The representative plots are shown in Fig. 3. Note that since the LQR design is "mismatched," we are not guaranteed that the sensitivity function is less than 1 for all frequencies. The corresponding complimentary sensitivity maximum singular value functions are shown in Fig. 4. Notice that the RLQR design has a higher closed-loop bandwidth than the mismatched LQR design, which implies that it is more sensitive to unstructured high-frequency model errors.

IV. Conclusions

We have presented an extension of the standard LQR called the Robust LQR (RLQR). It is derived using an overbounding technique known as Petersen–Hollot bounds. The result of this overbounding is a guarantee of stability in the presence of parametric uncertainty, and also guaranteed robustness in terms of MIMO gain and phase margins. The resulting full-state controller is designed by solving a single Riccati-type equation. This Robust Riccati equation is identical to ones which have appeared in the literature with this overbounding method.

The novelty presented in the derivation was the interpretation of the controller as an extension of LQR. In fact, it was shown that the RLQR design is equivalent to an LQR design with an intelligent choice of the state weighting matrix, or a modified full-state $\mathcal{H}_2/\mathcal{H}_\infty$ design. It is this choice of the state weighting matrix which makes the system robust to parametric uncertainty.

We were able to show analytically how the choice of the "equivalent state weighting matrix" added robustness to the system. In the standard LQR design, we minimize a cost functional which contains quadratic weights on the states and on the control. In the RLQR design, the state weighting matrix adds two more quadratic terms to this cost functional. For structural systems, the first is equivalent to a weighted sum of the potential energies of each uncertain stiffness element, and a weighted sum of the rate of dissipation of energy through each uncertain damping element. The second is a term which is the same as a worst-case disturbance in a direction defined by the specific uncertain parameters. These two terms were sufficient to guarantee robustness to the parametric uncertainty, as well as the additional robustness guarantees stated earlier. The RLQR design hedges for parameter uncertainty; however, its robustness to other types of uncertainty, e.g., high frequency model errors, must be evaluated separately.

In summary, we have examined a full-state controller which is robust to parametric uncertainty. It achieves its performance robustness by minimizing the effect of uncertain stored energy and uncertain power dissipation. It also provides the same guaranteed robustness to unstructured uncertainty as in standard LQR designs.

References

[1] D. S. Bernstein, "Robust static and dynamic output-feedback stabilization: Deterministic and stochastic perspectives," *IEEE Trans. Automat. Contr.*, vol. 32, no. 12, pp. 1072–1084, Dec. 1987.

[2] D. S. Bernstein and W. M. Haddad, "The optimal projection equations with Petersen–Hollot bounds: Robust stability and performance via fixed-order dynamic compensation for systems with structured real-valued parameter uncertainty," *IEEE Trans. Automat. Contr.*, vol. 33, no. 6, pp. 578–582, June 1988.

[3] J. Douglas, "Linear quadratic control for systems with structured uncertainty," SM thesis, Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, May 1991.

[4] J. Douglas and M. Athans, "Robust LQR control for the benchmark problem," in *Proc. 1991 Amer. Contr. Conf.*, Boston, MA, June 1991, pp. 1923–1924.

[5] J. Doyle, K. Glover, P. Khargonekar, and B. Francis, "State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems," *IEEE Trans. Automat. Contr.*, vol. 34, no. 8, pp. 831–847, Aug. 1989.

[6] P. Khargonekar, I. Petersen, and K. Zhou, "Robust stabilization of uncertain linear systems: Quadratic stabilizability and $\mathcal{H}_\infty$ control theory," *IEEE Trans. Automat. Contr.*, vol. 35, no. 3, pp. 356–361, Mar. 1990.

[7] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems.* New York: Wiley-Interscience, 1972.

[8] N. A. Lehtomaki, N. R. Sandell Jr., and M. Athans, "Robustness results in LQG based multivariable control designs," *IEEE Trans. Automat. Contr.*, vol. 26, no. 1, pp. 75–93, Feb. 1981.

[9] I. Petersen and D. McFarlane, "Optimal guaranteed cost control of uncertain linear systems," in *Proc. 1992 Amer. Contr. Conf.*, Chicago, IL, June 1992, pp. 2929–2930.

[10] I. R. Petersen, "A stabilization algorithm for a class of uncertain linear systems," *Syst. Contr. Lett.*, vol. 8, pp. 351–357, 1987.

[11] I. R. Petersen and C. V. Hollot, "A Riccati equation approach to the stabilization of uncertain linear systems," *Automatica*, vol. 22, no. 4, pp. 397–411, 1986.

[12] M. G. Safonov and M. Athans, "Gain and phase margins for multiloop LQG regulators," *IEEE Trans. Automat. Contr.*, vol. 22, no. 2, pp. 173–179, Apr. 1977.

[13] L. Xie and C. de Souza, "Robust $\mathcal{H}_\infty$ control for linear systems with norm-bounded time-varying uncertainty," in *Proc. 1990 Conf. Decision Contr.*, Honolulu, HI, Dec. 1990, pp. 1034–1035.