A Multiple Scales Approach to the Stability and Control of a Hypersonic Re-entry Glider

by

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ABSTRACT

A method is presented for the calculation of continuously varying gains for the control of 2nd order linear time varying systems. The approach used is to develop stability criteria based on the requirement for uniform decay of the GMS description of the response, and to select position and rate feedback gains to satisfy these criteria. The criteria are presented in the form of first order differential inequalities involving the system coefficients, and the solution of these yields the gain algorithms. In order to test the methods developed, they are applied to the stability and control analyses of the longitudinal dynamics of a hypersonic re-entry vehicle. The GMS-stability criterion is presented for a generally configured vehicle descending along an arbitrary trajectory. Simplification to a straight line ballistic trajectory yields an expression for critical altitude which is identical to the established result in the literature. The control example yields a variable gain controller for a Space Shuttle type vehicle descending along a typical Shuttle re-entry trajectory. In general, the results from numerical simulation are promising, though actual implementation of a practical flight control system would involve difficulties with real-time data measurement for gain calculation — especially if the system were to be completely autonomous (self sufficient).

Thesis Supervisor: Dr. R. V. Ramnath
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Nomenclature

The following symbol definitions are global unless otherwise stated. When a symbol has different meanings in different sections of the text, the local definitions are apparent and no confusion should arise. Where appropriate, the equation numbers where the symbols are defined are given in parenthisis. Note that S.I. units have been adopted throughout.

Symbols introduced in Chapters 1 and 2:

$\sim$	'$\cdots$ is asymptotic to $\cdots$' (2.1)
O	Order-of-Magnitude symbol (see Ref. [4])
ε	small parameter
y	dependent variable in generic ODE
y_0	zeroth order asymptotic approximation to y
y_1	first order term in asymptotic approximation to y
y_s	*slow part* of y_0 (2.9)
y_f	*fast part* of y_0 (2.9)
t	time; independent variable in generic ODE
p_0, p_1	coefficients of generic 2nd order LTV system
k	'clock function' (time varying *characteristic root*) (2.3)
I	expression used in definition of y_s (2.8)
ϕ, θ	dummy variables used in integration
τ_0, τ_1	time scales in *extended space*
D	'discriminant' used in GMS description of 2nd order dynamics (page 19)
R	'radius' used in GMS description of 2nd order dynamics (2.11)
Ω	'frequency' used in GMS description of 2nd order dynamics (2.13)
C, C_1, C_2	arbitrary constants
g	'transformation function' (2.33)
u	dependent variable in equivalent-canonical-system (2.34)

u_s	asymptotic approximation of *slow part* of u
u_f	asymptotic approximation of *fast part* of u
w_0	'equivalent-canonical-coefficient' (2.34)

Symbols introduced in Chapter 3:

v	dependent variable in generic 1st order differential inequality (3.1)
u	dependent variable of ODE associated with v-inequality (3.1)
f	arbitrary function in a differential inequality
z	dependent variable in a Bernoulli Equation (3.22)
$\delta(t)$	Dirac Delta function at $t = 0$
K, K_0, K_1	control gains

Symbols introduced in Chapters 4 and 5:

K_α	α-position feedback gain
L	characteristic length
L_v	vehicle length
L_{nc}	nose-cone length
f_d	fuselage diameter (nose cone base diameter)
L_F	flap length
S	wing area
S_F	flap area
$\bar{c}$	mean chord
c_{cl}	chord at centreline
b	wing span
δ_{nc}	nose-cone half-angle
m	vehicle mass
X_{cg}	position of $c.g.$(measured from nose)

I_x, I_y, I_y	principle moments of inertia
σ	non-dimensional mass parameter (4.5)
ν	non-dimensional inertia parameter (4.5)
ρ	free-stream density
δ	non-dimensional density parameter (4.5)
δ_F	total flap deflection angle $(\delta_{F_T} + \delta_{F_c})$
δ_{F_T}	flap-angle-to-trim
δ_{F_c}	control flap angle
ξ	non-dimensional distance variable (4.6)
V	velocity of vehicle $c.g.$(specified on trajectory); airspeed (in still air)
Ma	free-stream Mach Number (specified on trajectory)
XCP_{nc}	nose-cone centre-of-pressure (measured from nose)
XCP_w	wing centre-of-pressure (measured from nose)
XCP_F	flap centre-of-pressure (measured from nose)
$\overline{\alpha}$	total angle-of-attack; $\overline{\alpha} = \alpha_T + \alpha$
α_T	angle-of-attack-to-trim (specified on trajectory)
α	perturbation angle of attack
θ	aircraft pitch angle
γ	flight-path angle (specified on trajectory)
γ_0	constant flight-path angle on ballistic trajectory
q	aircraft pitch rate relative to inertial space
h	altitude (specified on trajectory)
r	distance from vehicle $c.g.$to planet centre (altitude + planet radius)
g	gravitational constant
f	inhomogeneous or forcing term (4.2c,4.8c)
C_L	lift coefficient
C_D	drag coefficient
C_M	pitching moment coefficient (nose-up positive)
C_{L_T}	lift coefficient evaluated at trim conditon (on trajectory)
C_{D_T}	drag coefficient evaluated at trim conditon (on trajectory)
C_{L_α}	lift stability derivative referred to trim condition (NACA convention[6])

C_{D_α}	drag stability derivative referred to trim condition (NACA convention[6])
C_{M_α}	moment stability derivative based on L (referred to trim)
$C_{M_{\dot\alpha}}$	moment stability derivative based on L (referred to trim)
C_{M_q}	'pitch damping' stability derivative based on L (referred to trim)
C_{M_δ}	control moment stability derivative based on L (referred to trim)
A, β	altitude–density parameters (4.18, 4.21)
C	density at sea level
g_1, g_2, g_3	constants along a ballistic trajectory (4.24a,4.24b,4.24c)
k_1, k_2, k_3	analagous constants used in Ref. [21] (see Table 4.1)
ξ_{tp}	ξ corresponding to a turning point (ballistic) (4.26)
h_{tp}	altitude of turning point (4.27)
ξ_{crit}	critical point (ballistic) (4.32)
h_{crit}	critical altitude (ballistic) (4.32)

Symbols introduced in Chapter 6:

β	sideslip angle
P	static pressure in atmosphere
T	static temperature in atmosphere
χ	tracking angle (heading in zero wind)

Subscripts (global unless otherwise stated:)

r	real part
i	imaginary part
s	corresponds to slow part of solution
f	corresponds to fast part of solution
o	pertains to initial conditions e.g. at start of re-entry
T	pertains to trim (trajectory) conditions

Superscripts (global unless otherwise stated:)

$\cdot$	denotes differentiation with respect to t
$'$	denotes differentiation with respect to ξ
$*$	pertains to *marginal* (neutral) GMS-stability condition
aug	pertains to augmented system

Chapter 1

Introduction to Linear Time Varying Control Systems

From ancient origins, feedback control has grown to become a vital part of modern civilisation. With the advent of flight, feedback control systems have proved invaluable in the capacity of stability augmenters and autopilots, by drastically reducing pilot workload and increasing flight safety. Modern fighter aircraft owe their unprecedented manoevrability to the feedback controller, which effortlessly stabilizes an inherently unstable vehicle (otherwise uncontrollable by a human pilot.)

1.1 LTV and LTI Models

In order to approach the problem of feedback control design in a mathematical manner, simplified descriptions (*mathematical models*) of the exact behavior of physical devices are required. Broadly speaking, there are two types of mathematical model used in the description of feedback control systems: linear systems and non-linear systems. Linear systems can be subclassified into the following types: linear time invariant (LTI), and linear time varying (LTV).

LTI systems are the easiest to analyze because the corresponding differential equations are linear with constant coefficients — hence the exact analytical solutions are available. Consequently there are many exact analytical techniques for designing LTI control systems. This forms the bulk of the 'linear control theory' found in abundance in the literature.

Unfortunately, many physical systems are not well represented by the LTI model. For example, in the analysis of the re-entry of a hypersonic vehicle, the variation in density with altitude renders the system highly time-varying. Similarly, in the transition from hover to cruise, a VTOL aircraft exhibits time-varying aerodynamic properties. Such systems are more accurately represented by LTV models. Alas, the exact analysis of LTV systems is not a trivial matter. With the exception of first order systems, there are no known general solutions to LTV differential equations.

The currently accepted method of analyzing an LTV system is to 'freeze' it over various time intervals and treat it as a constant coefficient system during each interval. The result is a controller incorporating stepwise gain adjustment, but with constant gains during the intervals. Engineers are confident with this approach because of the availability of proven LTI control design techniques.

However, it seems reasonable that an improvement on this 'frozen' method would be to treat the LTV system as a truly time varying system, and not as a series of constant-coefficient systems. The control systems will have continuously varying gains instead of constants which are updated every so often.

1.2 The GMS Solution: A Way Forward

In order to design such a variable gain controller, one must inevitably tackle the associated LTV differential equations. Although no exact solutions are known for the general case, 'Perturbation Methods'[14] are available for developing asymptotic approximate solutions. In particular, the method of Generalized Multiple Scales (GMS) has been formulated by Ramnath[15] for application to slowly varying linear systems. The method involves splitting the dynamics of the system into fast and slow components, and developing asymptotic approximate descriptions of each component. The 'slowly varying' implication[1] requires that the system dynamics are much more rapid than the rate of change of the coefficients of the differential equation. This is true of a large class of physical systems (such as the re-entry and VTOL transition problems cited earlier.)

1.3 Research Objectives

The main objective of this work, therefore, is to devise a procedure for the preliminary design of a variable gain controller for a slowly varying linear system, based on the GMS approximate description of the LTV dynamics.

This objective can be subdivided into the following goals:

1. The prime task of a controller is to provide or enhance stability, so the first step is to develop analytical stability criteria (using GMS theory) which, under appropriate conditions, are applicable to 2nd order LTV systems. This will be an improvement on the 'frozen' approach which adopts LTI techniques to perform stability analysis on LTV systems.
2. Devise analytical techniques for determining time-varying gains for a 2nd order LTV control system, such that the aforementioned stability criteria are satisfied.

[1] the concept of a 'slowly varying' system is defined mathematically in Ref.[15]

3. Demonstrate the use of the new stability and control techniques in a practical engineering application — namely the stability and control analysis of the longitudinal re-entry dynamics of a hypersonic vehicle.
4. Although the development of the theory is general, application to flight control systems is the underlying motivation. Consequently, the important practical implications associated with implementation of time varying flight control systems will be investigated — at least in a qualitative sense.

Only 2nd order systems will be considered in the quantitative stability and control analyses. Extension to higher order systems is complicated by other considerations, hence such systems will not be treated. In practical terms, this is not as drastic a restriction as it may seem, because it is a well established fact that many high order physical systems can be represented by equivalent second order systems, and this equivalent second order system should exhibit favorable characteristics.

1.4 Thesis Structure

Chapter 2 contains an analytical development of GMS-stability criteria for 2nd order LTV systems. The GMS description of the response serves as the starting point. The resulting GMS-stability criteria are in the form of ***differential inequalities***[2]. Chapter 3 contains an analytical development of time-varying gain calculation algorithms obtained by 'solving' the 'stability inequalities' of Chapter 2. The analysis involves the theory of differential inequalities and has the appearance of being mathematically complex — in fact the treatment is quite simple. Chapter 4 contains an example of application of the GMS-stability criteria of Chapter 2. In particular, the longitudinal stability of a hypersonic re-entry glider is investigated. Results are presented for arbitrary trajectories and vehicle configurations. The straight–line ballistic trajectory is chosen as a specific example, and it is shown that the general stability predictions reduce identically to the established results for this type of trajectory. Continuing the re-entry vehicle example, Chapter 5 consists of an LTV control system design demonstration using the results of Chapter 3. In particular, a time-varying control system (with variable gains) is devised to control the Space Shuttle along a typical re-entry trajectory. Numerical simulation is used to implement the control system (since the complicated nature of the trajectory profile precludes closed-form analytical solution.) Chapter 6 contains a short qualitative discussion on the practicalities of implementing a time-varying controller (with real-time gain computation) in a real flight control scenario. The main issue concerns on-board measurement and processing of the necessary flight and atmospheric information required for gain calculation. In particular, the practical implications behind a completely autonomous flight control system are investigated. Chapter 7 presents a summary of the findings and conclusions from the preliminary research into the design of LTV control systems, with suggestions for future development of the various concepts introduced.

[2]A differential inequality is like a differential equation but with the '=' symbol replaced by an inequality operator such as '$<$' or '$\geq$' etc

Chapter 2

Approximate Stability Analysis of LTV Systems

The aim of this chapter is to develop some stability criteria applicable to 2nd order LTV systems. The GMS description of the response will be used as a starting point, and the stability criteria will be based on the condition that the envelope of the GMS response must monotonically decrease with time if the system is 'stable'. It will be assumed that if the GMS response is stable, then the exact response will also be stable. Justification of this assumption would require a rigorous error analysis of the GMS method. This is unnecessarily complicated because experience has shown that the assumption is valid in many practical situations. This is sufficient assurance for present purposes, but it must be stressed that the resulting stability criteria are not exact. Hence the response will be characterized as being 'GMS-stable'.

2.1 The GMS Formulation

2.1.1 General Description

The Generalized Multiple Scales (GMS) method formulated by Ramnath[15] falls under the category of 'Perturbation Theory' which refers to a large collection of iterative techniques for obtaining approximate solutions to differential (and difference) equations involving a small parameter ε (see Refs.[14][15][16][4]). The approximations obtained are ***asymptotic*** to the exact solutions, and this asymptotic relationship is conventionally denoted by the $\sim$ symbol. For example, the notation:

$$y_0 \sim y, \quad \varepsilon \to 0^+ \tag{2.1}$$

means that "the relative error between y_0 (approximation) and y (exact) goes to zero as ε (positive) goes to zero". See Ref.[4] for a formal mathematical definition.

The simplest approach in Perturbation Theory involves direct substitution of a power series in ε into the differential equation. The terms in the series can be generated in succession, thus yielding an asymptotic approximate series solution to the problem. Normally only the first term is required for an accurate description. However, in many cases this approach fails to produce a uniformly valid description. Under such circumstances, the Multiple Scales techniques can be used to eliminate the non-uniformities[4][14][15].

In essence, the classical Multiple Scales method[4] involves *extension* of the independent variable (time, t) to a set of independent time scales $(\tau_0, \tau_1, \ldots)$, thus transforming the original ODE into a set of partial differential equations. Solutions of these PDE is made simple through prudent choice of the time scales, whereupon *restriction* to the t-domain yields the desired uniformly valid approximate solution. Choice of scales is effectively arbitrary, and Ramnath's *generalization* (in the GMS structure[15]) presents a systematic procedure for selecting the scales in terms of 'clock functions' (denoted k). These clock functions are generally complex, non-linear functions of time, representing the 'natural clocks' of the system. In this way, the transient response of a dynamical system can be split into seperate components, each varying on a different time scale. In a qualitative sense, this amounts to employing a number of independent observers each with a different clock (which runs at a different rate from the others.) For linear systems, asymptotic approximations to these component responses can be found in terms of simple transcendental functions, which combine to form the uniformly valid approximate solution.

2.1.2 GMS Solutions of 2nd order LTV Systems

Following Ramnath[15][17][16], a two-time-scale (*fast* and *slow*) approximation to the solution of a homogeneous 2nd order slowly-varying LTV system is constructed as follows.

Consider the 2nd order LTV system given by:

$$\ddot{y} + p_1\dot{y} + p_0 y = 0 \tag{2.2}$$

The clocks are chosen to satisfy the time-varying 'characteristic equation':

$$k^2 + p_1 k + p_0 = 0 \quad \Longrightarrow \quad k = -\frac{p_1}{2} \pm \frac{1}{2}\sqrt{(p_1^2 - 4p_0)} \tag{2.3}$$

For convenience, denote $(p_1^2 - 4p_0)$ by D (for 'discriminant'). Then k can be rewritten as:

$$k = -\frac{p_1}{2} \pm \frac{1}{2}\sqrt{|D|} \quad \text{if } D > 0, \tag{2.4}$$

$$k = -\frac{p_1}{2} \pm \frac{i}{2}\sqrt{|D|} \quad \text{if } D < 0. \tag{2.5}$$

If $D > 0$, the clocks will be real and the response will be non-oscillatory. Similarly, if $D < 0$, the response will be oscillatory. If D changes sign, the point of changeover is called a 'turning point'.

The *fast part* y_f is given by:

$$y_f = \exp\left[\int_0^t k d\phi\right] \tag{2.6}$$

and the *slow part* y_s is given by:

$$y_s = C \exp\left[-\int_0^t I d\phi\right] \tag{2.7}$$

$$\text{where} \quad I = \left[\frac{\dot{k}}{p_1 + 2k}\right] \tag{2.8}$$

Hence the complete approximate solution:

$$y \sim y_s y_f \tag{2.9}$$

takes the following forms, depending on the sign of D (after some algebra):

Non-oscillatory Response ($D > 0$)

$$y \sim y_0 = y_s y_f = |D|^{-1/4} \exp\left[-\int_0^t \frac{p_1}{2} d\phi\right] \{C_1 \exp[R(t)] + C_2 \exp[-R(t)]\} \tag{2.10}$$

where

$$R(t) = \frac{1}{2}\int_0^t \frac{|D| + \dot{p}_1}{|D|^{1/2}} d\phi \tag{2.11}$$

Oscillatory Response ($D < 0$)

$$y \sim y_0 = y_s y_f = |D|^{-1/4} \exp\left[-\int_0^t \frac{p_1}{2} d\phi\right] \{C_1 \cos\Omega(t) + C_2 \sin\Omega(t)\} \qquad (2.12)$$

where

$$\Omega(t) = \frac{1}{2}\int_0^t \frac{|D| - \dot{p}_1}{|D|^{1/2}} d\phi \qquad (2.13)$$

The foregoing analysis contains no mention of the small parameter ε. Infact it has been subtley introduced — via the ***slowly-varying*** assertion which allows the dynamics to be split into fast and slow components (see Refs.[15][17] for details) — and then removed from the final result on restriction to the time domain.

Note also that the approximation y_0 (eqn. 2.9) represents only the first term (leading behavior) of the perturbation series solution. This is accurate enough for most engineering purposes.

2.2 GMS-based Stability Criteria for 2nd order Systems

Let us proceed with the two-time-scale approximate solution of a homogeneous LTV system (equation (2.9))

$$y \sim y_0 = y_s y_f \qquad (2.14)$$

In order that this be GMS-stable, the magnitude of the envelope of the response, $|y_0|$, must be monotonically decreasing with time. Mathematically, this requires that:

$$\frac{d|y_0|}{dt} = |y_s|\frac{d|y_f|}{dt} + |y_f|\frac{d|y_s|}{dt} < 0 \quad \forall t \qquad (2.15)$$

From the previous section, y_f and y_s are given by:

$$y_f = \exp\left[\int_0^t k d\phi\right] \quad \text{and} \quad y_s = \exp\left[-\int_0^t I d\phi\right] \qquad (2.16)$$

In general, k and I (given by equations (2.3 & 2.8)) will be complex and can be written as:

$$\begin{aligned} k &= k_r + ik_i \\ I &= I_r + iI_i \end{aligned} \tag{2.17}$$

where subscripts r and i conventionally denote *real* and *imaginary* parts respectively.

Substitution into equation (2.16) yields the following expressions for $|y_f|$ and $|y_s|$:

$$|y_f| = \exp\left[\int_0^t k_r d\phi\right] \quad \text{and} \quad |y_s| = \exp\left[-\int_0^t I_r d\phi\right] \tag{2.18}$$

Substituting these into equation (2.15) yields the following approximate condition for stability:

$$k_r < I_r \quad \forall t \tag{2.19}$$

This will be referred to as the GMS-stability condition from now on. The preceding analysis can be extended in principle for higher order systems but there would be problems with determining the clocks (roots), k, from a high order 'characteristic equation'.

Returning to the generic 2nd order equation:

$$\ddot{y} + p_1\dot{y} + p_0 y = 0 \tag{2.20}$$

From equation (2.3) the clocks satisfy:

$$k^2 + p_1 k + p_0 = 0 \quad \Longrightarrow \quad k = -\frac{p_1}{2} \pm \frac{1}{2}\sqrt{p_1^2 - 4p_0} = k_r + ik_i \tag{2.21}$$

The expression for I_r given in equation (2.8) reduces to the following (after some algebra):

$$I_r = \frac{\dot{k}_r(p_1 + 2k_r) + 2\dot{k}_i k_i}{(p_1 + 2k_r)^2 + 4k_i^2} \tag{2.22}$$

The GMS-stability condition (2.19) becomes:

$$k_r < \frac{\dot{k}_r(p_1 + 2k_r) + 2\dot{k}_i k_i}{(p_1 + 2k_r)^2 + 4k_i^2} \tag{2.23}$$

Clearly this is a cumbersome and generally unmanageable expression. Complexity is reduced in the simplified case of the *canonical* equation.

2.2.1 2nd Order Canonical Systems

Consider the following 2nd order canonical equation ($p_1 = 0$):

$$\ddot{y} + p_0 y = 0 \tag{2.24}$$

The clocks are given by:

$$k^2 + p_0 = 0 \quad \Longrightarrow \quad k_{1,2} \ = \ \pm\sqrt{-p_0} \tag{2.25}$$

For convenience, consider the two seperate cases:

- Case(a): clocks are imaginary ($p_0 > 0$)
- Case(b): clocks are real ($p_0 < 0$)

Case(a): Canonical Equation with Imaginary Clocks

In this case, the clocks are given by:

$$k_{1,2} \ = \ k_{i_{1,2}} \ = \ \pm i\sqrt{p_0} \tag{2.26}$$

The GMS-stability condition beomes:

$$\frac{\dot{k}_i}{k_i} > 0 \tag{2.27}$$

Applying this to both clocks leads to the following criterion:

$$\dot{p}_o > 0 \qquad \forall t \tag{2.28}$$

The above result can be summarized as follows:

- GMS-stability Criterion (1): The multiple scales solutions of the slowly varying equation $\ddot{y} + p_0 y = 0$, (with $p_0 > 0$) will decay uniformly if $\dot{p}_0 > 0 \quad \forall t$

This system is representative of a mass/spring arrangement. The result concludes that the response is uniformly decaying following an initial disturbance if the 'spring stiffness' increases monotonically with time. Figure 2.1 shows the response of such a system. Clearly the amplitude is decaying, thus supporting the conclusions.

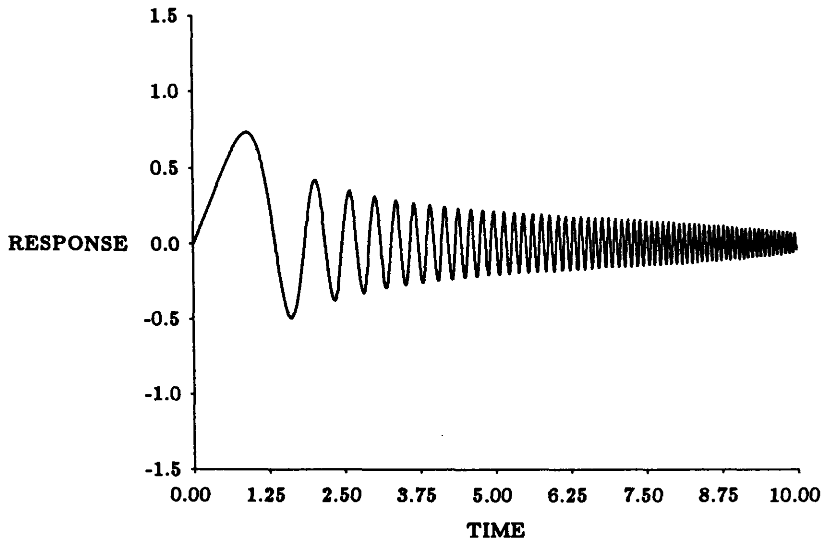


Figure 2.1: Response of the system: $\ddot{y} + 10t^3 y = \delta(t)$

Case(b): Canonical Equation with Real Clocks

In this case the clocks are given by:

$$k_{1,2} = k_{r_{1,2}} = \pm\sqrt{-p_0} \tag{2.29}$$

Modification of the GMS-stability condition (2.23) yields:

$$k_r < \frac{\dot{k}_r}{2k_r} \tag{2.30}$$

In terms of the coefficient p_0, this condition applied to both clocks becomes:

$$\dot{p}_0 < 4(-p_0)^{3/2} < -\dot{p}_0 \tag{2.31}$$

Now, it is difficult to find a function ($p_0 < 0$) which satisfies this differential inequality for all time (t) (although it could be satisfied over a limited time range). Hence it is unlikely that a 2nd order canonical system with real clocks will be stable. This is intuitively obvious when one imagines a mass/spring system with a 'negative spring' and no damping term. Such an arrangement is unlikely to be stable following an initial disturbance.

Let us now consider the question of stability of non-canonical 2nd order equations by converting them to canonical equations and adapting the results obtained above. This is a simpler approach than dealing with the non-canonical equation directly which would involve very cumbersome equations (exemplified by equation (2.23).)

2.2.2 Non-canonical 2nd Order Equations

Returning again to the generic 2nd order equation:

$$\ddot{y} + p_1\dot{y} + p_0 y = 0 \tag{2.32}$$

This can be converted to a canonical equation by introducing $y = gu$ such that:

$$g = C\exp\left[-\frac{1}{2}\int_0^t p_1 d\phi\right] \qquad \text{(chosen to satisfy} \quad \frac{2\dot{g}}{g} + p_1 = 0) \tag{2.33}$$

Substitution into equation (2.32) yields the canonical u–equation:

$$\ddot{u} + w_0 u = 0; \qquad \text{where} \quad w_0 = p_0 - \frac{p_1^2}{4} - \frac{\dot{p}_1}{2} \tag{2.34}$$

and the complete solution is given by:

$$y = gu \qquad \text{where } g \text{ is defined in equation (2.33.)} \tag{2.35}$$

From now on, the quantity w_0 will be referred to as the 'equivalent canonical coefficient'.

The stability of the response will depend on the behavior of both g and u. Consequently, the requirement for stability (uniform decay) becomes:

$$\frac{d|y|}{dt} = |g|\frac{d|u|}{dt} + |u|\frac{d|g|}{dt} < 0 \quad \forall t \tag{2.36}$$

Approximating u by the GMS solution, $u \sim u_s u_f$, the GMS-stability criteria for uniform decay of y are developed as follows:

Case(a)

In this case the clocks corresponding to u are purely imaginary, therefore $|u_f| = 1$ and $|u| \sim |u_s|$, and the rate of change of amplitude is given by:

$$\frac{d|y|}{dt} = |g|\frac{d|u_s|}{dt} + |u_s|\frac{d|g|}{dt} \tag{2.37}$$

Now,

$$|g| = C\exp\left[-\frac{1}{2}\int_0^t p_1 d\phi\right] \implies \frac{d|g|}{dt} = -\frac{p_1}{2}|g|$$

$$\text{and} \qquad |u_s| = C_2\exp\left[-\int_0^t I_r d\phi\right] \implies \frac{d|u_s|}{dt} = -I_r|u_s|$$

Substitution into equation (2.37) yields:

$$\frac{d|y|}{dt} = |g||u_s|\left[-I_r - \frac{p_1}{2}\right] \tag{2.38}$$

The GMS-stability condition requires:

$$\frac{d|y|}{dt} < 0 \Longrightarrow I_r > -\frac{p_1}{2} \tag{2.39}$$

Now for a 2nd order equivalent canonical equation with purely imaginary clocks I_r is given by:

$$I_r = \frac{\dot{k}_i}{2k_i} \quad \text{where} \quad k_i = \pm w_0^{1/2} \tag{2.40}$$

Finally, on substitution into condition (2.39) the GMS-stability criterion emerges as:

$$\frac{\dot{k}_i}{2k_i} > -\frac{p_1}{2}$$

$$\Longrightarrow \quad \frac{\pm\frac{1}{2}{w_0}^{-1/2}\dot{w}_0}{\pm 2w_0^{1/2}} > -\frac{p_1}{2}$$

$$\Longrightarrow \quad \frac{\dot{w}_0}{w_0} > -2p_1 \tag{2.41}$$

Case(b)

In this case the clocks of the equivalent canonical u-equation are real and $\frac{d|u|}{dt}$ is given by:

$$\frac{d|u|}{dt} = |u_s||u_f| \cdot k_r + |u_f| \cdot \frac{d|u_s|}{dt}$$

$$\text{since} \quad |u_f| = \exp\left[\int_0^t k_r d\phi\right] \tag{2.42}$$

Now,

$$\frac{d|u_s|}{dt} = -I_r \cdot |u_s|$$

$$\text{and} \quad \frac{d|g|}{dt} = -\frac{p_1}{2} \cdot g$$

$$\text{and} \quad \frac{d|u|}{dt} = |u_s||u_f|\left[k_r - I_r - \frac{p_1}{2}\right] \tag{2.43}$$

The rate of change of magnitude becomes:

$$\frac{d|y|}{dt} = |g||u_s||u_f|\left[k_r - I_r - \frac{p_1}{2}\right] \tag{2.44}$$

Applying the GMS-stability condition requires:

$$\frac{d|y|}{dt} < 0 \Longrightarrow k_r - I_r < \frac{p_1}{2} \tag{2.45}$$

Now since the roots of the equivalent canonical equation are real, the quantities k_r and I_r are given by:

$$
\begin{aligned}
k_r &= \pm(-w_0)^{1/2} \\
I_r &= \frac{\dot{k}_r}{2k_r}
\end{aligned}
\tag{2.46}
$$

Finally, substitution into the GMS-stability condition yields:

$$
\begin{aligned}
& k_r - \frac{\dot{k}_r}{2k_r} < \frac{p_1}{2} \\
\Longrightarrow \quad & \pm(-w_0)^{1/2} + \frac{\dot{w}_0}{4(-w_0)} < \frac{p_1}{2} \\
\Longrightarrow \quad & -(2p_1 w_0 + \dot{w}_0) > 4(-w_0)^{3/2}
\end{aligned}
\tag{2.47}
$$

These results can be summarized as follows:

- GMS-stability Criteria (2):
  The multiple scales solutions of the slowly varying equation $\ddot{y} + p_1\dot{y} + p_0 y = 0$ will decay uniformly if:

  case (a) $\dfrac{\dot{w}_0}{w_0} > -2p_1$ when $(p_0, w_0) > 0$

  case (b) $-(2p_1 w_0 + \dot{w}_0) > 4(-w_0)^{3/2}$ when $w_0 < 0$ and $p_0 > 0$

  where $w_0 = p_0 - \frac{p_1^2}{4} - \frac{\dot{p}_1}{2}$, the equivalent canonical coefficient.

The 'GMS-stability Criteria' developed above and on page 24 give conditions on the 'shapes' of the coefficient functions required for the GMS solutions to be stable. As mentioned at the begining of the chapter, it is assumed that under the appropriate conditions for the validity of the GMS method (see Refs.[15][17]), then GMS-stability ensures stability of the exact response — although mathematical justification has not been offered. The criteria (they will also be referred to as 'stability inequalities') are awkward to use because they consist of differential inequalities. However, by trial-and-error one can choose gains which make the coefficients satisfy the criteria.

The stability inequalities can be used in a more constructive manner — namely to assess 'degree of stability' and to help in the design of an LTV control system with variable gains. This is the subject of the next chapter.

Note that in the limiting case of 'constant coefficients', the derivative terms vanish and the GMS-stability conditions are exact.

Chapter 3

Determination of Variable Gains for 2nd order LTV Control Systems

The aim of this chapter is to devise some algorithms to determine variable gains for 2nd order LTV control systems. The approach taken is to utilize the information in the stability inequalities by 'solving' the associated differential inequalities.

3.1 Solution of 'Stability Inequalities'

So the issue at hand is to determine 'solutions' to a first order differential inequality of the form:

$$\dot{v}(t) \leq f(v(t), t) \qquad \text{or ' } \leq \text{ ' replaced by ' } \geq \text{ '}$$
$$\cdots \text{subject to the initial condition} \quad v(t_0) = v_0 \tag{3.1}$$

The 'corresponding' differential equation is:

$$\dot{u}(t) = f(u(t), t)$$
$$\cdots \text{subject to the initial condition} \quad u(t_0) = u_0 \tag{3.2}$$

A function v satisfying (3.1) with an '$\leq$' is called a *subfunction* of u. Similarly, v satisfying (3.1) with an '$\geq$' is called a *superfunction* of u.

The problem of 'solving' a differential inequality is tantamount to determining super- or sub- functions (depending on the direction of the inequality) of the 'corresponding' differential equation (3.1).

It is clear that in general, the inequality has an infinite number of solutions and it is therefore not possible to obtain strict bounds on the solutions $v(t)$. For example, one cannot hope to find a solution such as :

'$\cdots$any $v(t)$ such that $v(t) \leq \tan t$ satisfies the inequality$\cdots$'.

However, a widely used result from the theory of differential inequalities states that under certain circumstances, the solution of the corresponding differential equation, $u(t)$ gives an accurate absolute bound (upper or lower — depending on the direction of the inequality) on the solutions of the inequality. Unfortunately this bound gives a *necessary* condition which v must satisfy, but it does not give a *sufficient* one which ensures that v satisfies the inequality. In other words, the solution of the corresponding differential equation alone does not grant solutions to the differential inequality.

However, the solution of the differential equation is very useful because it gives an accurate indication of the form of the solution functions v, which must ultimately be obtained by finding super- or sub- functions. In terms of control system design, this translates into predicting the orders of magnitude of the time-varying gains by solving the differential equations corresponding to the 'stability inequalities'. In order to obtain actual expressions for the gains, the 'stability inequalities' must be solved by determining the super- or sub- functions.

The relationship between super- or sub- functions v and the solution of the corresponding differential equation u is contained in the following theorem[11][22][12][8]:[1]

- Theorem 1
  Consider an IVP (Initial Value Problem) $\dot{u} = f(t, u)$; $\quad u = u_0$ when $t = t_0$.
  Assume f is continuous and $f_2 = \frac{\partial f}{\partial u}$ is bounded in a plane domain D that contains the point (t_0, u_0). Let u be a solution of the IVP in an interval $[t_0, b]$ and let v be a differentiable function such that $v(t_0) \leq u_0$ and for each $t \in [t_0, b]$ the point $(t, v(t))$ belongs to D and $\dot{v} \leq f(t, v)$,
  then $v(t) \leq u(t)$ for each $t \in [t_0, b]$.
  The theorem remains true if '$\leq$' is replaced by '$\geq$' throughout. Proof of this theorem can be found in the references [11][22][12][8].

Stated in words, the theorem says that a subfunction v of u must be less than u itself as long as the continuity and boundedness conditions hold, and the initial value of v is less than the initial value of u (and vice versa for superfunctions.)

It is important to note that the converse is not generally true. Just because some arbitrary function v is less than u, this does not imply that v is a subfunction of u (and similarly for the superfunction case). This is a restatement of the fact that the theorem gives necessary but not sufficient conditions on a function satisfying a differential inequality.

In summary, the solution of the differential equation corresponding to a 'stability inequality' yields a strict upper (or lower) bound on the coefficient functions required for stability. This gives an initial idea of the magnitude and rate of change of gains

[1] This is a simplification of the general result given in Thm 1.2.3 ref [11]

required to stabilize a system. To obtain actual expressions for the coefficient functions, super- or sub- functions (solutions of the 'stability inequalities') must be determined.

Since there are an infinite number of super- or sub- functions to a given differential inequality, the process of finding one is not unique, but it can be done systematically and simply.

The super- or sub- function we seek depends on what we want. For example, if we desire to keep control gains as low as possible but large enough to ensure stability, then we should seek a function which is very close to the solution u of the associated differential equation. Alternatively, the function may be 'chosen' to yield specified response characteristics.

Determination of appropriate superfunctions pertaining to the 2nd order stability inequalities will now be discussed.

3.1.1 Super-(Sub-)functions for the 2nd order Stability Inequalities

The GMS-stability criteria on page 28 consist of two 'stability inequalities' associated with 2nd order non-canonical systems. For convenience, let us consider these separately since the results are quite different for each.

Case(a): Non-canonical 2nd order Systems with $w_0 > 0$

For this class of systems, characterized by large 'spring stiffness' and small 'damping' (such that $w_0 = p_0 - \frac{p_1^2}{4} - \frac{\dot{p}_1}{2} > 0$), the appropriate stability criterion requires that:

$$\dot{w}_0 > -2p_1 w_0$$

The corresponding differential equation is:

$$\dot{u} = -2p_1 u \tag{3.3}$$

Ther solution of this is direct and is given by:

$$u = u_0 \exp\left[-2\int_0^t p_1 d\phi\right] \tag{3.4}$$

Substituting $u = p_0 - \frac{p_1^2}{4} - \frac{\dot{p}_1}{2}$ gives:

$$p_0 - \frac{p_1^2}{4} - \frac{\dot{p}_1}{2} = u_0 \exp\left[-2\int_0^t p_1 d\phi\right] \tag{3.5a}$$

where:

$$u_0 = p_0(0) - \frac{p_1^2(0)}{4} - \frac{\dot{p}_1(0)}{2} \tag{3.5b}$$

Therefore if p_0, p_1 are coefficient functions satisfying this integro-differential equation, then the GMS approximate solution will be of constant amplitude i.e. $\frac{d|y_0|}{dt} = 0$. Such a system can thus be thought of as being on the division between stable (decaying response envelope) and unstable (growing response envelope) — analagous to an LTI system with characteristic roots lying on the imaginary axis only. Consequently, let these ***marginal*** values for the coefficient functions be denoted with an asterisk (*) superscript. From Theorem 1 (and asserting that $w_0(0) = u_0$), these marginal values represent an absolute lower bound on the coefficients of a GMS-stable system. In other words p_0^*, p_1^* serve as a guide to the order of magnitude of the coefficients (and their rates of change) required for stability i.e. we can say that w_0 must be at least of $O(w_0^*)$ for GMS-stability.[2]

The next step is to find a superfunction of u which, by the definition of a 'superfunction', will consist of p_0, p_1 satisfying the stability inequality. These values of p_0, p_1 will be 'better' than the marginal values p_0^*, p_1^* (in that the corresponding GMS response envelope will be monotonically decaying), and the difference between them will serve as some measure (at least qualitatively) of 'relative stability' (the degree to which the system is GMS-stable).

Clearly, a function $v(t)$ satisfying:

$$\dot{v}(t) = -2p_1 v(t) + f(t); \qquad f(t) > 0 \tag{3.6}$$

where $f(t)$ is a strictly positive-valued (but otherwise arbitrary) function, must be a superfunction of u. Equation (3.6) is a linear first order inhomogeneous ODE and consequently has the exact solution given by (consult ref. [4]):

$$v = C\exp\left[-2\int_0^t p_1 d\phi\right] + \exp\left[-2\int^t p_1 d\phi\right] \cdot \left\{\int^t f \cdot \left[2\int^\phi p_1 d\theta\right] d\phi\right\} \tag{3.7}$$

[2] where $w_0^* = u^* = u_0 \exp\left[-2\int_0^t p_1^* d\phi\right]$

where the constant of integration, C, is determined from the initial condition $v(0) = u(0) = w_0^*(0)$.

It is logical that the larger the magnitude and growth of $f(t)$, then the greater the margin by which the stability inequality is satisfied.

Substituting $v = p_0 - \frac{p_1^2}{4} - \frac{\dot{p}_1}{2}$ into equation (3.7) we have the expression which p_0, p_1 must satisfy to yield a GMS-stable system:

$$p_0 - \frac{p_1^2}{4} - \frac{\dot{p}_1}{2} = C\exp\left[-2\int_0^t p_1 d\phi\right] + \exp\left[-2\int^t p_1 d\phi\right]\cdot\left\{\int^t f\cdot\left[2\int^\phi p_1 d\theta\right]d\phi\right\}$$

$$\text{where } v(0) = p_0(0) - \frac{p_1^2(0)}{4} - \frac{\dot{p}_1(0)}{2} \tag{3.8a}$$

Obviously the superfunctions defined in equation (3.6) represent only one class of an infinite number of possible superfunctions. However, these were chosen because they are simple to determine analytically —i.e. from the solution of a linear first order inhomogeneous ODE.)

Use of the concepts introduced above is illustrated as follows.

Simple Control Example

Consider the plant described by the ODE:

$$\ddot{y} - \dot{y} + ty = \delta(t) \qquad \delta(t) = \text{ unit impulse at } \quad t = 0 \tag{3.9}$$

The response (determined numerically)[3] is shown in Figure (3.1). Clearly this plant is unstable.

Let us attempt to stabilize it using only position feedback with time-varying gain $K(t)$. The block diagram of the compensated system is shown in Figure (3.2).

The modified system equation is:

$$\ddot{y} - \dot{y} + (K + t)y = 0 \tag{3.10}$$

Under the standard notation, $p_1 = -1$; $p_0 = K + t$.

[3]... by the 4th order Runge-Kutta Method (see Ref.[7])

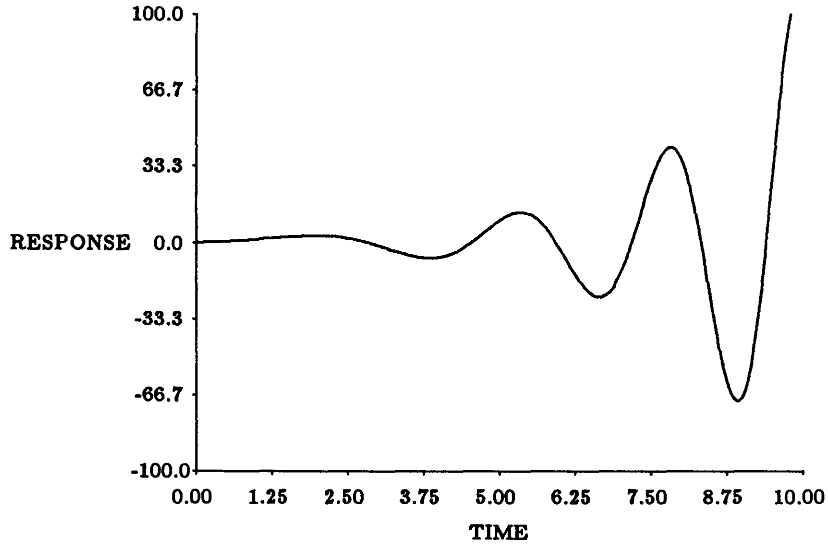


Figure 3.1: Response of the system: $\ddot{y} - \dot{y} + ty = \delta(t)$

The stability inequality becomes:

$$\dot{w}_0 > 2w_0 \quad \text{where} \quad w_0 = p_0 - \frac{1}{4} \tag{3.11}$$

The corresponding differential equation is:

$$\dot{u} = 2u \tag{3.12}$$

which has the solution:

$$u = u_0 e^{2t} \quad \Longrightarrow \quad p_0^* = e^{2t} + \frac{1}{4} \tag{3.13}$$

Hence, by Theorem 1, we know that $p_0 > O(e^{2t} + \frac{1}{4})$ for stability.

Now determine a superfunction given by:

$$\dot{v} = 2v + f(t) \tag{3.14}$$

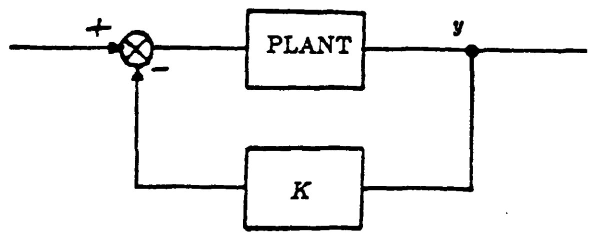


Figure 3.2: Block diagram of compensater in simple control example

which has the solution:

$$v = Ce^{2t} + e^{2t} \cdot \left[\int^{t} f(\phi) e^{-2\phi} d\phi \right] \tag{3.15}$$

Choose $f = e^{-t}$ such that the gain is kept low (to reduce expense, say.) Then v becomes:

$$v = \frac{4}{3} e^{2t} - \frac{1}{3} e^{-t} \quad \text{by setting} \quad v_0 = u_0 = 1 \tag{3.16}$$

And substituting $v = p_0 - \frac{1}{4}$ gives:

$$p_0 = \frac{4}{3} e^{2t} - \frac{1}{3} e^{-t} + \frac{1}{4} \tag{3.17}$$

and so the gain K is given by:

$$K = \frac{4}{3} e^{2t} - \frac{1}{3} e^{-t} + \frac{1}{4} - t \tag{3.18}$$

The response (determined numerically) of this modified system is shown in Figure (3.3). Clearly the system has been stabilized.

Furthermore, Figure (3.4) shows p_0 (eqn. 3.17) versus the marginal value p_0^* (eqn. 3.13) . The margin is positive (assuring stability) but is seen to be very small (almost imperceptible) initially. After a short time ($t \approx 4$) the margin has widened and the response correspondingly decays. Hence the relative stability increases with time—and so does the rate of decay.

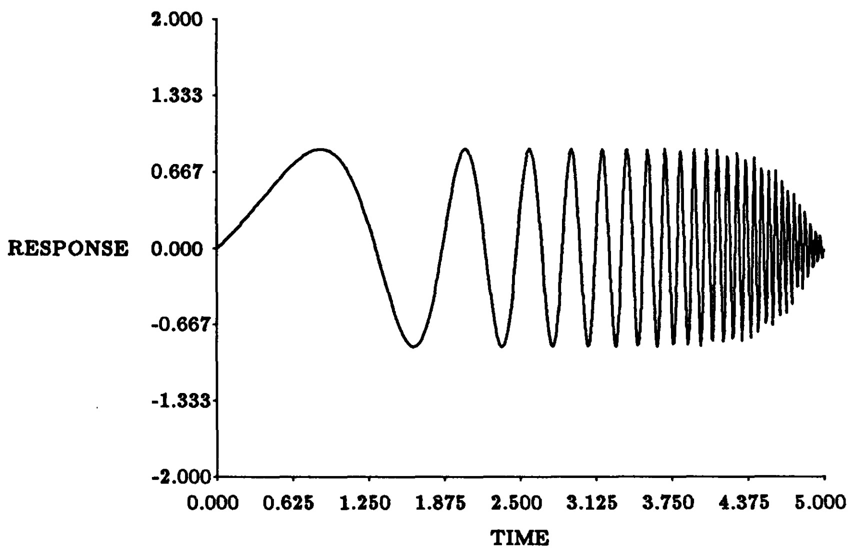


Figure 3.3: Response of the modified system: $\ddot{y} - \dot{y} + (\frac{4}{3}e^{2t} - \frac{1}{3}e^{-t} + \frac{1}{4})y = \delta(t)$

For the sake of comparison, consider another choice of $f(t)$, namely $f(t) = e^{2t}$. The superfunction (3.15) becomes:

$$\begin{aligned} v &= e^{2t}(1+t) \\ \Longrightarrow \quad p_0 &= e^{2t}(1+t) + \frac{1}{4} \\ \Longrightarrow \quad K &= e^{2t}(1+t) + \frac{1}{4} - t \end{aligned} \tag{3.19}$$

The corresponding response is contained in Figure (3.5). Clearly the decay is more rapid than before. The margin between p_0 and p_0^* is shown in Figure (3.6). Here the margin grows more rapidly with time, suggesting that the relative stability is enhanced accordingly. This is supported by the observed response. Obviously this response is more desirable but the cost is high — the gains are very large.

The above example has demonstrated how the stability inequality can be used to design an LTV control system, and how the choice of superfunction can affect the relative stability. The example was made simple by having a constant p_1 term. In general, this may not be the case, and the design problem is more tricky.

Now consider the stability inequality associated with case(b) mentioned on page 31. It will be seen that the analysis is more complicated than for case(a) because the stability inequality is more complex.

Case(b): Non-canonical 2nd order Systems with $w_0 < 0$

This class of 2nd order systems is characterized by large 'damping' and low 'spring stiffness' (such that $w_0 = p_0 - \frac{p_1^2}{4} - \frac{\dot{p}_1}{2} < 0$.) The appropriate stability inequality is given by:

$$\dot{w}_0 < -2p_1 w_0 - 4(-w_0)^{3/2} \quad \text{and} \quad w_0 < 0 \tag{3.20}$$

The corresponding differential equation is:

$$\dot{u} = -2p_1 u - 4(-u)^{3/2}; \qquad u(0) = u_0 \tag{3.21}$$

Substituting $z = -u$ gives:

$$\dot{z} = -2p_1 z + 4z^{3/2} \tag{3.22}$$

This is a Bernoulli equation (see ref. [4]) which has the following exact solution (with $u = -z$):

$$u = -\frac{\exp\left[-2\int^t p_1 d\phi\right]}{\left\{C + 2\int^t \exp\left[-\int^\phi p_1 d\theta\right] d\phi\right\}^2} \qquad C \text{ determined from } \quad u(0) = u_0 \tag{3.23}$$

For any non-trivial p_1, this is exceedingly cumbersome and numerical evaluation is necessary. As before, u can be written in terms of the marginal coefficient function values p_0^*, p_1^*:

$$p_0^* - \frac{p_1^{*2}}{4} - \frac{\dot{p}_1^*}{2} = -\frac{\exp\left[-2\int^t p_1^* d\phi\right]}{\left\{C + 2\int^t \exp\left[-\int^\phi p_1^* d\theta\right] d\phi\right\}^2} \tag{3.24}$$

In order to solve the stability inequality, we need to determine a subfunction of u. Using the same technique as in case(a), introduce a positive function $f(t)$ on the RHS. This yields the following ODE:

$$\dot{v} = -2p_1 v - 4(-v)^{3/2} - f(t); \qquad f(t) > 0 \quad \forall t \tag{3.25}$$

This is a Ricatti equation (see ref.[4]) and no analytical solution exists for general $f(t)$. Hence numerical methods must be employed. This makes the task of designing a control systems more awkward for this class of systems, although the underlying ideas are simple.

Let us now consider common types of compensation and formulate the gain calculation algorithms by solving the 'stability inequalities'.

3.1.2 Gain Algorithms for Simple Control Systems

Only the following types of compensation will be considered:

1. Position feedback
2. Rate feedback
3. Position and Rate feedback

Integral feedback will not be considered because this would raise the order of the system to three, and this is beyond the scope of the present analysis. For convenience, consider only case (a) problems $((w_0^{aug}, p_0^{aug}) > 0)$ since the methods of analysis are identical for both cases but the mathematical details of case (b) problems are more cumbersome. Note that the superscript 'aug' denotes augmented system parameters.

Let us develop the gain calculation algorithm for a general compensation configuration (position and rate feedback) then specialize to the simpler cases.

Gain Algorithms

The block diagram for the general case of position and rate feedback is shown in Figure (3.7). The corresponding differential equation is given by:

$$\ddot{y} + p_1^{aug}\dot{y} + p_0^{aug}y = 0 \tag{3.26}$$

where:

$$p_0^{aug} = p_0 + K_0; \qquad K_0 \text{ is the (time varying) position gain}$$

$$p_1^{aug} = p_1 + K_1; \qquad K_1 \text{ is the (time varying) rate gain}$$

$$p_0, p_1 = \text{unaugmented system coefficients}$$

The equivalent-canonical-coefficient, w_0^{aug} is given by:

$$\begin{aligned} w_0^{aug} &= p_0^{aug} - \frac{p_1^{aug^2}}{4} - \frac{\dot{p}_1^{aug}}{2} \\ &= (p_0 + K_0) - \frac{(p_1 + K_1)^2}{4} - \frac{(\dot{p}_1 + \dot{K}_1)}{2} \end{aligned} \tag{3.27}$$

and it is assumed that K_0, K_1 are chosen such that $w_0^{aug} > 0$ (augmented system is type (a)).

The appropriate 'stability inequality' from the GMS-stability Criteria on page 28 is given by:

$$\dot{w}_0^{aug} > -2p_1^{aug} \cdot w_0^{aug} \qquad \text{(where } (p_0^{aug}, w_0^{aug}) > 0\text{)} \tag{3.28}$$

and so the gains should satisfy:

$$\dot{p}_0 + \dot{K}_0 + \frac{(p_1 + K_1)(\dot{p}_1 + \dot{K}_1)}{2} - \frac{(\ddot{p}_1 + \ddot{K}_1)}{2} > -2(p_1 + K_1)\left[(P_0 + K_0) - \frac{(p_1 + K_1)^2}{4} - \frac{(\dot{p}_1 + \dot{K}_1)}{2}\right] \tag{3.29}$$

This is a non-linear, 2nd order differential inequality with two independent variables K_0, K_1 and there are no unique solutions. There is considerable flexibility in the choice of the gains, and it is suggested that K_1 is chosen first, which would then yield a simple 1st order differential inequality for K_0.

Note that if the gains are chosen to be constant, they should be chosen such that:

$$\dot{p}_0 + \dot{K}_0 + \frac{(p_1 + K_1)\dot{p}_1}{2} - \frac{\ddot{p}_1}{2} > -2(p_1 + K_1)\left[(P_0 + K_0) - \frac{(p_1 + K_1)^2}{4} - \frac{\dot{p}_1}{2}\right] \tag{3.30}$$

for all time.

If the unaugmented system parameters $p_0, p_1, \dot{p}_0, \dot{p}_1, \ddot{p}_1$ are known then K_0, K_1 can be determined easily.

Now the general variable gain case can be simplified as follows:

(1) Position Feedback

In this case, $K_1 = 0$ and the following relations hold:

$$\begin{aligned} p_0^{aug} &= p_0 + K_0 \\ p_1^{aug} &= p_1 \\ w_0^{aug} &= w_0 + K_0 \end{aligned} \tag{3.31}$$

and the stability inequality becomes:

$$(\dot{w}_0^{aug} + \dot{K}_0) > -2p_1 \cdot (w_0 + K_0) \tag{3.32}$$

which can be rearranged into:

$$\dot{K}_0 + 2p_1 K_0 > -(\dot{w}_0 + 2p_1 w_0) \tag{3.33}$$

This is a simple first order differential inequality in K_0. The *marginal* value of the gain is given by:

$$\dot{K}_o^* + 2p_1 K_0^* = -(\dot{w}_0 + 2p_1 w_0) \tag{3.34}$$

which can be solved (analytically or numerically) if p_0, p_1, w_0 are known.

The values of K_0^* give an initial idea as to what the required gain history will be. To actually determine the gain, one needs to find a superfunction of K_0^* and the simplest one results from:

$$\begin{aligned} \dot{K}_0 + 2p_1 K_0 &= -(\dot{w}_0 + 2p_1 w_0) + \text{some positive constant} \\ \text{and} \quad K_0(0) &\geq 0 \quad \text{to satisfy } w_0^{aug}(0) \geq w_0(0) \end{aligned} \tag{3.35}$$

This ODE can easily be solved for K_0. Note that the larger the value of the constant on the right hand side, the greater the degree of 'relative stability'. Qualitative evaluation of relative stability based on this premise will not be developed at this point in time, though the procedure should not be too complicated.

(2) Rate Feedback

Rate feedback alone would not be very common but it serves as a good example.

In this example, $K_0 = 0$ and the following relations hold:

$$\begin{aligned} p_0^{aug} &= p_0 \\ p_1^{aug} &= p_1 + K_1 \\ w_0^{aug} &= w_0 - \left[\frac{2p_1K_1 + K_1^2}{4} + \frac{\dot{K}_1}{2}\right] \end{aligned} \tag{3.36}$$

The stability inequality becomes (from equation (3.29)):

$$\dot{p}_0 + \frac{(p_1 + K_1)(\dot{p}_1 + \dot{K}_1)}{2} - \frac{(\ddot{p}_1 + \ddot{K}_1)}{2} > -2(p_1 + K_1)\left[P_0 - \frac{(p_1 + K_1)^2}{4} - \frac{(\dot{p}_1 + \dot{K}_1)}{2}\right] \tag{3.37}$$

This is generally complicated, and simplifying assumptions should be made to render the condition manageable. For example, K_1 could be made constant and the stability inequality becomes (after some algebra):

$$K_1^3 + 3p_1K_1^2 + (\dot{p}_1 + 2p_1^2 - 4w_0)K_1 < (\dot{w}_0 + 2p_1w_0) \tag{3.38}$$

(3) Position and Rate Feedback

This brings us back to the full-blown generalization with the complicated stability inequality (3.29). The simplest approach would be to set K_1 equal to a constant (limited by the maximum allowable from practical constraints) which effectively reduces the problem to a simple 1st order differential inequality for K_0.

Discussion of Results

In any realistic situation, these stability inequalities would be solved numerically because the coefficients p_0, p_1 would typically vary in a complicated manner (determined by the physics of the problem.)

In any case, the above analysis demonstrates how variable gains can be determined in a systematic manner to satisfy the GMS-stability criteria.

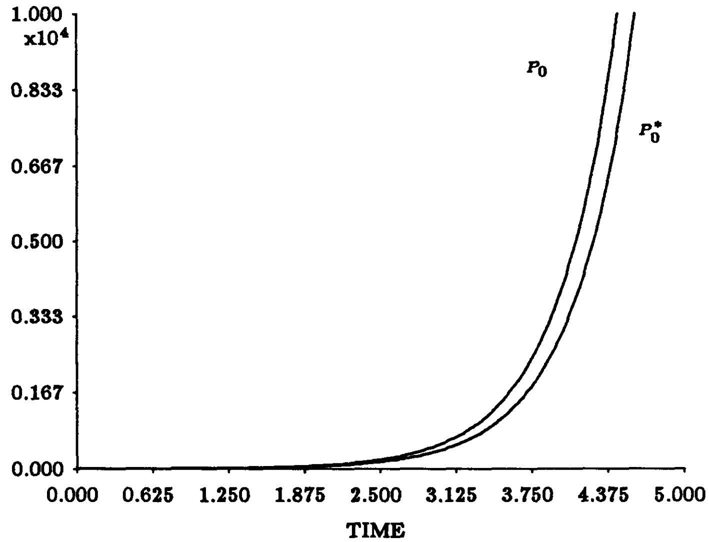


Figure 3.4: Coefficient plots: $p_0 = \frac{4}{3}e^{2t} - \frac{1}{3}e^{-t} + \frac{1}{4}$ and $p_0^* = e^{2t} + \frac{1}{4}$

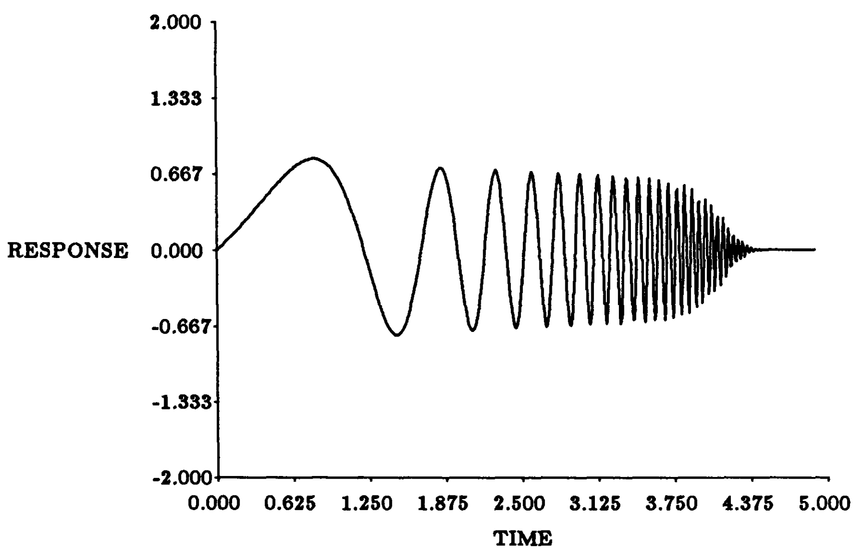


Figure 3.5: Response of the modified system: $\ddot{y} - \dot{y} + (e^{2t}(1+t) + \frac{1}{4})y = \delta(t)$

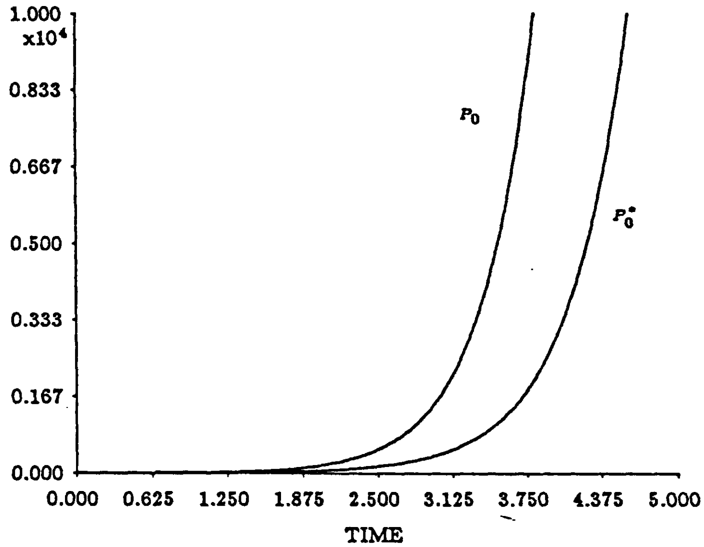


Figure 3.6: Coefficient plots: $p_0 = e^{2t}(1+t) + \frac{1}{4}$ and $p_0^* = e^{2t} + \frac{1}{4}$

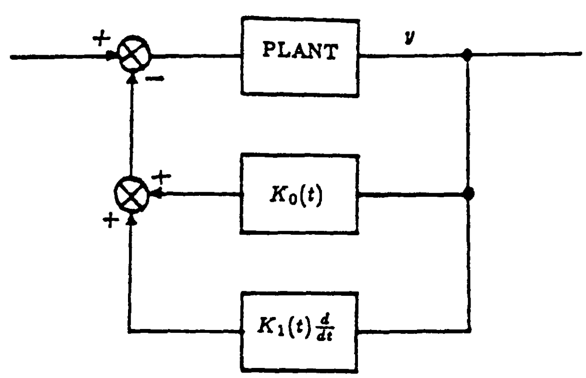


Figure 3.7: Block diagram for Position and Rate feedback

Chapter 4

Longitudinal Stability of a Hypersonic Re-entry Vehicle

There is considerable research activity in the field of hypersonic re-entry gliders owing to the current interest in fully recoverable single-stage-to-orbit launchers. The stability and control aspects require considerable attention particularly if the vehicle is to be unmanned, whereby an autonomous flight control system represents a desirable alternative to remote control.

The angle-of-attack dynamics of a re-entry glider can be modelled by a 2nd order non-linear ordinary differential equation. Due to the non-linearity, there are no known solutions. Nonetheless, many approximate solutions have been proposed, but these pertain only to specialized trajectories. For example, Allen[2] demonstrated that during the ballistic part of the entry when the deceleration is high, the aerodynamic forces dominate over the gravitational forces (which can thus be neglected), and the short-period angle-of-attack oscillations can be described in terms of Bessel functions. Etkin[6] considered the opposite extreme of very shallow flight paths.

Vinh and Laitone[21], on the other hand, developed a unified 2nd order linear differential equation describing the angle-of-attack variations valid for all entry trajectories. However, the coefficients in this equation are time-varying, and consequently no exact solutions are available, although Vinh and Laitone derived approximate solutions for two specific entry trajectories— a straight line ballistic entry for which approximate solutions are in terms of Bessel functions, and a shallow gliding entry for which the dynamics can be described by an inhomogeneous damped Mathieu equation.

Ramnath and Sinha[17] developed asymptotic solutions to Vinh and Laitones' unified equation using the GMS method. These solutions are valid for any trajectory, any vehicle configuration, any atmosphere (Earth's or other) and have inherently simple structures. Furthermore, the approximations have been shown to be accurate (enough for engineering purposes) via a strict error analysis (see Ref.[17].)

The natural continuation from where Ramnath and Sinha left off, is to analyze the stability of the re-entry vehicle, and to address the problem of controlling the angle-of-attack.

Consequently, this chapter deals with the stability of the uncontrolled vehicle (about an arbitrarily prescribed trajectory) using the criteria developed in Chapter 2. Restriction to the simplified ballistic trajectory through an isothermal atmosphere reveals that our results reduce identically to those of Vinh and Laitone.

Having analyzed the stability of the uncontrolled vehicle, Chapter 5 addresses the question of controlling the angle-of-attack. The Space Shuttle is used as a model vehicle, with longitudinal control provided by a simple flap driven with position feedback.

Let us start with the equations of motion of the re-entry vehicle.

4.1 Longitudinal Equations of Motion

Following Vinh and Laitone[21], the motion of a non-rolling, lifting vehicle in a resisting medium and subject to the gravity force of a non- rotating spherical planet (see Figure (4.1)), can be described by the following linear 2nd order differential equation:[1]

$$\ddot{\alpha} + p_1(t)\dot{\alpha} + p_0(t)\alpha = f(t) \tag{4.1}$$

where:

$$p_1(t) = \delta \frac{V}{L}\left[C_{L_\alpha} - \sigma\left(C_{M_{\dot{\alpha}}} + C_{M_q}\right)\right] \tag{4.2a}$$

$$\begin{aligned} p_0(t) &= \delta \frac{V}{L}\frac{\dot{\rho}}{\rho} C_{L_\alpha} - \delta\sigma\left(\frac{V}{L}\right)^2 \left[C_{M_\alpha} + \delta C_{M_q} C_{L_\alpha}\right] \\ &+ \frac{3g\nu}{r}\cos 2(\gamma + \alpha_T) - \delta\frac{g}{L}\cos\gamma C_{D_\alpha} \\ &- \delta^2\left(\frac{V}{L}\right)^2\left[C_{D_T}C_{L_\alpha} + C_{D_\alpha}C_{L_T}\right] \end{aligned} \tag{4.2b}$$

$$\begin{aligned} f(t) &= \delta^2\left(\frac{V}{L}\right)^2\left[C_{D_T}C_{L_T} + \sigma C_{M_q}C_{L_T}\right] + \delta\frac{g}{L}\cos\gamma\left[C_{D_T} - \sigma C_{M_q}\right] \\ &- \frac{3g}{2r}\nu\sin 2(\gamma + \alpha_T) - \left(\frac{3}{r} - \frac{2g}{V^2}\right) g\sin\gamma\cos\gamma \end{aligned}$$

[1] all symbols defined under 'Nomenclature' at the begining of the thesis (see page 10).

$$- \quad \delta \frac{V}{L} \frac{\dot{\rho}}{\rho} C_{L_T} \tag{4.2c}$$

where

$$\delta = \frac{\rho S L}{2m} \tag{4.3}$$

$$\nu = \frac{I_x - I_z}{I_y} \tag{4.4}$$

$$\sigma = \frac{mL^2}{I_y} \tag{4.5}$$

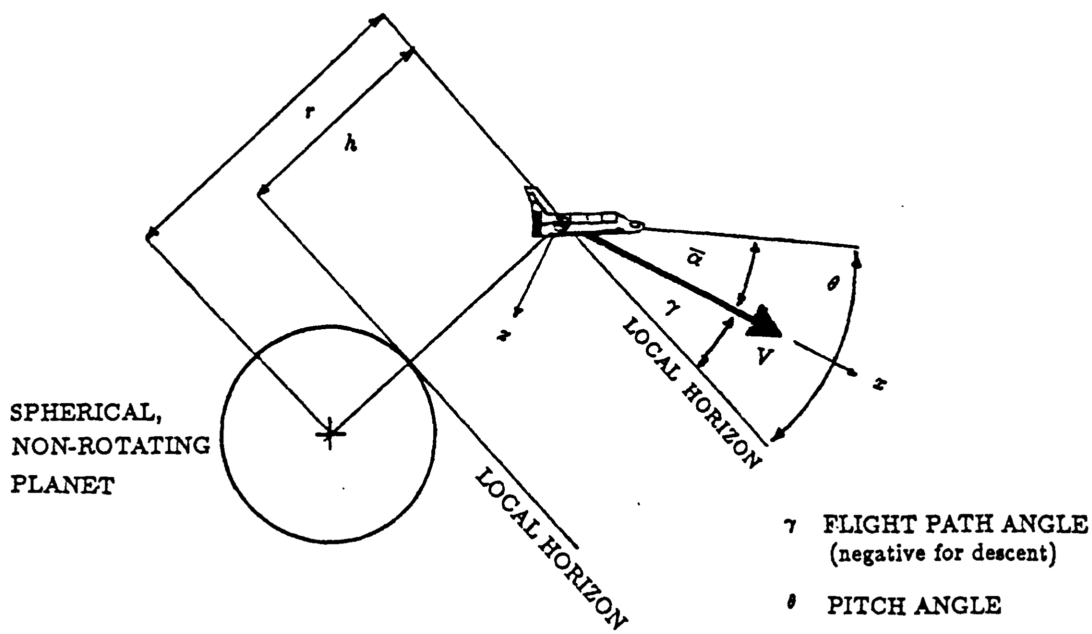


Figure 4.1: Axes notation for re-entry model

Equation (4.1) has been linearized by assuming that the *c.g.* of the vehicle follows a nominal trajectory along which it is in a 'trimmed state' (zero nett moment about the *c.g.*). Hence the dependent variable in equation (4.1) is the perturbation angle-of-attack, α measured relative to the trajectory-specified angle-of-attack, α_T. The prescribed trajectory may be chosen on a basis of 'minimum- thermal-protection weight' or to obey some optimum guidance law etc. The assumption is made that the prescribed trajectory parameters will influence the perturbation dynamics, but the perturbation dynamics—if stable—will not affect the prescribed trajectory[2]. Mathematically, this means that the coefficients p_0, p_1 will be trajectory dependent but unaffected by α (hence the equation is linear.)

[2]...the so called 'limited problem'

The independent variable in equation (4.1) is time (t). It is convenient to transform to 'non-dimensional distance travelled by the *c.g.*' (ξ) as the independent variable.

Using the transformation:

$$\xi = \frac{1}{L}\int_0^t V\,d\phi \qquad L = \text{characteristic length (e.g. chord length)} \tag{4.6}$$

where ξ thus represents the distance travelled by the *c.g.*(scaled by the characteristic length), the differential equation becomes:

$$\frac{d^2\alpha}{d\xi^2} + p_1(\xi)\frac{d\alpha}{d\xi} + p_0(\xi)\alpha = f(\xi) \tag{4.7}$$

where:

$$p_1(\xi) = \frac{V'}{V} + \delta\left[C_{L_\alpha} - \sigma\left(C_{M_{\dot\alpha}} + C_{M_q}\right)\right] \tag{4.8a}$$

$$\begin{aligned} p_0(\xi) &= \delta' C_{L_\alpha} - \delta\sigma\left[C_{M_\alpha} + \delta C_{M_q} C_{L_\alpha}\right] \\ &+ \frac{3g\nu}{r}\left(\frac{L}{V}\right)^2 \cos 2(\gamma + \alpha_T) - \delta g \frac{L}{V^2}\cos\gamma C_{D_\alpha} \\ &- \delta^2\left[C_{D_T} C_{L_\alpha} + C_{D_\alpha} C_{L_T}\right] \end{aligned} \tag{4.8b}$$

$$\begin{aligned} f(\xi) &= \delta^2\left[C_{D_T} C_{L_T} + \sigma C_{M_q} C_{L_T}\right] + \delta\frac{gL}{V^2}\cos\gamma\left[C_{D_T} - \sigma C_{M_q}\right] \\ &- \frac{3g}{2r}\nu\left(\frac{L}{V}\right)^2 \sin 2(\gamma + \alpha_T) - \left(\frac{3}{2r} - \frac{g}{V^2}\right)\left(\frac{L}{V}\right)^2 g\sin 2\gamma \\ &- \delta' C_{L_T} \end{aligned} \tag{4.8c}$$

where '$'$' denotes differentiation with respect to ξ.

Equation (4.7) is the unified differential equation of Vinh and Laitone[21].

This is a 2nd order LTV system with slowly varying coefficients. There is no known exact solution, but the GMS method yields good approximations.

4.2 GMS Solution of the Angle-of-attack Equation

Consider the homogeneous equation obtained by dropping the $f(\xi)$ term from (4.7), whence the differential equation becomes:

$$\frac{d^2\alpha}{d\xi^2} + p_1(\xi)\frac{d\alpha}{d\xi} + p_0(\xi)\alpha = 0 \tag{4.9}$$

and the GMS formulation[15][17] can be used to describe the response. The solution to the non-homogeneous equation can be approximated via the method of 'variation of parameters' [4] using the approximate homogeneous solutions. Ramnath and Sinha[17] have demonstrated this successfully. In any case, typical orders of magnitude of terms in f are of $O(10^{-4})$ which are subdominant compared to the terms on the left hand side of the equation which are typically of $O(1)$, hence the terms in f can effectively be neglected. The only cases when f could be significant is when it is of the same functional form as the homogeneous solutions (thus causing resonance) or when the planetary atmosphere is significantly denser than that of the Earth. Under these circumstances, variation of parameters should be used to include the forced response term.

Using equations (2.10 and 2.12), the angle-of-attack variations can be approximated by the following expressions, depending on whether the dynamics are non-oscillatory or oscillatory:

Non-oscillatory Dynamics:

$$\alpha \sim |D|^{-1/4} \exp\left[-\int_0^{\xi} \frac{p_1}{2} d\phi\right] \{C_1 \exp[R(\xi)] + C_2 \exp[-R(\xi)]\} \tag{4.10}$$

where

$$R(\xi) = \frac{1}{2}\int_0^{\xi} \frac{|D| + p_1'}{|D|^{1/2}} d\phi \tag{4.11}$$

and $D = p_1^2 - 4p_0 > 0$ (for non-oscillatory response)

Oscillatory Dynamics:

$$\alpha \sim |D|^{-1/4} \exp\left[-\int_0^{\xi} \frac{p_1}{2} d\phi\right] \{C_1 \cos\Omega(\xi) + C_2 \sin\Omega(\xi)\} \tag{4.12}$$

where

$$\Omega(\xi) = \frac{1}{2}\int_0^{\xi} \frac{|D| - p_1'}{|D|^{1/2}} d\phi \tag{4.13}$$

when $D < 0$ (for oscillatory response).

These equations (4.10-4.13) essentially summarize the work of Ramnath and Sinha[17].

In both cases, the independent variable can be changed from ξ to time (t), in which case expressions (4.2a-4.2c) for p_0, p_1 should be used (in place of expressions (4.8a-4.8c).) The expressions for α are valid for any vehicle configuration and any trajectory. All that is required is sufficient information to allow the coefficients p_0, p_1 to be evaluated.

It is mathematically feasible that the angle-of-attack dynamics are entirely oscillatory or entirely non-oscillatory. Furthermore, a turning point may occur such that the dynamics change from oscillatory to non-oscillatory (or vice versa) somewhere along the trajectory. This is unlikely to affect the stability and will not be considered further (although there are possible implications in control system design.)

4.3 Stability Analysis for Arbitrary Trajectories

The angle-of-attack dynamics have been modelled by a non-canonical second order homogeneous system (equation (4.9)). For typical re-entry vehicles (re-entry cone, Space Shuttle etc.) the dynamics will be oscillatory ($p_1^2 - 4p_0 < 0$) and the equivalent canonical coefficient (w_0 defined in Chapter 2) will typically be positive.

Under these circumstances, the results of Chapter 2 reveal the appropriate GMS-stability criterion to be:

- p_0, p_1 should satisfy:
  $\dot{w}_0 > -2p_1 w_0$ (where $(p_0, w_0) > 0$) and $w_0 = p_0 - \frac{p_1^2}{4} - \frac{\dot{p}_1}{2}$

where p_0, p_1, w_0 are given by:

$$p_1(t) = \delta \frac{V}{L}\left[C_{L_\alpha} - \sigma\left(C_{M_{\dot{\alpha}}} + C_{M_q}\right)\right]$$

$$
\begin{aligned}
p_0(t) &= \delta\frac{V}{L}\frac{\dot{\rho}}{\rho}C_{L_\alpha} - \delta\sigma\left(\frac{V}{L}\right)^2\left[C_{M_\alpha} + \delta C_{M_q}C_{L_\alpha}\right] \\
&+ \frac{3g\nu}{r}\cos 2(\gamma + \alpha_T) - \delta\frac{g}{L}\cos\gamma C_{D_\alpha} \\
&- \delta^2\left(\frac{V}{L}\right)^2\left[C_{D_T}C_{L_\alpha} + C_{D_\alpha}C_{L_T}\right]
\end{aligned}
$$

$$
w_0(t) = p_0 - \frac{p_1^2}{4} - \frac{\dot{p}_1}{2} \tag{4.14}
$$

Note that the above condition has been stated with time (t) as the independent variable. By substituting '$'$' for '$\cdot$' in the derivative terms, and using the appropriate p_0, p_1, w_0 expressions, the results are valid for ξ as the independent variable.

If the aerodynamic and trajectory parameters are known, the coefficients p_0, p_1, w_0 can be determined, and the above criterion can be used to assess the GMS-stability. Furthermore, by replacing the inequality operators ('$>$' or '$<$') by '$=$', the resulting equations can be solved to yield the 'critical time' or 'critical position on flight path' which correspond to the instance where the GMS response is no longer positively decaying. Mathematically this is tantamount to saying that the critical point occurs when the following equality is satisfied:

$$
\dot{w}_0 = -2p_1 w_0 \qquad \text{where} \quad p_1, w_0 \quad \text{defined above} \tag{4.15}
$$

If the trajectory profile is known in advance then the altitude (and density etc.) can be expressed in terms of time t and the equality will yield the critical altitude for any trajectory and vehicle configuration. The p_0, p_1 variations will generally be complicated and the above equation would require solution via numerical iteration. Nonetheless, the result is versatile.

This concludes the discussion of stability for arbitrary trajectories. The analysis has been approximate owing to the dependancy on GMS-based stability criteria. However, it will now be demonstrated how the results reduce to the exact criteria when applied to a vehicle descending along a straight–line ballistic re-entry trajectory. Vinh and Laitone[21] obtained approximate solutions in terms of Bessel functions. It will be shown that the GMS results reduce identically to these, with the advantage of being composed of simply calculable elementary transcendental functions.

4.4 Example: Straight–line Ballistic Re-entry Trajectory

For a straight–line ballistic re-entry trajectory, the following assumptions can be made (see Ref.[21]):

- Flight path angle $\gamma = \gamma_0 = \text{constant}$, $\gamma_0 \in [-30^o, -80^o]$ (negative for descent)
- Flight path angle is high enough such that the descent is ballistic, the lift can be neglected, and the drag is independent of α. i.e. C_L, $C_{D_\alpha} \approx 0$.
- The atmosphere is isothermal. This is a good model of Earth's stratosphere where the main portion of the re-entry occurs.
- $gL \ll V^2$ and $3g\nu L^2 \ll r$. Both are justified by considering the typical orders of magnitude.

Under these assumptions, the following simplifications can be made on the equations of motion.

4.4.1 Equations of Motion Pertaining to a Ballistic Trajectory

Following Vinh and Laitone[21], the 'X-force' equation in terms of ξ is given by:

$$\frac{V'}{V} = -\delta C_D - \frac{gL}{V^2}\sin\gamma \tag{4.16}$$

Under the above assumptions, this simplifies to:

$$\frac{V'}{V} \approx -\delta C_{D_T} \tag{4.17}$$

Assuming the stratosphere is isothermal, density variations with altitude (h) can be expressed as follows, based on the International Standard Atmosphere (I.S.A.)[18]:

$$\rho \approx Ce^{-\beta h} \qquad \text{where} \quad C \approx 2\text{kg/m}^3$$
$$\beta \approx 1.6 \times 10^{-4} \quad \text{(based on I.S.A.)} \tag{4.18}$$

To proceed further, the altitude h must be expressed in terms of ξ. This is a simple matter since the flight path angle (γ_0) is constant and the following relationship holds:

$$h - h_0 = \int^t V \sin\gamma_0 d\phi \qquad \text{where} \quad h_0 = \text{initial altitude} \tag{4.19}$$

Using expression (4.6) for ξ, h becomes:

$$h = h_0 + L\sin\gamma_0 \cdot \xi \qquad \text{and} \quad \gamma_0 < 0 \quad \text{for descent.} \tag{4.20}$$

Substituting this into the density relationship (4.18) yields:

$$\rho = \rho_0 e^{A\xi} \qquad \text{with constants} \quad \rho_0 = Ce^{-\beta h_0}; \quad A = -\beta L \sin\gamma_0 \tag{4.21}$$

For convenience, introduce the constant $\delta_0 = \frac{\rho_0 SL}{2m}$ such that δ can be written as:

$$\delta = \delta_0 e^{A\xi} \tag{4.22}$$

Based on the typical orders-of-magnitude of parameters associated with a conventional configuration, the constant δ_0 will be of $O(10^{-9})$ in the Earth's stratosphere.

Incorporating the above simplifications in the α-equation (4.7) and neglecting the forcing term yields:

$$\frac{d^2\alpha}{d\xi^2} + g_1\delta_0 e^{A\xi}\frac{d\alpha}{d\xi} + \left(g_2\delta_0 e^{A\xi} + g_3\delta_0^2 e^{2A\xi}\right)\alpha = 0 \tag{4.23}$$

where

$$g_1 = \left[C_{L_\alpha} - C_{D_T} - \sigma\left(C_{M_{\dot\alpha}} + C_{M_q}\right)\right] \tag{4.24a}$$

$$g_2 = -\left[\beta L \sin\gamma_0 C_{L_\alpha} + \sigma C_{M_\alpha}\right] \tag{4.24b}$$

$$g_3 = -C_{L_\alpha}\left[C_{D_T} + \sigma C_{M_q}\right] \tag{4.24c}$$

The coefficient functions p_0, p_1 are therefore given by:

$$\begin{aligned} p_1 &= g_1\delta_0 e^{A\xi} \\ p_0 &= g_2\delta_0 e^{A\xi} + g_3\delta_0^2 e^{2A\xi} \end{aligned} \tag{4.25}$$

$g_1 = \dfrac{4\beta \sin\gamma_0 m}{SC} \cdot k_1$
$g_2 = \dfrac{2m\beta^2 L \sin^2\gamma_0}{SC} \cdot k_2$
$g_3 = \left(\dfrac{2m\beta \sin\gamma_0}{SC}\right)^2 \cdot k_3$

Table 4.1: Conversion from g_1, g_2, g_3 to k_1, k_2, k_3 as used by Vinh and Laitone

The next assumption to be made is that g_1, g_2, g_3 are constant along the trajectory. This is tantamount to saying that the stability derivatives and drag coefficient are constant. This is a reasonable assumption for hypersonic flight at a low angle-of-attack (see Ref [20].)

Now, for the sake of making comparisons with the results of Vinh and Laitone, Table 4.1 contains the conversions from g_1, g_2, g_3 to the constants k_1, k_2, k_3 which appear in their paper[21].

Having developed the simplified α-equation (4.23), the GMS-stability analysis can be carried out. The dynamics will typically be oscillatory, but in the unlikely event that a turning point is encountered, the altitude at which this would occur (denoted h_{tp}) can be determined simply by solving the $D = p_1^2 - 4p_0 = 0$ equation for ξ. This is demonstrated as follows.

Determination of h_{tp} for an Unconventional Vehicle

Solving $D = p_1^2 - 4p_0 = 0$ for ξ, using equations (4.25) yields:

$$\xi_{tp} = -\frac{1}{\beta L \sin\gamma_0} \ln\left[\frac{4g_2}{\delta_0(g_1^2 - 4g_3)}\right] \tag{4.26}$$

Using equation (4.20), h_{tp} is given by:

$$h_{tp} = h_0 + L\sin\gamma_0 \cdot \xi_{tp} = \frac{1}{\beta} \ln\left[\frac{SLC}{8m} \cdot \frac{(g_1^2 - 4g_3)}{g_2}\right] \tag{4.27}$$

Note that h_{tp} only exists if the argument of the logarithm is greater than unity. This could only be true for unconventional configurations, and even then h_{tp} would be very close to ground level.

Let us now turn to the subject of stability for a conventional vehicle.

GMS-stability Analysis for a Conventional Vehicle

Assuming that the α-dynamics will be purely oscillatory and the GMS description is valid, assessment of the stability of the vehicle can be approached using the GMS-stability criteria stated on page 50. Consequently, the α-dynamics will be GMS-stable if the following condition is satisfied:

$$\frac{w_0'}{w_0} > -2p_1$$

where

$$w_0 = p_0 - \frac{p_1^2}{4} - \frac{p_1'}{2} > 0$$

Substituting p_0, p_1 from equations (4.25) yields:

$$w_0 = \delta_0^2 e^{2A\xi}\left[g_3 - \frac{g_1^2}{4}\right] + \delta_0 e^{A\xi}\left[g_2 - \frac{A}{2}g_1\right]$$

Differentiating with respect to ξ yields:

$$w_0' = 2A\delta_0^2 e^{2A\xi}\left[g_3 - \frac{g_1^2}{4}\right] + A\delta_0 e^{A\xi}\left[g_2 - \frac{A}{2}g_1\right]$$

Therefore $\frac{w_0'}{w_0} > -2p_1$ is given by (with some algebra):

$$\frac{w_0'}{w_0} > A\left[1 + \frac{1}{1 + \left(\frac{g_2 - \frac{A}{2}g_1}{g_3 - \frac{g_1^2}{4}}\right)\cdot\frac{1}{\delta_0 e^{A\xi}}}\right] \tag{4.28}$$

Now, from order–of–magnitude arguments g_1, g_2, g_3 are all of $O(1)$ and the term $\delta_0 e^{A\xi}$ ranges from very small (≈ 0) to very large ($O(10^9)$) as the descent proceeds. Infact if the vehicle starts at an altitude of 120km say, then $\delta_0 e^{A\xi}$ will be of $O(10)$ by the time the altitude drops to 50km. With this in mind, it is entirely reasonable to make the following simplification:

$$\left(\frac{g_2 - \frac{A}{2}g_1}{g_3 - \frac{g_1^2}{4}}\right) \cdot \frac{1}{\delta_0 e^{A\xi}} \approx 0$$

and $\frac{w_0'}{w_0}$ is approximated by:

$$\frac{w_0'}{w_0} \approx 2A \tag{4.29}$$

Now, the GMS-stability criterion therefore becomes:

$$\frac{w_0'}{w_0} > -2p_1 \quad \Longrightarrow \quad A > -\delta_0 g_1 e^{A\xi} \tag{4.30}$$

Recalling that $A = -\beta L \sin\gamma_0$ which is positive for descending flight, the GMS-stability condition (4.30) will certainly be satisfied if $g_1 > 0$. From equation (4.24a) this is equivalent to saying that:

$$g_1 = \left[C_{L_\alpha} - C_{D_T} - \sigma\left(C_{M_{\dot\alpha}} + C_{M_q}\right)\right] > 0 \quad \text{for stability} \tag{4.31}$$

The above condition is sufficient to ensure GMS-stability. In the event that the condition is violated (i.e. $g_1 < 0$), the oscillations will be GMS-stable at least down to a critical altitude (h_{crit}) below which an instability may occur. This critical altitude is determined by treating condition (4.30) as an equality and solving for ξ.

Determination of Critical Altitude

From condition (4.30), ξ_{crit} must satisfy:

$$w_0' = -2p_1 w_0\big|_{\xi=\xi_{crit}}$$

$$-\delta_0 g_1 e^{A\xi_{crit}} = A \quad \text{and} \quad g_1 < 0$$

$$\Longrightarrow \quad \xi_{crit} = \frac{1}{A} \ln \left(-\frac{A}{\delta_0 g_1} \right)$$

$$\Longrightarrow \quad h_{crit} = \frac{1}{\beta} \ln \left(\frac{SCg_1}{2\beta m \sin \gamma_0} \right) \qquad \text{(using eqns. (4.20-4.22))} \qquad (4.32)$$

The turning point and stability results pertaining to the ballistic trajectory can be summarized as follows:

- A re-entry vehicle descending along a near–ballistic straight–line trajectory may exhibit a turning point (oscillatory to non-oscillatory) at the altitude h_{tp} below which stability cannot be assured. h_{tp} is given by:
  $h_{tp} = h_0 + L \sin \gamma_0 \cdot \xi_{tp} = \frac{1}{\beta} \ln \left[\frac{SLC}{8m} \cdot \frac{(g_1^2 - 4g_3)}{g_2} \right]$ where g_1, g_2, g_3 are given by:
  $g_1 = \left[C_{L_\alpha} - C_{D_T} - \sigma \left(C_{M_{\dot{\alpha}}} + C_{M_q} \right) \right]$
  $g_2 = -\left[\beta L \sin \gamma_0 C_{L_\alpha} + \sigma C_{M_\alpha} \right]$
  $g_3 = -C_{L_\alpha} \left[C_{D_T} + \sigma C_{M_q} \right]$
- A re-entry vehicle descending along a near–ballistic straight–line trajectory will exhibit GMS-stable (uniform decay of GMS approximate response) angle-of-attack oscillations if the following condition holds:
  $g_1 = \left[C_{L_\alpha} - C_{D_T} - \sigma \left(C_{M_{\dot{\alpha}}} + C_{M_q} \right) \right] > 0$
  Should this be violated, then the oscillations may become unstable. The altitude at which this would occur is given by:
  $h_{crit} = \frac{1}{\beta} \ln \left(\frac{SC}{2\beta m \sin \gamma_0} \cdot g_1 \right)$

For the sake of comparison, Vinh and Laitone obatined the following expressions for h_{tp} and h_{crit}:

- $h_{tp} = \frac{1}{\beta} \ln \left[\frac{(k_1^2 - k_3)}{k_2} \right]$ assuming $\quad k_1 \gg k_2$
- $h_{crit} = \frac{1}{\beta} \ln (2k_1)$

which are identical to our results (on conversion from g_1, g_2, g_3 to k_1, k_2, k_3 using the relationships listed in Table 4.1).

This completes the ballistic trajectory example.

4.5 Comments on the GMS Analysis of the α-dynamics

The advantages of the GMS approach are clear. The GMS-stability criteria are applicable to a general vehicle descending along an arbitrary trajectory. All that is required is sufficient information on the vehicle, the trajectory, and the re-entry environment in order that the coefficient functions p_0, p_1 can be determined. Then it is a simple matter of substituting these into the generalized expressions thus yielding an accurate description of the α-dynamics for the given trajectory. It should be stressed that validity of the GMS approximate solution is a pre-requisite for application of the results.

The parameters corresponding to a practical re-entry trajectory will vary in a complicated manner (compared to the straight–line ballistic trajectory example) and it may not be possible to develop an accurate mathematical model of these variations. However, owing to the inherent simplicity of the GMS solution structure, numerical implementation is easy. This opens the door to the possibility of an on-board computer which calculates control gains (real– time) using the GMS-stability criteria.

On this note, attention will now be focused on the implementation of a time-varying control system for the hypersonic re-entry vehicle.

Chapter 5

Longitudinal Control of a Hypersonic Re-entry Vehicle

This chapter demonstrates how the GMS approach can be used to develop a time–varying control system in a flight control application. The system under consideration is a Space Shuttle type vehicle descending towards Earth along a typical Shuttle re–entry trajectory. Analysis will pertain to an idealised mathematical model of the Space Shuttle, with trajectory parameters extrapolated from actual flight data of the SSV 049 vehicle[17]. During most of the re-entry the Space Shuttle is longitudinally stable. However, there are regimes where it is statically unstable and artificial stability augmentation is required (see Ref. [3]). The aim here is to use the GMS approach to devise a control law which provides stability augmentation to the model vehicle during a portion of the re-entry where it is inherently unstable. Longitudinal control will be provided by a large body flap driven by position (α) feedback. The body flap also serves as the trim device. This is a simplification on the actual Shuttle design which uses the body flap for trim purposes only, and incorporates wing mounted elevons to provide control moments. Since the Shuttle re-entry trajectory is complicated, numerical simulation is used to implement the controller and study the response.

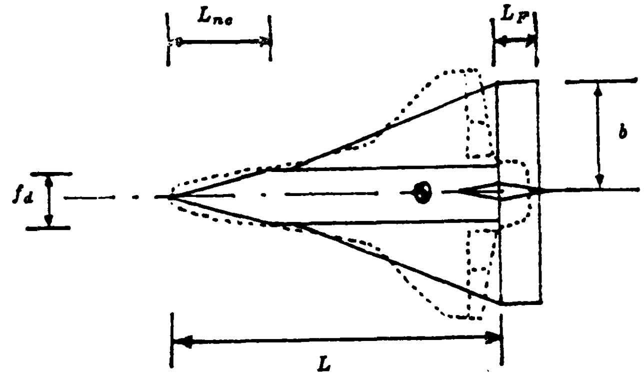
5.1 Longitudinal Stability Augmentation Example

The first step in the analysis is to define the vehicle configuration, trajectory characteristics, and aerodynamic characteristics.

5.1.1 Development of Mathematical Model

Vehicle Characteristics

As mentioned above, the vehicle model is based on the Space Shuttle. The physical characteristics are illustrated in Fig (5.1). The inertial and geometric data is contained in Table (5.1).



(Shuttle Orbiter shown in dotted lines)

Figure 5.1: Vehicle physical characteristics

The body flap has been enlarged (compared to that of the actual shuttle) to account for the omission of wing-mounted elevons.

Trajectory Parameters

The simulated trajectory is based on the actual trajectory of the SSV 049 vehicle which has been designed to minimize the thermal- protection-system-weight (TPS)[10]. The re-entry environment (Earth's stratosphere) is assumed isothermal and the I.S.A. model[18] is used to yield density and static pressure variations. The entry starts at the fringe of the atmosphere (~120km) at around Mach 27, and is assumed to terminate at about 30km (~Mach 3) afterwhich the vehicle initiates its approach procedure (following a short deceleration to subsonic speeds).

Figures (5.2)-(5.8) show the variations of altitude h, trim-angle-of-attack α_T, velocity V, flight path angle γ, Mach Number Ma, dynamic pressure $\frac{1}{2}\rho V^2$, and density ρ with the non-dimensional-distance-travelled variable (ξ). Figure (5.9) shows the non-linear relationship between ξ and real time t. All this data was extracted from Ref.[17]

Aerodynamic Characteristics

In order to simplify the mathematical model, Newtonian Impact Theory [20] has been used to predict the aerodynamic characteristics of the vehicle. Although this is a technically crude approach (due to its inherent simplicity), the results are remarkably close to experimental data derived from wind tunnel testing and actual flight measurements[3].

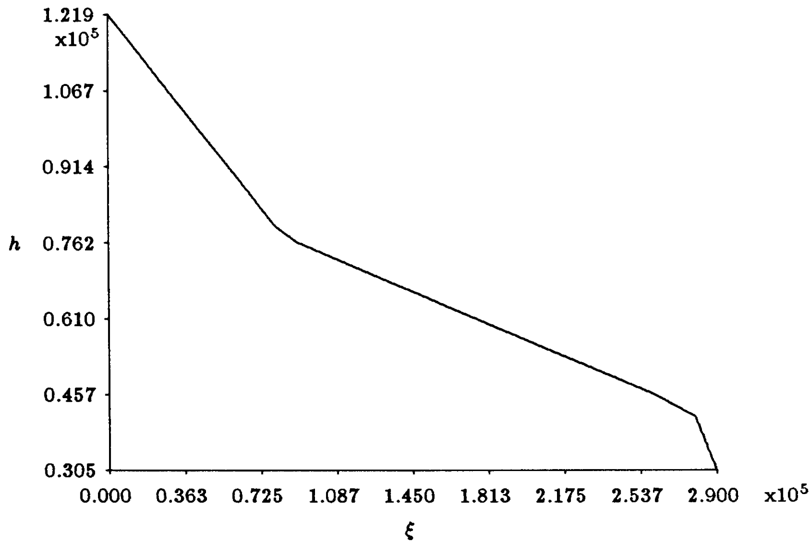


Figure 5.2: SSV 049 Trajectory: Altitude variation (m)

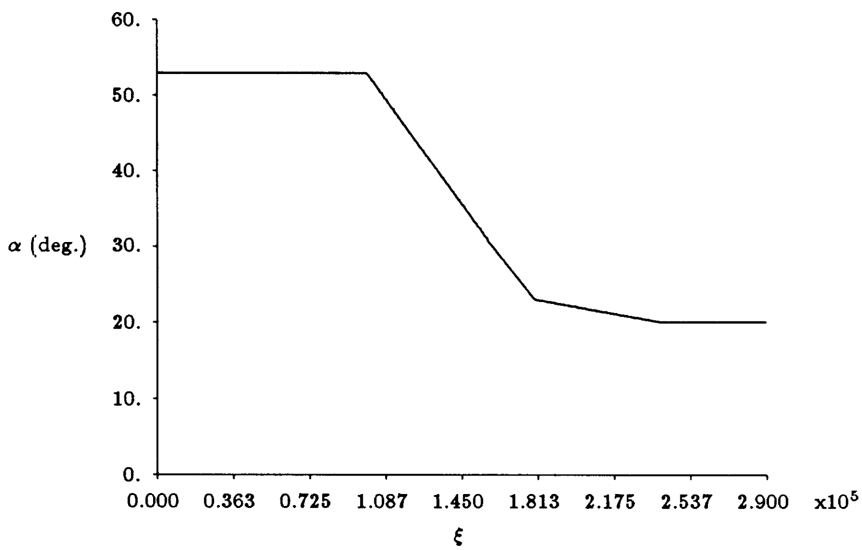


Figure 5.3: SSV 049 Trajectory: α_T variation (deg.)

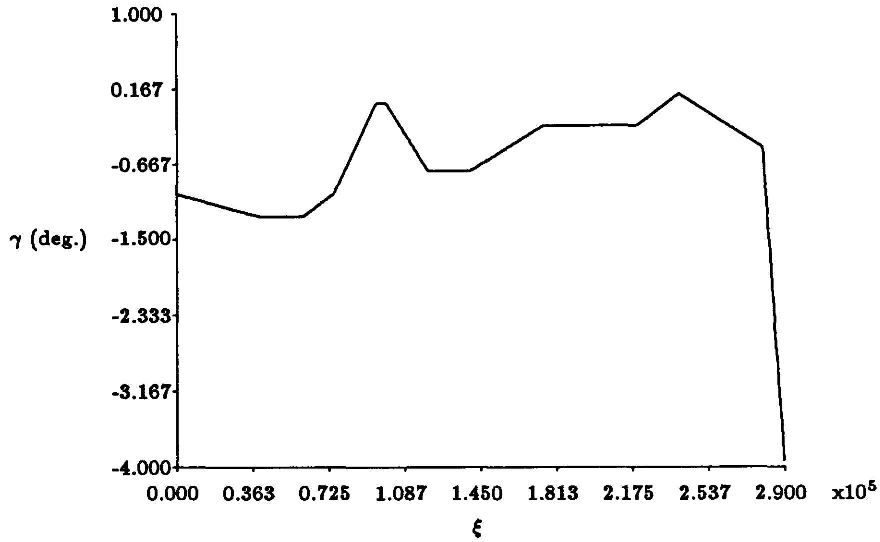


Figure 5.4: SSV 049 Trajectory: γ variation (deg.)

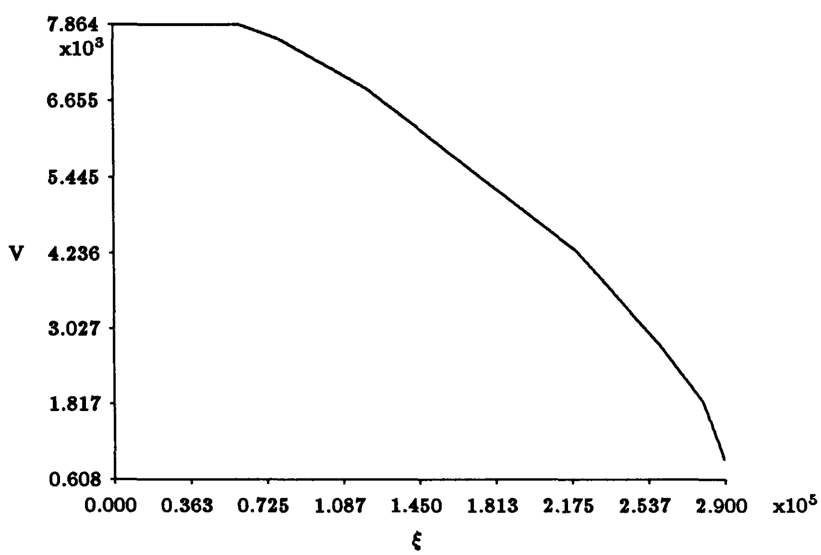


Figure 5.5: SSV 049 Trajectory: Velocity variation (m/s)

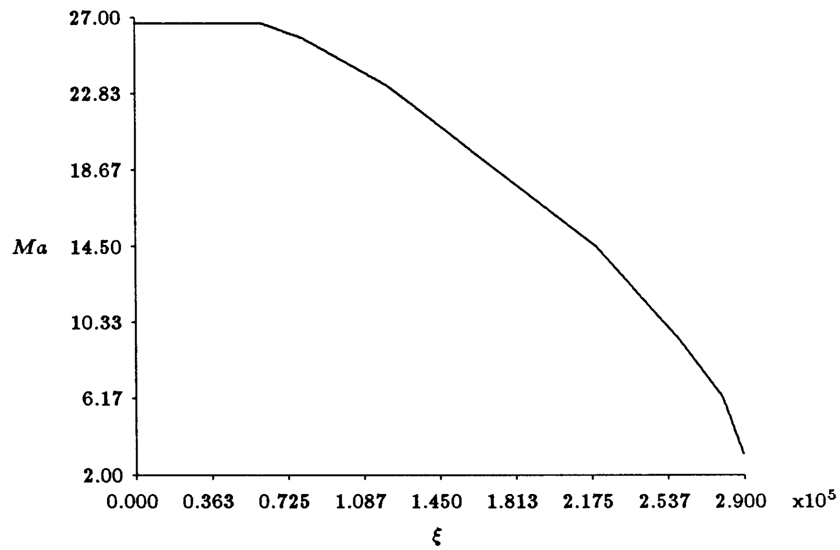


Figure 5.6: SSV 049 Trajectory: Mach Number variation

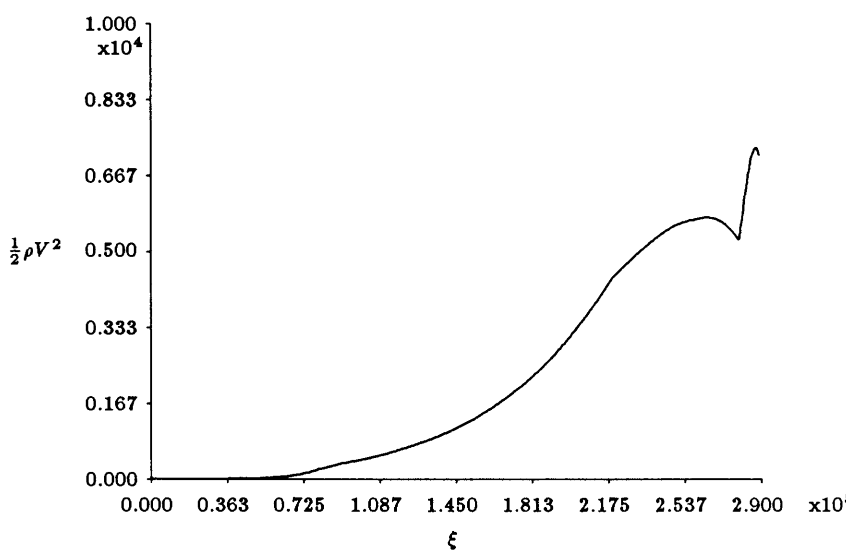


Figure 5.7: SSV 049 Trajectory: Dynamic Pressure variation (Nm^{-2})

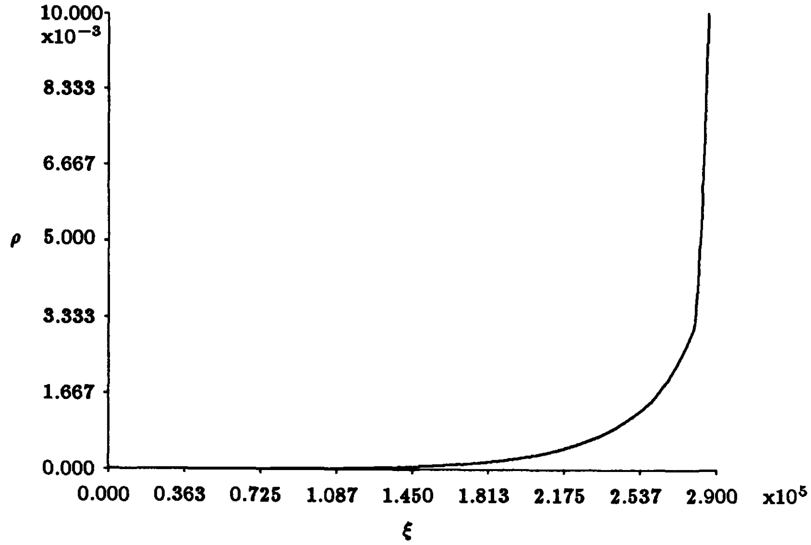


Figure 5.8: SSV 049 Trajectory: Density variation (kgm^{-3})

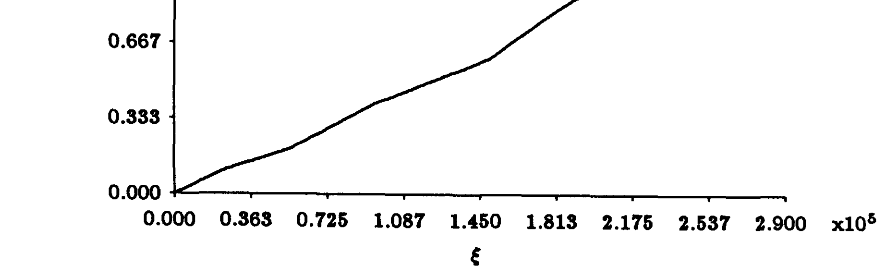


Figure 5.9: SSV 049 Trajectory: Time(sec.) versus ξ

Quantity	Value
Vehicle Length, L_v	33m
Characteristic Length, L	30m
Fuselage Diameter, f_d	5m
Mean Chord, $\bar{c}$	10m
Wing Span, b	20m
Wing Area, S	250m^2
Body Flap Area, S_F	80m^2
Vehicle Mass, m	75000kg
$\mathrm{I_x}$	77500000kgm^2
$\mathrm{I_y}$	67500000kgm^2
$\mathrm{I_z}$	10000000kgm^2
$\nu = \frac{\mathrm{I_x}-\mathrm{I_z}}{\mathrm{I_y}}$	1
$\sigma = \frac{mL^2}{\mathrm{I_y}}$	1
$c.g.$ position, X_{cg}	$0.7 \times L_v$

Table 5.1: Vehicle inertial and geometric data

The 'aerodynamic model' of the vehicle consists of three components (illustrated in Fig (5.10)):

1. Conical nose cone,
2. Flat plate delta wing,
3. Rectangular flat plate body flap.

The physical dimensions of these components are listed in Table (5.2). Newtonian Impact Theory[20] has been used to predict the lift, drag, and pitching moment coefficients for each component. From these, the associated stability derivatives are determined. Under this theory, the aerodynamic forces vary non-linearly with angle-of-attack and flap-deflection-angle. Linearizing about the trim state yields stability derivatives which vary with the trimmed trajectory parameters α_T (angle-of-attack at trim) and δ_{F_T} (flap angle required to trim the vehicle).

The stability derivatives pertaining to the nose cone have been neglected since they are subdominant to those pertaining to the wing and flap. It has also been assumed that the fuselage lies within the 'aerodynamic shadow' of the low-mounted wing such that the body lift and drag can be neglected. This assumption becomes increasingly more valid for higher Mach Numbers.

The centre of pressure of the wing is assumed to lie at $0.66 \times L$ which is forward of the $c.g.$ position ($0.77 \times L$). This is a physically realizable situation and the effect is

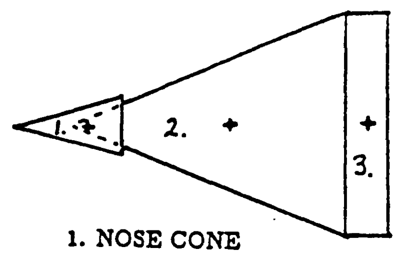


Figure 5.10: 'Aerodynamic model' of vehicle

to make the vehicle statically unstable at the start of the re-entry ($C_{M_\alpha} > 0$). As the descent proceeds, the angle-of-attack and flap-angle-to-trim vary such that eventually the contribution of C_{M_α} from the flap dominates the contribution from the wing and the vehicle becomes statically stable ($C_{M_\alpha} < 0$). It is during this initial regime of static instability that the angle-of- attack must be actively controlled.

Substitution of the trim angles α_T, δ_{F_T} into the Newtonian Impact equations yields the aerodynamic coefficient and stability derivative variations. The $C_{M_{\dot\alpha}}$ and C_{M_q} derivatives are assumed constant and equal to -0.001. This is a valid assumption because the *tail volume* is relatively small (see Ref. [5]) Figures (5.11)-(5.16) contain the variations in flap-angle-to-trim δ_{F_T}, lift coefficient C_L, drag coefficient C_D, and the stability derivatives C_{L_α}, C_{D_α}, and C_{M_α} with the non-dimensional distance variable (ξ). All quantities have been calculated using Newtonian Impact Theory applied to the simple aerodynamic model.

5.1.2 Dynamics of Unaugmented System

Substitution of the relevant trajectory parameters and aerodynamic data into the equations (4.8b) and (4.8a) yields values for the variable coefficients $p_0(\xi), p_1(\xi)$. These are plotted in Figures (5.17) and (5.18). The plots are not smooth because the extrapolated trajectory data is not continuous, and the numerical differentiation process (for calculating V' etc.) is not smooth. Such discontinuities may have important implications in real applications but will be ignored in this simulation. The fact that p_0 is initially negative (during the regime $\xi = 0$ to 1.3×10^5) reflects that C_{M_α} is positive and the vehicle is statically unstable during the first stage of the descent. Consequently the angle-of-attack should be divergent following an initial disturbance. This is sup-

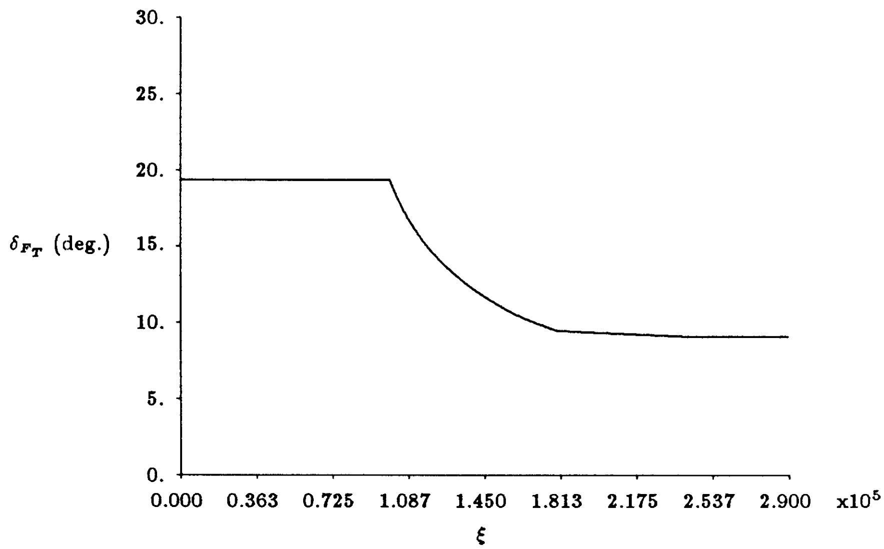


Figure 5.11: δ_{F_T} variation (deg.)

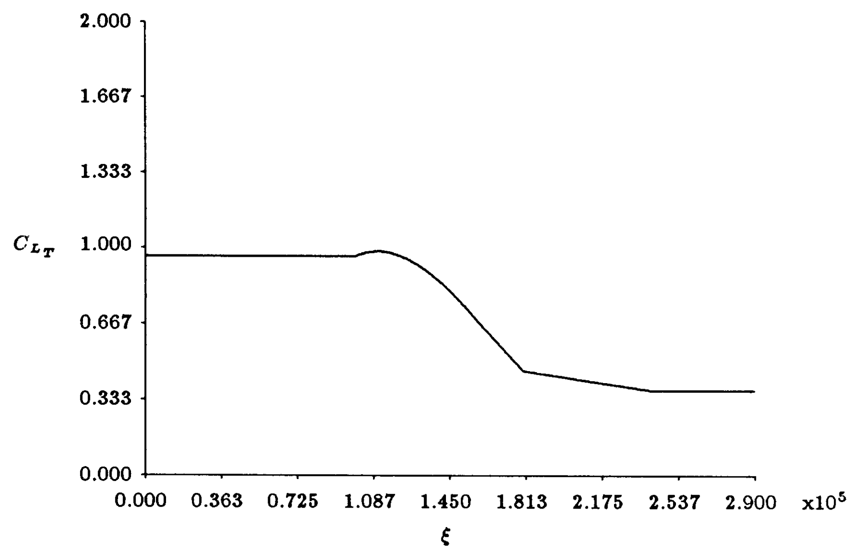


Figure 5.12: C_{L_T} variation

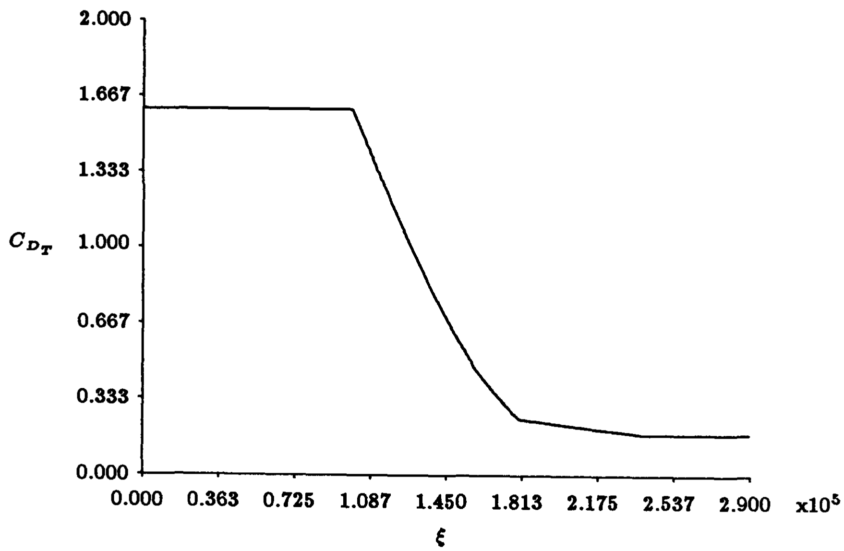


Figure 5.13: C_{D_T} variation

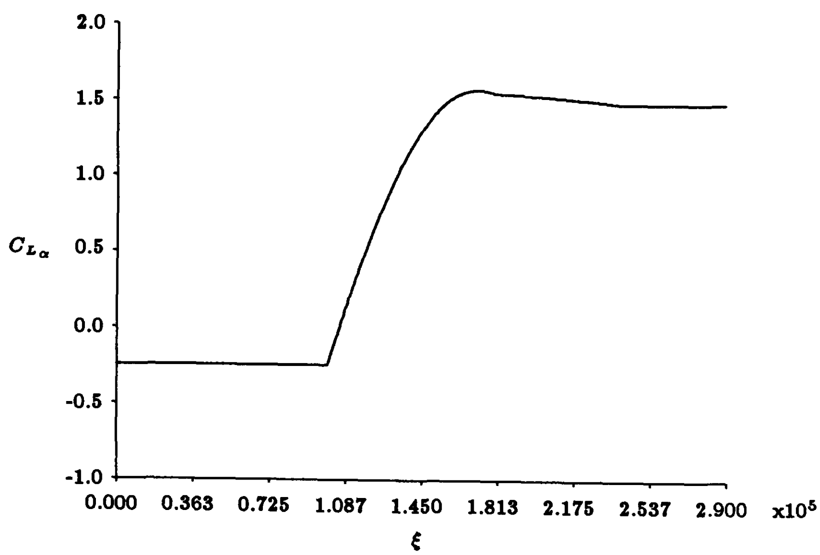


Figure 5.14: C_{L_α} variation

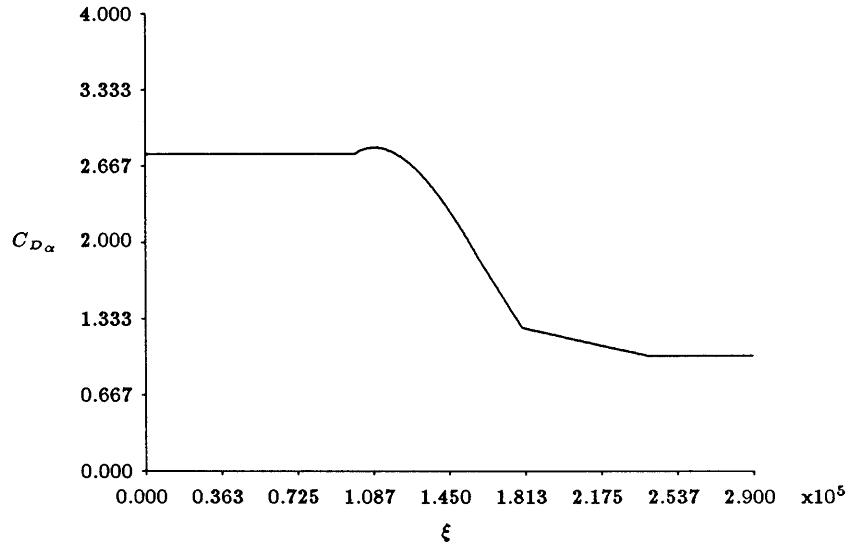


Figure 5.15: C_{D_α} variation

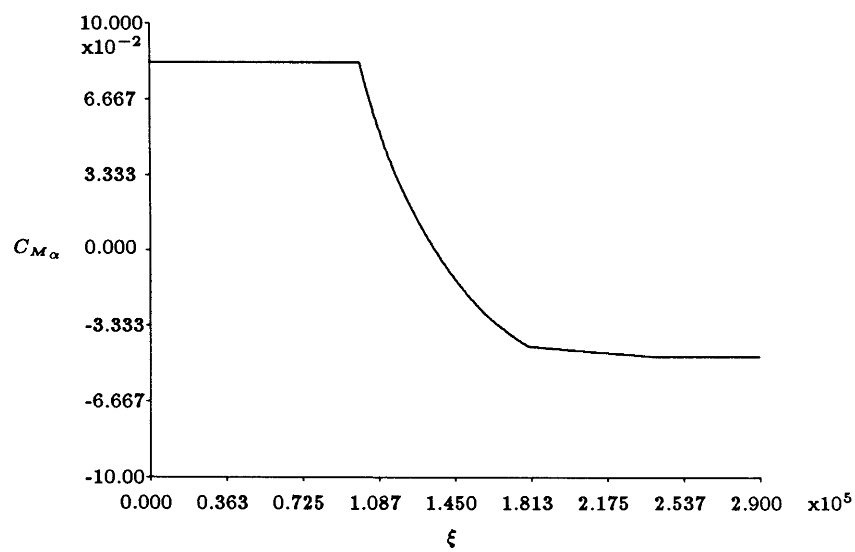


Figure 5.16: C_{M_α} variation

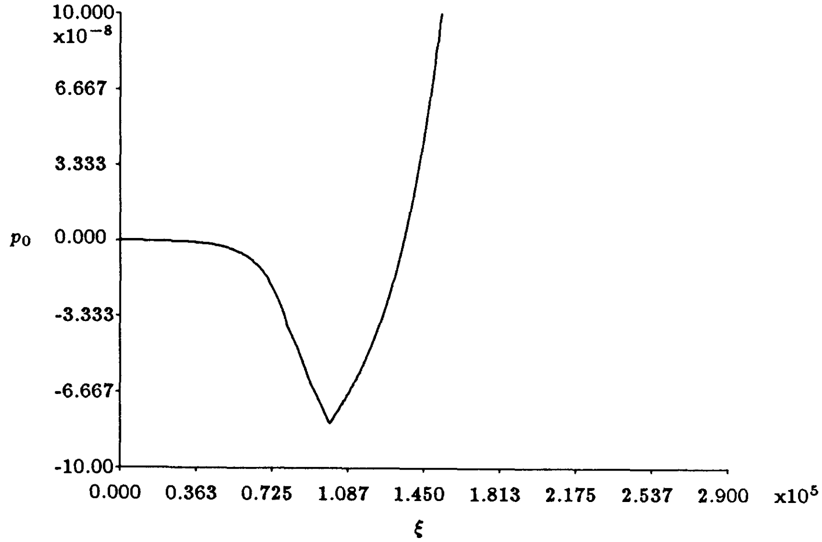


Figure 5.17: $p_0(\xi)$ variation

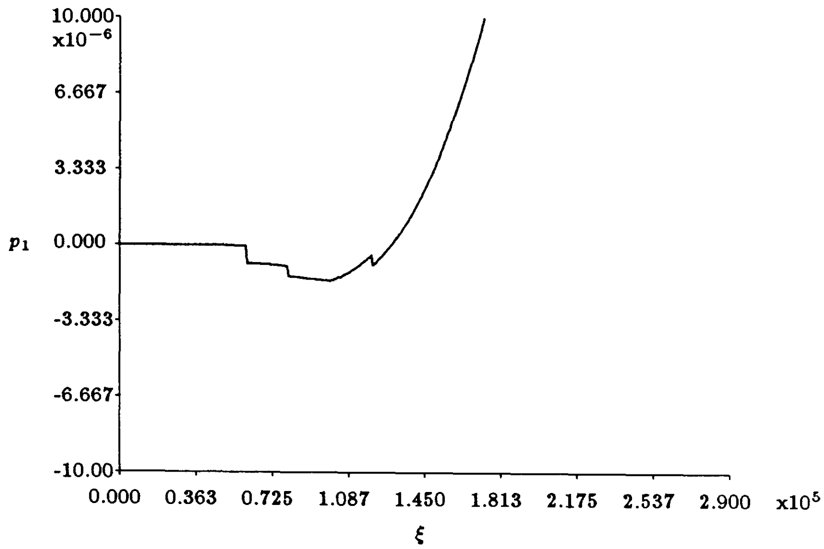


Figure 5.18: $p_1(\xi)$ variation

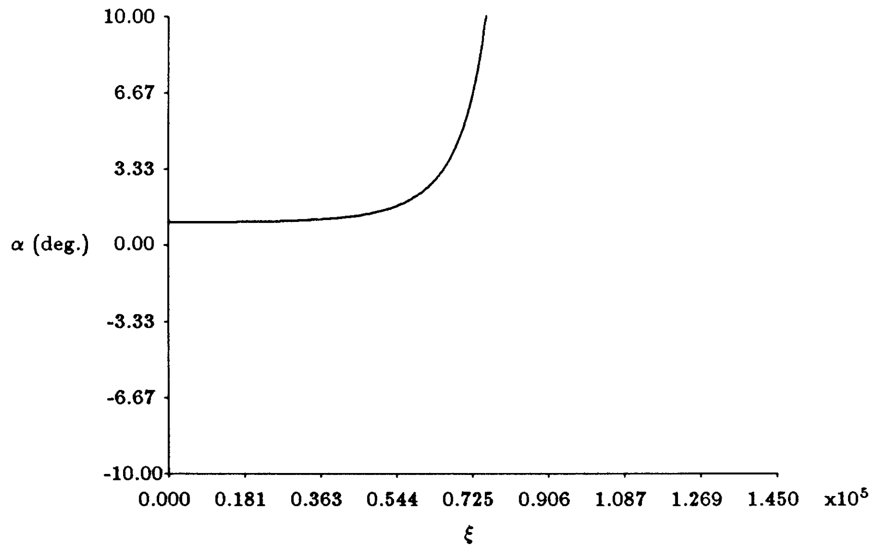


Figure 5.19: Response of unaugmented system

ported by the results of the numerical simulation[1] illustrated in Figure (5.19) which corresponds to an initial offset (disturbance) of 1^o from the trim angle-of-attack. The task at hand is to stabilize this system using angle-of-attack feedback.

5.1.3 Stability Augmentation using the GMS Approach

Consider a simple control system which incorporates α-position feedback to drive the body flap during the unstable phase.

Position Feedback Control

The body flap is the must be deflected from the angle-to-trim δ_{F_T} to provide the longitudinal correcting moments. The block diagram for a simple position feedback system is shown in Figure (5.20). Denoting the flap control deflection angle by δ_{F_c}, then the position feedback control law is given by:

$$\delta_{F_c} = K_\alpha \cdot \alpha \tag{5.1}$$

[1]Numerical solution obtained via the 4th order Runge-Kutta method (see Ref.[7])

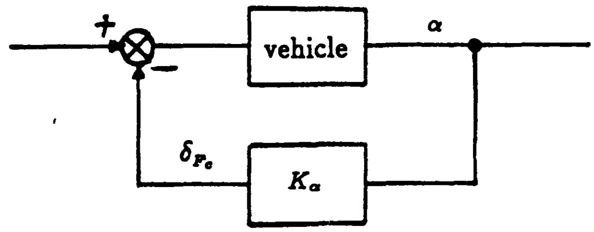


Figure 5.20: Block diagram of simple position feedback control system

where α is the perturbation angle-of-attack and K_α is the control gain (which will be time-varying).

Note that the total flap angle δ_F is given by:

$$\delta_F = \delta_{F_T} + \delta_{F_c} \tag{5.2}$$

and this cannot exceed the theoretical maximum deflection of $\pm 90^o$. The practical limits will be less, typically around $\pm 60^o$.

The effect of α-position feedback is to augment the C_{M_α} derivative. Denoting augmented system properties with the superscript 'aug', the modified C_{M_α} can be written as follows:

$$C_{M_\alpha}^{aug} = C_{M_\alpha} + K_\alpha \cdot C_{M_\delta} \tag{5.3}$$

where C_{M_δ} is the 'pitching–moment–due–to–flap–deflection' stability derivative which can be approximated by the following expression resulting from Newtonian Impact Theory (see Ref.[20]):

$$\begin{aligned} C_{M_\delta} &= \left.\frac{\partial C_M}{\partial \delta_F}\right|_{\delta_F=\delta_{F_T}} \\ &= -(\text{moment arm} \times \left|4\sin(\alpha_T+\delta_{F_T})\cos(\alpha_T+\delta_{F_T})\right| \end{aligned} \tag{5.4}$$

Replacing C_{M_α} by $C_{M_\alpha}^{aug}$ in the equations of motion (4.7), the aerodynamic spring coefficient ($p_0(\xi)$) becomes:

$$p_0^{aug}(\xi) = p_0 - \delta\sigma C_{M_\delta} \cdot K_\alpha \tag{5.5}$$

where p_0 is the unaugmented value.

The control gain K_α will now be determined using the GMS theory.

Determination of Gain $K_\alpha(\xi)$

To reduce computational complexity, assume from the outset that the gain will be chosen such that the 'equivalent-canonical-coefficient' (w_0) of the augmented system will be positive. Under this assumption, the appropriate GMS-stability inequality becomes (from equation (4.3)):

$$\frac{dw_0^{aug}}{d\xi} > -2p_1^{aug} \cdot w_0^{aug} \qquad (\text{where} \quad (p_0^{aug}, w_0^{aug}) > 0) \tag{5.6}$$

and:

$$w_0^{aug} = p_0^{aug} - \frac{p_1^{aug^2}}{4} - \frac{\frac{dp_1^{aug}}{d\xi}}{2} \tag{5.7}$$

Note that in this example $p_1^{aug} = p_1$ since there is no rate feedback.

Using the results of Chapters 2 and 3, the solution of this inequality (5.6) requires determination of a superfunction. A simple solution is w_0^{aug} which satisfies the following differential equation:

$$\frac{dw_0^{aug}}{d\xi} + 2p_1^{aug} w_0^{aug} = \text{arbitrary positive constant} \tag{5.8}$$

$$\text{with} \quad w_0^{aug}(0) \geq w_0(0) \quad (\text{initial condition})$$

Choice of the constant on the right hand side will depend essentially on the upper limit on the gain.

Solving this equation (5.9) yields w_0^{aug} from which the gain K_α is simply determined:

$$w_0^{aug} = p_0^{aug} - \frac{p_1^{aug^2}}{4} - \frac{\frac{dp_1^{aug}}{d\xi}}{2} \qquad \text{(from eqn. (5.7))}$$

$$\Longrightarrow \quad p_0^{aug} = w_0^{aug} + \frac{p_1^{aug^2}}{4} + \frac{\frac{dp_1^{aug}}{d\xi}}{2}$$

$$\Longrightarrow \quad p_0 - \delta\sigma C_{M_\delta} K_\alpha = w_0^{aug} + \frac{p_1^{aug^2}}{4} + \frac{\frac{dp_1^{aug}}{d\xi}}{2} \qquad \text{(from eqn. (5.5))}$$

$$\Longrightarrow \quad K_\alpha = -\frac{w_0^{aug} + \frac{p_1^{aug^2}}{4} + \frac{\frac{dp_1^{aug}}{d\xi}}{2}}{\delta\sigma C_{M_\delta}} \tag{5.9}$$

Equation (5.9) gives the gain required to stabilize the approximate GMS description of the system response. Note that C_{M_δ} will be negative (except for very high flap angles) and the gain K_α will typically be positive.

The 'degree of stability' is enhanced by increasing the value of the constant on the right hand side of the w_0^{aug} differential equation (5.9), but this is limited by the maximum gain available (which reflects the extent of the body flap control authority).

Dynamics of Augmented System

The controller was implemented numerically under the following initial conditions:

$$w_0^{aug}(0) = w_0(0) = 1 \times 10^{-14} \tag{5.10}$$

The right hand side of the w_0^{aug} differential equation (5.9) was set equal to 1×10^{-12} which ensured that the gain K_α would not exceed 100.

The computed gain history ($K_\alpha(\xi)$)and the corresponding control flap deflection angle ($\delta_{F_c}(\xi)$) are plotted in Figures (5.21) and (5.22). Note that when the system becomes inherently stable, the augmentation is shut off (i.e. $K_\alpha = 0$ for $\xi > 1.3 \times 10^5$.)

The numerical response of the augmented system following an initial offset of 1^o from trim is shown in Figure (5.23). Clearly the system has been stabilized.

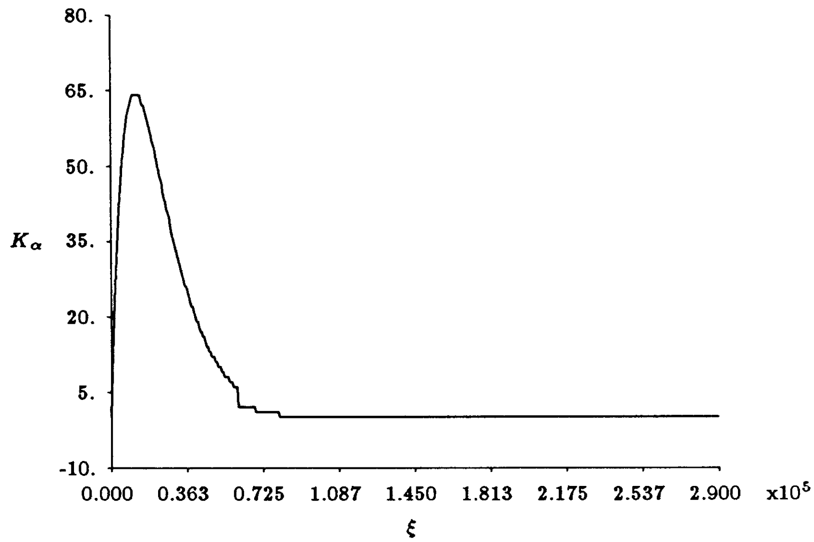


Figure 5.21: Gain variation ($K_\alpha(\xi)$)

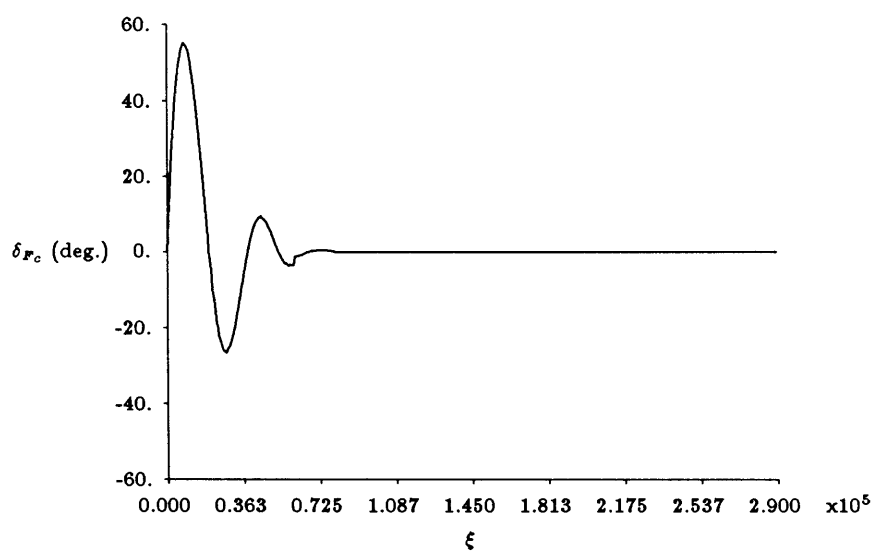


Figure 5.22: Control flap angle variation ($\delta_{F_c}(\xi)$)

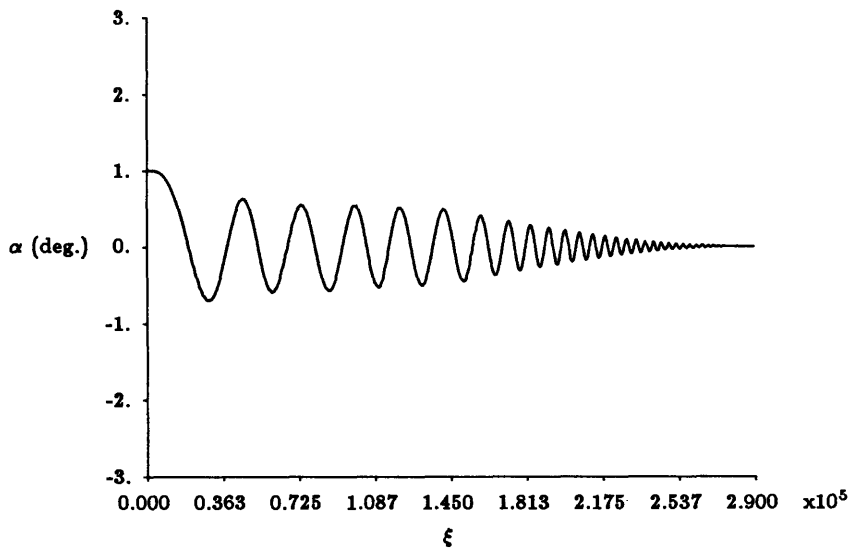


Figure 5.23: Response of augmented system

With reference to Figure (5.9), it can be deduced that the maximum 'instantaneous frequency' of the oscillations is around 0.02 hertz. Hence it is not necessary to add rate feedback which would have the effect of enhancing the the damping and reducing the 'instantaneous frequency' (see equation (4.13) for justification.)

5.2 Comments on the GMS-based Control Design

The above example has successfully demonstrated the application of the GMS time-varying control system design.

However, the practical limitations of this example are very important. In order to calculate the gains, information on the trajectory and the vehicle dynamics must be provided continuously. This was easy using a mathematical model of the vehicle dynamics (and aerodynamics) and predetermined trajectory parameters. In a practical situation involving an on-board computer to calculate the gains, there will be significant difficulties associated with accurate real-time determination of the trajectory and dynamics information. The next chapter discusses the practical considerations associated with actual implementation of time-varying flight control systems. The discussion is qualitative and general, pertaining to any flight control scenarios involving real-time flight data acqusition.

Quantity	**Value**
Characteristic Length, L	30m
Nose Cone Length, L_{nc}	9m
Nose Cone half-angle, δ_{nc}	16^o
Nose Cone Base Diameter, f_d	5m
Delta Wing Area, S	250m^2
Wing Span, b	20m
Mean Chord, $\bar{c}$	12.5m
Chord at centreline, c_{cl}	25m
Body Flap Length, L_F	4m
Body Flap Area, S_F	80m^2
Centres of Pressure:	
Nose Cone	$XCP_{nc} = 0.67 \times L_{nc}$ (from nose)
Delta Wing	$XCP_w = 0.66 \times L$ (from nose)
Body Flap	$XCP_F = 0.5 \times L_F + L$ (from nose)

Table 5.2: 'Aerodynamic Model' geometric data

Chapter 6

Practicalities Associated with Variable Gain Flight Control Systems

The simplified example of the previous chapter illustrates how a time-varying control system may be devised using the GMS theory. The results from numerical simulation are promising.

Although the development of the theory has been general, application to flight control systems has been the underlying motivation. In particular, the concept of a fully autonomous controller is generating considerable research interest.

However, success of such systems in practice would depend on how accurately and quickly the necessary information (for gain calculation) could be acquired and processed by an on-board computer.

The aim of this chapter is two-fold:

1. Identify typical information requirements for input to the gain algorithms
2. Summarize some practical methods for on-line determination of the quantities in (1).

The discussion will be qualitative and general, with the conclusions valid on a class of flight control systems requiring real-time measurement of flight and atmospheric data for use in the control algorithms — the hypersonic re-entry control problem being a typical example.

6..1 Information Required for Gain Calculations

Guided by the hypersonic glider control example of the previous chapter, the information which should be made available to the gain calculation algorithms for a general flight control system falls into five basic categories (with some overlapping):

- Air Data Information

- Kinematic Information
- Aerodynamic Forces Information
- Vehicle Configuration Information
- Vehicle Position and Attitude Information (relative to external reference frame)

Column 1 in Tables (6.1) to (6.5) contain typical variables in each category. These lists are not exhaustive (each flight control system will have different information requirements) but are rather intended to be representative of the general information requirements.

Note that the vehicle position and attitude information is required primarily for guidance purposes, but will invariably have implications in control due to guidance/control interactions. The process of linking the control system with a real-time trajectory (via a *command interface*) is an important aspect of autonomous system development, but is beyond the scope of the present analysis. Refer to Alexander[1] for discussion of this.

Having identified the type of information required, the next step is to summarize the practical methods of obtaining this information in real-time.

6..2 Information Acquisition Methods

Due to the lags associated with finite information retrieval and processing times, it is practically impossible to provide continuous real-time data streams to the gain calculation algorithms. Rather, it is more likely that the system would update the gains every few seconds, with the 'update frequency' depending on the rate at which the trajectory and environmental parameters are changing.

There are essentially three techniques available for practical on-line determination of the information discussed above:

1. Use of pre-determined values.
2. Direct measurement via on-board sensors.
3. Indirect measurement/calculation using 'look–up' databases compiled from analytical predictions and experimental/flight test results.

Furthermore, some of the measurements could be validated by cross-checking with tracking systems etc., but this would detract from the autonomous nature of the system.

All these techniques are subject to errors which must be tolerated by robust gain calculation algorithms such that the controller provides good system response despite erroneous information inputs. This aspect and the problem of lag in the information processing cycle will have implications in the control system design. Recognizing that these factors are important will suffice at this stage — it is beyond the scope of this discussion to approach these problems in a quantitative or analytical manner.

The process of mapping the information measurements to the information requirements, and ultimately to the gain calculations is illustrated in Figure (6.1.) Some practical methods of measurement of various relevant quantities are also listed in Column 2 of Tables (6.1) to (6.5). This information is derived from:

a) current techniques used in aircraft instrumentation and automatic control[19] (for measurement of quantities used in conventional flight control systems e.g. altitude)

b) methods employed in actual Space Shuttle Flight Data measurement [3](for measurements of quantities not common to conventional flight control systems but which may be required by autonomous (self-contained) control systems e.g. free stream air density.)

So it appears that the relevant data can be determined (at least approximately). The quality of the information acquisition process will depend on the degree of sophistication and ultimately there will be a compromise between cost and quality.

Having measured or calculated the necessary information, practical implementation of a time-varying-gain calculation algorithm (such as the GMS-based algorithms of the previous chapters) would typically require on-line numerical quadrature of the various measured quantities. This would require finite computation times resulting in additional system lag. Furthermore, errors in measurement would propagate and accumulate (with time) under the quadrature process. These factors must be taken into account in the design of the system — the main consequence being elevated hardware costs. There will be additional hardware costs associated with practical implementation of variable gain controls which would require programmable servos driven by the output from the gain algorithms.

In conclusion it seems that an on-line time-varying flight control system is feasible in practice. There will be logistical problems associated with rapid and accurate measurement of the relevant flight and atmospheric data required to determine the gains. The most likely result will be high development and hardware costs. With this in mind, a completely autonomous systems is perhaps far-sighted. Considerable reduction in system complexity and hardware requirements would result if the some of the on-board workload were delegated to external processors.

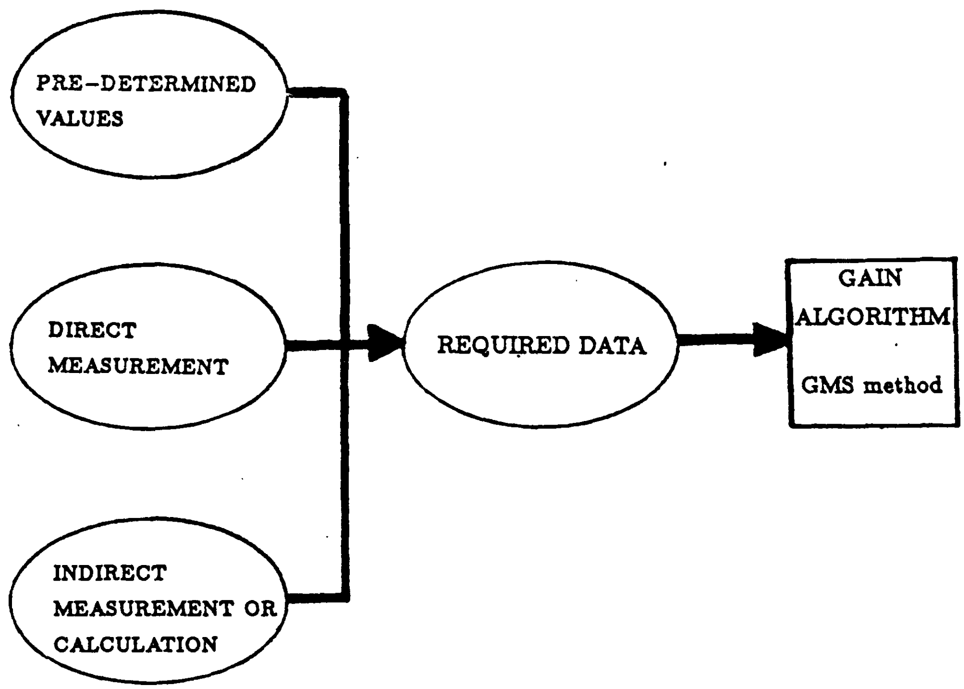


Figure 6.1: Real-time gain calculation process

Quantity	Methods of Measurement	Comments
Incidence angles (α, β)	Direct measurement via wind vanes or pressure tappings	See ref.[3] for discussion on Space shuttle air data system
Atmospheric properties (ρ, P, T)	Direct measurement for low speed flight or 'look-up' tables for	Based on Shuttle experience[3] direct measurement is not accurate for hypersonic flight. Best to use look-up tables such as I.S.A. models corrected for local metereological conditions
Air Speed (V)	Direct measurement via pitot tube etc.	Can be checked using inertial platform and wind data
Mach Number (Ma)	Direct measurement via Mach Meter etc.	

Table 6.1: Air Data Information and measurement techniques

Quantity	Methods of Measurement
Direction angles: (γ flight path angle) (χ tracking angle)	Direct measurement via gyroscopes etc.
Altitude (h)	Direct measurement via altimeter
Climb/descent rate ($\dot{h}$)	Direct measurement via barometric means or inertial platform

Table 6.2: Kinematic Information and measurement techniques

Quantity	Methods of Measurement	Comments
Aerodynamic force and moment coefficients	Calculations based on accelerations (and angular rates) measured by inertial platform or by 'look-up' data using (Ma, α,etc.) inputs. Data derived from experimentation, analysis, and accumulated flight data.	Used for Space Shuttle post flight analysis[3]
Stability Derivatives	Calculate from 'look-up' data derived from theory, experiment, and accumulated flight data.	Cannot be measured by a practical on-board system
Thrust coefficient	Calculation from measurements of base pressure (for rockets and turbines etc) or by 'look-up' data from engine performance charts etc.	

Table 6.3: Aerodynamic Forces Information and measurement techniques

Quantity	Methods of Measurement
Mass, moments and products of inertia	Pre-determined. Only likely to change in predicted manner (e.g. fuel burn, variable payloads)
Control Deflections	Direct measurement via control servo transducers.

Table 6.4: Vehicle Configuration Information and measurement techniques

Quantity	Methods of Measurement
Attitude (via Euler Angles etc.)	Directly measurable from gyroscopes and/or inertial platform
Position (w.r.t. external frame)	Calculated by integration of accelerations via inertial platform or external communication

Table 6.5: Position and Attitude Information and measurement techniques

Chapter 7

Conclusions and Recommendations

7.1 Conclusions

7.1.1 General Conclusions

The objective of this work was to devise a control system design methodology for 2nd-order LTV systems using the GMS approximate description of the dynamics.

To this end, GMS-stability criteria were developed by considering the requirement for uniform decay of the envelope of the GMS response. The resulting criteria are in the from of differential inequalities involving the coefficient functions and their first and second derivatives. Various simple examples have successfully demonstrated the application of these criteria although rigorous proof of validity has not been presented. It has also been shown how the 'solution' of these 'stability inequalities' (through 'super-(sub-)functions) can be used to determine the variable gains required to stabilize the system. The notion of relative stability is encompassed in the methodology — although systematic quantitative evaluation of this requires further development.

The GMS-stability criteria and the associated gain calculation algorithms have been combined in a modified 'trial-and-error' control system design procedure for LTV systems. Although basic and unrefined, this lays the foundation for the development of a more elaborate control design procedure.

7.1.2 Limitations

The limitations of the approach must be recognized.

To start with, the analysis was based entirely on the assumption that the GMS approximation is valid (which is true only for a class of slowly varying linear systems[15]), and that the GMS-stability criteria are applicable to the exact response. There is as yet no rigorous mathematical evidence to support the latter, and so, at best, the results should only be used for preliminary design — at least until the stability predictions have been mathematically validated (by comparison with exact boundedness theorems, for example.)

Furthermore, the entire control design process was based on the homogeneous (unforced) system. This represents only the transient behavior, whereas a complete control design methodology would necessarily include investigation of forced responses (to desirable and undesirable inputs.)

Finally, the practicalities associated with implementing controllers with continuously varying gains are considerable — especially in flight control applications. In reality, the parameters required for gain calculation would have to be measured or determined in real-time. This introduces logistical problems in addition to the inevitable effects of lag and error.

7.1.3 Conclusions Pertaining to the Hypersonic Re-entry Example

Using the aforementioned GMS-stability criteria, a method has been presented (4.14) for the assessment of the longitudinal stability of an arbitrary vehicle descending along an arbitrary re-entry trajectory. For the result to be of use, the trajectory parameters (and their rates of change) must be specified in advance and substituted into a differential inequality. Solution of the corresponding first order differential equation can be used to yield the critical altitude below which stability cannot be assured. In most practical situations, numerical solution would be required due to the complexity of the coefficient function variations.

Although the general stability criterion is approximate by nature, restriction to the simplified straight-line ballistic trajectory through an isothermal atmosphere yields identical stability predictions to those of Vinh and Laitone[21] (who reduced the system to Whittaker's equation and used an exact boundedness criterion.)

Similarly, the simple longitudinal control example (Chapter 6) proved successful. A single loop variable gain controller was developed (via the 'stability inequality' methods) to actively control a Space Shuttle type vehicle along a portion of the trajectory on which the vehicle is statically unstable.

Again, the practicalities associated with implementation of such a system must be recognized. In general, a completely autonomous flight control system is perhaps far-sighted considering the difficulties (and cost) associated with fast and accurate on-line acquisition of the flight data and environmental data required for gain calculation. Nonetheless, such a system is feasible.

7.2 Recommendations for Future Work

The ultimate aim of this line of research would be to yield a complete, analytically-based design methodology for the control of LTV systems. To this end, it is suggested that the following aspects be addressed in more detail:

- Validation of the GMS-stability criteria by further mathematical analysis.
- Development of dynamic performance measures (such as relative stability etc.) for time-varying systems. This is complicated by the fact that the response of an LTV system to a given stimulus changes depending on when the stimulus is applied.
- Refinement of time-varying gain algorithms by incorporating dynamic performance specifications in addition to the basic stability requirement.
- More detailed investigation of practical implications such as the problems associated with lags and errors (introduced in the gain calculation process) and the ramnifications of discretizing the gain algorithms (for digital control formulation.)

This list is not exhaustive but is representative of the types of problems to be faced.

Furthermore, the LTV control design procedure could be enhanced by employing optimal control methods based on 'Performance Indices (PI's)' constructed from the GMS description of the response — although this would inevitably involve considerable numerical analysis because the associated quadratures would, in general, have no closed form solutions.

Finally, there is a need to properly justify the use of variable gain controllers by highlighting the advantages over the established 'frozen' method.

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