Efficiency and Braess' Paradox under Pricing in General Network

by

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Abstract

Today's large scale networks such as the Internet emerge from the interconnection of privately owned networks and serve heterogeneous users with different service needs. The service providers of these networks are interested in maximizing their profit. Since the existing pricing scheme cannot satisfy their needs, the service providers are looking for new pricing mechanisms. However, designing a for-profit pricing scheme is not a trivial task. The network contains millions of users who have their own interests and they react differently to price. Given such variety, how should the service providers charge the network resources to maximize their profit? In the presence of profit maximizing price, how should we allocate resources among these heterogeneous users? Will the resulting system suffer from efficiency loss? In this thesis, we will study these fundamental questions of profit maximizing price.

We make three main contributions: First, we develop a framework to study profit maximizing prices in a general congested network. We study the flow control and routing decisions of self-interested users in the present of profit maximizing price. We define an equilibrium of the user choices and the monopoly equilibrium (ME) as the equilibrium prices set by the service provider and the corresponding user equilibrium. Second, we use the framework to analyze the networks containing different types of user utilities: elastic or inelastic. For a network containing inelastic user utilities, we show that the flow allocations at the ME and the social optimum are the same. For a network containing elastic user utilities, we explicitly characterize the ME and study its performance relative to the user equilibrium at 0 prices and the social optimum that would result from centrally maximizing the aggregate system utility. Third, we define Braess' Paradox for a network involving pricing and show that Braess' Paradox does not occur under monopoly prices.

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Chapter 1

Introduction

The Internet is ... built, operated, and used by multitude of diverse economic interests, in varying relationships of collaboration and competition with each other. This suggests that the mathematical tools and insights most appropriate for understanding the Internet may come from a fusion of algorithmic ideas with concepts and techniques from Mathematical Economics and Game Theory.

-Christos H. Papadimitriou, *Algorithms, Games, and the Internet [39]*

Today's large scale networks such as the Internet emerge from the interconnection of privately owned networks and serve heterogeneous users with different service needs. The service providers of these networks are interested in maximizing their profit. Since the existing pricing scheme cannot satisfy their needs as we will discuss in Section 1.1.2, the service providers are looking for new pricing mechanisms. However, designing a for-profit pricing scheme is not a trivial task. The network contains a large number of users who have their own interests and they react differently to price. Given such variety, how should the service providers charge the network resources to maximize their profit? In the presence of profit maximizing price, how should we allocate resources among these heterogeneous users? Will the resulting system suffer from efficiency loss? In this thesis, we will study these fundamental questions of profit maximizing price in the context of a data network such as the Internet. The

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analysis, however, can be applied to other kinds of large scale network systems (eg. transportation networks). Our objective is twofold. First, we will develop a framework to analyze and characterize profit maximizing prices. Second, we will use this framework to study the implications of pricing on various performance results.

In the remaining of this chapter, we first give the motivations behind our framework in Section 1.1. We then review some basic concepts of network design, economics, and game theory in Section 1.2. In Section 1.3, we do a high-level survey of the related literatures. Finally, we briefly outline the contributions of this thesis in Section 1.4.

1.1 Motivation

Comparing to the existing Internet, our model has two major differences: First, users have controls on their routing decisions. Second, the service provider charges the users depending on their usage of the network resources with the objective of maximizing his profits.

1.1.1 User-Directed Routing

Despite the significant increase in bandwidth, management of congestion is still a major problem in communication networks. Such management typically involves two elements: flow control, which is defined as the control of the amount of data sent by various users, and routing, which is defined as the control of the route choices of data transmitted in the network. Currently, the Internet has two levels of routing: Inter-domain routing and Intra-domain routing. The Internet contains a large number of Autonomous Systems (ASs) and each AS is a local area network under control of a single entity, typically an Local Internet Service Provider (Local ISP) or a large organization. Large service providers such as Regional Internet Service Providers (Regional ISPs) or Network Service Providers (NSPs) provide connections between the ASs. Finding a path to direct data between the routers within an AS is called
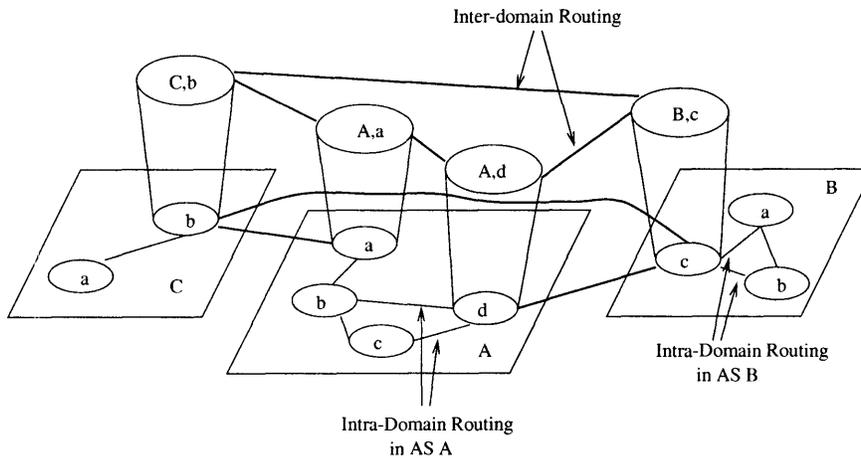
Figure 1-1: A hierarchy of Autonomous Systems (ASes).

Intra-domain routing and finding a path to send data between ASs is called Inter-domain routing. Since ASs are under different administrative controls and the Intra-domain routing is done within an AS, different ASs can use different Intra-domain routing protocols such as Routing Information Protocol (RIP), Open Shortest Path First (OSPF), and Interior Gateway Routing Protocol (IGRP). However, for Inter-domain routing, the ASs work together to delivery the data, therefore, all ASs need to use the same Inter-domain routing protocol which is Border Gateway Protocol (BGP). These two levels of routing are illustrated in Figure (1-1).

In today's Internet, both the Inter-domain and Intra-domain routing decisions are made by the network but not by the users. This, however, has several major drawbacks [46]:

1. **Sub-optimal Routing:** Since the routing are not controlled by users, the for-profit service providers will make the decisions based on their primary concern which is minimizing their cost but not increasing performance. In effect, this will lead to a sub-optimal routing. A well known example of this situation is "hot potato routing" [43] in which the service providers route the data to other service providers even though that might not be the optimal choice for the system. For example, we can consider Figure (1-1). Router (A, b) needs to send

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data to a router located in AS B. It finds out that there are two ways to do it: either through gateway router (A, a) or (A, d). Under the common practice of "hot potato routing", router (A, b) sends the data to the closest gateway router (i.e. (A, a)) since this will minimized the Intra-domain routing cost. However, the cost of sending data from router (A, a) to AS B can be much higher than the cost of sending data from router (A, d) to AS B. In this situation, the routing decisions made by (A, a) is sub-optimal.

2. **High Configuration and Maintaining Cost:** The Internet contains many local network systems that are operated by different service providers. The current routing system allows service providers to use their independent policies. Given such a wide variety of network setting, the system configuration process is costly and problematic. As a result, misconfiguration frequently disrupts the network performance. Moreover, these errors are often hard to debug and might cause serious consequences. [44] For example, we review the incident "AS 7007".[45] In April 1997, a small Internet Service Provider known as AS 7007 misconfigured its router that was connected to the Sprint network. The error propagated to many large network service providers such as Sprint, ANS, MCI, UUNet, and caused a significant portion of the Internet to be unreachable for more than one hour.

3. **Lack of Discrimination:** Under the current Internet routing protocol, all data is treated equally. However, the data from different applications or users have different quality of services (QoS) requirements. The lack of discrimination in routing decreases the level of services and the service providers' revenue. We will discuss how users and service providers can benefit from discrimination in Section 1.1.2.

Due to the above problems of current routing model, numerous papers [ex. 46, 49] have suggested that some routing decisions should be handed over to users. Such routing schemes are known as *user-directed routing*. In user-directed routing, the

route choices are decided by users and service providers will forward the data based on users' decisions. This simplifies the routing process and reduces the configuration and maintaining cost for service providers. By distributing the routing choices to users, service providers shift the balance of control and therefore they can route the data in a more reliable manner. User-directed routing also eliminates some of the sub-optimal routing scenarios such as "hot potato routing". However, since users are selfish, letting users make their own routing decisions still cannot lead to an optimal system. We will discuss the efficiency of user-directed routing in section 1.1.2. Nevertheless, user-directed routing is a promising method to increase the reliability and performance of the Internet flows.

1.1.2   Usage Based Profit Maximizing Price

As we mentioned in section 1.1.1, many efforts has been dedicated to study the flow control and routing of the Internet. The standard approach to both of these controls is the regulation of network traffic in a centralized manner, by a network manager (planner) with complete information about user needs and command over user actions, resulting in the so-called *system or social optimum.* However, in many real world networks, it is impossible and impractical to regulate the network traffic in such a central manner. This approach, moreover, assumes that the network manager knows the preferences of all users in the system and this is unrealistic for most large scale networks.

Consequently, a recent theoretical literature considers a distributed control paradigm in which some network control functions are delegated to users and studies the selfish flow choices and users' routing behavior in the absence of central planning (see, among others, [2]-[10]). In these models, users determine their own flow rates and the routes based on their knowledge of the network condition with goals of maximizing their own utilities. However, without regulation users do not take into account the congestion that they cause for the others. Therefore, when operating on their own, these selfish behaviors typically lead to allocations of traffic that are highly inefficient

from a system point of view (e.g., too much flow or the wrong routing choices).

In many real world networks information is indeed decentralized and users are selfish, but they do also face prices and restrictions set by the service provider of the network. Currently, the dominant pricing scheme in the Internet is flat pricing. Under this scheme, users pay flat monthly or yearly fees for unlimited access to the Internet. Flat pricing scheme simplifies the accounting process as well as encourages usage. However, there is strong momentum in both industry and academia to base Internet pricing on usage. This due to several drawbacks of flat pricing:

1. **Inefficiency:** The flat price does not induce users to take into account the congestion costs that they impose on other users. After paying flat fees, using the Internet is essentially free. Therefore users will tend to overexploit the common network resources and this causes the Internet to be congested. This phenomenon is known as the "tragedy of the commons". With the emerging flow-intensive Internet applications, the situation is expected to be even worse in the future. Let's review the P2P file sharing system as an example. A P2P file share software allows users to download/upload files from/to their peers. After the user identifies the files in which he is interested, the whole process is self-contained. Since the price does not depend on the usage, users will leave the application to run for days. A home DSL user can easily download gigabytes of data in one week by using a P2P software such as BitTorrent. Currently, the traffic generated by P2P systems is one of the major sources of congestion in the Internet. In fact, given the selfish nature of users, the "tragedy of the commons" is unavoidable under flat pricing. The Internet, therefore, cannot run efficiently.

2. **Lack of Incentive to Invest in Technology :** Under flat pricing, the service providers have limited economic incentives to invest in technology. A for-profit service provider will upgrade the services only if he expects that doing so will increase his profit. However, when the price is flat, the returns gained

from upgrading may not be proportional to the cost. Let's look at a capacity expansion example. A service provider increases the capacity of his network to decrease the congestion level and hopes that can attract some new users. However, he discovers that the performance improvement is marginal since the extra bandwidth will be quickly consumed by the existing users due to the "tragedy of the commons" we described previously and since upgrading is expensive, the service provider does not gain much revenue or may result in a net loss.

3. **Lack of Discrimination:** All data and users are treated equally under flat pricing. However, in some situations, discrimination can bring benefits to both users and service providers. First, different applications have varying QoS requirements. For example, delay might have little effects on email applications, but is significant to real-time video applications. Providing different services to email applications and real-time video applications can increase the performance of the overall network. However, if users are provided with the same price for all services, they will select the service with highest quality and then the system is the same as before. Second, some users are willing to pay more for better services. Under flat pricing, the movie data can take precedence over the video conference data sent by a Fortune 500 company even though the Fortune 500 company would be willing to pay more for better services. In this case, price discrimination can lead to a more efficient allocation of resources and generate more revenue for the service providers.

Given these downsides of flat pricing, people have proposed that the Internet pricing should be usage based, where the price charged will depend on the amount of data that the users sent or/and the quality of the services that they received. Service providers can charge users based on their objectives. For example, people show that pricing can be used as means of achieving social optimum in a distributed manner (ex. [4, 5]), can be used to prevent the "tragedy of the commons" (ex. [35]), or should be used to induce users to choose the service class that can satisfy their needs

on QoS (ex. [47]). We will outline some of these pricing schemes in Section 1.3. Despite the differences in objective, these methods use pricing to increase efficiency of the network system. However, in commercial networks, maximizing profit is the major goal for the service providers. Even though efficiency is important, we believe that price is more likely to be used for profit maximizing purpose. Therefore, in this thesis, we will study for-profit pricing and its effects on the system performance.

1.2 Preliminaries: Modeling Network, Users, and Service Providers

To study profit maximizing prices, we first need to model the network and the behaviors of users and service providers.

1.2.1 Modeling a Data Network

A data network can be modelled as a graph using graph theory. A graph is a collection of vertices and links connecting some subset of them. Mathematically, this can be written as $G = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ denotes the set of nodes and $\mathcal{E}$ denotes the set of links. A graph is *simple* if there are at most one link connecting any two vertices. The links in the graph can have directions. A graph in which links are undirected is said to be *undirected*. Otherwise, the graph is said to be *directed*. A path p connecting two vertices v_0 and v_n in a graph is a sequence of $\{v_0, v_1, \cdots, v_{n-1}, v_n\}$ such that $(v_0, v_1), (v_2, v_3), \cdots, (v_{n-1}, v_n)$ are the links of the graph. A link e lies along the path p if $e \in \{(v_0, v_1), (v_2, v_3), \cdots, (v_{n-1}, v_n)\}$. This can be written as $e \in p$.

1.2.2 User Utility and Payoff

A user in a data network is an entity that uses an application to transfer some data from one point to another point in the network. We can assume that each user only uses one application. A user who is using multiple applications can be viewed as
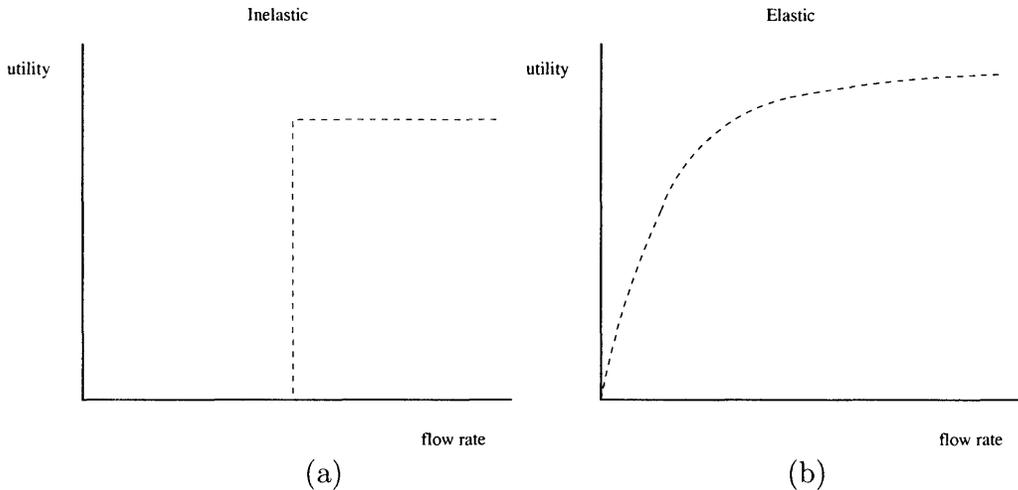
Figure 1-2: a) Inelastic utility as a function of flow rate. b) Elastic utility as a function of flow rate.

multiple users, each using one application. We identify each user j with a *utility function* $u_j(\Gamma_j)$, which defines the monetary value to j by sending $\Gamma_j \geq 0$ units of data. Depending on the application service requirements, the utility function takes different forms. Shenker [9] categorized applications into two main classes based on their service requirements: *inelastic and elastic applications*. Real-time voice and video applications require a fixed amount of bandwidth for adequate QoS, hence are inelastic in their demand for bandwidth. Therefore, it is reasonable to model their utility as a step function, see Figure 1-2(a). On the other hand, traditional applications such as e-mail and file transfer are elastic; they are tolerant of delay and can take advantage of even the minimal amounts of bandwidth. The utility function in this case can be represented as a nondecreasing and concave function, see Figure 1-2(b). Different users might have different utility functions even though they are using the same type of application, representing different preferences. A user with an inelastic (elastic) application has an inelastic (elastic) utility function.

Now, suppose a user j sends Γ_j units of data from a source point s to a destination point t and receives $u_j(\Gamma_j)$ utility. He will encounter some cost c_j of sending these data. The cost can come from many different sources[1] such as the latencies that

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[1]Mackie-Mason and Varian discuss the different types of costs for Internet in [35]

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the data experiences in the network and the price charged by the service providers for using the network resources. We assume that these different types of cost are comparable and the aggregate cost can be compared with the user's utility, then the net *payoff* of user j is:

$$u_j(\Gamma_j) - c_j$$

1.2.3 Wardrop Equilibrium and Nash Equilibrium

In the absence of central regulation, users can choose the paths to transmit their data. Different paths might have different congestion level and prices, and will yield different costs. When users are selfish, they will pick the paths with the lowest cost and also they will not take into account the additional congestion caused to other users. To model these selfish user behaviors, we can use the following two equilibrium concepts: *Wardrop Equilibrium* and *Nash Equilibrium*.

Let's consider a simple network that contains only two vertices v_1 and v_2. There are n directed links $e_1, \cdots, e_n$ start from v_1 and end at v_2 and J users want to send data from v_1 to v_2 through these links. If user j sends $f_j^i \geq 0$ units of data through link i, then the cost of using link i for user j will be

$$f_j^i g_i(f^i),$$

where f^i is the total amount of data travelling on (or the link load of) the link and $g_i(f^i)$ is the cost of sending one unit of data on the link when the link load is f^i.[2] Denote the vector of flows of user j on the links by $f_j = [f_j^1, \cdots, f_j^n]$ and the vector of total flows on the links by $\gamma = [f^1, \cdots, f^n]$. The payoff of user j is then given by:

$$v_j(f_j; \gamma) = u_j(\Gamma_j) - \sum_i f_j^i g_i(f^i),$$

[2]For this example, we assume that the cost function g_i per unit flow only depends on the link load f^i but not other parameters such as link price. A more complete model will be described in Chapter 2.

where $\Gamma_j = \sum_{i=1}^n f_j^i$. The first term is the utility to user j of sending Γ_j units of data; the second term is the total cost of using the links. The link load f^i, depends on the amount of data put on the link by all users. In particular,

$$f^i = \sum_{j=1}^J f_j^i. \tag{1.1}$$

When user j changes the amount of data, f_j^i, that he sends on link i, the link load f^i will change as well. However, if user j is small, he does not anticipate the effect of his flow on the link loads. Therefore, he takes the unit cost of each link as fixed and then chooses the routing strategy[3] to maximize his payoff. If there exists a set of routing strategies such that:

1. the link loads are defined by Eq. (1.1), and

2. all users has chosen their strategies to maximize their payoffs given the link loads,

then this set of strategies is called a *Wardrop equilibrium*. Mathematically, a set of strategy $\{f_1, \cdots, f_J\}$ is a Wardrop equilibrium if

$$f_j^* \in \arg \max_{0 \leq f_j^i \leq C^i} v_j(f_j, \gamma), \quad \forall\, j$$

where C^i is the capacity of link i and $f^i = \sum_{j=1}^J f_j^i$. Since each user takes the link loads as given, at Wardrop equilibrium, the unit cost of all the routes that the user uses is equal and less than the unit cost of the unused routes. Wardrop equilibrium is first introduced by Wardrop in 1952 [1] in the context of transportation network.

Wardrop equilibrium relies on the assumption that a user does not anticipate the effect of his flow on the link loads. For some networks, there are some users that account for a large portion of the total traffic. In these situations, the small user assumption is not applicable. A related definition in game theory that does

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[3]User j routing strategy includes the choice of the total amount of data the he will send, Γ_j, and the routing choices ,f_j^i's.

not require the small users assumption is *Nash equilibrium.* A network is at a Nash equilibrium if no user has incentive to vary their routing strategies. More precisely, if we denote the payoff of user j by

$$v_j(f_j; f_1, \cdots, f_{j-1}, f_{j+1}, \cdots f_J) = u_j(\Gamma_j) - \sum_i f_j^i g_i(\sum_{k \neq j} f_k^i + f_j^i),$$

then f_j^* is a Nash equilibrium if

$$f_j^* \in \arg \max_{0 \leq f_j^i \leq C^i} v_j(f_j; f_1^*, \cdots, f_{j-1}^*, f_{j+1}^*, \cdots f_J^*), \quad \forall j,$$

where C^i is the capacity of link i and $f^i = \sum_{j=1}^{J} f_j^i$. At the equilibrium, each user views the link costs as functions of the composite routing strategies of all users. Therefore, instead of taking the link costs as given, they anticipate the effect of their traffic on the link cost. Thus their payoffs do not only depend on their own routing strategies but also the routing strategies of all the other users. Standard arguments establish that Wardrop equilibria are obtained as estimates of Nash equilibria as the number of users go to ∞ (see, for example, [13] and [18]).

1.2.4 Stackelberg Games

A commercial data network normally will have two types of participants: service providers and network users. Service providers own the network resources and will price the usage of the network for profit. Depending on the prices, users will then choose their routing strategies to maximize their payoffs. If the prices are fixed and a user equilibrium (see Section 1.2.3) exists, the system will settle at a equilibrium. Knowing the behavior of the users, how should the service provider set the prices so that his profit is maximized? This problem can be formulated as a *Stackelberg Game.* In a Stackelberg game, one player acts as a *leader* (here the service provider who wants to maximize his profit) and the other players as *followers* (the selfish users). The problem is then to find a strategy (how to set the prices) that the leader can
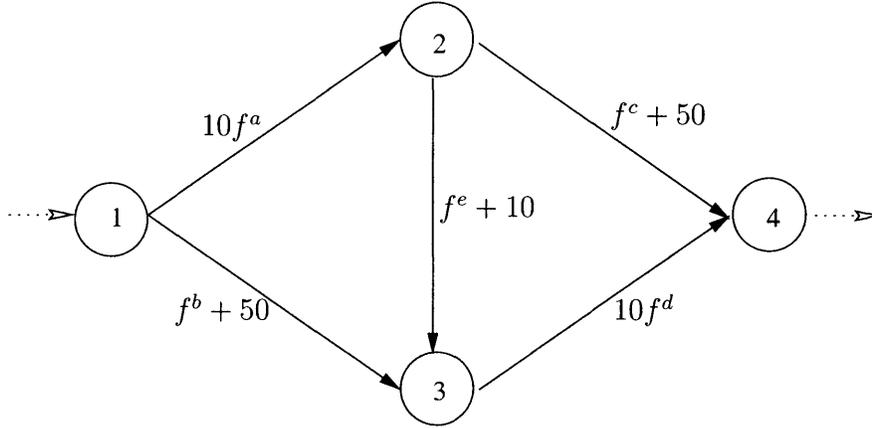
Figure 1-3: The original example of Braess' Paradox.

induce the followers to react in a way that will maximize his profit. The model we used in this thesis is based on Stackelberg games.

1.2.5 Braess' Paradox

We close this section with a classical example that demonstrates the inefficiency of selfish routing networks: Braess' Paradox. When a network manager installs a new link into a network, he expects the congestion level of the network to decrease. However, is this assumption always true? Let's consider the network in Figure 1-3.

The network contains 4 links: a, b, c, and d. The congestion level of each link is specified by its latency function:

$$l^a(f^a) = 10f^a, \quad l^b(f^b) = f^b + 50, \quad l^c(f^c) = f^c + 50, \quad l^d(f^d) = 10f^d.$$

A large number of users want to send data from node 1 to node 4 and their aggregate data are 6 units. By symmetry, if the users are selfish, we expect that at the equilibrium each of two paths to carry 3 units of traffic so that the latency for each user is 83. (We assume the users are small and the user equilibrium is a Wardrop equilibrium.) Now, a network manager would like to increase the network capacity

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and he installs a link e between node 2 and node 3:

$$l^e(f^e) = f^e + 10.$$

When link e is first installed, there is no traffic on the link. Thus the latency of path $\{a, e, d\}$ is

$$30 + 0 + 30 = 60,$$

which is less than latency of path $\{a, c\}$ and $\{b, d\}$ (which is 83). Therefore, some users will choose to send their data through the new path instead of the old one. This kind of behavior will continue as long as the latency of all the paths are not equal. Eventually, the system reaches a new equilibrium and all the paths will have the same latency. At the new equilibrium, each path will carry 2 units of data and the latency is 92. Adding the new link e actually negatively impact all of the traffic. We will study Braess' Paradox in more detail in Chapter 5.

1.3   Related Work

It has been known for a long time that selfish behaviors leads to inefficiency in performance [36]. There is a recent interest for quantifying this inefficiency, referred to as the *price of anarchy (POA)*, which is defined as the ratio of the performance of user equilibrium to the social optimum. In [37], Koutsoupias and Papadimitriou consider a two-links parallel-link network with users that have fixed demands. The uer equilibrium is defined as follows: each user chooses the probability that he will route all of his flow on a given link to minimize the expected delay that he will experience with the objective of minimizing the delay on the most congested link. They provide a tight analysis of the ratio of the worst-case Nash equilibrium and the social optimum. The tight analysis of a parallel-link network with arbitrary number of links is given by Czumaj and Vöcking [38]. A recent paper by Roughgarden and Tardos [8] studies the POA for a general topology network. They consider the fixed demand routing

case with the objective of minimizing the total latency experienced by all users. They show that when the latency functions are linear, the total latency of a user equilibrium is at most 4/3 of the minimum total latency (that is achieved at the social optimum). However, for more general latency functions, the total latency at the user equilibrium can be arbitrary large. Johari and Tsitsiklis [10] consider a different game among users and define the user equilibrium as follows: each user receives a utility that is similar to the one we defined in the Section 1.2.2 with objective of maximizing the total utility received by all the users. They show that the ratio of the total utility received at any user equilibrium is at least 3/4 of the total utility received maximum possible aggregate utility.

Using pricing to cope with the inefficiency created by selfish users is not a new idea. In [4], Kelly considers a model where R users share the network resources and each user tries to maximize his own utility. He shows that the network manager can use prices to induce the rate allocation that maximizes the total utility. Similar results are given by Low and Lapsley in [5] and Yaïche, Mazumdar, and Rosenberg in [48]. All of them propose a distributed algorithm to achieve the social optimal rate allocation. Yaïche, Mazumdar, and Rosenberg in [48] also consider a situation where each user submits a budget, which represent the maximum amount of money he is willing to pay. They show that the service provider can use price to achieve a system optimum subject to the budget constraint. They also give the explicit characterization of the social optimum price. Orda and Shimkin [47] consider a network that offers multi-class services and users can choose different service classes in order to maximize their own performance. They show that in this case the price can be used to induce the users to choose the service class that can satisfy their needs on QoS. There are many other works that study pricing as a tool to achieve efficiency. (see [42],[48] and the references therein) However, with a few exceptions ([7], [13], [14], and [15]) the game-theoretic interaction between users and service providers have largely been neglected. In [15], He and Walrand propose a fair revenue sharing pricing scheme for multiple service providers. In [13], Acemoglu and Ozdaglar analyze equilibrium

flows and routing in a parallel-link network and show how profit-maximizing prices from the viewpoint of the service provider typically also play the role of efficiently regulating data transmission. The model used in this thesis is an extension of the model used in [13].

An important problem in general network topologies is the potential for network performance to deteriorate as a result of increasing network resources, which is also referred to as Braess' paradox [16]. An example of Braess' paradox is given in Section 1.2.5. Previous research has focused on the detection of Braess' paradox on specific network topologies and restrictions on methods of network upgrade for preventing it (see Chapter 5 for a survey). In [23], Roughgarden shows that designing a large general network that is free from Braess' paradox is actually NP-hard. In Chapter 5 of this thesis, we will show that Braess' Paradox does not exist when the network is priced by a profit maximizing service provider.

1.4   Contributions of This Thesis

In this thesis, we analyze the equilibrium of a model that incorporates a self-interested service provider and study the performance gap between the equilibrium and the system optimum in a network with a general topology. Analysis of a general network is considerably more difficult than networks with parallel links. For a given price, we provide a characterization of the user equilibrium of flow rates and routing decisions under the standard Wardrop assumption that each user is small (thus ignores the effect of their decisions on aggregate congestion). Furthermore, we provide a full characterization of the "monopoly equilibrium", i.e., profit-maximizing prices from the viewpoint of service provider and the resulting allocations. We show that for the case of routing with participation control (see Chapter 3, which naturally corresponds to the inelastic user utilities), the monopoly equilibrium achieves the system optimum. This result contrasts with pervasive inefficiencies in the routing models with selfish agents, for example, as in [8]. For the case of elastic user utilities, monopoly pricing

introduces a distortion and induces users to reduce their flow rates. The performance of the monopoly equilibrium relative to a situation without prices and to the social optimal depends on the extent of the congestion effects (externalities). When these are important, the monopoly equilibrium, which forces users to internalize these effects, performs relatively well. At the end, we study the effects of profit-maximizing prices on Braess' paradox, and show that at the monopoly prices, there can never be Braess' paradox, so for-profit incentives appear sufficient to eliminate this type of paradoxical outcomes.

The rest of the thesis is organized as follows. Chapter 2 describes the network topology and user preferences, provides the definition of a user equilibrium, and monopoly equilibrium. Chapter 3 shows the efficiency of the monopoly equilibrium in the case of users with inelastic utility. Chapter 4 discusses the monopoly equilibrium in the case of users with elastic utility. It first analyzes the sensitivity of the equilibrium allocations to prices. Then, it defines and characterizes the monopoly equilibrium, and provides a comparison of the monopoly equilibrium with the social optimum. Finally, Chapter 5 discusses Braess' paradox under pricing.

Chapter 2

Model: User Equilibrium, Monopoly Equilibrium, and Social Optimum

In this chapter, we will propose a framework that can be used to study decentralized data networks with many heterogeneous selfish users and service provider charging prices for bandwidth and data transmission to maximize his profit.

2.1   Network Model

We consider a directed network $G = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ denotes the set of nodes and $\mathcal{E}$ denotes the set of links. We assume that there are m origin-destination (OD) node pairs $\{s_1, t_1\}, ..., \{s_m, t_m\}$, and we denote the set of OD pairs by $\mathcal{W}$. For each OD pair $\{s_k, t_k\} \in \mathcal{W}$, there are J_k users, belonging to set $\mathcal{J}_k$, that send data from node s_k to node t_k through paths that connect s_k and t_k. We also denote the set of paths between s_k and t_k by $\mathcal{P}_k$ and the set of all paths in the network by $\mathcal{P} = \cup_{k \in \mathcal{W}} \mathcal{P}_k$.

In the absence of central regulation, we assume that each user in the network is interested in his own payoff. This payoff should reflect the tradeoff between the utility of sending data and the disutility of incurred delays and monetary costs during

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transmission. We next formalize the user payoff function.

We use the term flow to represent the data stream that the user sends. Let $f_{k,j}^p$ denote the flow of user j on path p where $j \in \mathcal{J}_k$ and $p \in \mathcal{P}_k$. Then the total flow rate of user j will be:

$$\Gamma_{k,j} = \sum_{p \in \mathcal{P}_k} f_{k,j}^p.$$

We assume each user $j \in \mathcal{J}_k$ receives a utility of $u_{k,j}(\Gamma_{k,j})$. As we describe in Section 1.2.2, we can classify the users into two categories: users with inelastic and elastic utility function. Both utility classes can be analyzed within the framework introduced here.

To model delays incurred during transmission, we assume that each link e has a flow-dependent latency function $l^e(f^e)$, where

$$f^e = \sum_k \sum_{j \in \mathcal{J}_k} \sum_{\{p|e \in p, p \in \mathcal{P}_k\}} f_{k,j}^p$$

is the total flow (link load) on link e. Let $f^{\mathcal{E}} = [f^1, \cdots, f^{|\mathcal{E}|}]$ denote the vector of total flows on links. The latency cost of sending one unit of flow on path p is given by

$$\sum_{e \in p} l^e(f^e) \tag{2.1}$$

and the latency of sending $f_{k,j}^p$ units of flow along path p is given by

$$\sum_{e \in p} l^e(f^e) f_{k,j}^p.$$

For the cost of services, we assume that the service provider charges a price q^p per unit of bandwidth for path p. We denote q to be the price vector $[q^1, \cdots, q^{|\mathcal{P}|}]$. Given the prices set by the service provider, the goal of each user in the network is to maximize his own payoff. Note that an alternative model is one in which the service provider charges prices for links rather than paths. However, it can be seen that the service provider can make more profit by charging prices for the paths. We

will discuss the difference between link pricing and path pricing in Section 2.3.

We will adopt the following assumptions on utility and link latency functions.

Assumption 1 Assume that for each user $j \in \mathcal{J}_k$, the utility function $u_{k,j}$ is non-decreasing. For elastic user utility functions, we further assume that the functions are strictly concave, continuously differentiable, and $0 < u'_{k,j}(0) < \infty$. We also assume that for each link e, the latency function l^e is continuous and strictly increasing.

We next define the user payoff function: For a given price q, each user j chooses $f_{k,j} = [f^1_{k,j}, \cdots, f^{|\mathcal{P}_k|}_{k,j}]$ to maximize his payoff function

$$v_{k,j}(f_{k,j}; f^{\mathcal{E}}, q) = u_{k,j}(\Gamma_{k,j}) - \sum_{p \in \mathcal{P}_k} \left(\sum_{e \in p} l^e(f^e) \right) f^p_{k,j} - \sum_{p \in \mathcal{P}_k} q^p f^p_{k,j}. \tag{2.2}$$

2.2 Wardrop Equilibrium, Monopoly Equilibrium, and Social Optimum

As is common in traffic equilibrium models used in transportation and communication networks, we assume that each user is small, thus focus on Wardrop Equilibria, where the individual user does not anticipate the effect of his flow on the total level of congestion. [1, 8, 13] This appears as a realistic assumption in today's large-scale data networks such as the Internet. In fact, in some network, a large user is indeed a group of small users who have one aggregate utility function. Even though these small users have the same objective, it is still hard to coordinate them to act as a single user. In these cases, the small user assumption is still applicable. [1]

Definition 1 Let $f = [f_{k,j}]_{j \in \mathcal{J}_k, k \in \mathcal{W}}$ denote the vector of flows of all users in the network. For a given price vector $q \geq 0$, a flow vector f^* is a Wardrop equilibrium

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[1]In the later sections of this thesis, we will have example in which the network only contains a small number of users. These users can be thought as group of small users who do not coordinate with each other.

(WE) of the user game if

$$f_{k,j}^* \in \arg\max_{f_{k,j} \geq 0} v_{k,j}(f_{k,j}; f^{\mathcal{E}}, q), \ \forall \ j \in \mathcal{J}_k, k \in \mathcal{W},$$

$$f^e = \sum_k \sum_{j \in \mathcal{J}_k} \sum_{p|e \in p, p \in \mathcal{P}_k} (f^*)_{k,j}^p, \qquad \forall \ e \in \mathcal{E}.$$

Hence, each price vector induces a WE among the users. The service provider (monopolist) chooses the price vector to maximize his profit. The maximization problem can be written as:

$$\max_{q \geq 0} \quad \sum_p q^p f^p(q), \tag{2.3}$$

where $f^p(q)$ is the flow on path p at a WE for a given price vector q. We will show in later sections that under Assumption (1), problem (2.3) has an optimal solution, which we denote by q^*. We refer to q^* as the *monopoly price*. Let $f^* = f(q^*)$ be the flow vector at a WE for price q^*. Then we call (q^*, f^*) the *monopoly equilibrium* (ME) of the problem.

To study the performance of the ME, we compare the total system utility at the equilibrium with the total system utility at the network's *social optimum*. A flow f is a social optimum if it maximizes the total system utility:

$$\sum_{k \in \mathcal{W}} \sum_{j \in \mathcal{J}_k} \left(u_{k,j}(\Gamma_{k,j}) - \sum_{p \in \mathcal{P}_k} \left(\sum_{e \in p} l^e(f^e) \right) f_{k,j}^p \right). \tag{2.4}$$

We can view the social optimum as the allocation that would be chosen by a network planner, which has full information and control over the network. The allocation at an ME is not necessarily the same as the social optimum. In the following, we analyze the performance of the ME relative to the social optimum for both inelastic and elastic user utilities. The different structure of the utility functions introduces significant differences in the analysis and the resulting performances of these utility classes.

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2.3 Path Pricing versus Link Pricing

We have used path price in our model. However, in stead of setting a price for each path, the service provider can set a price, q^e, for each link e. Thus the price for a user to route one unit of data on a path, p, will be the summation of the prices of the links in that path: $\sum_{e \in p} q^e$. Therefore, the maximization problem (2.3) can be rewritten as:

$$\max \quad \sum_p \left(f^p(q^{\mathcal{E}}) \sum_{e \in p} q^e \right) \tag{2.5}$$
$$\text{s.t.} \quad \sum_{e \in p} q^e \geq 0, \forall\, p \in \mathcal{P}$$

where $f^p(q^{\mathcal{E}})$ is the flow on path p at a WE for a given link price vector $q^{\mathcal{E}}$. Note that the price of a link can be negative as long as the price for each of the paths is nonnegative.

From the service provider viewpoint, there is a clear advantage of using path pricing over link pricing: the service provider can collect more profit.

Proposition 1 Let q and $q^{\mathcal{E}}$ be the monopoly price vectors under path pricing and link pricing. Then

$$\sum_p f^p q^p \geq \sum_p \left(f^p \sum_{e \in p} q^e \right)$$

Proof: Let f be a WE under link price $q^{\mathcal{E}}$. Now consider the path price vector q where

$$q^p = \sum_{e \in p} q^e, \ \forall\, p \in \mathcal{P}$$

Clearly, f is also a WE under the path price q. Therefore, for every feasible solution $(f, q^{\mathcal{E}})$ of problem (2.5), we can find a feasible solution (f', q) of problem (2.3) such that:

$$f = f'$$
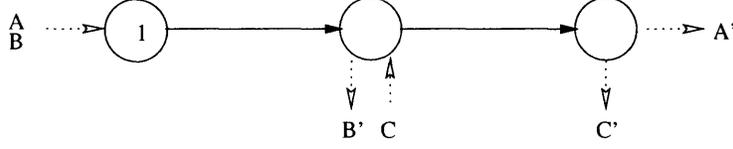
Figure 2-1: A network that the service provider can collect more profit from path pricing than link pricing.

and

$$\sum_p \left(f^p \sum_{e \in p} q^e \right) = \sum_p f^p q^p.$$

Hence, the result follows.   **Q.E.D.**

The preceding proposition shows that from the profit maximizing viewpoint, path pricing is as good as link pricing. To see whether it is better, we can look at the following example. Let's consider the simple network shown in Figure (2-1). The network contains two links and three users:

$$u_A(\Gamma_A) = 160\Gamma_A^{0.5}, u_B(\Gamma_B) = 160\Gamma_B^{0.25},$$

$$u_C(\Gamma_C) = 160\Gamma_C^{0.25},$$

$$l^a(f^a) = 0.1f^a, l^b(f^b) = 10f^b.$$

User A sends his data from node 1 to node 3. User B sends his data from node 1 to node 2. And user C sends his data from node 2 to node 3. Under path pricing, the ME of this network is

$$f^{\{a,b\}} = 1.3865, f^{\{a\}} = 8.5838, f^{\{b\}} = 0.2123$$

$$q^{\{a,b\}} = 50.9551, q^{\{a\}} = 6.9792, q^{\{b\}} = 111.9160$$

and the total profit is 154.3170. Note that in the ME under path pricing, $q^{\{a,b\}} < q^{\{b\}}$. Now, let's consider the link pricing. Since the path $\{a,b\}$ contains the path $\{b\}$ and the prices of both links can not be negative, $q^{\{a,b\}}$ must be greater than or equal to

$q^{\{b\}}$ in link pricing. We find that the ME is

$$f^{\{a,b\}} = 1.0172, f^{\{a\}} = 9.0834, f^{\{b\}} = 0.4595$$

$$q^a = 6.6349, q^b = 56.9093$$

and the total profit is 151.0518 which is less than 154.3170. Therefore, in this network, the service provider can get more profit from path pricing. In fact, path pricing is more flexible than link pricing in that path pricing does not have the restriction that the price of path $\{a, b\}$ must be greater than or equal to the price of path $\{b\}$. As the result, the service provider has more freedom not only in setting the prices, but also in maximizing the usage of his resources. Under certain conditions, the ME under link pricing scheme is consist with the ME under path pricing scheme which means the link loads are the same and the prices for the path with positive flow are the same. Therefore, the maximum profit that can be collected are the same as well. We will show this consistency in Chapter 4.3 after we give the characterization of monopoly prices (Proposition 5).

Chapter 3

Inelastic User Utility (Routing with Participation Control)

In this chapter, we analyze a network containing users with inelastic utility functions. When a user has an inelastic utility function, it can be seen from Eq. (2.2) that at a given price vector, he either sends a fixed amount of data or decides not to participate in the network. Hence, the problem with inelastic utility functions is a routing problem, where user j is interested in choosing the paths to send his fixed amount of data, say $t_{k,j}$ units; but he also has the option of not sending any data when it is costly to do so. This is also a natural model to study the routing problem in the presence of service providers since it prevents the service provider from charging infinite prices. We refer to this problem as *the routing problem with participation control*. This problem was studied for parallel link networks in [13]. Here, we extend this analysis to general networks.

The problem can be modelled using the following utility function for user j

$$u_{k,j}(x) = \begin{cases} 0, & \text{if } 0 \leq x < t_{k,j}, \\ t_{k,j}, & \text{if } x \geq t_{k,j}, \end{cases} \tag{3.1}$$

together with binary variables $z_{k,j}$ which indicate whether user j chooses to participate

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or not, i.e., $z_{k,j} = 1$ if user j decides to send $t_{k,j}$ units of data, and $z_{k,j} = 0$ if he decides not to send any data. The user equilibrium of this problem can be defined as follows.

Definition 2 For a given price vector $q \geq 0$, a vector $(f^*, z^*) = (f^*_{k,j}, z^*_{k,j})_{\{j \in \mathcal{J}_k,\ k \in \mathcal{W}\}}$, is a WE of the routing problem with participation control if for all k and all $j \in \mathcal{J}_k$,

$$(f^*_{k,j}, z^*_{k,j}) \in \arg \max_{f_{k,j} \geq 0, z_{k,j} \in \{0,1\}} \left\{ u_{k,j}(\Gamma_{k,j} z_{k,j}) - \sum_{p \in \mathcal{P}_k} \left(\sum_{e \in p} (l^e(f^e)) + q^p \right) f^p_{k,j} \right\}, \quad (3.2)$$

$$f^e = \sum_k \sum_{j \in \mathcal{J}_k} \sum_{p | e \in p, p \in \mathcal{P}_k} (f^*)^p_{k,j}, \quad \forall\ e \in \mathcal{E},$$

where $u_{k,j}$ is given by Eq. (3.1).

Since the utility function [Eq. (3.1)] is not concave, we cannot guarantee the existence of a WE for any price vector. In fact, a WE may not exist for some price vectors. For example, consider a network that consists of one directed link where two users, A and B, send data through this link. Assume that $t_A = 1$, $t_B = 1.5$, $l(x) = \frac{1}{2}x$. It can be seen that if the price of the link is 0, a WE does not exist. In the same example, however, one can also show that the profit-maximizing price set by the monopolist is 0.5, and at this price, there exists a WE in which A sends his data and B does not. In the following, we show that at the monopoly price, there exists a WE, which moreover achieves the social optimum. (i.e., the flow allocations at any ME and the social optimum are the same). For consistency, we define the social optimum for the inelastic utility case as a vector (f, z) that maximizes the total system utility:

$$\sum_{k \in \mathcal{W}} \sum_{j \in \mathcal{J}_k} \left(u_{k,j}(\Gamma_{k,j} z_{k,j}) - \sum_{p \in \mathcal{P}_k} \left(\sum_{e \in p} l^e(f^e) \right) f^p_{k,j} \right). \quad (3.3)$$

Proposition 2 Consider a routing problem with participation control.

1. There exists a monopoly price q, and a WE (f, z) at price q.

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2. A vector (f, z) is a social optimum if and only if there exists a price vector q such that $(q, (f, z))$ is an ME.

Proof: To establish this proposition, we first prove two lemmas. The first lemma gives a characterization of a WE at any price vector and the second one gives an explicit characterization of the monopoly price. The first lemma is proved by exploiting the linear structure of problem (3.2).

Lemma 1 For a given price vector $q \geq 0$, a vector $(f_{k,j}, z_{k,j})_{\{j \in \mathcal{J}_k, \ k \in \mathcal{W}\}}$, with $f_{k,j} \geq 0$, $z_{k,j} \in \{0, 1\}$, is a WE if and only if it satisfies the following conditions:

1. $f^e = \sum_k \sum_{j \in \mathcal{J}_k} \sum_{p \mid e \in p, p \in \mathcal{P}_k} f_{k,j}^p$, $\forall \ e \in \mathcal{E}$.

2. If $z_{k,j} = 1$, $\sum_{p \in \mathcal{P}_k} f_{k,j}^p = t_{k,j}$.

3. If $z_{k,j} = 0$, $f_{k,j}^p = 0$ for all $p \in \mathcal{P}_k$.

Define the set

$$\overline{\mathcal{P}_k} = \left\{ p \mid p \in \mathcal{P}_k \text{ and } \sum_{e \in p} l^e(f^e) + q^p \leq \min\{1, \min_{m \in \mathcal{P}_k}\{\sum_{e \in m} l^e(f^e) + q^m\}\} \right\}.$$

4. If $p \notin \overline{\mathcal{P}_k}$, then $f_{k,j}^p = 0$, $\forall \ j \in \mathcal{J}_k$.

5. If $\min_{m \in \mathcal{P}_k}\{\sum_{e \in m} l^e(f^e) + q^m\} < 1$, then $z_{k,j} = 1$ for all $j \in \mathcal{J}_k$ and k.

Proof: The proof of the necessity of conditions (1) - (5) is immediate. We show that these conditions are sufficient. We rewrite problem (3.2) as:

$$(f_{k,j}^*, z_{k,j}^*) \in \arg \max_{f_{k,j} \geq 0, z_{k,j} \in \{0,1\}} \left\{ t_{k,j} z_{k,j} - \sum_{p \in \mathcal{P}_k} \left(\sum_{e \in p} l^e(f^e) + q^p \right) f_{k,j}^p \right\} \qquad (3.4)$$

$$\text{s.t.} \quad \sum_{p \in \mathcal{P}_k} f_{k,j}^p = t_{k,j}, \ if \ z_{k,j} = 1. \qquad (3.5)$$

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Let $(\bar{f}_{k,j}, \bar{z}_{k,j})_{\{j \in \mathcal{J}_k,\ k \in \mathcal{W}\}}$ be a vector satisfying conditions (1) - (5). To show that this vector is a WE, we show that for all j,

$$\left\{ t_{k,j} \bar{z}_{k,j} - \sum_{p \in \mathcal{P}_k} \left(\sum_{e \in p} l^e(f^e) + q^p \right) \bar{f}_{k,j}^p \right\} > \left\{ t_{k,j} z_{k,j} - \sum_{p \in \mathcal{P}_k} \left(\sum_{e \in p} l^e(f^e) + q^p \right) f_{k,j}^p \right\},$$

where

$$f^e = \sum_k \sum_{j \in \mathcal{J}_k} \sum_{p | e \in p, p \in \mathcal{P}_k} \bar{f}_{k,j}^p, \quad \forall\, e \in \mathcal{E},$$

and $(f_{k,j}, z_{k,j})$ is any feasible solution of problem (3.4). Consider an arbitrary $j \in \mathcal{J}_k$. There are two cases:

Case 1: $z_{k,j} \neq \bar{z}_{k,j}$.

First consider the case $\bar{z}_{k,j} = 0$ and $z_{k,j} = 1$. By condition (3), $\bar{z}_{k,j} = 0$ implies that $(\bar{f})_{k,j}^p = 0 \ \forall\, p$. Therefore, user j's payoff is 0 at $(\bar{f}_{k,j}, \bar{z}_{k,j})$. By condition (5), we further have

$$\min_{m \in \mathcal{P}_k} \left\{ \sum_{e \in m} l^e(f^e) + q^m \right\} \geq 1.$$

Since $z_{k,j} = 1$, this shows that user j's payoff is less than or equal to 0 at $(f_{k,j}, z_{k,j})$. Next assume that $\bar{z}_{k,j} = 1$ and $z_{k,j} = 0$. Condition (4) implies that user j's payoff is greater than or equal to 0 at $(\bar{f}_{k,j}, \bar{z}_{k,j})$. However, $z_{k,j} = 0$ implies by problem (3.4) that user j's payoff is less than or equal to 0 at $(f_{k,j}, z_{k,j})$. Therefore, for both cases, user j's payoff at $(\bar{f}_{k,j}, \bar{z}_{k,j})$ is greater than or equal to his payoff at $(f_{k,j}, z_{k,j})$.

Case 2: $z_{k,j} = \bar{z}_{k,j}$.

For the case where $z_{k,j} = \bar{z}_{k,j} = 0$, user j's payoff is 0 at $(\bar{f}_{k,j}, \bar{z}_{k,j})$ [cf. condition (3)] and is less than or equal to 0 at $(f_{k,j}, z_{k,j})$. Next, we look at the case where $z_{k,j} = \bar{z}_{k,j} = 1$. By condition (4), it follows that for all paths p such that $\bar{f}_{k,j}^p > 0$, we have

$$\sum_{e \in p} l^e(f^e) + q^p = \min \left\{ 1, \min_{m \in \mathcal{P}_k} \{ \sum_{e \in m} l^e(f^e) + q^m \} \right\}.$$

In view of the linear structure of the problem, this shows that user j's payoff at $(\bar{f}_{k,j}, \bar{z}_{k,j})$ is greater than or equal to his payoff at $(f_{k,j}, z_{k,j})$.   **Q.E.D.**

For the second lemma, we consider the monopoly problem for the routing problem with participation control,

$$\max_{} \quad \sum_{p \in \mathcal{P}} q^p f^p \tag{3.6}$$

$$\text{subject to} \quad f^p = \sum_{j \in \mathcal{J}_k} f^p_{k,j}, \quad \forall\, p \in \mathcal{P},$$

$$q \geq 0,$$

$$(f, z) \in G(q),$$

where $G(q)$ is the set of vectors (f, z) that satisfy conditions (1)-(5) of Lemma 1.

Lemma 2 Let $(q, (f, z))$ be an ME. Then, for all p with $f^p > 0$, we have

$$q^p = 1 - \sum_{e \in p} l^e(f^e). \tag{3.7}$$

Proof: Since (f, z) is a feasible solution of problem (3.6), (f, z) is a WE. Let p be a path in $\mathcal{P}_k$ with positive flow ($f^p > 0$). By condition (4) in Lemma 1, we have $p \in \overline{\mathcal{P}_k}$. Therefore, by condition (3) we have

$$q^p + \sum_{e \in p} l^e(f^e) \leq 1.$$

Now, assume $q^p + \sum_{e \in p} l^e(f^e) < 1$, then for every $p' \in \mathcal{P}_k$ with $f^{p'} > 0$ we have

$$q^p + \sum_{e \in p} l^e(f^e) \;=\; q^{p'} + \sum_{e \in p'} l^e(f^e)$$

$$<\; \min\{1, \min_{m \notin \mathcal{P}_k} \sum_{e \in m} l^e(f^e) + q^m\}, \forall\, p' \in \overline{\mathcal{P}_k}.$$

Hence, there exists some $\epsilon > 0$ such that

$$q^{p'} + \sum_{e \in p'} l^e(f^e) + \epsilon < \min\left\{1, \min_{m \notin \mathcal{P}_k} \sum_{e \in m} l^e(f^e) + q^m\right\}, \forall\, p' \in \overline{\mathcal{P}_k}.$$

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Now, let $q' = q + e_m$, where e_m is an $|\mathcal{P}|$ dimensional vector with value ϵ in the mth component if $m \in \overline{\mathcal{P}_k}$, and 0 otherwise. We can verify that, given price vector q', (f, z) satisfies all of the conditions in Proposition 1. Therefore, (f, z) is a WE at price q'. However, $(q', (f, z))$ has a strictly higher objective value than $(q, (f, z))$, which contradicts the fact that $(q, (f, z))$ is an ME. Therefore, $q^p = 1 - \sum_{e \in p} l^e(f^e)$ for every p with $f^p > 0$. Now if $f^p = 0$, condition (5) implies $q^p \geq 1 - \sum_{e \in p} l^e(f^e)$. Q.E.D.

We now return to the proof of Proposition 2. We first consider the following problem.

$$\max \quad \sum_{p \in \mathcal{P}} q^p f^p \tag{3.8}$$

$$\text{subject to} \quad f^p = \sum_{j \in \mathcal{J}_k} f^p_{k,j}, \quad \forall\, p \in \mathcal{P},$$

$$q^p = 1 - \sum_{e \in p} l^e(f^e), \text{ if } f^p = 0,$$

$$q^p \geq 0, \text{ if } f^p > 0,$$

$$(f, z) \in G(q),$$

where $G(q)$ is the set of vectors (f, z) that satisfy conditions (1)-(5) of Lemma 1. It can be shown that $(q, (f, z))$ is an optimal solution of problem (3.6) if and only if there exists a price $\bar{q}$ such that $(\bar{q}, (f, z))$ is an optimal solution of problem (3.8).

Now, we can rewrite problem (3.8) as

$$\max \quad \sum_{p \in \mathcal{P}}(1 - \sum_{e \in p} l^e(f^e)) f^p$$

$$\text{subject to} \quad f^p = \sum_{j \in \mathcal{J}_k} f^p_{k,j}, \quad \forall\, p \in \mathcal{P}$$

$$\sum_{p \in \mathcal{P}_k} f^p_{k,j} = t_{k,j}, \text{ if } z_{k,j} = 1,$$

$$f^p_{k,j} = 0, \, \forall\, p \in \mathcal{P}_k, \text{ if } z_{k,j} = 0,$$

$$f_{k,j} \geq 0, \; z_{k,j} \in \{0, 1\}, \; \forall\, j \in \mathcal{J}_k, \forall\, k,$$

or equivalently,

$$\max_{f_{k,j} \geq 0, z_{k,j} \in \{0,1\}} \quad \sum_{k} \sum_{j \in \mathcal{J}_k} \left(z_{k,j} t_{k,j} - \sum_{p \in \mathcal{P}_k} \sum_{e \in p} l^e(f^e) f_{k,j}^p \right) \tag{3.9}$$

$$\text{subject to} \quad f^p = \sum_{j \in \mathcal{J}_k} f_{k,j}^p, \quad \forall \, p \in \mathcal{P},$$

$$\sum_{p \in \mathcal{P}_k} f_{k,j}^p = t_{k,j}, \text{ if } z_{k,j} = 1$$

$$f_{k,j} \geq 0, \;\; z_{k,j} \in \{0,1\}, \; \forall \, j \in \mathcal{J}_k, \forall \, k.$$

This problem has an optimal solution (since for each z, the objective function is continuous and the constraint set is compact). This proves part (1) of Proposition 2. For part (2), we notice that problem (3.9) is the same as the social problem that maximizes the aggregate system utility as defined by Eq. (3.3). Hence, the result in part (2) of Proposition 2 follows. **Q.E.D.**

Proposition 2 shows that the monopolist service provider can achieve the social optimum in the case of network containing users with inelastic utilities. This result contrasts with pervasive inefficiencies observed in the routing models with selfish users, for example, as in [8].

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Chapter 4

Elastic User Utility

In this section, we study a network containing users with elastic utility functions.

4.1 Existence, Essential Uniqueness, and Price Sensitivity

Each price vector q defines a user subgame. Given the price vector, users play this subgame by choosing the flow rates and path flows that maximize their payoffs. If a WE exists, then at this WE, no user can increase his payoff by any deviation, so he does not have any incentive to deviate. We make a further assumption on link latency functions:

Assumption 2 Assume $l^e(f^e) \to \infty$ as $f^e \to C^e$, where C^e denotes the available capacity on link e.

This assumption on the latency functions serves to guarantee that no individual has an infinite demand. This assumption could be relaxed by assuming that, for each j, there exists a nonzero scalar B_j such that $u'_j(B_j) = 0$, which holds for the inelastic utility case.

Proposition 3 (Existence-Essential Uniqueness) Let Assumptions (1) and (2) hold. For a given price vector q, let the payoff function for each user in the network be defined as Eq. (2.2). Then for any $q \geq 0$, the user game has a WE. Moreover, the user flow rates and link loads at any WE are unique.

Proof: Given a price vector q, consider the following optimization problem

$$\text{maximize} \quad \sum_k \sum_j u_{k,j}(\Gamma_{k,j}) - \sum_{e \in \mathcal{E}} \int_0^{f^e} l^e(x)dx - \sum_p q^p f^p$$

$$\text{subject to} \quad \Gamma_{k,j} = \sum_{p \in \mathcal{P}_k} f_{k,j}^p, \quad \forall\, j \in \mathcal{J}_k, k \in \mathcal{W}$$

$$f^e = \sum_k \sum_{j \in \mathcal{J}_k} \sum_{p|e \in p, p \in \mathcal{P}_k} f_{k,j}^p, \quad \forall\, e \in \mathcal{E}$$

$$f^p = \sum_{j \in \mathcal{J}_k} f_{k,j}^p, \quad \forall\, p \in \mathcal{P}_k, k \in \mathcal{W}$$

$$f_{k,j}^p \geq 0, \quad \forall\, p \in \mathcal{P}_k, j \in \mathcal{J}_k, k \in \mathcal{W} \tag{4.1}$$

Notice that the first order necessary and sufficient conditions for problem (4.1) are exactly the same as those for WE at price q. Hence, both problems have the same set of optimal solutions. By Assumption (1), the objective function of problem (4.1) is continuous and the feasible set is compact. Therefore, problem (4.1) has an optimal solution which shows the existence of WE.

Now, consider the objective function of problem (4.1). Let Γ be the user flow rate vector $[\cdots, \Gamma_{k,j}, \cdots]$ and $f^{\mathcal{P}}$ be the path flow vector $[\cdots, f^p, \cdots]$. For a given q, since $u_{k,j}$ is strictly concave for all k and j and l^e is strictly increasing for all e, the objective function of problem (4.1) is a strictly concave function of Γ and $f^{\mathcal{P}}$. Therefore, the WE under q is unique for Γ and $f^{\mathcal{P}}$. Since $f^e = \sum_{p|e \in p} f^p$, the WE is also unique on $f^{\mathcal{E}} = [\cdots, f^e, \cdots]$.   **Q.E.D.**

Essential uniqueness of a WE is important for our analysis, since it implies that total flows on each path are uniquely defined. This result does not, however, imply the uniqueness of a WE. In fact, it is easy to establish that when there is one OD

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pair with at least two users with positive equilibrium flows and at least two paths with positive total flows, then there are infinitely many WEs.[1] We use this property of a WE in proving the following result, which will be essential in our subsequent analysis.

Lemma 3 Let Assumptions (1) and (2) hold. Given any price $q \geq 0$, let f be a WE, and Γ be the flow rate at price q. Define $f^p = \sum_{j \in \mathcal{J}_k} f^p_{k,j}$ as the flow on path p. Also define $\overline{\mathcal{P}_k} = \{p \mid f^p > 0, \ p \in \mathcal{P}_k\}$ and $\overline{\mathcal{J}_k} = \{j \mid \Gamma_{k,j} > 0, \ j \in \mathcal{J}_k\}$ for every k. Then

1. For every k, if $p \in \overline{\mathcal{P}_k}$ and $j \in \overline{\mathcal{J}_k}$,

$$u'_{k,j}(\Gamma_{k,j}) - \sum_{e \in p} l^e(f^e) - q^p = 0.$$

2. There exists a WE f such that $f^p_{k,j} > 0$ for all $p \in \overline{\mathcal{P}_k}$, $j \in \overline{\mathcal{J}_k}$, and for all k.

Proof: 1) Let $p \in \overline{\mathcal{P}_k}$, and $j \in \overline{\mathcal{J}_k}$. Since $\Gamma_{k,j} > 0$, there exists some path s such that $f^s_{k,j} > 0$, which implies by the first order conditions that

$$u'_{k,j}(\Gamma_{k,j}) - \sum_{e \in s} l^e(f^e) - q^s = 0 \tag{4.2}$$

and

$$u'_{k,j}(\Gamma_{k,j}) - \sum_{e \in s'} l^e(f^e) - q^{s'} \leq 0, \ \forall \ s' \in \mathcal{P}_k.$$

Combining the preceding two relations, we obtain

$$\sum_{e \in s'} l^e(f^e) - q^{s'} \geq \sum_{e \in s} l^e(f^e) - q^s, \ \forall \ s' \in \mathcal{P}_k.$$

Therefore,

$$\sum_{e \in s} l^e(f^e) - q^s = \min_{s' \in \mathcal{P}_k} \left\{ \sum_{e \in s'} l^e(f^e) - q^{s'} \right\}. \tag{4.3}$$

---

[1] This is because, for such user game, we can construct a new WE from a given WE by interchanging ϵ units of user j_1's flow on path p_1 with ϵ units of user j_2's flow on path p_2 (where p_1 and p_2 belong to the same OD pair and ϵ is less than or equal to the minimum of j_1's flow on path p_1 and j_2's flow on path p_2).

Now, since $f^p > 0$, there exists some j' such that $f_{k,j'}^p > 0$. Then,

$$u'_{k,j'}(\Gamma_{k,j'}) - \sum_{e \in p} l^e(f^e) - q^p = 0$$

and

$$u'_{k,j'}(\Gamma_{k,j'}) - \sum_{e \in s'} l^e(f^e) - q^{s'} \leq 0, \ \forall \ s' \in \mathcal{P}_k.$$

So, we have

$$\sum_{e \in p} l^e(f^e) - q^p = \min_{s' \in \mathcal{P}_k} \left\{ \sum_{e \in s'} l^e(f^e) - q^{s'} \right\}. \tag{4.4}$$

From equations (4.3) and (4.4), we get

$$\sum_{e \in p} l^e(f^e) - q^p = \sum_{e \in s} l^e(f^e) - q^s. \tag{4.5}$$

Substituting equation (4.5) into equation (4.2) yields the result

$$u'_{k,j}(\Gamma_{k,j}) - \sum_{e \in p} l^e(f^e) - q^p = 0.$$

2) Let f be a WE at the price q. We construct a new flow $\tilde{f}$ in the following way: If $j \notin \overline{\mathcal{J}_k}$ or $p \notin \overline{\mathcal{P}_k}$, set $\tilde{f}_{k,j}^p = 0$. Otherwise, set

$$\tilde{f}_{k,j}^p = \frac{\Gamma_{k,j} f^p}{\sum_{p \in \mathcal{P}_k} f^p}$$

which is > 0 because $j \in \overline{\mathcal{J}_k}$ and $p \in \overline{\mathcal{P}_k}$. Now, since

$$\tilde{f}^p = \sum_{j \in \overline{\mathcal{J}_k}} \tilde{f}_{k,j}^p = f^p, \ \forall \ p \in \mathcal{P}$$

and

$$\tilde{\Gamma}_{k,j} = \sum_{p \in \overline{\mathcal{P}_k}} \tilde{f}_{k,j}^p = \Gamma_{k,j}, \ \forall \ j \in \mathcal{J}_k, k \in \mathcal{W}.$$

$\tilde{f}$ is a WE such that $\tilde{f}_{k,j}^p > 0$ for all $p \in \overline{\mathcal{P}_k}$, $j \in \overline{\mathcal{J}_k}$, and for all k.   Q.E.D.

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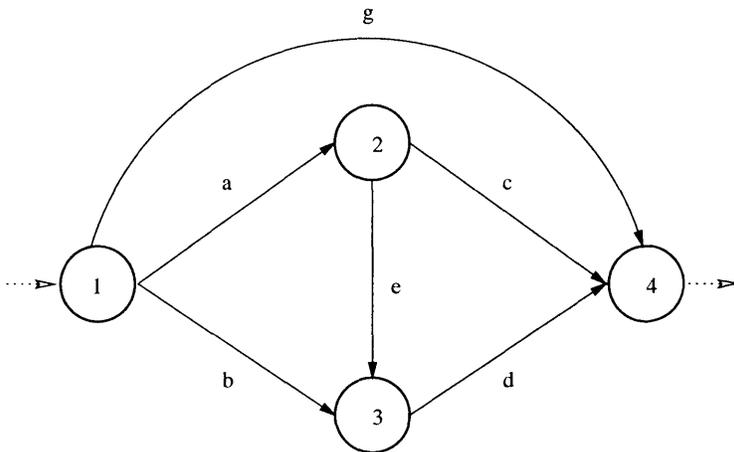
Figure 4-1: A network that violates the monotonicity of flow rates and the monotonicity of link loads.

It is informative to understand how link loads and users' flow rates change with prices. There are two natural conjectures in this context: As the price of a particular path increases, the amount of data transmitted on the other paths increase. Similarly, the flow rate of each user is a nondecreasing function of the price vector. These results were proven for networks with parallel links in [13]. The same results do not generalize to a general network topology, however.

In a general network where there are no prices and users have fixed demands, improving the latency function of one link (i.e., replacing $l^e(x)$ with $\overline{l}^e(x)$ such that $\overline{l}^e(x) \leq l^e(x)$ $\forall$ x for some link e) while keeping the rest unchanged, may cause all users to encounter higher latency costs. This phenomenon is known as the Braess' Paradox. We next demonstrate such a counterintuitive phenomenon in a network with users with elastic utilities. Consider the example in Figure 4-1, where a single user sends flow from node 1 to node 4. Assume that the user's utility function is $u(\Gamma) = 184\sqrt{8}\Gamma^{0.5}$, and the link latency functions are given by

$$l^a(f^a) = 10f^a, \quad l^b(f^b) = f^b, \quad l^c(f^c) = f^c,$$

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$$l^d(f^d) = 10f^d, \quad l^e(f^e) = f^e, \quad l^g(f^g) = f^g.$$

Given the price vector q, where $q^{\{a,c\}} = 50$, $q^{\{b,d\}} = 50$, $q^{\{a,e,d\}} = 10$, $q^{\{g\}} = 90$, the path flows at the WE are $f^{\{a,c\}} = f^{\{b,d\}} = f^{\{a,e,d\}} = f^{\{g\}} = 2$. Consider another price vector $\bar{q}$ where we increase the price of path $\{a, e, d\}$ to 14. Given $\bar{q}$, the path flows at the new WE are $\bar{f}^{\{a,c\}} = \bar{f}^{\{b,d\}} \approx 3.032$, $\bar{f}^{\{a,e,d\}} \approx 0.792$, $\bar{f}^{\{g\}} \approx 1.2721$. However,

$$\bar{f}^{\{g\}} < f^{\{g\}} = 2, \text{ and } \bar{\Gamma} \approx 8.1281 > \Gamma = 8.$$

This shows that at a higher price vector, the flow on an alternative path decreases and the total flow rate of the user increases. We will study Braess' paradox in general networks in more detail in Chapter 5.

4.2 Monopoly Price, Social Optimum, and Performance

In this section, we provide an explicit characterization of the monopoly price and compare the system performance at the monopoly equilibrium with the social optimum. Recall that the monopoly problem is

$$\max_{q \geq 0} \sum_p q^p f^p(q),$$

where $f^p(q)$ is the flow on path p at a WE for a given price vector q. Under Assumptions 1 and 2, we can assume

$$0 \leq q^p \leq \min_{j \in \mathcal{J}_k} u'_{k,j}(0), \ \forall \ p \in \mathcal{P}_k, \ k,$$

and by an argument similar to the one given in [13], we can show that $f^p(q)$ is continuous in q for all p. Therefore, problem (2.3) has an optimal solution. We now look at the following proposition which is essential to our analysis.

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Proposition 4 Let Assumptions (1) and (2) hold, and let (q, f) be an ME. Let $\overline{\mathcal{P}} = \cup_k \overline{\mathcal{P}_k}$ where $\overline{\mathcal{P}_k} = \{p | p \in \mathcal{P}_k, f^p > 0\}$. Then $q^p > 0$, $\forall p \in \overline{\mathcal{P}}$.

Proof: To arrive at a contradiction, we assume that there exists a path $p' \in \overline{\mathcal{P}_{k'}}$ and $q^{p'} = 0$. From Lemma 3, we know that since $p' \in \overline{\mathcal{P}_k}$, there exists some WE such that for all $j \in \overline{\mathcal{J}_{k'}}$, $f^{p'}_{k',j} > 0$. Since $u_{k',j}$ is strictly concave and $u'_{k',j}$ is continuous, we can pick an $\epsilon > 0$ such that for every $j \in \overline{\mathcal{J}_{k'}}$, there exist a $0 < \delta_j < f^{p'}_{k',j}$ satisfies the following equation.

$$u'_{k',j}(\Gamma_{k',j} - \delta_j) - u'_{k',j}(\Gamma_{k',j}) = \epsilon$$

Notice that for every $e \in p'$, $f^e > \sum_{j \in \overline{\mathcal{J}_{k'}}} \delta_j$. We define a new price vector $\overline{q}$ as

$$\overline{q}^p = q^p + \sum_{e | e \in p, e \in p'} \left(l^e(f^e) - l^e(f^e - \sum_{j \in \overline{\mathcal{J}_{k'}}} \delta_j) \right) + \epsilon, \text{if } p \in \mathcal{P}_{k'};$$

$$\overline{q}^p = q^p + \sum_{e | e \in p, e \in p'} \left(l^e(f^e) - l^e(f^e - \sum_{j \in \overline{\mathcal{J}_{k'}}} \delta_j) \right), \text{otherwise.} \tag{4.6}$$

Since l is strictly increasing, $\overline{q}^p > q^p$ if $p \in \mathcal{P}_{k'}$ and $\overline{q}^p \geq q^p$ otherwise. Now consider the flow $\overline{f}$ that satisfies the following conditions:

$$\begin{cases} \overline{f}^{p'}_{k',j} = f^{p'}_{k',j} - \delta_j, & \forall j \in \overline{\mathcal{J}_{k'}}; \\ \overline{f}^p_{k,j} = f^p_{k,j}, & \text{otherwise.} \end{cases} \tag{4.7}$$

Now, we will show $\overline{f}$ is a WE for the price vector $\overline{q}$. From Eqs. (4.7), we have

$$\begin{cases} \overline{f}^e = f^e - \sum_{j \in \overline{\mathcal{J}_{k'}}} \delta_j, & \text{if } e \in p'; \\ \overline{f}^e = f^e, & \text{otherwise.} \end{cases} \tag{4.8}$$

$$\begin{cases} \overline{\Gamma}_{k',j} = \Gamma_{k',j} - \delta_j, & \forall j \in \overline{\mathcal{J}_{k'}}; \\ \overline{\Gamma}_{k,j} = \Gamma_{k,j}, & \text{otherwise.} \end{cases} \tag{4.9}$$

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From Eqs. (4.6), (4.8), and (4.9), we see that for every $p \in \mathcal{P}_{k'}$ and $j \in \overline{\mathcal{J}}_{k'}$

$$
\begin{aligned}
u'_{k',j}(\overline{\Gamma}_{k',j}) &= u'_{k',j}(\Gamma_{k',j} - \delta_j) \\
&= u'_{k',j}(\Gamma_{k',j}) + \epsilon \\
&\leq \sum_{e \in p} l^e(f^e) + q^p + \epsilon \\
&= \sum_{e \in p} l^e(f^e) - \sum_{e | e \in p, e \in p'} \left(l^e(f^e) - l^e(f^e - \sum_{j \in \overline{\mathcal{J}}_{k'}} \delta_j) \right) + \overline{q}^p \\
&= \sum_{e \in p} l^e(f^e) - \sum_{e | e \in p, e \in p'} \left(l^e(f^e) - l^e(\overline{f}^e) \right) + \overline{q}^p \\
&= \sum_{e \in p} l^e(\overline{f}^e) + \overline{q}^p. \tag{4.10}
\end{aligned}
$$

We know $\overline{f}^p_{k',j} > 0$ iff $f^p_{k',j} > 0$. Hence, the equality of Eq. (4.10) holds if $\overline{f}^p_{k',j} > 0$. Similarly, for every $p \in \mathcal{P}_k$, $j \in \overline{\mathcal{J}}_k$, and $k \neq k'$

$$
\begin{aligned}
u'_{k,j}(\overline{\Gamma}_{k,j}) &= u'_{k,j}(\Gamma_{k,j}) \\
&\leq \sum_{e \in p} l^e(f^e) + q^p \\
&= \sum_{e \in p} l^e(f^e) - \sum_{e | e \in p, e \in p'} \left(l^e(f^e) - l^e(f^e - \sum_{j \in \overline{\mathcal{J}}_{k'}} \delta_j) \right) + \overline{q}^p \\
&= \sum_{e \in p} l^e(\overline{f}^e) + \overline{q}^p. \tag{4.11}
\end{aligned}
$$

Again, the equality of Eq. (4.11) holds if $\overline{f}^p_{k,j} > 0$. For price $\overline{q}$, Eqs. (4.10) and (4.11) show that $\overline{f}$ satisfies the first order necessary and sufficient conditions:

$$
u'_{k',j}(\overline{\Gamma}_{k',j}) - \sum_{e \in p} l^e(\overline{f}^e) - \overline{q}^p
\begin{cases}
= 0, & \text{if } \overline{f}^p_{k,j} > 0; \\
\leq 0, & \text{if } \overline{f}^p_{k,j} = 0.
\end{cases}
$$

Therefore, $\overline{f}$ is a WE with price $\overline{q}$. However, since $\overline{q}^{p'} > q$ and $\overline{f}^{p'} > 0$

$$
\overline{f}^{p'} \overline{q}^{p'} > 0. \tag{4.12}
$$

Then equation (4.12) together with $\bar{q} \geq q$ and $q^{p'} = 0$

$$\sum_{p \in \mathcal{P}} \bar{f}^p \bar{q}^p = \bar{f}^{p'} \bar{q}^{p'} + \sum_{p \neq p'} \bar{f}^p \bar{q}^p > \sum_{p \neq p'} f^p q^p = \sum_{p \in \mathcal{P}} f^p q^p.$$

Therefore, (f, q) is not an ME and this yields a contradiction. Hence, the result follows. **Q.E.D.**

Now, we can derive an explicit characterization of the monopoly prices. Let (f, q) be a ME and for each k, let $\mathcal{I}_k = \{1, \cdots, |\mathcal{P}_k|\}$ be the set of indices of $\mathcal{P}_k$ and $\{1, \cdots, |\mathcal{J}_k|\}$ be the set of indices of $\mathcal{J}_k$. Without loss of generality, we assume that user $1 \in \overline{\mathcal{J}_k}$ and path $1 \in \overline{\mathcal{P}_k}$ for every k such that $\mathcal{J}_k \neq \emptyset$. Using the necessary and sufficient optimality conditions of a WE at a price vector q together with Lemma 3, we can see that if (f, q) is a ME, then $([f^p]_{p \in \overline{\mathcal{P}}}, [\Gamma_j]_{j \in \mathcal{J}}, q)$ is an optimal solution of the following problem.

$$\text{maximize} \quad \sum_{p \in \overline{\mathcal{P}}} q^p f^p \tag{4.13}$$

$$\text{subject to} \quad u'_{k,1}(\Gamma_{k,1}) - \sum_{e \in p_k^i} l^e\left(\sum_{p | e \in p} f^p\right) - q^{p_k^i} = 0, \ \forall \ p_k^i \in \overline{\mathcal{P}_k}, k \in \mathcal{W} \tag{4.14}$$

$$u'_{k,1}(\Gamma_{k,1}) - \sum_{e \in p_k^i} l^e\left(\sum_{p | e \in p} f^p\right) - q^{p_k^i} \leq 0, \ \forall \ p_k^i \notin \overline{\mathcal{P}_k}, k \in \mathcal{W} \tag{4.15}$$

$$u'_{k,j}(\Gamma_{k,j}) - \sum_{e \in p_k^1} l^e\left(\sum_{p | e \in p} f^p\right) - q^{p_k^1} = 0, \ \forall \ j \in \overline{\mathcal{J}_k} - \{1\}, k \in \mathcal{W} \tag{4.16}$$

$$u'_{k,j}(\Gamma_{k,j}) - \sum_{e \in p_k^1} l^e\left(\sum_{p | e \in p} f^p\right) - q^{p_k^1} \leq 0, \ \forall \ j \notin \overline{\mathcal{J}_k}, \forall \ k \in \mathcal{W} \tag{4.17}$$

$$\sum_{p \in \overline{\mathcal{P}_k}} f^p = \sum_{j \in \overline{\mathcal{J}_k}} \Gamma_{k,j}, \forall \ k \in \mathcal{W} \tag{4.18}$$

$$\Gamma_{k,j} \geq 0, \ \forall \ j \in \overline{\mathcal{J}_k}, k \in \mathcal{W},$$

$$f^p \geq 0, \ \forall \ p \in \mathcal{P},$$

$$q^p \geq 0, \ \forall \ p \in \mathcal{P}.$$

Note that we use the necessary and sufficient optimality conditions for a WE to write

problem (4.13) in variables $([f^p]_{p\in\overline{\mathcal{P}}}, [\Gamma_j]_{j\in\mathcal{J}}, q)$ instead of variables $(f^p_{k,j}, q)$ and use Lemma 3 to eliminate the redundant constraints. This reduction in the dimension of the feasible set allows us to show that the regularity constraint qualification is satisfied (i.e., the constraint gradients of problem (4.13) are linearly independent at the optimal solution). Thus, the nonconvex problem (4.13) admits Lagrange multipliers, which will be the key in proving the subsequent proposition. This is stated in the following Lemma.

Lemma 4 The constraint gradients of problem (4.13) at any feasible solution $([f^p], [\Gamma], q)$ are linearly independent.

Proof: Let (f^p, Γ, q) be a feasible solution of problem (4.13). Let $\mathcal{C}_k$ be the set of constraints gradients for OD pair k at (f^p, Γ, q), and $\mathcal{C} = \cup_k \mathcal{C}_k$. We first show that the vectors in $\mathcal{C}_k$ are linearly independent. Consider the matrix R, where each row corresponds to a vector in $\mathcal{C}_k$ (for simplification, we do not include the entries that are 0 in all rows).

$$
\begin{pmatrix}
u''_{k,1} & \cdots & 0 & S_{1\times|\mathcal{P}|} & -1 & \cdots & 0 \\
u''_{k,1} & \cdots & 0 & & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & M_{(|\mathcal{P}_k|-1)\times|\mathcal{P}|} & \vdots & \vdots & \vdots \\
u''_{k,1} & \cdots & 0 & & 0 & \cdots & -1 \\
& \cdots & 0 & & -1 & \cdots & 0 \\
\vdots & \ddots & \vdots & N_{|\mathcal{J}_k|\times|\mathcal{P}|} & \vdots & \vdots & \vdots \\
0 & \cdots & u''_{k,|\mathcal{J}_k|} & & -1 & \cdots & 0 \\
-1 & \cdots & -1 & T_{1\times|\mathcal{P}|} & 0 & \cdots & 0
\end{pmatrix}
$$

Let r_i be the ith row vector of R and $r_i(x)$ be the entry in r_i corresponding to variable x. Note that M can be a arbitrary matrix, but N is a matrix with each of its row equal to vector S.

We first show that the vectors $\{r_1, \cdots, r_{|\mathcal{P}_k|+|\mathcal{J}_k|}\}$ are linearly independent. Let

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$r_i \in \{r_2, \cdots, r_{|\mathcal{P}_k|+|\mathcal{J}_k|}\}$, then there exists x such that $r_i(x) \neq 0$ but $r_j \neq 0$ for all $j \neq i$. Therefore, $\{r_2, \cdots, r_{|\mathcal{P}_k|+|\mathcal{J}_k|}\}$ are linearly independent. Suppose that r_1 can be written as a linear combination of vectors in $\{r_2, \cdots, r_{|\mathcal{P}_k|+|\mathcal{J}_k|}\}$. Again, we let $r_i \in \{r_2, \cdots, r_{|\mathcal{P}_k|+|\mathcal{J}_k|}\}$, then there exists x such that $r_i(x) \neq 0$ but $r_j(x) = 0$ for all $j \neq i$. Therefore, $r_1 = 0$. However, $r_1 \neq 0$ and therefore it cannot be written as a linear combination of vectors in $\{r_2, \cdots, r_{|\mathcal{P}_k|+|\mathcal{J}_k|}\}$. As a result, vectors $\{r_1, \cdots, r_{|\mathcal{P}_k|+|\mathcal{J}_k|}\}$ are linearly independent.

We then consider the last row, $r_{|\mathcal{P}_k|+|\mathcal{J}_k|+1}$, of R. We assume it can be written as a linear combination of the vectors in $\{r_1, \cdots, r_{|\mathcal{P}_k|+|\mathcal{J}_k|}\}$:

$$r_{|\mathcal{P}_k|+|\mathcal{J}_k|+1} = y_1 r_1 + \sum_{i=2}^{|\mathcal{P}_k|} y_i r_i + \sum_{j=1}^{|\mathcal{J}_k|} t_j r_{|\mathcal{P}_k|+j}.$$

For each $r_{i'} \in \{r_2, \cdots, r_{|\mathcal{P}_k|}\}$, $\exists\, x$ such that $r_{i'}(x) \neq 0$ but $r_{j'}(x) = 0$ for all $i' \neq j'$. Therefore, y_i is 0 for all $i = 2, \cdots, |\mathcal{P}_k|$ and

$$r_{|\mathcal{P}_k|+|\mathcal{J}_k|+1} = y_1 r_1 + \sum_{j=1}^{|\mathcal{J}_k|} t_j r_{|\mathcal{P}_k|+j}.$$

We also see that

$$r_{|\mathcal{P}_k|+|\mathcal{J}_k|+1}(q^{p_k^1}) = 0, \text{ and}$$

$$r_{k,i}(q^{p_k^1}) = -1, \text{ if } i = 1, |\mathcal{P}_k| + 1, \cdots, |\mathcal{P}_k| + |\mathcal{J}_k|.$$

Therefore,

$$y_1 + \sum_{j=1}^{|\mathcal{J}_k|} t_j = 0 \tag{4.19}$$

However, since each row of N is identical to S, Eq. (4.19) implies all entries in T are 0. This yields a contradiction. Therefore, $r_{|\mathcal{P}_k|+|\mathcal{J}_k|+1}$ can not be written as a linearly combination of the vectors in $\{r_1, \cdots, r_{|\mathcal{P}_k|+|\mathcal{J}_k|}\}$. As a result, the vectors in $\mathcal{C}_k$ are linearly independent.

We next show that the vectors in $\mathcal{C}$ are linearly independent. Let $\mathcal{W}_k$ be the

subspace spanned by the vectors in $\mathcal{C}_k$. We will show $\mathcal{W}_k \cap \mathcal{W}_{k'} = \{0\}$ for $k \neq k'$. Assume there exist a vector $w \in \mathcal{W}_k, \mathcal{W}_{k'}$, and $w \neq 0$. Since $\mathcal{J}_k \cap \mathcal{J}'_k = \emptyset$ and $\mathcal{P}_k \cap \mathcal{P}'_k = \emptyset$ if $k \neq k'$. Therefore,

$$w(u''_{k,j}) = 0, \forall\ j; \quad w(u''_{k,j'}) = 0, \forall\ j'; \quad w(q^p) = 0, \forall\ p. \qquad (4.20)$$

Now, we write w as a linear combination of the gradients of $\mathcal{C}_k$:

$$w = y'_1 r_1 + \sum_{i=2}^{|\mathcal{P}_k|} y'_i r_i + \left(\sum_{j=1}^{|\mathcal{J}_k|} t'_j r_{|\mathcal{P}_k|+j} \right) + t'_0 r_{|\mathcal{P}_k|+|\mathcal{J}_k|+1}.$$

For each $r_{i'} \in \{r_2, \cdots, r_{|\mathcal{P}_k|}\}$, $\exists\ x$ such that $r_{i'}(x) \neq 0$ but $r_{j'}(x) = 0$ for all $i' \neq j'$. Therefore, y'_i is 0 for all $i = 2, \cdots, |\mathcal{P}_k|$. Therefore,

$$w = y'_1 r_1 + \left(\sum_{j=1}^{|\mathcal{J}_k|} t'_j r_{|\mathcal{P}_k|+j} \right) + t'_0 r_{|\mathcal{P}_k|+|\mathcal{J}_k|+1}.$$

Also, since

$$w(q^{p^1_k}) = r_{k,|\mathcal{P}_k|+|\mathcal{J}_k|+1}(q^{p^1_k}) = 0;$$

$$r_{k,i}(q^{p^1_k}) = -1, \text{ for } i = 1, |\mathcal{P}_k| + 1, \cdots, |\mathcal{P}_k| + |\mathcal{J}_k|$$

then,

$$y'_1 + \sum_{j=1}^{|\mathcal{J}_k|} t'_j = 0 \qquad (4.21)$$

From (4.20), (4.21) and the fact that each row of N is identical to S, we can see

$$w = [0, \cdots, 0, T, 0, \cdots, 0]$$

By applying the same argument to OD pair k', we also have

$$Sw = [0, \cdots, 0, T', 0, \cdots, 0]$$

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However, since $k \neq k'$, $T \neq T'$. So by contradiction, $\mathcal{W}_k \cap \mathcal{W}_{k'} = \{0\}$ for $k \neq k'$. Since the vectors in $\mathcal{C}_k$ form a basis of $\mathcal{W}_k$, we conclude that the vectors in $\mathcal{C}$ are linearly independent. **Q.E.D.**

Proposition 5 Let Assumptions (1) and (2) hold. Assume further that $u_{k,j}$ is twice continuously differentiable for each j and k, and l^e is continuously differentiable for each e. Let (f, q) be an ME, then for every path p in $\overline{\mathcal{P}_k}$, we have

$$q^p = \left(\sum_{e \in p} f^e (l^e)'(f^e) \right) + \frac{\sum_{p \in \overline{\mathcal{P}_k}} f^p}{-\sum_{j \in \mathcal{J}_k} \frac{1}{u''_{k,j}(\Gamma_{k,j})}} \ . \tag{4.22}$$

Proof: Let $(q, [f^p]_{p \in \overline{\mathcal{P}}}, [\Gamma_j]_{j \in \mathcal{J}_k})$ be an optimal solution of problem (4.13). Define $\mathcal{I}_k$ to be the set of the indices of the paths in $\mathcal{P}_k$ and $\overline{\mathcal{I}}_k$ to be the set of indices of the paths in $\overline{\mathcal{P}}_k$. By Lemma 4, there exist Lagrange Multipliers for problem (4.13). We assign λ_k^i to the constraints (4.14) and (4.15), $\mu_{k,j}$ to the constraints (4.16) and (4.18), and finally $\xi_{k,j}$ to the constraints (4.17). The Lagrangian function $L(q, f, \lambda, \mu, \xi)$ can be written as

$$
\begin{aligned}
L(q, f, \lambda, \mu, \xi) = \sum_{p \in \overline{\mathcal{P}}} q^p f^p &+ \sum_k \sum_{i \in \mathcal{I}_k} \lambda_k^i [u'_{k,1}(\Gamma_{k,1}) - \sum_{e \in p_k^i} l^e(f^e) - q^{p_k^i}] \\
&+ \sum_k \sum_{j \in \mathcal{J}_k - \{1\}} \mu_{k,j} [u'_{k,j}(\Gamma_{k,j}) - \sum_{e \in p_k^1} l^e(f^e) - q^{p_k^1}] \\
&+ \sum_k \sum_{j \notin \mathcal{J}_k} \xi_{k,j} [u'_{k,j}(\Gamma_{k,j}) - \sum_{e \in p_k^1} l^e(f^e) - q^{p_k^1}] \\
&+ \sum_k \mu_{k,1} [\sum_{p \in \mathcal{P}_k} f^p - \sum_{j \in \mathcal{J}_k} \Gamma_{k,j}].
\end{aligned}
$$

If the monopoly price vector q is not greater than 0, we can find another monopoly price vector q' such that

$$(q')^p = q^p, \text{ if } p \in \overline{\mathcal{P}};$$
$$(q')^p > 0, \ \forall \ p.$$

Therefore, without loss of generality we can assume that the ME price vector q satisfies $q > 0$. So, for each OD pair k,

$$\frac{\partial L}{\partial q^{p_k^1}} = 0 \rightarrow f^{p_k^1} = \lambda_k^1 + \sum_{j \in \mathcal{J}_k - \{1\}} \mu_{k,j} + \sum_{j \notin \mathcal{J}_k} \xi_{k,j}, \qquad (4.23)$$

$$\frac{\partial L}{\partial q^{p_k^i}} = 0 \rightarrow f^{p_k^i} = \lambda_k^i, \ \forall \ i \in \overline{\mathcal{I}_k}, \ i \neq 1, \qquad (4.24)$$

$$\frac{\partial L}{\partial q^{p_k^i}} = 0 \rightarrow 0 = \lambda_k^i, \ if \ i \notin \overline{\mathcal{I}_k}. \qquad (4.25)$$

Recall that $\overline{\mathcal{P}} = \{p \mid p \in \mathcal{P}, f^p > 0\}$ and problem (4.13) is defined on $\overline{\mathcal{P}}$ but not $\mathcal{P}$. Therefore, for each $f^p \in \overline{\mathcal{P}}$, we have

$$\frac{\partial L}{\partial f^{p_k^i}} = 0 \rightarrow q^{p_k^i} - \sum_m \left[\sum_{n \in \mathcal{I}_m} \lambda_m^n \left(\sum_{e \mid e \in p_k^i, e \in p_m^n} (l^e)'(f^e) \right) \right] -$$

$$\sum_m \left(\sum_{j \in \overline{\mathcal{J}_m} - \{1\}} \mu_{m,j} + \sum_{j \notin \overline{\mathcal{J}_m}} \xi_{m,j} \right) \left(\sum_{e \mid e \in p_k^i, e \in p_m^1} (l^e)'(f^e) \right) + \mu_{k,1} = 0.$$

Simplifying the preceding equation, we get

$$q^{p_k^i} - \sum_m \left[\sum_{n \in \overline{\mathcal{I}_m} - 1} \lambda_m^n \left(\sum_{e \mid e \in p_k^i, e \in p_m^n} (l^e)'(f^e) \right) + \right.$$

$$\left. \left(\lambda_m^1 + \sum_{j \in \overline{\mathcal{J}_m} - \{1\}} \mu_{m,j} + \sum_{j \notin \overline{\mathcal{J}_m}} \xi_{m,j} \right) \sum_{e \mid e \in p_k^i, e_1 p_m^1} (l^e)'(f^e) \right] + \mu_{k,1} = 0.$$

Substituting Eqs. (4.25) into the equation above, we have

$$q^{p_k^i} - \sum_m \left[\sum_{n \in \overline{\mathcal{I}_m} - 1} f^{p_m^n} \left(\sum_{e \mid e \in p_k^i, e \in p_m^n} (l^e)'(f^e) \right) + f^{p_m^1} \left(\sum_{e \mid e \in p_k^i, e \in p_m^1} (l^e)'(f^e) \right) \right] + \mu_{k,1} = 0$$

and then

$$q^{p_k^i} - \sum_{e \in p_k^i} \left[\left(\sum_m \sum_{\{n \mid n \in \overline{\mathcal{I}_m}, e \in p_m^n\}} f^{p_m^n} \right) (l^e)'(f^e) \right] \mu_{k,1} = 0.$$

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Therefore,

$$q^{p_k^i} - \sum_{e \in p_k^i} f^e(l^e)'(f^e) + \mu_{k,1} = 0, \ \forall \ k, i \in \overline{\mathcal{I}_k}. \tag{4.26}$$

Also for the set of flow rate variables, we have:

$$\frac{\partial L}{\partial \Gamma_{k,1}} = 0 \rightarrow u''_{k,1}(\Gamma_{k,1}) \left(\sum_i \lambda_k^i \right) - \mu_{k,1} = 0, \ \forall \ k, \tag{4.27}$$

$$\frac{\partial L}{\partial \Gamma_{k,j}} = 0 \rightarrow \mu_{k,j} u''_{k,j}(\Gamma_{k,j}) - \mu_{k,1} = 0, \ \forall \ j \in \overline{\mathcal{J}_k} - 1, k, \tag{4.28}$$

$$\frac{\partial L}{\partial \Gamma_{k,j}} = 0 \rightarrow \xi_{k,j} u''_{k,j}(\Gamma_{k,j}) \leq 0, \ \forall \ j \notin \overline{\mathcal{J}_k}, \forall \ k. \tag{4.29}$$

Since $\xi_{k,j} \leq 0$ for all k and j, and $u''_{k,j}(\Gamma_{k,j}) \leq 0$ for all k and j, Eq. (4.29) implies that $\xi_{k,j} = 0$ for all k and j. Therefore, summing all the equations in (4.28), we get

$$\sum_{j \in \overline{\mathcal{J}_k} - 1} \mu_{k,j} = \mu_{k,1} \sum_{j \in \overline{\mathcal{J}_k} - 1} \frac{1}{u''_{k,j}(\Gamma_{k,j})}, \ \forall \ k. \tag{4.30}$$

From Eqs. (4.27) and (4.30), we obtain

$$\sum_{j \in \overline{\mathcal{J}_k} - 1} \mu_{k,j} + \left(\sum_i \lambda_k^i \right) = \mu_{k,1} \sum_{j \in \overline{\mathcal{J}_k}} \frac{1}{u''_{k,j}(\Gamma_{k,j})}, \ \forall \ k. \tag{4.31}$$

Eqs. (4.23), (4.24), (4.25), and (4.31) imply that

$$\sum_{p \in \overline{\mathcal{P}_k}} f^p = \mu_{k,1} \sum_{j \in \overline{\mathcal{J}_k}} \frac{1}{u''_{k,j}(\Gamma_{k,j})}, \ \forall \ k. \tag{4.32}$$

Substituting Eq. (4.32) into Eq. (4.26) we

$$q^p = \left(\sum_{e \in p} f^e(l^e)'(f^e) \right) + \frac{\sum_{p \in \overline{\mathcal{P}_k}} f^p}{- \sum_{j \in \overline{\mathcal{J}_k}} \frac{1}{u''_{k,j}(\Gamma_{k,j})}} \ .$$

Q.E.D.
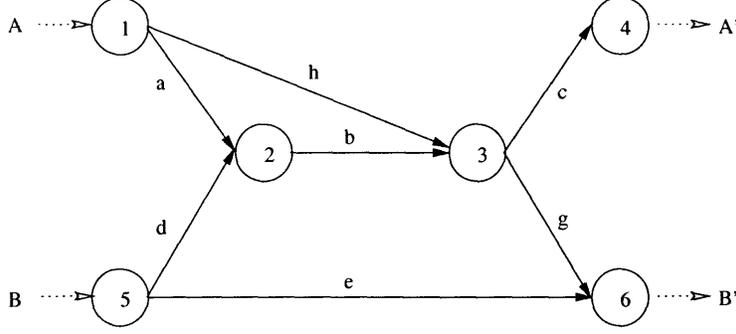
Figure 4-2: A simple general network

This proposition shows that the monopoly price is given by two terms: The first term is the "marginal congestion cost" (which corresponds to a Pigovian tax on the externality created by the users[11]). This amounts to charging every user the marginal increase in congestion by sending an extra unit of data. It is well-known that this is the price that a network planner maximizing the total system performance would charge in order to force users to internalize the congestion effects (resulting in the social optimum) [1213]. The second term is a markup above this given by the profit-maximizing objective of the service provider. Which of these two terms is dominant will determine the relative performance of the monopoly equilibrium compared to a situation without prices and to the social optimum.

Example 1: We consider a simple general network as given in Figure (4-2). We have two users (A and B) and 4 paths ($\{h, c\}, \{a, b, c\}, \{d, b, g\}, \{e\}$) in the network. The utility functions of the users and the latency functions of the links are given by

$$u_A(\Gamma_A) = 200(\Gamma_A)^\alpha, \quad u_B(\Gamma_B) = 200(\Gamma_B)^\alpha,$$

$$l^e(f^e) = (f^e)^\beta, \quad \forall e \in \mathcal{E}.$$
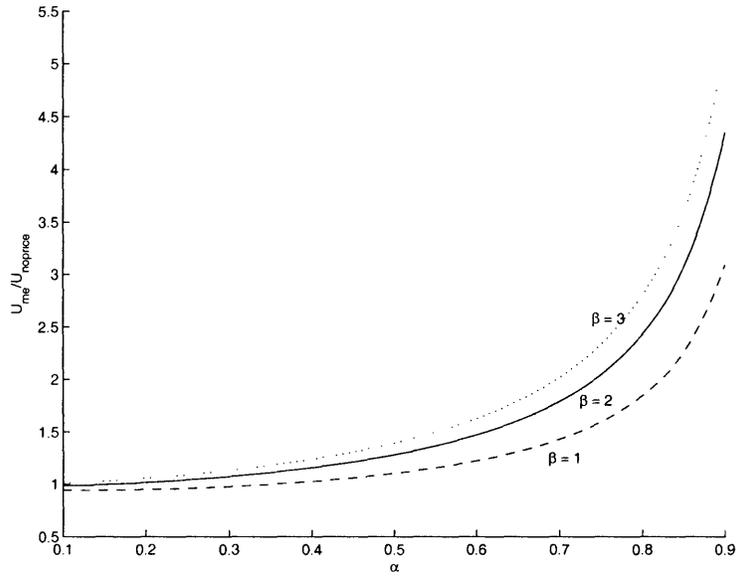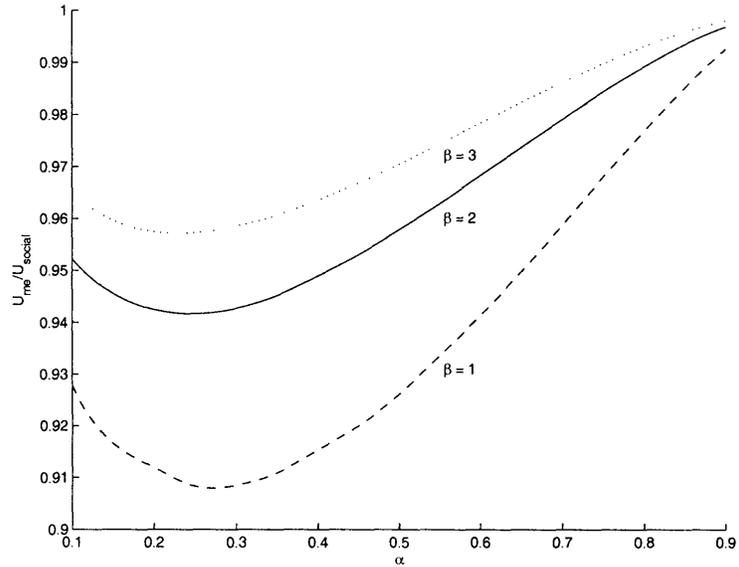
Figure 4-3: a) Performance of ME over WE at price 0 b) Performance of ME over Social Optimum

Let U_{me}, U_{social}, and U_0 be the total system utility,

$$\sum_k \left[\sum_{j \in \mathcal{J}_k} u_{k,j}(\Gamma_{k,j}) - \sum_{p \in \mathcal{P}_k} \left(\sum_{e \in p} l^e(f^e) \right) f^p \right]$$

at the monopoly equilibrium, social optimum, and at the WE at 0 prices, respectively. The plot of the ratios U_{me}/U_0 and U_{me}/U_{social} as a function of different values of α and β are given in Figures 4-3(a) and (b), respectively.

The results shown in Figure 4-3 are intuitive. The first panel shows that as β increases, performance of the monopoly equilibrium improves relative to an equilibrium without any prices (e.g., as in [13]). This is because higher values of β imply that latencies are more sensitive to link load and thus correspond to greater congestion effects (externalities), which are internalized in the monopoly equilibrium, but not in the equilibrium without prices. It also shows that performance improves as α increases. Greater α corresponds to a more linear utility function, and as Eq. (4.22) shows the markup is smaller when the utility function is less concave, reducing the monopoly distortions. The second panel is similar, however, it shows that the performance of the monopoly equilibrium relative to social optimal with respect to α is non-monotonic. The reason why values of α close to 1 improve the performance of the monopoly equilibrium is the same as above. However, the monopoly equilibrium also performs relatively well for very small values of α. This is because, in this case, even though the markup is substantial, individuals have a very high marginal utility of data transmission at low flow rates and choose not to reduce their flow rates much in response to this high markup, thus system performance does not suffer much.

4.3 Consistency between Path Pricing and Link Pricing

Given the monopoly price characterization, we can show that under certain conditions the ME under path pricing is consist with the ME under link pricing.

Proposition 6 In a network, if each path contains at least one link that is not used by the paths from other OD pair, then the ME (f, q) under path pricing is consist with the ME $(f', q^{\mathcal{E}})$ under link pricing which means the link loads in both equilibrium are the same and the prices of the paths with positive flow are the same.

Proof: To see this, we consider a network G satisfying the condition stated above. Let (f, q) be an ME under path pricing. The service provider can use the following link price, $q^{\mathcal{E}}$:

- For each path $p \in \overline{\mathcal{P}}$ in (f, q), let k be the OD pair such that $p \in \mathcal{P}_k$. Pick one and only one link e in p which is not used by any path from other OD pairs. Set its price to be

$$f^e(l^e)'(f^e) + \frac{\sum_{p \in \overline{\mathcal{P}_k}} f^p}{-\sum_{j \in \mathcal{J}_k} \frac{1}{u''_{k,j}(\Gamma_{k,j})}}$$

- For each path $p \notin \overline{\mathcal{P}}$ in (f, q), let k be the OD pair such that $p \in \mathcal{P}_k$. Pick a link e in p which is not used by any path from other OD pairs. Set its price to be ∞.

- For the remaining links, if the link has a positive flow in (f, q), set its price to be $f^e(l^e)'(f^e)$, otherwise, set its price to be ∞.

Let f' be a WE under link price $q^{\mathcal{E}}$. If a path $p \in \overline{\mathcal{P}}$ in (f, q), then in $(f', q^{\mathcal{E}})$ its price is

$$\overline{q^p} = \sum_{e \in p} \overline{q^e} = \left(\sum_{e \in p} f^e(l^e)'(f^e) \right) + \frac{\sum_{p \in \overline{\mathcal{P}_k}} f^p}{-\sum_{j \in \mathcal{J}_k} \frac{1}{u''_{k,j}(\Gamma_{k,j})}} \ .$$

From Proposition (5), we can see that $\overline{q^p} = q^p$ for every $p \in \overline{\mathcal{P}}$. Now, if a path $p \notin \overline{\mathcal{P}}$ in (f, q), then in $(f', q^{\mathcal{E}})$ its price is ∞ and its flow is 0. Therefore, $f = f'$ and

$$\sum_p \left(f^p \sum_{e \in p} q^e \right) = \sum_p f^p q^p.$$

Hence, the ME, (f, q), under path pricing is consist with the ME, $(f', q^{\mathcal{E}})$ under link pricing. **Q.E.D.**

Chapter 5

Braess' Paradox

In this chapter, we address an important performance problem in general topology network: Braess' Paradox.

5.1 Classical Braess' Paradox and Generalized Braess' Paradox

Service providers often face the problem of how to increase the network capacity. For example, where should they add a new link? One might expect that adding a new link can always increase or at least does not decrease the performance of the network. This assumption, however, is not true in a general topology network. The first example of this counterintuitive phenomenon is given by Braess in 1968 [16] and therefore known as Braess' Paradox. Braess' Paradox states the counterintuitive fact that adding a link to a network might cause all users to be worse off than in the previous equilibrium. This phenomenon is due to the non-cooperative nature of the selfish users, as each user only wants to minimize his travel cost without considering the travel costs of other users. Braess' Paradox has been recognized and studied in different kinds of networks. For example, Hagstrom and Abrams [28] outlined a characterization of Braess' Paradox in traffic networks. Steinberg and Zangwill

[33] gave necessary and sufficient conditions for the existence of Braess' Paradox in a transportation network under limited assumptions. Cohen and Kelly [26] also studied an example of Braess' Paradox in a queueing network. A detailed survey of research on Braess' Paradox can be found in [23] and [31].

The observation of Braess' Paradox motivated research in methods of upgrading the network capacity without degrading network performance. Some proposed methods were:

1. Multiplying the capacity of each link by some constant factor $\alpha > 1$ [29, 30] or a link dependent factor $\alpha_l >$ number of users [30].

2. Adding a direct link between the source and the destination [29, 30, 32].

3. Increasing the capacity of a direct link [30].

These methods emerged as results of studies in sensitivity analysis. In particular, methods (1)-(2) are motivated by the sensitivity result that states that the equilibrium cost of an OD pair is a monotone non-decreasing function of the corresponding demand [22, 30, 34]. Method (3) is motivated by the sensitivity result that states that improving the link latency function on only one link results in a decrease of the latency on that link [22]. The methods given above have some major drawbacks:

1. A large-scale network contains many users and links, therefore updating the capacities of all the links or the capacity of a link by a factor that is larger than the number of users is a very expensive operation.

2. The source and destination usually are very far away from each other or/and locate in different geographic regions. Therefore, in reality, there are very few links connecting the source and destination directly and also building such directed links is often infeasible.

The methods proposed above are further constrained by assumptions on link latency functions or users. However, whether any assumption has been made or not, we can see that these methods are limited.

Braess' Paradox can be arbitrarily severe in many networks [23]. Most of the network design problems related to Braess' Paradox, such as the ones mentioned above, focus on finding ways to avoid this undesired but common phenomenon. Therefore, in the remainer of this section, we will examine the implications of profit maximizing prices on Braess' Paradox.

Hagstrom and Abrams [28] gave a definition of Braess' Paradox in a network without pricing: A Braess' Paradox occurs if there exists some other distribution of flows for which some flow have improved travel costs and no flow has worse travel cost than in the equilibrium. This is a generalization of the classical Braess' Paradox which refers to change in network performance by adding/deleting a link. In [28], Hagstrom and Abrams showed a network which experiences a generalized Braess' Paradox but no classical Braess' Paradox. The example is given in Figure 5-1. The network is similar to the original Braess' Paradox given in Section 1.2.5 but with different link latency functions:

$$l^a(f^a) = 10f^a, \ l^b(f^b) = 10f^b + 32, \ l^c(f^c) = 10f^c + 32,$$

$$l^d(f^d) = 10f^d, \ l^e(f^e) = f^e + 10.$$

A user want to send 6 units of data from node 1 to node 4. Assume the prices of all the paths are 0, then at the WE, the flow on the paths are:

$$f^{\{a,c\}} = f^{\{b,d\}} = f^{\{a,e,d\}} = 2,$$

and the latencies of all the paths is 92. Now consider a new flow distribution where

$$f^{\{a,c\}} = f^{\{b,d\}} = 2.5, \ f^{\{a,e,d\}} = 1.$$

The latencies of the paths then become

$$l^{\{a,c\}} = l^{\{b,d\}} = 92, \ l^{\{a,e,d\}} = 81.$$

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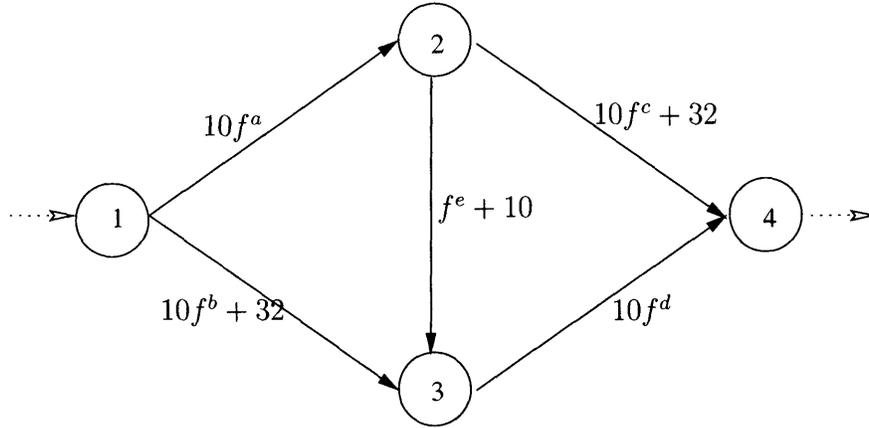
Figure 5-1: A generalized Braess' Paradox for which no link can be deleted.

The flow on path $\{a, e, d\}$ have lower latency cost while the other flow have the same latency cost. Therefore, the generalized Braess' Paradox exists. However, if any link is deleted, all flow will experience higher latency cost. Thus, the classical Braess' Paradox does not occur.

5.2 Braess' Paradox under Pricing

In a network without pricing, at a WE, all flows on the paths that belong to the same OD pair experience the same latency cost. Therefore we can restate the generalized definition of Braess' Paradox given above as: A Braess' Paradox occurs in a network if there exists some other distribution of flows for which *some paths* have improved latency costs and *no path* has a worse latency cost than in the equilibrium. At a WE with prices, flows on different paths may have different latency costs. Therefore, there might exist some other flow distribution for which some paths have improved latency costs and no path has worse latency cost, but some flow which switched from one path to another has worse latency cost than in the equilibrium. Such a situation should not be considered as a Braess' Paradox. For an example, let's consider the network in Figure (5-2).
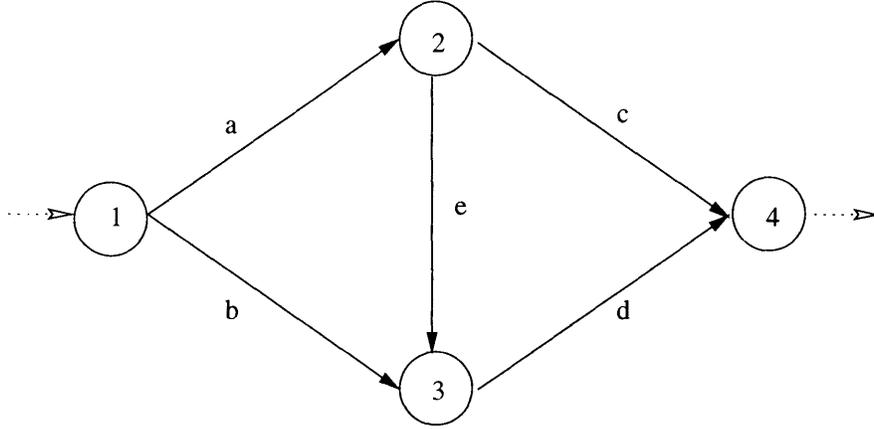
Figure 5-2: An example of Braess' Paradox under pricing.

A single user sends data from node 1 to node 4.

$$u(\Gamma) = 368\sqrt{6}\Gamma^{0.5}$$

The link latency functions and path prices are as follows:

$$l^a(f^a) = (f^a)^2, \quad l^b(f^b) = 5f^b, \quad l^c(f^c) = 5f^c,$$

$$l^d(f^d) = (f^d)^2, \quad l^e(f^e) = 0, q^{\{a,c\}} = 182.5619,$$

$$q^{\{b,d\}} = 182.5619, \quad q^{\{a,e,d\}} = 193.5619$$

The path flows at the WE are

$$f^{\{a,c\}} = f^{\{b,d\}} = 2, \quad f^{\{a,e,d\}} = 1.$$

The latency costs of the paths are

$$l^{\{a,c\}} = l^{\{b,d\}} = 19, \quad l^{\{a,e,d\}} = 18.$$

We next consider moving 0.5 units of flow from path $\{a, e, d\}$ to paths $\{a, c\}$ and

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$\{b, d\}$. The resulting path flows are

$$f^{\{a,c\}} = f^{\{b,d\}} = 2.5, \ f^{\{a,e,d\}} = 0,$$

and the corresponding latency costs are

$$l^{\{a,c\}} = l^{\{b,d\}} = 18.75, \ l^{\{a,e,d\}} = 12.5.$$

Hence, the flow that is moved from $\{a, e, d\}$ to alternative paths experiences a higher latency cost. It can be seen that there is no flow distribution in which all flows experience improved latency costs.

We next give two alternative definitions of Braess' Paradox under pricing. The following notation will be useful in the definitions. Consider two feasible flow distributions f and $\overline{f}$ such that

$$\Gamma_{k,j} = \overline{\Gamma}_{k,j}, \forall \ k, j.$$

Let $\gamma = [f^p]_{p \in \mathcal{P}_k, k \in \mathcal{W}}$ be the path flow vector and Δ be a transformation matrix such that

$$\Delta \cdot \gamma = \overline{\gamma}. \tag{5.1}$$

Hence, $\Delta_{i,j} f^j$ represents the amount of flow that is moved from path j to path i. Note that there are infinitely many transformation matrices Δ satisfying Eq. (5.1).

Definition 3 *(Strong Braess' Paradox):* Let G be a general network. Given a price q, let f be a WE. Let $l^p(\gamma)$ be the latency cost of routing one unit of flow on path p as defined in Eq. (2.1). A Strong Braess' Paradox occurs if there exists some other distribution of path flows, $\overline{\gamma}$, and a transformation Δ such that

$$\Delta \cdot \gamma = \overline{\gamma}$$

$$\Gamma_{k,j} = \overline{\Gamma}_{k,j}, \forall \ k, j$$

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$$l^p(f) \geq l^{p'}(\overline{f}), \quad \text{for all } p, p' \text{ with } \Delta_{p',p} \neq 0, \tag{5.2}$$

with strict inequality for some p, p', where $\Delta_{p',p}$ is the (p', p) entry of matrix Δ.

Under condition (5.2), no flow experiences a higher latency cost than in the WE. For a price vector in which the prices of all the paths that belong to an OD pair are the same, Definition (3) is consistent with the definition of Braess' Paradox in a network without pricing. For an example of Strong Braess' Paradox, we can consider the same network shown in Figure (5-2). The user sends data from node 1 to node 4. The user's utility function, link latency functions, and path prices are given as:

$$\Gamma = 184\sqrt{6}\Gamma^{0.5}$$

$$l^a(f^a) = 10f^a, \ l^b(f^b) = f^b, \ l^c(f^c) = f^c,$$

$$l^d(f^d) = 10f^d, \ l^e(f^e) = f^e, \ q^{\{a,c\}} = 50,$$

$$q^{\{b,d\}} = 50, \ q^{\{a,e,d\}} = 10.$$

The path flows at the WE are

$$f^{\{a,c\}} = f^{\{b,d\}} = f^{\{a,e,d\}} = 2.$$

The latency costs of the paths are

$$l^{\{a,c\}} = l^{\{b,d\}} = 42; \ l^{\{a,e,d\}} = 82.$$

We move one unit of flow from path $\{a, e, d\}$ to each of paths $\{a, c\}$ and $\{b, d\}$ in order to get a new flow distribution: $f^{\{a,c\}} = f^{\{b,d\}} = 3$ and $f^{\{a,e,d\}} = 0$. In this flow distribution, the latency costs of the paths are

$$l^{\{a,c\}} = l^{\{b,d\}} = 33.$$

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Each unit of flow experiences a latency cost equal to 33, which is less than the latency cost at the WE. Note that this price vector is not a monopoly price vector. Later, we will show that under monopoly prices, Strong Braess' Paradox does not occur.

Conditions in Definition (3) state that when Strong Braess' Paradox occur, at the new flow distribution, some flows have lower latency cost and no flow has a higher latency cost. We will next relax these conditions so that some flow may encounter higher latency costs at the new flow distribution, but on average the latency encountered by the total flow will decrease. This leads to the following definition.

Definition 4 *(Weak Braess' Paradox):* Let G be a general network. Given a price q, let f be a WE. Let $l^p(\gamma)$ be the latency cost of routing one unit of flow on path p under a path flow γ and $l(\gamma) = [l^1(f), \cdots, l^{|\mathcal{P}|}(f)]$ be the path latency vector. A Weak Braess' Paradox occurs if there exists some other distribution of path flows, $\overline{\gamma}$, under price q and a transformation Δ such that

$$\Delta \cdot \gamma = \overline{\gamma}$$

$$\Gamma_{k,j} = \overline{\Gamma}_{k,j}, \forall\, k,j \tag{5.3}$$

for some p'

$$l^p(\gamma) \geq \Delta'_p \cdot l(\overline{\gamma}), \forall\, p, \tag{5.4}$$

with strict inequality for some p', where Δ_p is the pth column of Δ.

Condition (5.2) in Definition (3) imply Condition (5.4) in Definition (4). Therefore, if Strong Braess' Paradox occurs, then Weak Braess' Paradox also occurs. The following example shows that the reverse implication is not true. Consider the network in Figure (5-2) with different functions:

$$\Gamma = 368\sqrt{6}\Gamma^{0.5}$$

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$$l^a(f^a) = (f^a)^2, \ l^b(f^b) = 3f^b, \ l^c(f^c) = 5f^c,$$

$$l^d(f^d) = (f^d)^2, \ l^e(f^e) = 0, \ q^{\{a,c\}} = 193.4649$$

$$q^{\{b,d\}} = 201.7149, \ q^{\{a,e,d\}} = 197.2149$$

The path flows at the WE are $f^{\{a,c\}} = 2, \ f^{\{b,d\}} = 1.5, \ f^{\{a,e,d\}} = 1$ and the path latency costs are:

$$l^{\{a,c\}} = 19; \ l^{\{b,d\}} = 10.75; \ l^{\{a,e,d\}} = 15.25.$$

Next, we move 0.5 units of flow from path $\{a, e, d\}$ to each of the paths $\{a, c\}$ and $\{b, d\}$ to get a new flow distribution: $f^{\{a,c\}} = 2.5, \ f^{\{b,d\}} = 2$ and $f^{\{a,e,d\}} = 0$. In this flow distribution, the latency costs of the paths are

$$l^{\{a,c\}} = 18.75; \ l^{\{b,d\}} = 10; \ l^{\{a,e,d\}} = 10.25.$$

We see that

$$18.75 < 19; \ 10 < 10.75$$

$$0.5 \times 18.75 + 0.5 \times 10 = 14.375 < 15.25.$$

Therefore, in this example, Weak Braess' Paradox occurs. However, it can be seen that there exists no flow distribution in which all flows will encounter lower latencies than at the WE. Therefore, Strong Braess' Paradox does not occur.

We next show that under monopoly prices, Weak Braess' Paradox does not occur, which also implies that under monopoly prices, there can be no Strong Braess' Paradox.

Proposition 7 Weak Braess' Paradox does not occur under monopoly prices.

Proof: We consider a general network G. Let (f, q) be an ME. Suppose that Weak Braess' Paradox occurs under the monopoly price q. Then there exists another flow distribution $\overline{f}$ satisfying Conditions (5.3) - (5.4). Now let us consider the price

vector $\bar{q}$ defined by

$$\bar{q}^p = u'_{k,j'}(\overline{\Gamma}_{k,j'}) - l^p(\bar{\gamma}) \text{ for some } j' \in \mathcal{J}_k, \text{ if } p \in \overline{\mathcal{P}}$$

$$\bar{q}^p = \infty, \text{ if } p \notin \overline{\mathcal{P}}.$$

It can be seen that $\bar{f}$ is a WE at price $\bar{q}$. In the following, we will examine the profit that the service provider makes under price $\bar{q}$.

$$
\begin{aligned}
\sum_p \bar{f}^p \bar{q}^p &= \sum_k \sum_{p \in \mathcal{P}_k} \bar{f}^p \bar{q}^p \\
&= \sum_k \sum_{p \in \mathcal{P}_k} \bar{f}^p \left(u'_{k,j'}(\overline{\Gamma}_{k,j'}) - l^p(\bar{\gamma}) \right) \\
&= \left(\sum_k u'_{k,j'}(\overline{\Gamma}_{k,j'}) \sum_{p \in \mathcal{P}_k} \bar{f}^p \right) - \sum_k \sum_{p \in \mathcal{P}_k} \bar{f}^p l^p(\bar{\gamma}) \\
&= \left(\sum_k u'_{k,j'}(\overline{\Gamma}_{k,j'}) \sum_{j \in \mathcal{J}_k} \overline{\Gamma}_{k,j} \right) - \sum_k \sum_{p \in \mathcal{P}_k} \left[\left(\sum_{p' \in \mathcal{P}_k} \Delta_{p,p'} \bar{f}^{p'} \right) l^p(\bar{\gamma}) \right] \\
&= \left(\sum_k u'_{k,j'}(\overline{\Gamma}_{k,j'}) \sum_{j \in \mathcal{J}_k} \overline{\Gamma}_{k,j} \right) - \sum_k \sum_{p' \in \mathcal{P}_k} \left[\bar{f}^{p'} \left(\sum_{p \in \mathcal{P}_k} \Delta_{p,p'} l^p(\bar{\gamma}) \right) \right] \\
&= \left(\sum_k u'_{k,j'}(\overline{\Gamma}_{k,j'}) \sum_{j \in \mathcal{J}_k} \overline{\Gamma}_{k,j} \right) - \sum_k \sum_{p' \in \mathcal{P}_k} \left(\bar{f}^{p'} \Delta'_{p'} \cdot l(\bar{\gamma}) \right) \\
&> \left(\sum_k u'_{k,j'}(\Gamma_{k,j'}) \sum_{j \in \mathcal{J}_k} \Gamma_{k,j} \right) - \sum_k \sum_{p' \in \mathcal{P}_k} \left(f^{p'} l^{p'}(\gamma) \right) \\
&= \left(\sum_k u'_{k,j'}(\Gamma_{k,j'}) \sum_{p' \in \mathcal{P}_k} f^{p'} \right) - \sum_k \sum_{p' \in \mathcal{P}_k} f^{p'} l^{p'}(\gamma) \\
&= \sum_k \sum_{p' \in \mathcal{P}_k} f^{p'} \left(u'_{k,j'}(\Gamma_{k,j'}) - l^{p'}(\gamma) \right) \\
&= \sum_{p'} f^{p'} q^{p'}
\end{aligned}
$$

The inequality follows from Eqs. (5.3)-(5.4). The service provider can make more profit by setting price $\bar{q}$ than q. Therefore, (f, q) is not an ME, which is a contradiction and shows that Braess' Paradox does not occur under monopoly prices. **Q.E.D.**

From Proposition 7 we can see that under monopoly prices, the service provider does not need to worry about Braess' Paradox when he upgrades the network capacity.

Chapter 6

Conclusions and Future Directions

A major topic in data network research is to characterize and regulate the flows sent by selfish users so as to increase the efficiency of the system. A number of studies shows that the service providers may achieve this goal by utilizing some pricing mechanisms. In reality, however, most networks are owned by for-profit entities and the service providers are most interested in setting the prices to maximize their profit instead of improve the efficiency. Therefore, service providers' interest should also be considered when studying network designs. Unfortunately, with a few exceptions, this issue has not been addressed in detail.

In this thesis, we proposed a tractable framework that may be used to analyze a general data network with many heterogeneous selfish users and for-profit service provider. We show that the user equilibrium for any given price is unique. We also gave the explicit characterizations of profit maximizing prices that a help of the service provider to set the prices. More importantly, we find that under a monopoly setting, the service provider can also increase the efficiency of the system while setting the prices to maximize his profit. In some important special cases, the monopoly equilibrium can even achieve the full information social optimum. Finally, setting monopoly prices also solve the Braess' Paradox, which is a critical performance problem in many network systems.

This research provides an unified approach to study flow control and routing in

the presence of for-profit service provider. The framework is general and can be applied to different types of large scale commercial networks such as the Internet and transportation networks. The work in this thesis also open the doors to many research directions:

1. A natural extension of the model in this thesis is to consider the case of oligopolistic service providers. Acemoglu and Ozdaglar [40] have shown that when service providers are competing for users in a parallel-link network, the user equilibrium can not achieve the social optimum. However, the performance gap between the user equilibrium and social optimum is bounded by 5/6 and the bound is tight. One can expect that a similar situation will be observed in a general network.

2. To apply the framework to a real world network system, we need a decentralized algorithm to calculate the monopoly prices. The problem of finding monopoly prices is indeed a Stackelberg game and can be approached by using bilevel programming technique. At the user level, for any given price, the flow distribution is a solution of the optimization problem of finding the Wardrop equilibrium. At the service provider level, the monopoly price is a solution of the optimization problem of finding the price so that at the corresponding Wardrop equilibrium the profit of the service provider is maximized. Bilevel programming problem is a hard problem and even the simplest version of it (which is the linear bilevel programming problem) is NP-hard [42]. However, with the monopoly price characterization given in this thesis, we might able to find an efficient algorithm to calculate these prices. Besides the complexity issue, the algorithm also need to be decentralized to have any practical use. Each path should be able to calculate its own price bases on its local information. The decentralized algorithms for calculating Wardrop equilibrium [41] can be a starting point of this future research.

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