LOW COMPLEXITY OPTIMAL MULTIPLE ACCESS JOINT DETECTION FOR LINEARLY DEPENDENT USER SETS

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ABSTRACT

The general problem of joint detection of linearly dependent users in an uncoded multiple access (MA) system is N-P hard. We look to exploit the existing structure in our problem so that low complexity algorithms may be devised to yield the optimal solution. In this paper advantage is taken over the design of user signatures in a typical MA communication system. By imposing a hierarchical cross-correlation structure on the user signature waveforms, the receiver design problem is reduced so that it is no longer N-P complete. A tree joint detection algorithm which takes advantage of such a cross-correlation structure is presented. The tree detector gives the *optimal* estimate with an extremely low computational complexity that is typically low-order-polynomial in the number of users. This is an enormous savings in computations over the $O(2^K)$ computations needed if the signatures did not exhibit any structure.

1. THE MA JOINT DETECTION PROBLEM

Due to natural limitations of any multiple access (MA) communication system, the user waveforms lie in a finite dimensional vector space.[1] For some set of signature waveforms represented in signal space by the set of signature vectors, $\{\mathbf{s}_k\}_1^K$, the detection problem is to compute an estimate $\mathbf{b}$ from an observation $\mathbf{r} \in \mathbb{R}^N$

$$\mathbf{r} = \sum_{k=1}^{K} b_k \mathbf{s}_k + \sigma \mathbf{n} = \mathbf{S}\mathbf{b} + \sigma \mathbf{n}, \qquad (1)$$

where

- K is the number of users.
- $\mathbf{b} \in \{[b_1 \cdots b_K]^T \mid b_i \in P_i\}$, where P_i is some finite set of real amplitudes and b_i is iid uniform. For P_i having M elements, this is M-ary pulse amplitude modulation (PAM).[2]
- $\mathbf{S} = [\mathbf{s}_1, \ldots, \mathbf{s}_K]$ is an $N \times K$ matrix whose columns are user signatures. The column vectors $\mathbf{s}_k$ of $\mathbf{S}$ span the space $\mathbb{R}^N$.
- $\mathbf{n}$ is a real Gaussian vector of mean zero and identity covariance.
- σ is the noise standard deviation.

The joint detection of uncoded interfering users in a multiple access communication system has been the topic of much work over the past several years. Verdú has shown in [1] that for the general MA joint detection problem stated as above for $K = N$, i.e. for an arbitrary non-singular signature matrix, $\mathbf{S}$, the optimal solution is N-P hard for $\sigma > 0$. In other words, the maximum likelihood estimate of $\mathbf{b}$ cannot be found by an algorithm having polynomial complexity in the number of users, K.

A strong effort has been made in the area of finding good *suboptimal* low complexity joint detectors for arbitrary sets of linearly independent user signature waveforms; the foundation has been set by Lupas and Verdù [1,2], Varanasi and Aazhang [3], and Duel-Hallen [4]. The only requirement for the existing works on MA joint detection is that the user signature waveforms form a linearly independent set. In other words, the user signatures may be correlated, but the number of users is not to exceed the number of signal space dimensions.

The topic of this paper differs from the existing literature in that we examine the case for which uncoded user signatures are linearly *dependent*. Such an "over-saturated" system is attractive in that it would permit service to more users than afforded by traditional systems. It can be shown that in the case of *arbitrary* $\mathbf{S}$ and $K > N$, the MA joint detection problem is N-P hard even when $\sigma = 0$ (no noise). Since the user waveforms are assigned, the set of signatures is not at all arbitrary. Furthermore, we assume the set of user signature waveforms seen at the receiver to be entirely known and within our control. Specifically, we assume the channel is flat (no distortion or multipath) and relative timings, phases and powers of user transmissions are within our control.[3]

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[1] Specifically, a user waveform is bandwidth and time localized and a limited bandwidth is available for total system operation.

[2] We have included binary phase shift keying as a subset of PAM; namely, it is antipodal modulation $P = \{+1, -1\}$.

[3] These assumptions may or may not be valid depending on the system of interest. Contemporary satellite communications, for example, impose many of the assumptions we have made for this paper.

For this work, therefore, the common assumption of arbitrary $\mathbf{S}$ in the MA scenario is relinquished to transform the over-saturated MA joint detection problem into the following:
Solve the problem as stated in Equation (1) with the following additional requirements:

- Maintain more users than dimensions $K > N$.
- Signature set $\mathbf{S}$ may exhibit chosen structure.
- Find low complexity algorithm that will give the optimal solution.

Method: incorporate signature set design to create structure that will advantage optimal detection.

2. STRUCTURE OF THE SIGNATURE SET

A low complexity optimal joint detector has been developed for any signature set $\mathbf{S}$ having tree-structured cross-correlations. The cross-correlation structure is satisfied if the signature vectors can be assigned to the nodes of a tree like the one shown in Figure 1. The tree pictorially conveys the following required relationships among user signature vectors.

- Each vector at a given level of the tree is orthogonal to all other vectors at that level.
- A signature vector is correlated only with its ancestor vectors (parent, grandparent, etc.) and its descendent vectors (children, grandchildren, etc.).

Both linearly dependent and linearly independent sets of signature vectors may be created to have tree-structured cross-correlations. The detector detailed in this paper finds the optimal solution for both cases.

Below, a linearly dependent signature set is built on a tree.

1. Choose the first N signatures to be an orthogonal basis of $\mathbb{R}^N$. Assign them to the N nodes at the lowest level of the tree.
2. Divide the signature vectors at this level into groups.
3. Create the signature vectors for the next highest level of the tree by choosing for each group a new signature vector to be a linear combination of its group members.
4. Represent the parent-child-sibling relation among signature vectors by assigning each new vector to the parent node of the group vectors used to create it.
5. If no more sibling vectors exist at this new parent level, stop, otherwise return to step 2.

This structure, as presented above, is quite general. We expect the tree structure to be satisfied by numerous vector sets. The authors are aware of two specific examples of tree correlated user signature sets which have appeared in recent literature. Linearly dependent wavelet packet waveform sets were developed in [5] and minimum distance sets were developed in [6].

Wavelet Packet Sets
Wavelet packet waveforms may be generated from a tree-type creation algorithm in which a parent basis function is

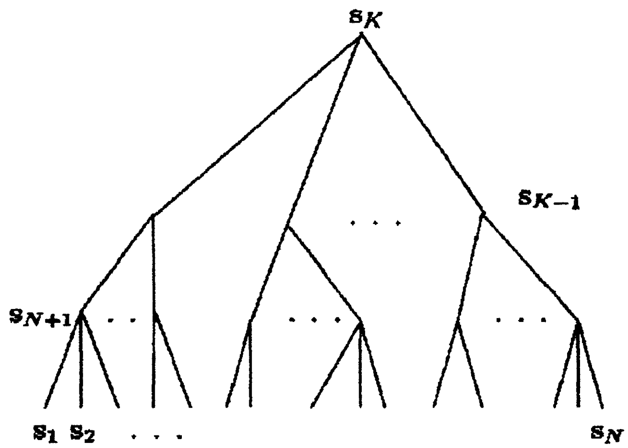

Figure 1: This example of a general tree shows the correlation structure needed among signature vectors within the signature set.

decomposed into many orthogonal parts, where each part is another wavelet packet waveform.[4] The redundant set of basis functions in a wavelet packet dictionary offers a rich set of signals from which to select many tree-structured signature vector sets. See [5] for more details on wavelet packet sets.

Minimum Distance Sets
Another example is the minimum distance sets developed by Ross and Taylor in [8]. Ross and Taylor begin with N orthogonal users in N dimensions that result in a given minimum distance between possible received points in an M-ary PAM MA system; more users are added so that the minimum distance between received points is preserved.

A specific example detailed in [8] is repeated below.

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 1/2 \end{bmatrix} \qquad (2)$$

The tree for this set is shown in Figure 2. The signature vector associated with the top of our correlation tree is the right-most column of $\mathbf{S}$. The first 4 columns would be associated with the bottom of our tree. These sets were designed for their minimum distance property; the tree hierarchy is a by-product.

3. THE TREE JOINT DETECTION ALGORITHM

3.1. Overview of the Algorithm

The optimum joint detector for the general problem stated in Section 1 chooses the weight vector estimate, $\hat{\mathbf{b}}$, according to the nearest neighbor or minimum distance rule.

[4] For a tutorial treatment of wavelet packets see the paper by Coifman and Wickerhauser [7].

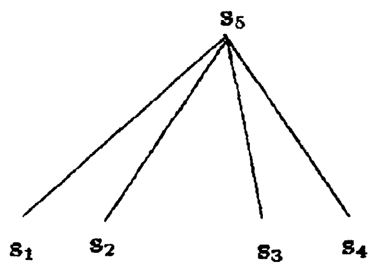

Figure 2: Example of a minimum distance set tree.

$$\hat{\mathbf{b}} = arg \min_{\mathbf{b} \in P^K} ||\mathbf{r} - \mathbf{S}\mathbf{b}||^2 \quad (3)$$

For ease of discussion, we assume each user employs M-ary PAM for the remainder of this paper where $b_i \in P$, $\forall i$, $|P| = M$; this assumption is not essential to the operation of the tree algorithm. An MA system employing an arbitrary set of signature vectors, $\mathbf{S}$, can achieve the optimal detection of the above detector through an exhaustive search, i.e. the detector needs to perform $M^K - 1$ comparisons to find the best estimate [1].

If the signature set has been constructed to have the tree cross-correlation structure the optimum detector for Equation (3) can be achieved through a tree-structured algorithm that offers a huge reduction in the number of comparisons. Recall that a signature at a given node is correlated with all signatures at its ancestor and descendent nodes and is orthogonal to all other signatures on the tree. The weight estimate at a given node, therefore, will depend only upon the estimates at descendent and ancestor nodes and will be independent of all other estimates on the tree.

The tree detection algorithm takes advantage of this structure and sweeps through the tree from bottom to top, creating a *conditional* weight estimate table at each node. The table of decisions at a given node is conditioned on weight decisions of the ancestors and is a function of weight decisions of the descendents.

The number of computations needed to create a table and the size of the table decreases exponentially as the algorithm moves from the bottom to the top of the tree until there is only one decision associated with the top node of the tree. The full weight vector estimate for all user weights is a by-product of this last decision.

3.2. The Algorithm

A simple notation is established in order to convey the algorithm details. The weight estimate and signature vector associated with a node, n, of the tree is denoted by $\hat{b}_n$ and $\mathbf{s}_n$, respectively. Collect the weight estimates and signature vectors of all ancestors of node n into a weight vector, $\hat{\mathbf{b}}_{an}$, and corresponding signature matrix, $\mathbf{S}_{an}$, respectively; collect the weight estimates and signature vectors of all descendants to node n into a weight vector, $\hat{\mathbf{b}}_{dn}$, and corresponding signature matrix, $\mathbf{S}_{dn}$.

A node at level l of the tree has $l-1$ ancestors, and, therefore, has M^{l-1} possible ancestor weight vectors, i.e. $\hat{\mathbf{b}}_{an} \in P^{l-1}$. At each node in the tree, starting from the bottom and progressing upward until reaching the root, create a table of M^{l-1} conditional decisions for $\hat{b}_n$. The conditional weight estimator is stated mathematically as

$$\hat{b}_n(\mathbf{r}|\hat{\mathbf{b}}_{an}) = arg \min_{b_n \in P} ||\mathbf{r} - \mathbf{s}_n b_n - \mathbf{S}_{an}\hat{\mathbf{b}}_{an} - \mathbf{S}_{dn}\hat{\mathbf{b}}_{dn}(\mathbf{r}|b_n, \hat{\mathbf{b}}_{an})||^2.$$

For each possible realization of the ancestor weight vector, choose $\hat{b}_n$ to be the value of $b_n \in P$ that minimizes the above distance measure. The values used for the descendant weight estimates, $\hat{\mathbf{b}}_{dn}(\mathbf{r}|b_n, \hat{\mathbf{b}}_{an})$, would have already been calculated. For example, since the algorithm begins at the bottom of the tree, the weight estimate of a child to node n has been calculated for each possible realization of $\{b_n\ \hat{\mathbf{b}}_{an}^T\}$.

Note that at the bottom level, there are no descendents to take into account. Conversely, at the top of the tree there are no ancestors; this leads to a table of one decision at the root node. This single decision at the top of the tree is built on all of the conditional decisions of its descendents. This last decision, therefore, inherently deduces all of the decisions for the rest of the tree. Proof of optimality of the tree detector is left for another paper.

It is not necessary to implement the algorithm as we have chosen to describe it. Implementation of the tree algorithm, if computations are to be conserved, should have no redundant calculations. In addition, the implementation may be tailored to the computer or chip architecture available. It is clear from the above example that there are possibilities for implementation employing parallel processing.

4. COMPUTATIONAL COMPLEXITY

For the simplicity of calculation of computational complexity, we restrict the tree to be of uniform composition in that there are exactly Q children emanating from each node. Recall that N is the number of signal space dimensions available (the number of nodes at the bottom of the tree) and M is the number of levels that can be modulated by each user. We measure complexity as the number of compares, c, needed to perform the tree algorithm; c is stated below and derived later in this section.

$$c(N,Q,M) = \frac{M-1}{QM-1} NQM^{\log_Q N+1} - 1 \quad (4)$$

For example, if a system were to employ antipodal modulation, $P = \{+1,-1\}$, $M = 2$, and signature sets having quad-tree ($Q = 4$) structure, the number of comparisons needed for the tree detector estimate, $\hat{\mathbf{b}}$, is

$$c(N, Q=4, M=2) = \frac{8N^{3/2} - 1}{7}. \quad (5)$$

The computational complexity is polynomial in the number of dimensions. The number of users, K, in this special case is

$$K = \frac{4}{3}N - \frac{1}{3},$$

hence, the tree detector is also polynomial in the number of users, resulting in a computational complexity of $O(K^{3/2})$.

Derivation of complexity

The algorithm creates a conditional bit estimate table for each node. For a given $\hat{\mathbf{b}}_{an}$ we must choose the best of M possible values of $\hat{b}_n$. This requires $M-1$ comparisons for a single configuration of $\hat{\mathbf{b}}_{an}$. Since there are $l-1$ ancestors of node n, there are M^{l-1} possible configurations of $\hat{\mathbf{b}}_{an}$. The tree detector, therefore, creates a single table at level l, node n, with $(M-1)M^{l-1}$ comparisons. There are Q^{l-1} tables needed for level l of the tree and there are a total of $log_Q N+1$ levels in the tree. It follows that the total number of comparisons needed for the tree algorithm is

$$\begin{aligned} c(N,Q,M) &= \sum_{l=1}^{log_Q N+1} Q^{l-1}(M-1)M^{l-1} \\ &= (M-1)\frac{(QM)^{log_Q N+1}-1}{QM-1} \\ &= \frac{(M-1)}{(QM-1)}(NQM^{log_Q N+1}-1) \end{aligned}$$

5. CONCLUSION

This paper addresses the problem of uncoded MA joint detection for the case in which user signatures are linearly dependent. The linearly dependent scenario occurs when the number of users in a communication system is increased beyond the dimension of the signal space available for transmissions. This "over-saturated" scenario is very attractive to the field of communications since it would allow a system to serve more users than previously thought. Oversaturated MA communications is largely unexplored due to the non-polynomial (N-P) complexity of optimal joint detection in the general case.

In this paper advantage is taken of the control that exists over user signatures through signature design. With the introduction of structure in the linearly dependent user signature sets, a low complexity optimal joint detector is possible. Specifically, by imposing a hierarchical cross-correlation constraint on the user signature sets, the joint detection problem is reduced so that it is no longer N-P hard. A tree joint detection algorithm which takes advantage of the cross-correlation structure is presented. The tree detector gives the *optimal* estimate with an extremely low computational complexity. For one example tree-structured signature set for which $K=\frac{4}{3}N-\frac{1}{3}$, where N is the number of signal dimensions and K is the number of users, the computational complexity of the tree algorithm is $O(K^{3/2})$. This is extremely low when compared to the solution of the optimal joint detection problem for an arbitrary set of linearly dependent user signatures, namely an exhaustive search of 2^K values. A more thorough treatment of this work is forthcoming.

The tree-correlated signature structure is a broad requirement that is easily fulfilled by "bad" signature sets, i.e. the received vector sets have small to no separation between points. Arbitrary choice of tree structured signature sets is, therefore, not an option for an actual system. The Ross/Taylor sets are, perhaps, the best tree-structured sets if minimum distance is the only measure, but due to strict power control and synchronization requirements, these appear to be incompatible with implementation. The field of signature set design is considered to be a complex problem and is a topic of our future work.

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