

Branching From K To M For Split Classical Groups

by

Nicholas McCarthy

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of
Master of Science in Mathematics

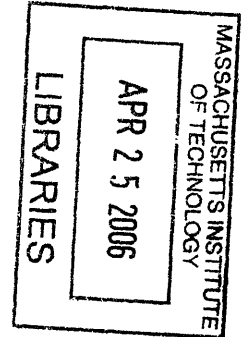
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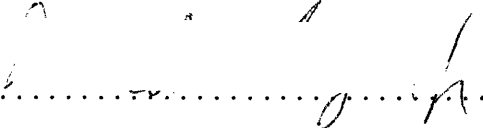
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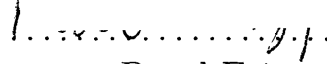
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Abstract

We provide two algorithms to solve branching from K to M for the real split reductive group of type A_n , one inductive and one related to semistandard Young tableaux. The results extend to branching from K_e to $M \cap K_e$ for the real split reductive groups of type B_n and D_n .

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1 Introduction

Principal series representations play an important and well-developed role in representation theory of reductive algebraic groups. Perhaps much of this importance, certainly much of this development, stems from the relationship between these representations and representations of simpler, better understood classes of groups. The underlying spaces for principal series representations depend only on the restrictions of these representations to compact groups, and the structures of these representations as modules for reductive groups depend on the structures of modules for parabolic subgroups of these reductive groups. In the absence of theory necessary to speak freely of general representations of reductive groups, those categories of representations to which the current state of theory does apply rise somewhat in prominence. Those categories rise both for lack of any more inventive mathematics and, justifiably, in the hope studying such categories will lead to more inventive mathematics.

In order better to understand the decomposition of principal series representations into irreducible components, we turn to the study of branching laws. Specifically, for any real reductive group, we consider branching from K , the maximal compact subgroup of that group, to M , the intersection of a chosen maximal torus with K . Frobenius reciprocity relates branching to the decomposition of principal series. We may gain significant information further narrowing our project to the consideration of branching from the identity component K_e of K to the intersection of M with K_e . Our results deal only with branching from K_e to $M \cap K_e$ for the split real reductive groups of classical type.

Kostant's multiplicity formula solves the problem of branching from real compact connected reductive groups to connected closed subgroups, so this formula seems a natural point of departure for our project. Two problems arise. In general groups M and $M \cap K_e$ tend toward disconnectedness of a nature severe enough to make very difficult or even impossible the task of massaging the situation into compliance with the conditions Kostant's formula requires. On the other hand, the classical

groups lend themselves to induction, and, making up somewhat for the subgroup's stubborn disconnectedness, M is finite and abelian for the classical groups. We take, for example, branching from $SO(n, \mathbb{R})$ to M where M is the intersection of some maximal torus for $SL(n, \mathbb{R})$ with $SO(n, \mathbb{R})$. Kostant's formula solves branching from $SO(n, \mathbb{R})$ to $SO(n-2, \mathbb{R}) \times SO(2, \mathbb{R})$, and for i equal to $n-2$ and 2 , we could hope to understand branching from $SO(i, \mathbb{R})$ to $M(SO(i))$ inductively where $M(SO(i))$ is the intersection of some maximal torus for $SL(i, \mathbb{R})$ with $SO(i, \mathbb{R})$. Use of this process would put us in good stead to solve branching from $SO(n, \mathbb{R})$ to M . The restriction map from the set of representations for $SO(n, \mathbb{R})$ to the set of representations for $M(SO(n-2)) \times M(SO(2))$ factors through the set of representations for M . We would have only to investigate this factorization. The second problem with utilizing Kostant's formula now presents itself. The combinatorics behind Kostant's formula, involving as it does Weyl groups and partition functions, can become complicated quite quickly. Kostant's formula provides a sure solution to a very nonspecific class of problems. The formula need not take advantage of simplifying assumptions pertaining to any specific subclass of these problems. Although Kostant's formula serves as a foundation for cited results critical to our approach, we eschew the known territory involving direct use of the formula, favoring instead a different tack.

In section 2, we give an overview of the structure theory for split real reductive algebraic groups and we provide definitions central to the discussion of our project. We recall a result from the study of algebraic groups affirming the uniqueness up to isomorphism of any split reductive algebraic group with a particular root datum. Up to isomorphism, then, there exists only one real split reductive group of each classical type. We then discuss briefly the principal series representations and Frobenius reciprocity, clearing the way for our focus on branching from K_e to $M \cap K_e$.

We determine the real split reductive group of type A_n in section 3. Applying

the structure theory from section 2, we determine K and M . We go on to classify the irreducible representations of M , associating the isomorphism class of each representation to some subset of $\{1, \dots, n\}$. Finally, we state and prove a lemma equating the multiplicities in any irreducible representation for K of any two irreducible representations for M corresponding to subsets of $\{1, \dots, n\}$ having the same cardinality.

Section 4 contains a thorough description of the group $K = SO(n, \mathbb{R})$ for the real split reductive group of type A_n . We study the Lie algebra for K , establishing notation and conventions such as a choice for a Cartan subalgebra and a choice for a set of positive roots. We establish a correspondence between the irreducible representations for any real reductive Lie group and the analytically integral forms on the Cartan subalgebra related to highest weight representations for the Lie algebra. Determining explicitly the set of analytically integral forms for K , we then state Murnaghan's theorem describing the branching law from $SO(n, \mathbb{R})$ to $SO(n-1, \mathbb{R})$. Making use of Murnaghan's result, we establish an inductive algorithm to solve branching from K to M .

The algorithm from section 4 is fairly simple, moreso probably than the algorithm a direct application of Kostant's theorem might have produced. On the other hand, even this relatively simple recursion appears computationally costly when applied to branching from K to M for the real split reductive group of type A_n when n is large. We search for a computationally simpler algorithm, one more intimately related to the structure of representations for $K = SO(n, \mathbb{R})$, hoping the methods we encounter might have applications to more general cases. We look in section 5 at the relationship between semistandard Young tableaux and representations for K . Semistandard Young tableaux serve as a sort of common language for the expression of results concerning representations, especially for matrix groups. These tableaux appear in the

study of combinatorics and algebraic geometry, so representation-theoretic results making use of semistandard Young tableaux stand to benefit from and to benefit theory related to these two fields. We define a subset of the set of semistandard Young tableaux, referring to this subset as the set of admissible semistandard Young tableaux. We categorize admissible semistandard Young tableaux according to depth and to the length of each row, and we subdivide the set of admissible Young tableaux into types corresponding to the highest weights of K representations. Finally, we develop a process for altering, or decorating, these admissible Young tableaux in such a way that the number of decorated admissible semistandard Young tableaux of type corresponding to some highest weight for K equates to the dimension of the representation for K of that highest weight.

The correspondence between admissible semistandard Young tableaux and representations for $K = SO(n, \mathbb{R})$ from section 5 identifies each decorated admissible semistandard Young tableaux of type corresponding to some highest weight for K with a line in the representation of that highest weight. In section 6, we study how the action of M for the real split reductive group of type A_n on this representation affects this set of lines. We use calculations within the universal enveloping algebra for K and the tenets of highest weight theory to study this action. The lines corresponding to decorated admissible semistandard Young tableaux do not, in general, span one-dimensional representations for M , but they do span one-dimensional representations for a large subgroup of M . The remainder of M acts on these lines in such a way as to make possible the identification of a basis for the highest weight representation comprising vectors each of which spans a one-dimensional representation for all of M . By studying the action of M on these vectors, we determine to which isomorphism class belongs the irreducible representation for M spanned by any one of these vectors. Moreover, using the connection between these vectors and the lines

corresponding to decorated admissible tableaux, we establish a bijection between the decorated admissible tableaux and the isomorphism classes of irreducible representations for M such that this bijection determines the multiplicity of any irreducible representation for M within the highest weight representation for K .

In section 7, we show how to extend our solution for branching from K to M for the real split reductive group of type A_n to solutions for branching from K_e to $M \cap K_e$ for the real split reductive groups of type B_n and D_n . We do not mention the group of type C_n , as branching from K to M for this group has a solution in terms of classical theory.

This paper concerns results only about branching for split classical groups, but the methods used in this paper suggest possible approaches to the pursuit of branching theorems in other cases. For instance, an understanding of the branching law from $SP(n, \mathbb{R})$ to $SP(n-1, \mathbb{R})$ in terms of semistandard Young tableaux could help to determine a branching law from K to M for the covering groups of type B_n and D_n . The linear real covering group of type A_n coincides with the real split group, so this paper does comment indirectly on branching for the covering groups of classical type. Calculations within universal enveloping algebras such as those we use to study the action of M on vectors in highest weight representations of K for the split real reductive group of type A_n could help to solve branching from K to M for a broad array of cases. Specifically, the split groups of exceptional type might benefit from careful analysis in terms of universal enveloping algebras and the tenets of highest weight theory.

2 Some Structure Theory for Split Real Reductive Algebraic Groups

The results of this paper focus entirely on the classical groups, but the motivation behind these results stems from somewhat more general theory. We deal first with affine algebraic groups as a context for introducing the notion of a split group. By restricting our attention in the following sections to split groups, we will allow for the exploration of representation theoretic results through the use of root data. Subsequent to our discussion of affine algebraic groups, we develop notation for the discussion of real reductive Lie groups. We show how well-worn theory applying to semisimple Lie groups extends to the reductive case and reduces certain questions concerning principal series representations for real reductive Lie groups to questions concerning branching laws for compact groups.

Letting k be any field, we start with the definition of an affine algebraic k -group.

Definition 2.1 *An affine algebraic k -group is a functor G' from the category of k -algebras to the category of groups such that there exists a finitely generated k algebra $k[G']$ with the property $G'(R) \cong \text{Hom}_{k\text{-alg}}(k[G'], R)$ for any k -algebra R .*

We call $k[G']$ the *coordinate ring* for G' .

We may take $k[G'] = k[X_1 \dots X_n]/\mathcal{I}(G')$ where $\mathcal{I}(G')$ is some radical ideal in $k[X_1 \dots X_n]$ and n is some natural number. In this case, the group $G'(k[G'])$ has the structure of a variety, and we can identify G' with $G'(k[G'])$ in order to bridge the gap between the definition for an affine algebraic group given above with the perhaps more familiar definition given in such texts as [2].

We will consider reductive algebraic groups in the entirety of the sequel.

Definition 2.2 *An algebraic group is reductive if it has a trivial unipotent radical.*

We now turn to the notion of a split algebraic group. A *split torus* for G' is an algebraic subgroup isomorphic to a direct product of copies of \mathbb{G}_m where \mathbb{G}_m is the usual multiplicative algebraic group defined over k . Considering G' as an affine variety, we may extend scalars for G' to an algebraically closed field as follows. If \bar{k} is the algebraic closure of k , then we define G'_k to be the affine algebraic \bar{k} -group with coordinate ring $\bar{k} \otimes_k k[G']$. Since any \bar{k} -algebra is a k algebra via the inclusion $k \hookrightarrow \bar{k}$, we see

$$G'_k(R) = \text{Hom}_{\bar{k}\text{-alg}}(\bar{k} \otimes_k k[G'], R)$$

is well-defined for any \bar{k} -algebra R . (In fact, as outlined on [7] p.15, we can extend scalars in this manner to any separable extension of k .) Using this extension of scalars, we can define a *torus* for G' to be an algebraic subgroup T of G' such that the extension T_k is a split torus for G'_k . By [7] 11.31, an algebraic group is reductive if and only if any connected normal abelian subgroup is also a torus. Thus, G' is a reductive group if and only if G'_k is a reductive group. A *maximal* torus for G' is a torus T such that T_k is properly contained in no torus for G'_k .

Definition 2.3 *A reductive algebraic group is split if it contains a split maximal torus.*

We restrict our attention to split reductive algebraic groups primarily because the study of representations over such groups reduces to the study of root data. The following definition is [7] 17.1.

Definition 2.4 *A root datum is a quadruple $\Psi = (\Delta, \Delta^\vee, X, X^\vee)$ where X and X^\vee are free \mathbb{Z} -modules of finite rank in duality via a pairing $\langle \cdot, \cdot \rangle: X \times X^\vee \rightarrow \mathbb{Z}$, Δ and Δ^\vee are in bijection via a map sending α in Δ to α^\vee in Δ^\vee and $(\Delta \times \Delta^\vee)$ is a finite subset of $X \times X^\vee$. In addition, the following properties must hold.*

1. We have $\langle \alpha, \alpha^\vee \rangle = 2$.

2. If $s_\alpha: X \rightarrow X$ is the homomorphism given by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$$

for x in X and α in Δ , we have $s_\alpha(\Delta) = \Delta$.

3. The group of automorphisms of X generated by $\{s_\alpha \mid \alpha \in \Delta\}$ is finite.

To each split, reductive, affine algebraic group G' together with a choice T of a split torus for G' , we associate a root datum $\Psi(G', T)$. We define $\Psi(G', T)$ through these four identifications:

$$\Delta = \Delta(\mathfrak{g}', T),$$

$$\Delta^\vee = \{\text{coroots for } \Delta\},$$

$$X = \text{Hom}(T, \mathbb{G}_m), \text{ and}$$

$$X^\vee = \text{Hom}(\mathbb{G}_m, T),$$

where \mathfrak{g}' is the Lie algebra for G' and $\Delta(\mathfrak{g}', T)$ are the roots of \mathfrak{g}' with respect to T . [7] 17.20 shows the root datum $\Psi(G', T)$ actually determines (G', T) up to isomorphism.

If $A = \bar{k}[X]$ is any affine \bar{k} -algebra, we follow [8] 1.3.7 to define a k -structure on X as a k -subalgebra A_0 of A of finite type over k such that the map

$$k \otimes_k A_0 \rightarrow A$$

given by multiplication of the left and right coordinates is an isomorphism. If \tilde{G}' is any affine algebraic \bar{k} -group with coordinate ring $\bar{k}[X]$, and if $k[X]$ is a k -structure for $\bar{k}[X]$, we define the affine algebraic k -group \tilde{G}'_k of k -rational points for our k -structure to be the affine algebraic k group with coordinate ring $k[X]$. In the sequel, we consider the groups of real points for complex matrix groups, meaning the subgroups

of matrices with entries in \mathbb{R} . Such subgroups are, in fact, \mathbb{R} -groups of \mathbb{R} -rational points for complex affine algebraic \mathbb{C} -groups defined with respect to an obvious \mathbb{R} structure.

If \tilde{G}' is a reductive affine algebraic \bar{k} -group, we wish to comment on the unipotent radical of \tilde{G}'_k . From the definition given above, we know there exists an injective map

$$\phi: \bar{k}[\tilde{G}'] \rightarrow \bar{k} \otimes_k k[\tilde{G}'_k],$$

thus, we get a surjective homomorphism

$$\Phi: (\tilde{G}'_k)_{\bar{k}} \rightarrow \tilde{G}'.$$

By [8] 12.4.3, we know the kernel of Φ contains no non-trivial, closed, normal subgroups of $(\tilde{G}'_k)_{\bar{k}}$. As a result, if (\tilde{G}'_k) has a non-toral, normal, abelian subgroup N , the image of N under Φ is also a non-toral, normal, abelian subgroup. Hence, if \tilde{G} is reductive, \tilde{G}'_k is also reductive.

We now confine our focus from general affine algebraic groups to real affine algebraic groups and encounter the Lie structure serving as a basic framework for this paper. Suppose \tilde{G}' is a connected, complex, reductive affine algebraic group. Then the group $\tilde{G}'_{\mathbb{R}}$ of \mathbb{R} -rational points for \tilde{G}' is a real, reductive affine algebraic group. This group has additional structure. The following definition comes from [5] p.446.

Definition 2.5 *A real reductive Lie group is a 4-tuple (G, K, θ, B) consisting of a real Lie group, G , a compact subgroup K of G , a Lie algebra involution for the Lie algebra \mathfrak{g}_0 of G , and an $\text{Ad}(G)$ -invariant, θ -invariant, bilinear form B on \mathfrak{g}_0 such that*

- (i) \mathfrak{g}_0 is a reductive Lie algebra,

- (ii) the decomposition of \mathfrak{g}_0 into $+1$ and -1 eigenspaces under θ is $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, where \mathfrak{k}_0 is the Lie algebra for K ,
- (iii) \mathfrak{k}_0 and \mathfrak{p}_0 are orthonormal under B , and B is positive definite on \mathfrak{p}_0 and negative definite on \mathfrak{k}_0 ,
- (iv) multiplication, as a map from $K \times \exp \mathfrak{p}_0$ into G , is a diffeomorphism onto, and
- (v) every automorphism $\text{Ad}(g)$ of $\mathfrak{g} = (\mathfrak{g}_0)^\mathbb{C}$ is given by some x in $\text{Int } \mathfrak{g}$.

According to [6] p. 245, the group $\tilde{G}'_{\mathbb{R}}$ is a real, reductive Lie group. Henceforth, we shall refer to $\tilde{G}'_{\mathbb{R}}$ as G .

For k in K and X in \mathfrak{p}_0 , we can define $\Theta: G \rightarrow G$ as

$$\Theta(k \exp X) = k \exp(-X).$$

By [5] 7.21, we know Θ is an automorphism of G with differential θ . We refer to Θ as the *global Cartan involution* for G . From Definition 2.5, iv, we conclude K is the subgroup G^Θ of G fixed by Θ . Again by Definition 2.5, iv, K is a maximal compact subgroup of G .

From Definition 2.5, i, we know $\mathfrak{g}_0 = Z_{\mathfrak{g}} \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$ where $Z_{\mathfrak{g}}$ denotes the center of \mathfrak{g}_0 and $[\mathfrak{g}_0, \mathfrak{g}_0]$ is the derived subalgebra of \mathfrak{g}_0 . In particular, $[\mathfrak{g}_0, \mathfrak{g}_0]$ is semisimple. We consider the complexified Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. If V is the underlying space for some representation π of \mathfrak{g} , then, for any α in \mathfrak{g}^* , we let V_α be

$$\{v \in V \mid \text{for every } X \in \mathfrak{g} \text{ there exists an } n \in \mathbb{N} \text{ with } ((\pi(X)v - \alpha(X) \cdot 1)^n = 0)\}.$$

If $V_\alpha \neq 0$, we call α a *weight*, and we refer to V_α as the *generalized weight space* of weight α . We define a subalgebra \mathfrak{h}_0 of \mathfrak{g}_0 to be a *Cartan subalgebra* if the complex-

ification \mathfrak{h} of \mathfrak{h}_0 is precisely the generalized weight space of weight 0 for the adjoint action $\text{ad}_{\mathfrak{g}} \mathfrak{h}$ of \mathfrak{h} on \mathfrak{g} . Using the theory of semisimple Lie algebras defined over algebraically closed fields, we can form a set of roots $\Delta([\mathfrak{g}, \mathfrak{g}], \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}])$. We extend these roots to all of \mathfrak{h} by defining $\alpha(H) = 0$ for any α in $\Delta([\mathfrak{g}, \mathfrak{g}], \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}])$ and any H in $Z_{\mathfrak{g}}$. This extended set of roots, $\Delta(\mathfrak{g}, \mathfrak{h})$, allows for a decomposition of \mathfrak{g} into root spaces

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})} \mathfrak{g}_{\alpha} \quad (1)$$

where $\mathfrak{g}_{\alpha} = \{X \in \mathfrak{g} \mid \text{ad}(H)X = \alpha(H)X\}$. If \mathfrak{h}_0 is any Cartan subalgebra for \mathfrak{g}_0 , there exists an element ϕ of $\text{Int}(\mathfrak{g}_0)$ such that $\phi(\mathfrak{h}_0)$ is a θ -stable Cartan subalgebra. ([5] p.457)

The *Cartan subgroup* corresponding to a Cartan subalgebra \mathfrak{h}_0 of \mathfrak{g}_0 is $\text{Cent}_G(\mathfrak{h}_0)$, the centralizer in G of \mathfrak{h}_0 via the adjoint action. We choose \mathfrak{h}_0 to be θ -stable, so the Cartan subgroup H of \mathfrak{h}_0 is Θ -stable. ([5] p.487)

Theorem 2.6 *If G is the group of \mathbb{R} -rational points of a connected, affine algebraic \mathbb{C} -group, let H be a subgroup. The following are equivalent.*

- (i) *The subgroup H is a Cartan subgroup.*
- (ii) *The subgroup H is a maximal torus.*

Proof omitted ■

We choose H , a Θ -stable Cartan subgroup of G . There exists natural action $\text{Ad}_{\mathbb{C}}(H)$ on \mathfrak{g} realized as an extension of $\text{Ad}_{\mathbb{C}}(h)(X \otimes z) = \text{Ad}(h)(X) \otimes z$ for h in H and $(X \otimes z)$ a simple tensor in \mathfrak{g} . Since H centralizes \mathfrak{h}_0 , and since the root system $\Delta(\mathfrak{g}, \mathfrak{h})$ depends only on the action of \mathfrak{h} , we deduce $\text{Ad}_{\mathbb{C}}(H)$ preserves each weight space \mathfrak{g}_{α} . Since each weight space has dimension one over \mathbb{C} , the adjoint action of H

on any given weight space must correspond to a homomorphism from H to \mathbb{C} . As a result, to each α in $\Delta(\mathfrak{g}, \mathfrak{h})$ we may assign a character

$$\alpha: H \rightarrow \mathbb{C}^\times.$$

(We note $\Delta(\mathfrak{g}_0, H) = \Delta(\mathfrak{g}, \mathfrak{h})$.) We next define $\Theta\alpha$ for any α in $\Delta(\mathfrak{g}, \mathfrak{h})$ to be the character for H given by

$$\Theta\alpha(h) = \alpha(\Theta(h))$$

for every h in H .

Lemma 2.7 *The character $\Theta\alpha$ is among the characters in $\Delta(\mathfrak{g}, \mathfrak{h})$. In other words, Θ permutes the set $\Delta(\mathfrak{g}, \mathfrak{h})$.*

Proof For any X in \mathfrak{g}_α , we know

$$\text{Ad}(h)(X) = \alpha(h)X, \quad \text{and}$$

$$\text{Ad}(\Theta h)(\theta X) = \alpha(h)\theta X.$$

Letting h' in H be such that $h = \Theta(h')$, we have

$$\text{Ad}(h')(\theta X) = \alpha(\Theta h')\theta X, \quad \text{so}$$

$$\text{Ad}(h')(\theta X) = \Theta\alpha(h')(\theta X).$$

Since θ stabilizes \mathfrak{h}_0 , we know θ maps \mathfrak{g}_α to some other root space \mathfrak{g}_β . We may conclude $\beta = \Theta\alpha$. ■

Supposing G is split, we choose a Θ -stable, split Cartan subgroup H for G . In this case, H is a product of copies of \mathbb{R}^\times , and \mathfrak{h}_0 is a product of copies of the additive group \mathbb{R} . With H thus chosen, the root space decomposition from Equation 1 for \mathfrak{g}

over \mathbb{C} restricts to a root space decomposition

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \sum_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})} (\mathfrak{g}_\alpha)_0$$

over \mathbb{R} . For each root α in $\Delta(\mathfrak{g}, \mathfrak{h})$, we see the corresponding character for H takes the form

$$\alpha: H \rightarrow \mathbb{R}^\times \subset \mathbb{C}^\times.$$

Theorem 2.8 *If G is a split real reductive Lie group with global Cartan involution Θ and H is a split Θ stable Cartan subgroup for G , then any root α in $\Delta(\mathfrak{g}, \mathfrak{h})$ has the property $\Theta\alpha = -\alpha$.*

Proof By Lemma 2.7, it suffices to show $\text{ad}(Y)(\theta X) = -\alpha(Y)(X)$ for any X in \mathfrak{g}_α and any Y in \mathfrak{h} . Considering the case $H = \mathbb{R}^\times$, we notice $\Theta(h) = h^{-1}$ for each h in H . Indeed, with Θ thus defined, Θ is a Cartan involution for the real reductive group (H, K_H, θ_H, B_H) : here $K_H = \{1, -1\}$, and B_H is ordinary multiplication. We see $\mathfrak{h}_0 \cap \mathfrak{k}_0 = 0$ where

$$\mathfrak{h}_0 = \mathfrak{h}_0 \cap \mathfrak{k}_0 \oplus \mathfrak{h}_0 \cap \mathfrak{p}(H)_0$$

is a decomposition of the Lie algebra \mathfrak{h}_0 for H taking the form of the decomposition in Definition 2.5, ii. Indeed, $\mathfrak{h}_0 \cap \mathfrak{k}_0$ is the Lie algebra for a finite group. Hence, θ_H is multiplication by -1 . We notice $\mathfrak{h}_0 = Z_{\mathfrak{g}_0}(\mathfrak{h}_0)$. By [5] 7.25, θ_H is the restriction of θ to $Z_{\mathfrak{g}_0}(\mathfrak{h}_0)$. For any X in \mathfrak{g}_α and any Y in \mathfrak{h} , we have

$$\begin{aligned} [Y, \theta X] &= \theta[\theta Y, X] = \theta(-\alpha(Y)X) \\ &= -\alpha(Y)(\theta X). \end{aligned}$$

Since H is a product of copies of \mathbb{R}^\times , the result follows. ■

We define \mathfrak{a}_0 to be $\mathfrak{h}_0 \cap \mathfrak{p}_0$. Additionally, we assign a notion of positivity to the root system $\Delta(\mathfrak{g}, \mathfrak{h})$, and we define \mathfrak{n}_0 to be $\bigoplus_{\alpha \in \Delta^+} (\mathfrak{g}_\alpha)_0$.

Theorem 2.9 (Iwasawa Decomposition) *The Lie algebra \mathfrak{g}_0 has a decomposition into the direct sum $\mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$. We let A and N be the analytic subgroups of G with Lie algebras \mathfrak{a}_0 and \mathfrak{n}_0 respectively. Then the map $K \times A \times N \rightarrow G$ given by $(k, a, n) \mapsto kan$ is a diffeomorphism. The groups A and N are simply connected.*

Proof [5]6.43 proves the first remark when \mathfrak{g}_0 is semisimple. If we replace [5]6.40(c) with Theorem 2.8, the arguments translate directly to the split real reductive case. The second statement amounts to [5]7.31. We need only note the first statement implies our definition for \mathfrak{n}_0 matches the one used in [5]7.31 precisely. ■ Since A is simply connected, exponentiation

$$\exp: \mathfrak{a}_0 \rightarrow A$$

is a diffeomorphism. For any a in A , we refer to the preimage of a as $\log a$.

Remark 2.10 [5]7.31 proves Theorem 2.9 for the more general real reductive case. This proof relies on the notion of restricted roots, and that notion proves unnecessary to our discourse.

We define the closed subgroup M of K to be $Z_K(\mathfrak{a}_0)$. Being closed in K , the subgroup M is compact, and, since AN is closed in G , the Borel subgroup MAN is closed in G . (Setting $B = MAN$, we refer to MAN as the *Langlands decomposition* for B with respect to θ . We refer to M as the *Langlands subgroup* of K .) By Theorem 2.9, we know $G = KMAN$. For any real reductive Lie group L , we denote by \hat{L} the set of representations for L . Since M is compact abelian, any representation δ in \hat{M} is a character

$$\delta: M \rightarrow \mathbb{C}^\times.$$

As exponentiation gives a diffeomorphism between A and \mathfrak{a}_0 , any representation ν in \hat{A} differentiates to a representation ν' for \mathfrak{a}_0 . The Lie algebra \mathfrak{a}_0 is a maximal abelian subalgebra for \mathfrak{p}_0 , so any representation ν in \hat{A} is one dimensional and corresponds to a character

$$\nu: A \rightarrow \mathbb{R}^\times.$$

We denote by ρ the half-sum of positive roots

$$\frac{1}{2} \sum_{\Delta(\mathfrak{g}, \mathfrak{h})^+} \alpha.$$

From [4] chapter 7, section 1, we develop a notion of induced representations. [4] handles induced representations for semisimple groups, and we extend the notion to real reductive groups consisting of the real points for semisimple complex groups. To each pair (δ, ν) , if V^δ is the underlying space for the irreducible representation δ in \hat{M} and ν is in \hat{A} , we associate an induced representation for G

$$\text{Ind}_{MAN}^G(\delta \otimes \nu' \otimes 1)$$

defined as follows. If

$$W := \{F: G \rightarrow V^\delta \text{ continuous} \mid F(xman) = e^{-(\nu'+\rho)\log a} \delta(m)^{-1} F(x)\},$$

we provide W with the L^2 norm over K :

$$\|F\|^2 = \int_K |F(k)|^2.$$

The underlying space for $\text{Ind}_{MAN}^G(\delta \otimes \nu' \otimes 1)$ is the completion \bar{W} of W under the L^2 norm, so W is dense in \bar{W} . We may define an action of G on \bar{W} via continuous

extension of the action

$$gF(x) = F(g^{-1}x)$$

on W . We refer to

$$\{\text{Ind}_{MAN}^G(\delta \otimes \nu' \otimes 1) \mid \delta \in \hat{M}, \nu' \in \hat{A}\}$$

as the *principal series* representations for G . (In the expression $\text{Ind}_{MAN}^G(\delta \otimes \nu' \otimes 1)$, 1 refers to the trivial representation of N .) Clearly, for any F in W , the restriction $F|_K$ completely determines F . Also, since $G = KMAN$, the representations $\text{Ind}_{MAN}^G(\delta \otimes \nu' \otimes 1)$ correspond one-to-one with the representations $\text{Ind}_{MAN}^G(\delta \otimes \nu' \otimes 1)|_K$. This observation allows us to study principal series representations using theory concerning representations of compact groups. For the real split reductive group of type A_n , the maximal compact subgroup K is connected. For types B_n and D_n , the subgroup K is not connected. According to [5] 7.33, M meets every component of K . We denote the identity component of K by K_e and we consider the closed subgroup $M \cap K_e$ of K_e . Since M meets every component of K , we can write any element of K as $k_e m$ for some k_e in K_e and some m in M . As such, the restriction $F|_{K_e}$ completely determines F for any F in W . Furthermore, $G = K_e MAN$, so the representations $\text{Ind}_{MAN}^G(\delta \otimes \nu' \otimes 1)$ correspond one-to-one with the representations $\text{Ind}_{MAN}^G(\delta \otimes \nu' \otimes 1)|_{K_e}$. Our attentions restrict further to the study of representations over compact connected groups.

The theory of representations over compact groups allows for its own notion of induced representations. We use the description provided in [1] chapter 3, section 6. If $M \cap K_e$ in K_e is any closed subgroup of the compact connected group K_e , and if δ is a representation for $M \cap K_e$ with underlying space V^δ , we define $\text{Ind}_{M \cap K_e}^{K_e}(\delta)$ as follows. We set

$$W: =$$

$$\{F: K_e \rightarrow V^\delta \text{ continuous} \mid F(km) = \delta(m)^{-1}F(k) \text{ for } k \in K_e \text{ and } m \in M \cap K_e\},$$

and we endow W with an action of K_e via

$$kF(x) = F(k^{-1}x) \text{ for } k \in K_e.$$

Evidently,

$$\text{Ind}_{MAN}^G(\delta \otimes \nu' \otimes 1)|_{K_e} = \text{Ind}_{M \cap K_e}^{K_e}(\delta).$$

Theorem 2.11 (Frobenius Reciprocity) *If δ is an $M \cap K_e$ module and μ is a K_e module, there is a canonical isomorphism*

$$\text{Hom}_{K_e}(\mu, \text{Ind}_{M \cap K_e}^{K_e} \delta) \cong \text{Hom}_{M \cap K_e}(\mu|_{M \cap K_e}, \delta)$$

Proof [1] 6.2 proves this theorem for general compact groups K_e with closed subgroups $M \cap K_e$. ■ If we consider an irreducible K_e module μ and an irreducible $M \cap K_e$ module δ , then 2.11 tells us the multiplicity of μ in $\text{Ind}_{M \cap K_e}^{K_e} \delta$ is exactly the multiplicity of δ in $\mu|_{M \cap K_e}$.

With this structure theory in mind, studying the branching law over $M \cap K_e$ for irreducible representations of K_e becomes a meaningful project. Any information shedding light on this branching law sheds light on the decomposing restrictions of principal series representations for real reductive groups into irreducible representations.

3 The Split Real Group of Type A_n

Eventually, we will comment on principal series representations for every split real classical group through studying these representations for the split real group of type A_n . As usual, we define $GL(n, \mathbb{R})$ to be the set of n by n matrices with nonzero

determinant and with entries in \mathbb{R} while we define the group $SL(n, \mathbb{R})$ to be

$$SL(n, \mathbb{R}) = \{g \in M(n, \mathbb{R}) \mid \det g = 1\}.$$

As the differential of the determinant morphism $\det: GL(n, \mathbb{R}) \rightarrow \mathbb{R}$ is trace, we see $SL(n, \mathbb{R})$ has Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ where

$$\mathfrak{sl}(n, \mathbb{R}) = \{X \in M(n, \mathbb{R}) \mid \operatorname{tr} X = 0\}.$$

The algebra $\mathfrak{sl}(n, \mathbb{R})$ has as a spanning set

$$\{X_{i,j}, Y_{i,j}, H_{i,j} \mid i < j \leq n\} \quad (2)$$

where the entries for the basis vectors in the i th and j th rows and columns take the form

$$X_{i,j} = \begin{matrix} & i & j \\ i & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ j & \begin{pmatrix} 0 & 0 \end{pmatrix} \end{matrix}, \quad Y_{i,j} = \begin{matrix} & i & j \\ i & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ j & \begin{pmatrix} 1 & 0 \end{pmatrix} \end{matrix}, \quad \text{and} \quad H_{i,j} = \begin{matrix} & i & j \\ i & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ j & \begin{pmatrix} 0 & -1 \end{pmatrix} \end{matrix},$$

and all other entries take the value 0.

We know $\mathfrak{sl}(n, \mathbb{R})$ is semisimple, hence this algebra has a nondegenerate Killing form

$$B(X, Y) = \operatorname{tr}(\operatorname{ad} X \operatorname{ad} Y).$$

Considering the involution θ on $\mathfrak{sl}(n, \mathbb{R})$ given by taking the negative transpose of each element, we wish to see this involution is a Cartan involution for B . We know B is symmetric because $\operatorname{tr}(xy) = \operatorname{tr}(yx)$ for any endomorphisms x and y of a finite-

dimensional vector space. Certainly, θ respects multiplication, as

$$\theta[X, Y] = -{}^t[X, Y] = -[{}^tY, {}^tX] = [-{}^tX, -{}^tY] = [\theta X, \theta Y].$$

For any automorphism ϕ of an arbitrary semisimple, finite-dimensional Lie algebra \mathfrak{g} over \mathbb{C} or \mathbb{R} , we note

$$[\phi X, Y] = \phi[X, \phi^{-1}Y] = \phi(\text{ad } X)\phi^{-1}Y$$

for any X and Y in \mathfrak{g} ([5] 1.118) As a result, the arbitrary automorphism ϕ holds the Killing form $B_{\mathfrak{g}}$ for \mathfrak{g} invariant:

$$\begin{aligned} B_{\mathfrak{g}}(\phi X, \phi Y) &= \text{tr}(\text{ad}(\phi X) \text{ad}(\phi Y)) \\ &= \text{tr}(\phi(\text{ad } X)\phi^{-1}\phi(\text{ad } Y)\phi^{-1}) \\ &= \text{tr}(\text{ad } X \text{ad } Y) \\ &= B_{\mathfrak{g}}(X, Y). \end{aligned}$$

([5] 1.119) If we define the inner product $\langle \cdot, \cdot \rangle: \mathfrak{sl}(n, \mathbb{R}) \rightarrow \mathbb{R}$ given by $\langle X, Y \rangle = \text{tr}(X {}^tY)$, we notice

$$\langle (\text{ad } Y)X, Z \rangle = \langle X, (\text{ad}({}^tY))Z \rangle.$$

Indeed, the equations

$$\begin{aligned} {}^t[{}^tY, Z] &= -{}^t[Z, {}^tY] = -[Y, {}^tZ] \text{ and} \\ \text{tr}(X, -[Y {}^tZ]) &= \text{tr}(-[X, Y], {}^tZ) = \text{tr}([Y, X], {}^tZ) \end{aligned}$$

hold, so we conclude

$$\text{tr}([Y, X], {}^tZ) = \text{tr}(X, {}^t[{}^tY, Z]).$$

As a result, we see

$$\text{ad}({}^tY) = {}^t(\text{ad } Y).$$

Clearly, $\mathfrak{sl}(n, \mathbb{R})$ has a decomposition

$$\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

where \mathfrak{k}_0 is the $+1$ and \mathfrak{p}_0 is the -1 eigenspace for θ . In fact, if X is in $\mathfrak{sl}(n, \mathbb{R})$, we can write X as a sum of unique elements from \mathfrak{k}_0 and \mathfrak{p}_0 in the following manner:

$$X = \frac{1}{2}(X - {}^tX) + \frac{1}{2}(X + {}^tX).$$

Now the form \langle, \rangle on $\mathfrak{sl}(n, \mathbb{R})$ has the property

$$\begin{aligned} \langle X, X \rangle &= -B(X, \theta X) = B(X, {}^tX) \\ &= \text{tr}((\text{ad } X)(\text{ad } {}^tX)) = \text{tr}((\text{ad } X)({}^t \text{ad } X)) \geq 0. \end{aligned}$$

([5] p.355) Since \langle, \rangle is positive definite, B is positive definite on \mathfrak{p}_0 and negative definite on \mathfrak{k}_0 . By [5] 6.31, the analytic subgroup K of $SL(n, \mathbb{R})$ having \mathfrak{k}_0 as its Lie algebra is compact, and the mapping $K \times \mathfrak{p}_0 \rightarrow SL(n, \mathbb{R})$ given by $(k, X) \mapsto k \exp X$ is a diffeomorphism. As $SL(n, \mathbb{R})$ is semisimple and connected, $\text{Ad}(SL(n, \mathbb{R})) = \text{Int}(\mathfrak{g}_0) \subset \text{Int}(\mathfrak{g})$. We have shown $(SL(n, \mathbb{R}), K, \theta, B)$ satisfies Definition 2.5.

From our definition for θ , we know

$$\mathfrak{k}_0 = \{X \in \mathfrak{gl}(n, \mathbb{R}) \mid X + {}^tX = 0, \text{ and } \text{tr } X = 0\}.$$

Hence, $\mathfrak{k}_0 = \mathfrak{so}(n, \mathbb{R})$. The analytic subgroup of $SL(n, \mathbb{R})$ having \mathfrak{k}_0 as its Lie algebra

is

$$K = SO(n, \mathbb{R}) = \{g \in GL(n, \mathbb{R}) \mid g^t g = 1, \text{ and } \det g = 1\}.$$

The group K is connected, so $K = K_e$. We examine the maximal abelian subalgebra \mathfrak{h}_0 for $\mathfrak{sl}(n, \mathbb{R})$ spanned by the elements $\{H_{i,j} \mid 1 \leq i < j \leq n\}$. Clearly, $\mathfrak{h}_0 = \mathfrak{sl}(n, \mathbb{R}) \cap D(n, \mathbb{R})$ where $D(n, \mathbb{R})$ denotes the diagonal matrices in $\mathfrak{gl}(n, \mathbb{R})$. The diagonal entries in the elements of \mathfrak{h}_0 have only one restriction, namely, the sum of these entries must equal 0. The Cartan subgroup H of $SL(n, \mathbb{R})$ having \mathfrak{h}_0 as its Lie algebra has the defining property

$$H = \{g \in D(n, \mathbb{R}) \mid \prod_{1 \leq i \leq n} g_i = 1\}$$

where g_i is the i^{th} diagonal entry of g . The group H is an $n - 1$ - dimensional split torus for $SL(n, \mathbb{R})$. Moreover, Θ stabilizes H where Θ , the global Cartan involution for $SL(n, \mathbb{R})$ must act as $\Theta g = ({}^t g)^{-1}$ by our definition for θ . The group $M = H \cap K$ is the group of self-inverse diagonal matrices with determinant 1. Thus, M has the description

$$M = \{g \in D(n, \mathbb{R}) \mid g_{i,i} = \pm 1, \text{ and } \prod_{1 \leq i \leq n} g_{i,i} = 1\}.$$

Being a finite (hence compact) group, M has an easily manageable set of irreducible representations \hat{M} . Since M is abelian, any (complex) irreducible representation V has dimension 1. We suppose the representation V has character $\chi_V: M \rightarrow \mathbb{C}$. Now $\chi_V(m) = \text{tr}(l_m)$ where l_m is the endomorphism of V given by the action of m in M . Every element of M is self inverse, so $\chi_V(m) = \pm 1$ for any m in M . In particular, $\chi_V(m)$ is a real number for any m in M , so any real irreducible representation for M has dimension 1. There exist only finitely many such characters for M , hence, up to isomorphism, there exist only finitely many irreducible representations. For any

subset S of $\{1, 2, \dots, n\}$, we define an irreducible character χ_S for M by

$$\chi_S(g) = \prod_{i \in S} g_{i,i}.$$

Lemma 3.1 *Every homomorphism $\chi: M \rightarrow \pm 1$ is isomorphic to χ_S for some subset S of $\{1, 2, \dots, n\}$.*

Proof For $n = 1$, the statement is obvious. We proceed by induction. For any j with $1 \leq j < n$, we define m^j to be the element of M with -1 in the j^{th} and $j + 1^{\text{st}}$ diagonal entries and 1 in all other diagonal entries. Then $\{m^j \mid 1 \leq j < n\}$ generate M . (Clearly the set of all elements of M with exactly two entries equal to -1 generates all of M . If $m^{i,j}$ is such an element of M with i^{th} and j^{th} diagonal entries equal to -1 where $i < j$, we see $m^{i,j} = m^i \cdot m^{i+1} \cdot \dots \cdot m^{j-1}$.) If $M(n-1)$ is the subgroup of M with n, n^{th} entries equal to 1 , then, by induction, there exists a subset S' of $\{1, 2, \dots, n-1\}$ such that $\chi|_{M(n-1)} = \chi_{S'}$. If $\chi(m^{n-1}) = -1$, then let

$$S = \begin{cases} S' & \text{if } n-1 \in S', \\ S' \cup n & \text{otherwise.} \end{cases}$$

If $\chi(m^{n-1}) = 1$, then let

$$S = \begin{cases} S' \cup n & \text{if } n-1 \in S', \\ S' & \text{otherwise.} \end{cases}$$

The character χ_S agrees with χ on a generating set for M , so the two homomorphisms must agree on all of M . ■

In fact, we can describe completely \hat{M} . We use the notation $[i/j]$ for i and j in \mathbb{Z} to denote the integer closest to i/j with absolute value less than or equal to i/j . To

indicate the cardinality of a set S , we write $|S|$. We define the set F_n as follows:

$$F_n := \{S \subset \{1, 2, \dots, n\} \mid |S| \leq \lfloor n - 1/2 \rfloor\} \\ \cup \{S \subset \{1, 2, \dots, n\} \mid |S| = n/2, \text{ and } 1 \in S\}.$$

Theorem 3.2 *The irreducible characters*

$$\{\chi_S \mid S \in F_n\}.$$

represent each of the isomorphism classes of representations in \hat{M} .

Proof By Lemma 3.1, we know every irreducible character for M is χ_S for some subset S of $\{1, 2, \dots, n\}$. We know $M \subset SL(n, \mathbb{R})$, so, for each m in M and each choice for S , we have

$$\prod_{j \in S} m_j = \prod_{i \in S^c} m_i$$

where S^c is the complement of S in $\{1, 2, \dots, n\}$. As such, we conclude $\chi_S = \chi_{S^c}$ for any subset S . Taking complements, we find any irreducible character χ for M is χ_S for some subset S in the set F_n . It remains to see $\chi_S \neq \chi_{S'}$ if $S \neq S'$ and each of S and S' are elements of the set F_n . Clearly, there exists an integer j with $1 \leq j \leq n$ such that j is neither an element of S nor an element of S' . Also, the symmetric difference $S \ominus S'$ of S and S' must contain an integer i with $1 \leq i \leq n$. Swapping the roles of S and S' if necessary, we suppose i is an element of S but not an element of S' . Using notation from the previous lemma, we consider the element $m^{i,j}$ of M . The equations $\chi_S(m^{i,j}) = -1$ and $\chi_{S'}(m^{i,j}) = 1$ both hold, so $\chi_S \neq \chi_{S'}$. ■

In much of the sequel, we will examine irreducible representations of $SO(n, \mathbb{R})$, attempting to decompose such representations into sums of irreducible representations χ_S for M . (Here, we see representations of $SO(n, \mathbb{R})$ as representations of M via

inclusion: $M \hookrightarrow SO(n, \mathbb{R})$.) The following result will prove useful. If p is any permutation on n elements, we define the $n \times n$ *permutation matrix* g_p corresponding to p as the result of permuting the rows of the identity matrix via p . If p is an even permutation, the permutation matrix corresponding to p has determinant 1 whereas the permutation matrix corresponding to an odd permutation has determinant -1 .

Theorem 3.3 *If S and S' are two elements of the set F_n having the same cardinality, then the multiplicity of χ_S in $\delta|_M$ is equal to the multiplicity of $\chi_{S'}$ in $\delta|_M$ for any irreducible representation δ of $SO(n, \mathbb{R})$.*

Proof We consider $S \ominus S'$. Choosing some bijection ϕ between $S \cap (S \ominus S')$ and $S' \cap (S \ominus S')$, we define the permutation t_i of the set $\{1, 2, \dots, n\}$ to be the transposition of i and $\phi(i)$. The permutation p given by

$$p = \prod_{i \in S \cap (S \ominus S')} t_i$$

maps S to S' .

First we suppose p is an even permutation, so g_p is an element of $SO(n, \mathbb{R})$. if V^S is the underlying space for a copy of χ_S in $\delta|_M$, we wish to see $g_p^{-1}V^S$ is a copy of $\chi_{S'}$. We choose some v in V^S . By the definition of g_p , conjugation of any element m of M by g_p permutes the diagonal entries of m via the permutation p . We know $g_p^{-1}v$ generates a copy of χ_{S^*} for some element S^* of F_n . The equations

$$\begin{aligned} (g_p m g_p^{-1})v &= (g_p m)(g_p^{-1}v) \\ &= g_p(\chi_{S^*}(m))(g_p^{-1}v) \\ &= (\chi_{S^*}(m))g_p(g_p^{-1}v) \\ &= (\chi_{S^*}(m))v \end{aligned}$$

certainly hold. For any m in M , if $\chi_{S'}(m) = \epsilon$, then $(g_p m g_p^{-1})v = \epsilon v$ because the permutation p maps S to S' . We conclude $S^* = S'$.

Furthermore, if V_1^S, \dots, V_r^S are the underlying spaces for linearly independent copies of χ_S in $\delta|_M$, then $g_p^{-1}V_1^S, \dots, g_p^{-1}V_r^S$ are underlying spaces for linearly independent copies of $\chi_{S'}$. We get an injective map λ from the set of copies of χ_S in $\delta|_M$ to the set of copies of $\chi_{S'}$ in $\delta|_M$. Reversing the roles of S and S' in the arguments above yields an injective map in the reverse direction, so λ is a bijection.

We now suppose p is an odd permutation. Then g_p is not an element of $SO(n, \mathbb{R})$, but

$$\tilde{g}_p = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} g_p$$

is an element of $SO(n, \mathbb{R})$. It is easy to see conjugation of any element m in M by \tilde{g}_p permutes the diagonal entries of m via the permutation p , so we may apply the arguments above replacing g_p with \tilde{g}_p . ■

4 An Inductive Method to Determine Branching From $SO(n, \mathbb{R})$ to M

Using highest weight theory and a thorough understanding of the branching law from $\mathfrak{so}(n, \mathbb{C})$ to $\mathfrak{so}(n-1, \mathbb{C})$, it is possible to determine theoretically the branching law for $SO(n, \mathbb{R})$ over M .

To begin this section, we establish some conventions regarding the root system for $\mathfrak{so}(n, \mathbb{C})$. These conventions come verbatim from [5] chapter II, section 1. For $n = 2k + 1$, we choose a maximal abelian subalgebra \mathfrak{h} to be the algebra of matrices

H such that

$$H = \begin{pmatrix} \begin{pmatrix} 0 & ih_1 \\ -ih_1 & 0 \end{pmatrix} & & & & \\ & \begin{pmatrix} 0 & ih_2 \\ -ih_2 & 0 \end{pmatrix} & & & \\ & & \dots & & \\ & & & \begin{pmatrix} 0 & ih_k \\ -ih_k & 0 \end{pmatrix} & \\ & & & & 0 \end{pmatrix}. \quad (3)$$

For $n = 2k$, we choose \mathfrak{h} to be the algebra of matrices H such that

$$H = \begin{pmatrix} \begin{pmatrix} 0 & ih_1 \\ -ih_1 & 0 \end{pmatrix} & & & & \\ & \begin{pmatrix} 0 & ih_2 \\ -ih_2 & 0 \end{pmatrix} & & & \\ & & \dots & & \\ & & & \begin{pmatrix} 0 & ih_k \\ -ih_k & 0 \end{pmatrix} & \\ & & & & \end{pmatrix}. \quad (4)$$

In both cases, h_j takes an arbitrary value in \mathbb{C} for each j such that $1 \leq j \leq k$. Taking either maximal torus H described above, we make the following definitions:

$$\mathfrak{h}_0 := \{H \in \mathfrak{h} \mid h_j \in \mathbb{R} \text{ for } 1 \leq j \leq k\}, \text{ and}$$

$$e_j := \text{the element of } H^* \text{ such that } e_j(H) = h_j \text{ for } 1 \leq j \leq k.$$

Furthermore, for $n = 2k + 1$, we define

$$\Delta := \{\pm e_i \pm e_j \mid 1 \leq i < j \leq k\} \cup \{\pm e_l \mid 1 \leq l \leq k\},$$

whereas for $n = 2k$, we define

$$\Delta := \{\pm e_i \pm e_j \mid 1 \leq i < j \leq k\}.$$

We let α in \mathfrak{h}^* be $\pm e_i \pm e_j$ for i and j such that $1 \leq i < j \leq k$, and we define each entry of E_α to be 0 except for the entries with indices among the i^{th} and j^{th} pairs of indices. (Those entries in either of the $2i - 1^{\text{th}}$, $2i^{\text{th}}$, $2j - 1^{\text{th}}$, or $2j^{\text{th}}$ columns and in either of the $2i - 1^{\text{th}}$, $2i^{\text{th}}$, $2j - 1^{\text{th}}$, or $2j^{\text{th}}$ rows.) The remaining sixteen entries take values according to the following equation:

$$E_\alpha = \begin{matrix} & i & j \\ i & \begin{pmatrix} 0 & X_\alpha \\ -{}^t X_\alpha & 0 \end{pmatrix} \\ j & \end{matrix} \quad (5)$$

where

$$X_{e_i - e_j} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad X_{e_i + e_j} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},$$

$$X_{-e_i + e_j} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad \text{and} \quad X_{-e_i - e_j} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

If $n = 2k + 1$, we also consider $\alpha = \pm e_l$ for l such that $1 \leq l \leq k$. With α as described, we let each entry of E_α equal 0 with the exception of the two entries in the n^{th} row and the $2l - 1^{\text{st}}$ or $2l^{\text{th}}$ column and the two entries in the n^{th} column and the $2l - 1^{\text{st}}$ or $2l^{\text{th}}$ row. These four entries take values as described by the following

equations:

$$E_{e_l} = \begin{matrix} & 2l-1 & 2l & 2k+1 \\ 2l-1 & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{pmatrix} & & \\ 2l & & & \\ 2k+1 & & & \end{matrix}, \text{ and} \tag{6}$$

$$E_{-e_l} = \begin{matrix} & 2l-1 & 2l & 2k+1 \\ 2l-1 & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ -1 & -i & 0 \end{pmatrix} & & \\ 2l & & & \\ 2k+1 & & & \end{matrix}.$$

For each choice of α in Δ , the matrix E_α spans a space of weight α for the adjoint action of \mathfrak{h} on $\mathfrak{so}(n, \mathbb{C})$. In particular, each α in Δ is a root. Moreover,

$$\mathfrak{so}(n, \mathbb{C}) = \mathfrak{h} \oplus_{\alpha \in \Delta} \mathbb{C}E_\alpha,$$

so Δ describes the full set of roots for $n = 2k + 1$ and for $n = 2k$. In accordance with the theory of abstract root systems, we choose sets of positive roots within the sets of roots. For $n = 2k + 1$, we define

$$\Delta^+ := \{e_i \pm e_j \mid 1 \leq i < j \leq k\} \cup \{e_l \mid 1 \leq l \leq k\}, \tag{7}$$

and for $n = 2k$, we define

$$\Delta^+ := \{e_i \pm e_j \mid 1 \leq i < j \leq k\}. \tag{8}$$

We briefly discuss integral forms for a compact connected Lie group G with max-

imal torus H . Writing $\mathfrak{h}_0 \subset \mathfrak{g}_0$ for the Lie algebras of $H \subset G$, we refer to the complexifications of these lie algebras as $\mathfrak{h} \subset \mathfrak{g}$. We write Δ to denote the root system $\Delta(\mathfrak{g}, \mathfrak{h})$, and we refer to one particular choice of positive roots as Δ^+ . For complex Cartan subalgebra \mathfrak{h} of a reductive Lie algebra \mathfrak{g} , we suppose \mathfrak{g} has root system Δ with respect to \mathfrak{h} , and we select a set Δ^+ of positive roots. If the compact connected Lie group G has \mathfrak{g}_0 as its Lie algebra, and if the complexification of \mathfrak{g}_0 is \mathfrak{g} , we formulate the subsequent definitions. An element λ of \mathfrak{h}^* in the subspace of \mathfrak{h}^* spanned by Δ is *dominant* if $\langle \lambda, \alpha \rangle \geq 0$ for each α in Δ^+ . Here, the pairing \langle, \rangle takes its definition from the identity

$$\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)}$$

for any α and β in Δ where $(,)$ is the given positive definite symmetric bilinear form on the space spanned by Δ . The form λ is *algebraically integral* if $\langle \lambda, \alpha \rangle$ is an integer for each α in Δ . If there exists a multiplicative character ξ_λ of H such that $\xi_\lambda(\exp X) = e^{\lambda(X)}$ for each X in \mathfrak{h}_0 , the form λ is *analytically integral*. According to [5] 4.58, a form is analytically integral if and only if whenever $\exp(X) = 1$ for some X in \mathfrak{h}_0 , we have $\lambda(X) = 2\pi ia$ for some a in \mathbb{Z} . Any analytically integral form is, in particular, algebraically integral by [5] 4.59.

The following points come from [5] p.254. Being compact, G is a reductive group, and we get a root decomposition for \mathfrak{g} of the form given by Equation 1. Now $\text{Ad}(H)$ acts by orthogonal transformations on \mathfrak{g}_0 with respect to the symmetric bilinear form B in Definition 2.5. Extending B to a Hermitian inner product on \mathfrak{g} , and extending $\text{Ad}(H)$ complex linearly to \mathfrak{g} , we see $\text{Ad} H$ is a commuting family of unitary operators, hence simultaneously diagonalizable. Since $\text{Ad}: H \rightarrow \text{Aut}(\mathfrak{h}_0)$ has differential $\text{ad}: \mathfrak{h}_0 \rightarrow \text{End}(\mathfrak{h}_0)$, we see the weight space decomposition for \mathfrak{g} agrees with the diagonalization of $\text{Ad} H$. Thus, each root space for \mathfrak{h} is also a root space for H , and for

each root α , we get a multiplicative character ξ_α of H such that

$$\text{Ad}(X)(Y) = \xi_\alpha(\exp X)(Y) = e^{\alpha(X)}(Y)$$

for each X in \mathfrak{h}_0 and each Y in \mathfrak{g}_α .

Algebraically integral, dominant forms in \mathfrak{h}^* are in one-to-one correspondence with irreducible, finite dimensional representations of \mathfrak{g} , where we associate to any algebraically integral form λ the unique irreducible representation of highest weight λ . ([3] section 21) If we start with a finite-dimensional irreducible representation Φ for G , we get a finite-dimensional irreducible representation ϕ for \mathfrak{g}_0 by differentiating Φ . We can extend ϕ to \mathfrak{g} complex linearly. The result is a finite-dimensional irreducible representation of a reductive complex Lie algebra corresponding to some dominant algebraically integral form λ in \mathfrak{h}^* . Since this representation comes from a representation Φ of G , we know the weight λ is actually analytically integral. On the other hand, if we start with any finite-dimensional irreducible representation for \mathfrak{g} corresponding to an algebraically integral form, we can restrict to a representation for \mathfrak{g}_0 . As long as this algebraically integral form is analytically integral for G , [5] 5.110 proves there exists a finite-dimensional, irreducible representation Φ of G with differential $\phi_{\mathfrak{g}_0}$. These processes establish a one-to-one correspondence between analytically integral dominant forms in \mathfrak{h} and finite-dimensional irreducible representations for G .

Returning to the language we have established to describe $SO(n, \mathbb{R})$, we wish to identify all possible analytically integral forms in \mathfrak{h} . The matrix

$$Y = \begin{pmatrix} 0 & ih \\ -ih & 0 \end{pmatrix}$$

exponentiates to

$$\exp Y = \begin{pmatrix} \cos(ih) & \sin(ih) \\ -\sin(ih) & \cos(ih) \end{pmatrix},$$

so $\exp Y = 1$ if and only if $h = 2\pi il$ for some l in \mathbb{Z} . We see X in \mathfrak{h} exponentiates to 1 if and only if for each j such that $1 \leq j \leq k$, we have $h_j = 2\pi il$ for some l in \mathbb{Z} where the complex numbers h_1, \dots, h_k define X as described in Equations 3 and 4. Choosing some arbitrary form $a_1 e_1 + \dots + a_k e_k$ in \mathfrak{h}^* where a_j is complex for each j with $1 \leq j \leq k$, we see

$$\lambda(X) = a_1 h_1 + \dots + a_k h_k.$$

If X in \mathfrak{h} exponentiates to 1, we see $\lambda(X) = 2\pi il$ for some l in \mathbb{Z} if and only if a_j is in \mathbb{Z} for each j with $1 \leq j \leq k$. Hence, the set of analytically integral weights for $SO(n, \mathbb{R})$ is

$$\{a_1 e_1 + \dots + a_k e_k \mid a_i \in \mathbb{Z} \text{ for } 1 \leq i \leq k\}.$$

Determining which of these analytically integral forms satisfies the definition of dominance proves easier still. We need only test whether a form λ satisfies $\langle \lambda, \alpha \rangle \geq 0$ for each α in Δ^+ , hence it suffices to test whether $(\lambda, \alpha) \geq 0$ for each α in Δ^+ using the positive definite, symmetric form $(\ , \)$. Referring to Equations 7 and 8, we see an analytically integral form $\lambda = a_1 e_1 + \dots + a_k e_k$ is dominant if

$$a_1 \geq \dots \geq a_k \geq 0 \quad \text{for } n = 2k + 1, \text{ or if} \quad (9)$$

$$a_1 \geq \dots \geq a_{k-1} \geq |a_k| \quad \text{for } n = 2k. \quad (10)$$

We realize the embedding $\iota_{n-1}: SO(n-1, \mathbb{R}) \hookrightarrow SO(n, \mathbb{R})$ via the mapping

$$A \mapsto \begin{pmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

When no confusion will result, we drop the subscript and write ι for ι_{n-1} . For any irreducible representation Φ of $SO(n, \mathbb{R})$, we can ask for an understanding of the decomposition of $\Phi|_{SO(n-1, \mathbb{R})}$ into irreducible representations for $SO(n-1, \mathbb{R})$. This branching law has a relatively simple and complete description.

Theorem 4.1 (*Murnaghan*) *For $n = 2k + 1$, the irreducible, finite-dimensional representation of $SO(n, \mathbb{R})$ with highest weight $a_1e_1 + \dots + a_k e_k$ decomposes under restriction to $SO(n-1, \mathbb{R})$ into a sum of representations with highest weight $c_1e_1 + \dots + c_k e_k$ such that*

$$a_1 \geq c_1 \geq a_2 \geq c_2 \geq \dots \geq a_{k-1} \geq c_{k-1} \geq a_k \geq |c_k|, \quad (11)$$

each such representation having multiplicity one in the decomposition.

For $n = 2k$, the representation with highest weight $a_1e_1 + \dots + a_k e_k$ decomposes into a sum of representations with highest weight $c_1e_1 + \dots + c_{k-1}e_{k-1}$ such that

$$a_1 \geq c_1 \geq a_2 \geq c_2 \geq \dots \geq a_{k-1} \geq c_{k-1} \geq |a_k|, \quad (12)$$

each such representation having multiplicity one in the decomposition.

Proof A proof based on Kostant's multiplicity formula exists in [5] pp.581-584. ■

We now consider the irreducible character $\chi_{n,S}$ of $M(SO(n))$, the Langlands subgroup of $SO(n, \mathbb{R})$, corresponding to the subset S of $\{1, 2, \dots, n\}$. By our choice for

the embedding $\iota: SO(n-1, \mathbb{R}) \hookrightarrow SO(n, \mathbb{R})$, we see

$$\chi_{n,S}|_{M(SO(n-1))} = \chi_{n-1, S \setminus \{n\}}. \quad (13)$$

If F is any reductive algebraic group and E is any closed subgroup of F , we refer to the multiplicity of an irreducible representation κ of E in the restriction of an irreducible representation μ of F to the subgroup E by $m(\kappa, \mu)$. We consider the closed subgroup $\iota(SO(n-1, \mathbb{R})) \subset SO(n, \mathbb{R})$. For any analytically integral dominant form λ corresponding to an irreducible representation μ_λ of $SO(n, \mathbb{R})$, we define \mathcal{A}_λ to be the set of analytically integral dominant forms γ such that $m(\mu_\gamma, \mu_\lambda) \neq 0$ where μ_γ is the irreducible representation of $SO(n-1, \mathbb{R})$ corresponding to the form γ . Theorem 4.1 describes \mathcal{A}_λ entirely for any analytically integral dominant form λ , and, for any γ in \mathcal{A}_λ , we know $m(\mu_\gamma, \mu_\lambda) = 1$.

If $\mu(n)_\lambda$ is the irreducible representation for $SO(n, \mathbb{R})$ with highest weight λ , the ruminations in section 2 motivate our attempting to ascertain the value $m(\chi_{n,S}, \mu_\lambda)$ for each character $\chi_{n,S}$ of $M(SO(n))$. To finish this section, we give an inductive process to ascertain these very multiplicities. The base case for such an inductive process arising from the case $n = 1$ is trivial since the group $SO(1, \mathbb{R})$ is trivial. For any algebraic group G , we refer to the category of representations for G as $\text{Rep}(G)$.

Lemma 4.2 *The equation*

$$\sum_{\gamma \in \mathcal{A}_\lambda} m(\chi_{n-1, \hat{S}}, \mu_\gamma) = m(\chi_{n, \hat{S}}, \mu_\lambda) + m(\chi_{n, (\hat{S} \cup \{n\})}, \mu_\lambda) \quad (14)$$

holds for any subset \hat{S} of $\{1, 2, \dots, n-1\}$.

Proof If each map in the diagram

$$\begin{array}{ccc} \text{Rep } SO(n, \mathbb{R}) & \longrightarrow & \text{Rep } SO(n-1, \mathbb{R}) \\ \downarrow & & \downarrow \\ \text{Rep } M(SO(n)) & \longrightarrow & \text{Rep } M(SO(n-1)) \end{array}$$

represents restriction, then the diagram commutes by the definition of the restriction morphism. As such, we may calculate $m(\chi_{n-1, \hat{S}}, \mu_\lambda)$ either to be

$$\sum_{\gamma \in \mathcal{A}_\lambda} m(\mu_\gamma, \mu_\lambda) \cdot m(\chi_{n-1, \hat{S}}, \mu_\gamma) \quad *$$

or to be

$$\sum_{\chi_{n,S} \text{ with } S \subset \{1,2,\dots,n\}} m(\chi_{n,S}, \mu_\lambda) \cdot m(\chi_{n-1, \hat{S}}, \chi_{n,S}). \quad **$$

Yet $m(\mu_\gamma, \mu_\lambda) = 1$ for each γ in \mathcal{A}_λ , so the quantity $*$ is

$$\sum_{\gamma \in \mathcal{A}_\lambda} m(\chi_{n-1, \hat{S}}, \mu_\gamma).$$

By Equation 13, we know

$$\chi_{n, \hat{S}}|_{M(SO(n-1))} = \chi_{n-1, \hat{S}} = \chi_{n, (\hat{S} \cup \{n\})}|_{M(SO(n-1))}.$$

Using Equation 13 together with Theorem 3.2, we conclude $\chi_{n, \hat{S}}$ and $\chi_{n, (\hat{S} \cup \{n\})}$ are the only characters of $M(SO(n))$ up to isomorphism with restriction to $M(SO(n-1))$ equal to $\chi_{n-1, \hat{S}}$. Hence $m(\chi_{n-1, \hat{S}}, \chi_{n,S})$ is nonzero if and only if $\chi_{n,S} = \chi_{n, \hat{S}}$ or $\chi_{n,S} = \chi_{n, (\hat{S} \cup \{n\})}$, in which case, $m(\chi_{n-1, \hat{S}}, \chi_{n,S}) = 1$. The quantity $**$ is

$$m(\chi_{n, \hat{S}}, \mu_\lambda) + m(\chi_{n, (\hat{S} \cup \{n\})}, \mu_\lambda)$$

■

For the following lemma, we suppose $k = n/2$ and $\lambda = a_1e_1 + \cdots + a_ke_k$ is the highest weight for some irreducible representation Φ of $SO(n, \mathbb{R})$.

Lemma 4.3 *If n is even and if $m(\chi_S, \Phi)$ is nonzero, then $|S|$ has the same parity as $a_1 + \cdots + a_k$.*

Proof Since n is even, the scalar matrix m_{-1} with all diagonal entries equal to -1 is an element of $M(SO(n))$. Moreover, m_{-1} is a central element. We choose some highest weight vector v in V^Φ for the representation Φ . Then v generates Φ in the sense that any vector for the representation Φ takes the form $c(gv)$ for some scalar c and some element g in $SO(n, \mathbb{R})$. We consider the action of m_{-1} on v . The dominant weight λ is an analytic integral form, so there exists a character ξ_λ of the maximal torus H in $SO(n, \mathbb{R})$ such that $\xi_\lambda(\exp X) = e^{\lambda(X)}$ for any X in \mathfrak{h}_0 . We define M_{-1} in \mathfrak{h}_0 by letting $h_j = \pi$ for each $1 \leq j \leq k$. Then $\exp(M_{-1}) = m_{-1}$. We see $\lambda(M_{-1}) = a_1 + \cdots + a_k$. Hence, according to the discussion preceding this proof,

$$m_{-1}v = (\xi_\lambda(m_{-1}))v = (-1)^{(a_1 + \cdots + a_k)}v.$$

If we let $\epsilon = (-1)^{(a_1 + \cdots + a_k)}$, we see

$$m_{-1}(c(gv)) = c(g(m_{-1}v)) = \epsilon(c(gv))$$

for any scalar c and any element g in $SO(n, \mathbb{R})$. Hence, viewing $\Phi(m_{-1})$ as an element of $GL(V^\Phi)$, we interpret $\Phi(m_{-1})$ as the diagonal matrix with each diagonal entry equal to ϵ . By Theorem 3.2, each irreducible subrepresentation of $\Phi|_M$ has character χ_S for some subset S of $\{1, 2, \dots, k\}$. Choosing S such that χ_S is the character for some such subrepresentation with underlying space $V^S \subset V^\Phi$, we denote the trace of

$\Phi(m_1)$ restricted to V^S by $\text{tr}_S(m_{-1})$. Then

$$\chi_S(m_{-1}) = \text{tr}_S(m_{-1}) = \epsilon.$$

As $\chi_S(m_{-1})$ is exactly $(-1)^{|S|}$, we see $|S|$ has the same parity as $a_1 + \dots + a_k$. ■

We have broached all the information necessary to prove the existence of an inductive method for the branching law from $SO(n, \mathbb{R})$ to M .

Proposition 4.4 *For any n in \mathbb{N} and any analytically integral dominant form λ corresponding to an irreducible representation μ_λ for $SO(n, \mathbb{R})$, Equation 14 allows us to determine $m(\chi_{n,S}, \mu_\lambda)$ for any subset S of $\{1, 2, \dots, n\}$ provided we already know $m(\chi_{n-1, \hat{S}}, \mu_\gamma)$ for any subset \hat{S} of $\{1, 2, \dots, n-1\}$ and any analytically integral dominant form γ corresponding to an irreducible representation μ_γ for $SO(n-1, \mathbb{R})$.*

Proof By Theorem 3.2, it suffices to show Equation 14 determines $m(\chi_{n,S}, \mu_\lambda)$ for each S in F_n , where we recall

$$\begin{aligned} F_n: &= \{S \subset \{1, 2, \dots, n\} \mid |S| \leq [n-1/2]\} \\ &\cup \{S \subset \{1, 2, \dots, n\} \mid |S| = n/2, \text{ and } 1 \in S\}. \end{aligned}$$

Furthermore, by Theorem 3.3, we need only determine $m(\chi_{n,S}, \mu_\lambda)$ when $S = S_i$ for $S_i = \{1, 2, \dots, i\}$ and for i such that $0 \leq i \leq [n/2]$. Using Theorem 4.1, we may determine completely the set \mathcal{A}_λ , so, by our inductive hypothesis, we may assume we know the value of the left-hand side of Equation 14 for any subset \hat{S} of $\{1, 2, \dots, n-1\}$.

We split the proof into two cases. First, we suppose $n = 2k$. We write $\lambda =$

$a_1e_1 + \cdots + a_ke_k$. For any i such that $1 \leq i \leq k$, Lemma 4.3 allows us to see

$$m(\chi_{n,S_{i-1}}, \mu_\lambda) = 0, \text{ and}$$

$$m(\chi_{n,(S_{i-1} \cup \{n\})}, \mu_\lambda) = \sum_{\gamma \in \mathcal{A}_\lambda} m(\chi_{n-1,S_{i-1}}, \mu_\gamma)$$

if the parity of $i - 1$ does not match the parity of $a_1 + \cdots + a_k$. If the parity of $i - 1$ matches the parity of $a_1 + \cdots + a_k$,

$$m(\chi_{n,S_{i-1}}, \mu_\lambda) = \sum_{\gamma \in \mathcal{A}_\lambda} m(\chi_{n-1,S_{i-1}}, \mu_\gamma), \text{ and}$$

$$m(\chi_{n,(S_{i-1} \cup \{n\})}, \mu_\lambda) = 0.$$

Using Theorem 3.3 again, we see

$$m(\chi_{n,(S_{i-1} \cup \{n\})}, \mu_\lambda) = m(\chi_{n,S_i}, \mu_\lambda).$$

Hence, we can determine $m(\chi_{n,S_i}, \mu_\lambda)$ for any i such that $0 \leq i \leq k$.

Now, we suppose $n = 2k + 1$. Theorem 3.2 tells us

$$\chi_{n,S_k \cup \{n\}} = \chi_{n,(S_k \cup \{n\})^c},$$

and $(S_k \cup \{n\})^c$ has cardinality k , as does S_k . By Theorem 3.3, we surmise

$$m(\chi_{n,S_k}, \mu_\lambda) = m(\chi_{n,(S_k \cup \{n\})}, \mu_\lambda).$$

As a result

$$\frac{\sum_{\gamma \in \mathcal{A}_\lambda} m(\chi_{n-1,S_k}, \mu_\gamma)}{2} = m(\chi_{n,S_k}, \mu_\lambda).$$

We induct on the quantity $k - i$. For $k - i = 0$, we have determined $m(\chi_{n,S_i}, \mu_\lambda)$. For

i such that $0 \leq i < k$, Equation 14 tells us

$$\sum_{\gamma \in \mathcal{A}_\lambda} m(\chi_{n-1, S_i}, \mu_\gamma) = m(\chi_{n, S_i}, \mu_\lambda) + m(\chi_{n, (S_i \cup \{n\})}, \mu_\lambda). \quad *$$

Turning once again to Theorem 3.3, we see

$$m(\chi_{n, (S_i \cup \{n\})}, \mu_\lambda) = m(\chi_{n, S_{i+1}}, \mu_\lambda),$$

and our inductive hypothesis allows us to assume we have determined the value $m(\chi_{n, S_{i+1}}, \mu_\lambda)$. As such, Equation * allows us to determine $m(\chi_{n, S_i}, \mu_\lambda)$. ■

Practically speaking, this inductive understanding of the branching law from $SO(n, \mathbb{R})$ to M makes actually calculating multiplicities a daunting task. Relying on this algorithm means computing multiplicities for the branching law from $SO(j, \mathbb{R})$ to M for each j such that $1 \leq j < n$. We would like to establish a more manageable algorithm for computing these multiplicities. Tableaux will serve as a tool to help us meet that objective.

5 Young Tableaux and $\widehat{SO}(n, \mathbb{R})$

Supposing β is some subset of \mathbb{Z}^2 , we will refer to a *tableau with shape β* as a map $\Omega: \beta \rightarrow \mathbb{Z}$. Young tableaux have visual representations gotten by considering \mathbb{Z} as a lattice in the plane \mathbb{R}^2 and dividing the plane into unit boxes. To each (i, j) in the subset β , we associate the unit box in \mathbb{R}^2 having (i, j) in the lower right-hand corner. We then fill the unit box corresponding to each point (i, j) in β with the

integer $\Omega(i, j)$. For instance, if

$$\begin{aligned}\beta &= \{(i, -1) \in \mathbb{Z}^2 \mid 1 \leq i \leq 4\} \\ &\cup \{(i, -2) \in \mathbb{Z}^2 \mid 1 \leq i \leq 3\} \\ &\cup \{(i, -3) \in \mathbb{Z}^2 \mid 1 \leq i \leq 1\},\end{aligned}$$

we may define a tableau with shape β by allowing the following diagram to illustrate the value of $\Omega(i, j)$ for any point (i, j) in β :

1	1	2	3
2	3	3	
3			

(15)

We assign to each n -tuple of natural numbers, $a = (a_1, \dots, a_n)$, a subset β_a of \mathbb{Z}^2 by letting β_a be the set of pairs $(i, -j)$ with $1 \leq j \leq n$ and with $0 \leq i \leq a_j$. To simplify notation, we will say a tableau with shape β_a has shape a .

A *partition* in n parts of a natural number m is an n -tuple of natural numbers (a_1, \dots, a_n) such that

$$a_1 + a_2 + \dots + a_n = m, \text{ and } a_1 \geq a_2 \geq \dots \geq a_n.$$

(We allow 0 to be a natural number.) We refer to a_i for $1 \leq i \leq n$ as a *part* of a . If a tableau has shape a for some partition a , we refer to the tableau as a *Young tableau*. A Young tableau Ω is *semistandard* if $1 \leq \Omega(i, j) \leq q$ for some natural number q , and if $\Omega(i, j) < \Omega(i, j-1)$ for each (i, j) in β_a with $(i, j-1)$ in β_a while $\Omega(i, j) \leq \Omega(i+1, j)$ for each (i, j) in β_a with $(i+1, j)$ in β_a . We refer to q as the *bound* for Ω . Justified by this definition, we say a semistandard Young tableau *increases weakly* along its rows and *increases strictly* down its columns. If a tableau Ω has shape a for some partition

a with n parts, we say the j^{th} row of Ω has *length* a_j . We refer to m , the number of nonzero parts of a and, hence, the number of rows in Ω , as the *depth* of Ω . Tableau 15 is semistandard. The following two tableaux are not semistandard. The tableau on the left-hand side does not increase weakly along its first row, and the tableau on the right-hand side does not increase strictly along its second and third columns:

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 2 \\ \hline 2 & 3 & & \\ \hline 3 & & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 3 \\ \hline 2 & 2 & 3 & \\ \hline 3 & & & \\ \hline \end{array}.$$

Furthermore, the length of the second row for the tableau on the left-hand side is 2 whereas the length of the second row for the tableau on the right-hand side is 3. Each tableau has depth 3.

As a point of reference, we illustrate the use of Young-diagrammatic methods in studying irreducible, finite-dimensional representations of $U(n)$ where

$$U(n) = \{g \in GL(n, \mathbb{C}) \mid g \cdot g^* = 1\}.$$

This compact (hence reductive) group has as its Lie algebra

$$\mathfrak{u}_0(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X + X^* = 0\}.$$

The complexification,

$$\mathfrak{u}(n) = \mathfrak{u}_0(n) \oplus i\mathfrak{u}_0(n),$$

is simply $\mathfrak{gl}(n, \mathbb{C})$. In order to study finite-dimensional representations of $U(n)$, the commentary in section 4 allows for us to restrict our attention to analytically integral dominant weights for $U(n)$ with respect to the root system for $\mathfrak{gl}(n, \mathbb{C})$ formed after

choosing some Cartan subalgebra. For a Cartan subalgebra \mathfrak{h} , we choose

$$\mathfrak{h} = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X \text{ is diagonal}\}.$$

We define e_i in \mathfrak{h}^* for each i such that $1 \leq i \leq n$ by

$$e_i \left(\begin{array}{ccc} h_1 & & \\ & \ddots & \\ & & h_n \end{array} \right) = h_i.$$

Furthermore, we define the set

$$\Delta := \{e_i - e_j \mid i \neq j, 1 \leq i \leq n, \text{ and } 1 \leq j \leq n\}.$$

If we choose E_{ij} to be the matrix in $\mathfrak{gl}(n, \mathbb{C})$ with 1 in the i, j^{th} entry and 0 in all other entries, we see E_{ij} spans a space in $\mathfrak{gl}(n, \mathbb{C})$ with weight $e_i - e_j$ for the adjoint action $\text{ad}_{\mathfrak{g}} \mathfrak{h}$. Furthermore,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbb{C}E_{ij},$$

so Δ is the set of roots for \mathfrak{g} with respect to our choice \mathfrak{h} of Cartan subalgebra. We choose a notion of positivity for our root system by setting

$$\Delta^+ := \{e_i - e_j \mid 1 \leq i < j \leq n\}.$$

With these definitions in place, calculations similar to those in section 4 show the analytically integral dominant weights for $U(n)$ are exactly

$$\{a_1 e_1 + \cdots + a_n e_n \mid (a_1, \dots, a_n) \in \mathbb{Z}^n, \text{ and } a_1 \geq a_2 \geq \cdots \geq a_n\}. \quad (16)$$

We realize the embedding $\iota_{n-1}: U(n-1) \hookrightarrow U(n)$ via the mapping

$$A \mapsto \begin{pmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

When no confusion will result, we drop the subscript and write ι for ι_{n-1}

Theorem 5.1 (Weyl) *The irreducible, finite-dimensional representation for $U(n)$ with highest weight $a_1e_1 + \dots + a_n e_n$ for (a_1, \dots, a_n) in \mathbb{Z}^n decomposes under restriction to $U(n-1)$ into a sum of representations with highest weight $c_1e_1 + \dots + c_{n-1}e_{n-1}$ such that*

$$a_1 \geq c_1 \geq a_2 \geq \dots \geq a_{n-1} \geq c_{n-1} \geq a_n, \quad (17)$$

each such representation having multiplicity one in the decomposition.

Proof A proof exists in [5] pp. 577-580. ■

The structure of semistandard Young tableaux relates directly to valuable information regarding the irreducible, finite-dimensional representations of $U(n)$. To begin to realize this relationship, we reproduce the following well-known result. We let a be a partition with n parts (a_1, \dots, a_n) for the natural number $a_1 + \dots + a_n$ (so (a_1, \dots, a_n) is in \mathbb{N}^n).

Theorem 5.2 *The number of semistandard Young tableaux with shape a and bound n is the dimension of the irreducible representation of $U(n)$ with highest weight $a_1e_1 + \dots + a_n e_n$.*

Proof We prove this result by induction, the case $n = 1$ being trivial. If Ω_a is a semistandard Young tableau having shape a , then we define the Young tableau

$\pi(\Omega_a)$ to be the tableau given by removing every occurrence of the number n in the visual representation of Ω_a . If $\Omega_a(i, j) = n$, then, since semistandard Young tableaux increase strictly down their columns, we know $(i, j-1)$ is not an element of β_a . Also, we know $\Omega_a(i, -n) = n$ if $(i, -n)$ is an element of β_a . Hence, $\pi(\Omega_a)$ has depth at most $n-1$, and, for each j such that $1 \leq j \leq n-1$, we see the j^{th} row of $\pi(\Omega_a)$ has length c_j for some c_j such that $a_j \geq c_j \geq a_{j+1}$. Since Ω_a increases weakly along its rows and strictly down its columns, $\pi(\Omega_a)$ increases weakly along its rows and strictly along its columns as well. We have shown the mapping $\Omega_a \mapsto \pi(\Omega_a)$ gives a well-defined map π from the set of semistandard Young tableaux having shape a to the set of semistandard Young tableaux having shape c for some partition

$$c = (c_1, \dots, c_{n-1}) \text{ such that } a_1 \geq c_1 \geq a_2 \geq \dots \geq a_{n-1} \geq c_{n-1} \geq a_n. \quad *$$

We now denote $\pi(\Omega_a)$ by Ω_c . The map π is clearly injective: if Ω_a and Ω'_a both map to Ω_c then $\Omega_a(i, j) = \Omega'_a(i, j)$ if $\Omega_a(i, j) < n$ and (i, j) is an element of β_a . This equation shows $\Omega_a(i, j) = \Omega'_a(i, j)$ for each (i, j) in β_a since both Ω_a and Ω'_a have the same shape a and map to the same tableau under π . Any semistandard Young tableau $\tilde{\Omega}_c$ with shape c for some partition c satisfying Equation $*$ has a completion to a semistandard Young tableau Ω_a of shape a by allowing $\Omega_a(i, j) = n$ for each (i, j) such that $-1 \geq j \geq -(n-1)$ and $c_{-j} \leq i \leq a_{-j}$ or $j = -n$ and $0 \leq i \leq a_n$. Then

$$\pi(\Omega_a) = \Omega_c = \tilde{\Omega}_c,$$

and we have shown π is surjective.

By our inductive hypothesis, the number of semistandard Young tableaux having shape c is the dimension of the irreducible representation for $U(n-1)$ with highest weight $c_1 e_1 + \dots + c_{n-1} e_{n-1}$. Since the map π is bijective, we know the number of

semistandard Young tableaux having shape a is the sum of the dimensions of the representations for $U(n-1)$ with highest weight $c_1e_1 + \cdots + c_{n-1}e_{n-1}$ such that the n -tuple of coefficients $c = (c_1, \dots, c_n)$ satisfies Equation *. Finally, by Theorem 5.1, we see the sum of the dimensions of the representations for $U(n-1)$ with highest weight $c_1e_1 + \cdots + c_{n-1}e_{n-1}$ such that the n -tuple of coefficients $c = (c_1, \dots, c_n)$ satisfies Equation * is exactly the dimension of the representation for $U(n)$ with highest weight $a_1e_1 + \cdots + a_n e_n$. ■

The proof of Theorem 5.2 establishes a correlation between the irreducible, finite-dimensional representations of $U(n)$ and semistandard Young tableaux. The semistandard Young tableaux make plain the relationship between the dimensions of irreducible representations of $U(n)$ and the dimensions of irreducible representations of $U(n-1)$. We wish to find an analagous picture for irreducible, finite-dimensional representations of $SO(n, \mathbb{R})$, a structure encoding information about these representations in such a way as to tie this information to corresponding facts about representations of $SO(n-1, \mathbb{R})$.

Definition 5.3 *A semistandard Young tableau Ω is admissible for $SO(n, \mathbb{R})$ if the following conditions hold:*

- (i) Ω is a semistandard Young tableau having shape a for some partition $a = (a_1, \dots, a_n)$, and
- (ii) $\Omega(i, -j) \geq 2j$ for each j such that $1 \leq j \leq n$ and each i such that $1 \leq i \leq a_j$.

We consider two semistandard Young tableaux:

2	2	3	3	4
5	5	6		
6	6	7		

, and

2	2	3	3	4
5	5	6		
6	6	7		
7				

The tableau on the left-hand side is admissible with shape $(5, 3, 3, 0, 0, 0, 0)$. The tableau on the right-hand side is admissible for no shape as its 4th row contains a number smaller than 8. In fact, it is easy to see an admissible semistandard Young tableau must have a shape $a = (a_1, \dots, a_n)$ such that $a_j = 0$ for all $j > \lfloor n/2 \rfloor$.

If a is a partition (a_1, \dots, a_n) such that $a_j = 0$ for $j > \lfloor n/2 \rfloor$, we refer to the set of admissible semistandard Young tableaux having shape a as Ψ_a . We define

$$\bar{\Psi}_a := (\Psi_a \times \{\epsilon_1, \dots, \epsilon_{\lfloor n/2 \rfloor} \mid \epsilon_i = \pm 1 \text{ for } 1 \leq i \leq \lfloor n/2 \rfloor\}) / \sim$$

where

$$(\Omega_a, (\epsilon_1, \dots, \epsilon_{\lfloor n/2 \rfloor})) \sim (\Omega'_a, (\epsilon'_1, \dots, \epsilon'_{\lfloor n/2 \rfloor}))$$

if and only if $\Omega_a = \Omega'_a$ and $\epsilon_j = \epsilon'_j$ for each j such that $1 \leq j \leq \lfloor n/2 \rfloor$ and $\Omega_a(1, -j) = 2j$. In other words, $\bar{\Psi}_a$ is the set of admissible semistandard Young tableau having shape a decorated with a choice of parity ϵ_j for each row j with $1 \leq j \leq \lfloor n/2 \rfloor$ such that every occurrence of the number $2j$ on row j has parity ϵ_j . Henceforth, we identify the pair $(\Omega_a, (\epsilon_1, \dots, \epsilon_{\lfloor n/2 \rfloor}))$ with its equivalence class in $\bar{\Psi}_a$ when context allows for no confusion. We refer to elements of the set $\bar{\Psi}_a$ as *decorated* admissible semistandard Young tableaux. If $\bar{\Omega}_a = (\Omega_a, (\epsilon_1, \dots, \epsilon_{\lfloor n/2 \rfloor}))$ is an element of $\bar{\Psi}_a$, we refer to Ω_a as the admissible semistandard Young tableau *associated* to $\bar{\Omega}_a$.

If we let Ω be the admissible semistandard Young tableau

2	2	3	3	4
5	5	6		
6	6	7		

then Ω is associated to four decorated tableaux, namely

$$\begin{array}{|c|c|c|c|c|} \hline 2 & 2 & 3 & 3 & 4 \\ \hline 5 & 5 & 6 & & \\ \hline 6 & 6 & 7 & & \\ \hline \end{array} , \quad \begin{array}{|c|c|c|c|c|} \hline -2 & -2 & 3 & 3 & 4 \\ \hline 5 & 5 & 6 & & \\ \hline -6 & -6 & 7 & & \\ \hline \end{array} , \\
 \begin{array}{|c|c|c|c|c|} \hline -2 & -2 & 3 & 3 & 4 \\ \hline 5 & 5 & 6 & & \\ \hline 6 & 6 & 7 & & \\ \hline \end{array} , \text{ and } \quad \begin{array}{|c|c|c|c|c|} \hline 2 & 2 & 3 & 3 & 4 \\ \hline 5 & 5 & 6 & & \\ \hline -6 & -6 & 7 & & \\ \hline \end{array} .$$

The pair $(\Omega, (1, -1, 1))$ is equivalent to the pair $(\Omega, (1, 1, 1))$.

We select an irreducible, finite-dimensional representation μ_λ for $SO(n, \mathbb{R})$ with highest weight $\lambda = a_1 e_1 + \cdots + a_{[n/2]} e_{[n/2]}$, and we let a be the partition $(|a_1|, \dots, |a_n|)$ such that $a_j = 0$ for each j such that $j > [n/2]$. We define a map π_n following exactly the steps used to define the map π from Theorem 5.2 such that $\pi_n(\Omega_a)$ is a semistandard Young tableau for any Ω_a in Ψ_a . In fact, $\pi_n(\Omega_a)$ is admissible. This tableau has Property ii from Definition 5.3 because Ω_a has Property ii. Since $a_j = 0$ for each $j > [n/2]$, the proof of Theorem 5.2 shows $\pi_n(\Omega_a)$ has shape $c = (c_1, \dots, c_{n-1})$ where

$$a_1 \geq c_1 \geq a_2 \geq \cdots \geq c_{[n/2]-1} \geq a_{[n/2]} \geq c_{[n/2]}, \text{ and} \quad (18)$$

$$c_j = 0 \text{ for each } j > [n - 1/2]. \quad (19)$$

Equation 19 comes from the equality $[n - 1/2] = [n/2]$ if n is odd and, if n is even, from the fact $\Omega_a(i, -j) = 2j$ if $j = n/2$ and $1 \leq i \leq a_j$. Whenever possible, we drop the subscript and write π for π_n .

We make the following definition: for even n ,

$$\bar{\Psi}_\lambda := \{ (\Omega_a, (\epsilon_1, \dots, \epsilon_{[n/2]})) \in \bar{\Psi}_a \mid \epsilon_{[n/2]} |a_{[n/2]}| = a_{[n/2]} \},$$

and for odd n , we have the equality $\bar{\Psi}_\lambda = \bar{\Psi}_a$. We will refer to the set $\bar{\Psi}_\lambda$ as the set of decorated admissible tableaux of *type* λ .

Theorem 5.4 *The dimension of the irreducible representation μ_λ is $|\bar{\Psi}_\lambda|$.*

Proof Just as we relied on Theorem 5.1 for giving structure to our proof of Theorem 5.2, we now rely on Theorem 4.1. We proceed by induction, the case $n = 1$ being trivial. Given an equivalence class

$$\bar{\Omega}_\lambda = (\Omega_a, (\epsilon_1, \dots, \epsilon_{[n/2]}))$$

in $\bar{\Psi}_\lambda$, we define $\bar{\pi}_n(\bar{\Omega}_a)$ to be $(\pi_n(\Omega_a), (\epsilon_1, \dots, \epsilon_{[n-1/2]}))$. As with the map π_n , we drop the subscript from $\bar{\pi}_n$ and write simply $\bar{\pi}$ whenever doing so will not cause confusion. According to the arguments immediately preceding this theorem, $\bar{\pi}(\bar{\Omega}_a)$ is an element of $\bar{\Psi}_c$ for some partition $c = (c_1, \dots, c_{n-1})$ satisfying Equation 18 with $c_j = 0$ for each j such that $j \geq n/2$ when n is even. When n is odd, $\bar{\pi}(\bar{\Omega}_a)$ is an element of $\bar{\Psi}_c$ for some partition $c = (c_1, \dots, c_{n-1})$ satisfying Equations 18 and 19. In fact, $\bar{\pi}(\bar{\Omega}_a)$ is an element of $\bar{\Psi}_\gamma$ where γ is the highest weight

$$c_1 e_1 + \dots + c_{[n-1/2]} e_{[n-1/2]}$$

for $SO(n-1, \mathbb{R})$ when n is even and where γ is the highest weight

$$c_1 e_1 + \dots + c_{[n-1/2]-1} e_{[n-1/2]-1} + \epsilon_{[n-1/2]} (c_{[n-1/2]} e_{[n-1/2]})$$

for $SO(n-1, \mathbb{R})$ when n is odd. From Theorem 4.1, we see each such integral form γ has the property $m(\mu_\gamma, \mu_\lambda) = 1$. Thus, we get a well-defined map

$$\bar{\pi}: \bar{\Psi}_\lambda \rightarrow \cup_{\gamma \in \mathcal{A}_\lambda} \bar{\Psi}_\gamma$$

where, as in section 4, \mathcal{A}_λ is the set of analytically integral forms γ for $SO(n-1, \mathbb{R})$ such that $m(\mu_\gamma, \mu_\lambda) \neq 0$. Given two elements,

$$(\Omega_a, (\epsilon_1, \dots, \epsilon_{[n/2]})) \text{ and } (\Omega'_a, (\epsilon'_1, \dots, \epsilon'_{[n/2]}))$$

of $\bar{\Psi}_\lambda$, we suppose $\bar{\pi}$ maps each element to the same element of $\bar{\Psi}_\gamma$ for some integral form γ . Then $\pi(\Omega_a) = \pi(\Omega'_a)$, and, since π is injective, $\Omega_a = \Omega'_a$. We know $\epsilon_{[n/2]} = \epsilon'_{[n/2]} = \epsilon$ where $\epsilon|_{a_{[n/2]}} = a_{[n/2]}$. Furthermore, for j such that $1 \leq j < [n/2]$, if $\epsilon_j \neq \epsilon'_j$, we may assume $\Omega_a(1, -j) > 2j$. Hence, $\bar{\pi}$ is injective. We let \mathcal{A}_a be the set of partitions $c = (c_1, \dots, c_{n-1})$ such that

$$a_1 \geq c_1 \geq a_2 \geq \dots \geq c_{n-1} \geq a_n.$$

From the proof of Theorem 5.2, we know

$$\pi: \Psi_a \rightarrow \{\Psi_c \mid c \in \mathcal{A}_a\}$$

is surjective. By Theorem 4.1, we know

$$\{(\Omega_c, (\epsilon_1, \dots, \epsilon_{[n-1/2]})) \mid c \in \mathcal{A}_a\} = \{\Psi_\gamma \mid \gamma \in \mathcal{A}_\lambda\}.$$

Hence, $\bar{\pi}$ is surjective.

By our inductive hypothesis, we know $|\Psi_\gamma| = \text{Dim } \mu_\gamma$ for each integral form γ of

$SO(n-1, \mathbb{R})$. By Theorem 4.1,

$$\text{Dim } \mu_\lambda = \sum_{\gamma \in \mathcal{A}_\lambda} \text{Dim } \mu_\gamma.$$

Since $\bar{\pi}$ is a bijection, we derive the equation

$$|\Psi_\lambda| = \sum_{\gamma \in \mathcal{A}_\lambda} |\Psi_\gamma|.$$

The theorem follows readily. ■

We make use of the map $\bar{\pi}$ from Theorem 5.4 throughout the remainder of the paper.

6 Decorated Admissible Tableaux and Branching Over M

In section 5, we established a connection between the irreducible representations of $SO(n, \mathbb{R})$ having finite dimension and the set of decorated admissible tableaux. Namely, we showed the representation with highest weight λ has dimension equal to the cardinality of the set of decorated admissible tableaux of type λ . In proving this result, we saw each decorated admissible tableau of type λ corresponds to a subspace of the representation with highest weight λ , itself an irreducible representation under restriction to $SO(n-1, \mathbb{R})$ via the embedding $\iota: SO(n-1, \mathbb{R}) \hookrightarrow SO(n, \mathbb{R})$ defined in section 4. Applied recursively to Theorem 4.1, this reasoning allows us to associate to each decorated admissible tableau of type λ a unique line within the representation of $SO(n, \mathbb{R})$, itself an irreducible representation under restriction to $SO(2, \mathbb{R})$ via the

embedding

$$SO(2, \mathbb{R}) \xrightarrow{\iota_2} SO(3, \mathbb{R}) \xrightarrow{\iota_3} \dots \xrightarrow{\iota_{n-1}} SO(n, \mathbb{R}).$$

We may realize these lines somewhat more explicitly as follows. As in section 4, we denote by μ_λ the irreducible representation corresponding to λ , and we denote by $m(\mu_\gamma, \mu_\lambda)$ the multiplicity of the $SO(n-1, \mathbb{R})$ -representation μ_γ in the restriction of μ_λ to $SO(n-1, \mathbb{R})$. In the interest of developing notation for this section, we select a highest weight $\nu_n(\lambda)$ for μ_λ . To each decorated admissible tableau $\bar{\Omega}$ of type λ , we apply the map $\bar{\pi}_n$ to arrive at a decorated admissible tableau of type γ_{n-1} for some highest weight γ_{n-1} of $SO(n-1, \mathbb{R})$ such that $m(\mu_{\gamma_{n-1}}, \mu_\lambda) = 1$. We select a highest weight vector $\nu_{n-1}(\gamma_{n-1})$ for the irreducible $SO(n-1, \mathbb{R})$ -representation $\mu_{\gamma_{n-1}} \subset \mu_\lambda$. We repeat this process, applying $\bar{\pi}_{n-1}$ to $\bar{\pi}_n \bar{\Omega}$ and replacing λ with γ_{n-1} to arrive at a highest weight γ_{n-2} for $SO(n-2, \mathbb{R})$ and to select a highest weight vector $\nu_{n-2}(\gamma_{n-1}, \gamma_{n-2})$ of highest weight γ_{n-2} for $\mu_{\gamma_{n-2}}$. Recursively, we continue in this manner until we have arrived at a highest weight γ_2 and selected a highest weight vector $\nu_2(\gamma_{n-1}, \dots, \gamma_2)$ of highest weight γ_2 for $\mu_{\gamma_{n-2}}$. We denote $\nu_2(\gamma_{n-1}, \dots, \gamma_2)$ by $\nu_2(\bar{\Omega})$. The vector $\nu_2(\bar{\Omega})$ is unique only up to multiplication by a scalar. As in section 4, we use the symbol \mathcal{A}_λ to refer to the set of analytically integral forms γ for $SO(n-1, \mathbb{R})$ such that $m(\mu_\gamma, \mu_\lambda) \neq 0$. According to Theorem 4.1,

$$\mu_\lambda|_{SO(n-1, \mathbb{R})} = \bigoplus_{\gamma \in \mathcal{A}_\lambda} \mu_\gamma.$$

Theorem 5.4 proves $\nu_2(\bar{\Omega}) \neq \nu_2(\bar{\Omega}')$ if $\bar{\Omega} \neq \bar{\Omega}'$ and

$$\mu_\lambda|_{SO(2, \mathbb{R})} = \bigoplus_{\bar{\Omega} \in \bar{\Psi}_\lambda} \mathbb{C} \nu_2(\bar{\Omega}). \quad (20)$$

In this section, we determine exactly how the decomposition in Equation 20 allows us to find the multiplicities of irreducible representations for M in irreducible represen-

tations for $SO(n, \mathbb{R})$ without first having to determine the multiplicities of irreducible representations for M in irreducible representations for $SO(n-1, \mathbb{R})$.

Ultimately, we want to provide a decomposition

$$\mu_\lambda|_M = \bigoplus_{1 \leq i \leq \text{Dim } \mu_\lambda} \mathbb{C}\nu(i)$$

and to understand the action of M on $\nu(i)$ for each i such that $1 \leq i \leq \text{Dim } \mu_\lambda$. We cannot assume $\mathbb{C}\nu_2(\bar{\Omega})$ is a representation of M for every element $\bar{\Omega}$ in $\bar{\Psi}_\lambda$. Nonetheless, we adopt as a provisional goal our understanding the manner in which M acts on the vectors in set

$$\mathcal{L} = \{\nu_2(\bar{\Omega}) \mid \bar{\Omega} \in \bar{\Psi}_\lambda\}$$

via the representation of M on μ_λ obtained by restriction. In order to gain a first foothold on a path toward this understanding, we delve into the details of the recursion defining the vectors in \mathcal{L} .

Supposing $n = 2k + 1$, we recall our notation from section 4, Equations 5 and 6, for the root vectors E_α in $\mathfrak{so}(n, \mathbb{C})$ where α takes values from among the positive roots

$$\Delta^+(\mathfrak{so}(n, \mathbb{C}), \mathfrak{h}) = \{e_i \pm e_j, e_l \mid 1 \leq i < j \leq k, \text{ and } 1 \leq l \leq k\}$$

chosen with respect to our designation \mathfrak{h} of a Cartan subalgebra for $\mathfrak{so}(n, \mathbb{C})$. To simplify notation, we refer to $\Delta^+(\mathfrak{so}(n, \mathbb{C}))$ as Δ^+ , just as in section 4. We impose an ordering on $\mathcal{N}_+ := \{E_\alpha \mid \alpha \in \Delta^+\}$ such that

$$E_{e_k} < E_{e_{k-1}} < \cdots < E_{e_1} < E_{e_i \pm e_j}$$

for any i and j such that $1 \leq i < j \leq k$. The set \mathcal{N}_+ comprises an ordered basis for \mathfrak{n}_+ , the Lie subalgebra of $\mathfrak{so}(n, \mathbb{C})$ spanned by the set \mathcal{N}_+ of positive root vectors. Next,

we take the union \mathcal{B} of a basis for \mathfrak{h} and the set of negative root vectors. We extend \mathcal{N}_+ to an ordered basis $\mathcal{N}_+ \cup \mathcal{B}$ for $\mathfrak{so}(n, \mathbb{C})$ by choosing an arbitrary ordering for \mathcal{B} and stipulating $E_\alpha < b$ for any b in \mathcal{B} and any α in Δ^+ . This ordering determines a *Poincaré-Birkhoff-Witt* or *PBW* basis for the universal enveloping algebra $\mathfrak{U}(\mathfrak{so}(n, \mathbb{C}))$ of $\mathfrak{so}(n, \mathbb{C})$. We let \mathfrak{b} be the Borel subalgebra with basis \mathcal{B} . Now we specify a decorated admissible tableau, $\bar{\Omega}$ of type λ , and we define $\nu_{n-1}(\gamma_{n-1})$ as above. We write

$$\begin{aligned}\lambda &= \lambda_1 e_1 + \cdots + \lambda_k e_k, \text{ and} \\ \gamma_{n-1} &= (\gamma_{n-1})_1 e_1 + \cdots + (\gamma_{n-1})_k e_k,\end{aligned}$$

and we recall $n = 2k + 1$.

Lemma 6.1 *For some k -tuple of natural numbers $(a_k, a_{k-1}, \dots, a_1)$ and some scalar d , We have*

$$E_{e_k}^{a_k} E_{e_{k-1}}^{a_{k-1}} \cdots E_{e_1}^{a_1} \nu_{n-1}(\gamma_{n-1}) = d \nu_n(\lambda)$$

for some scalar d and where $(\lambda_j - (\gamma_{n-1})_j) = a_j$ for each j such that $1 \leq j \leq k$.

Proof Since $\nu_n(\lambda)$ is a highest weight for μ_λ , we know $E_1 E_2 \cdots E_p \nu_{n-1}(\gamma_{n-1}) = c \nu_n(\lambda)$ for some natural number p , some scalar C , and some E_1, E_2, \dots, E_p such that E_i is an element of \mathcal{N}_+ for each i such that $1 \leq i \leq p$. By the Poincaré-Birkhoff-Witt theorem ([3] p.92), we know $\mathfrak{U}(\mathfrak{so}(n, \mathbb{C}))$ is a free \mathfrak{n}_+ -module with basis

$$\{1\} \cup \bigcup_{\substack{j \in \mathbb{N} \\ j \geq 1}} \{b_1 b_2 \cdots b_j \mid b_i \in \mathcal{B} \text{ for } 1 \leq i \leq j \text{ and } b_1 \leq b_2 \leq \cdots \leq b_j\}.$$

Hence, we can express $E_1 E_2 \cdots E_p$ as

$$\sum_{i=1}^m c(i) E_{i_1} E_{i_2} \cdots E_{i_{p(i)}}$$

for some natural number m and an m -tuple of scalars $(c(1), \dots, c(m))$ where for $1 \leq i \leq m$ and $1 \leq j \leq p(i)$ the expression E_{i_j} is an element of \mathcal{N}_+ and where $E_{i_j} \leq E_{i_{j+1}}$ if $j < p(i)$. We have determined

$$\sum_{i=1}^m c(i) E_{i_1} E_{i_2} \cdots E_{i_{p(i)}} \nu_{n-1}(\gamma_{n-1}) = c \nu_n(\lambda). \quad (21)$$

For any i and j such that $1 \leq i < j \leq k$, we know $E_{e_i \pm e_j}$ is the image of a positive root vector for $\mathfrak{so}(n-1, \mathbb{C})$ under the differential of ι mapping $\mathfrak{so}(n-1, \mathbb{C})$ to $\mathfrak{so}(n, \mathbb{C})$, and $\nu_{n-1}(\gamma_{n-1})$ is a highest weight for the irreducible representation $\mu_{\gamma_{n-1}}$ of $\mathfrak{so}(n-1, \mathbb{C})$. Hence, in the sum from Equation 21, we may exclude all terms such that $E_{i_{p(i)}} = E_{e_i \pm e_j}$ for some i and j such that $1 \leq i < j \leq k$. According to our ordering for \mathcal{N}_+ we may express the sum of the remaining terms as

$$\sum_{i=1}^m c(i) E_{e_k}^{a_{i,k}} E_{e_{k-1}}^{a_{i,k-1}} \cdots E_{e_1}^{a_{i,1}}$$

where, for each i such that $1 \leq i \leq m$, we know $a_{i,j}$ is a natural number for each j with $1 \leq j \leq k$. Furthermore,

$$E_{e_k}^{a_{i,k}} E_{e_{k-1}}^{a_{i,k-1}} \cdots E_{e_1}^{a_{i,1}} \nu_{n-1}(\gamma_{n-1})$$

has weight $\gamma_{n-1} + a_{i,k} e_k + a_{i,k-1} e_{k-1} \cdots + a_{i,1} e_1$, so we may remove from Equation 21 any term such that $\gamma_{n-1} + a_{i,k} e_k + a_{i,k-1} e_{k-1} \cdots + a_{i,1} e_1$ is not λ . We arrive at the equation

$$c(i) (E_{e_k}^{a_{i,k}} E_{e_{k-1}}^{a_{i,k-1}} \cdots E_{e_1}^{a_{i,1}} \nu_{n-1}(\gamma_{n-1})) = c \nu_n(\lambda)$$

for some i such that $1 \leq i \leq m$ and some nonzero scalar $c(i)$. Letting $d = c/c(i)$ and defining $a_{i,j} = a_j$ for each j with $1 \leq j \leq k$ gives the desired result. \blacksquare

We must venture one step further into the recursive process used to define the

vectors in \mathcal{L} . Considering again the terminology in Equation 5, and recalling $n = 2k + 1$ is odd, we enumerate the positive roots of $\mathfrak{so}(n - 1, \mathbb{C})$:

$$\Delta^+(\mathfrak{so}(n - 1, \mathbb{C}), \mathfrak{h}) = \{e_i \pm e_j \mid 1 \leq i < j \leq k\}.$$

We choose a somewhat less intuitive basis for the Lie subalgebra $\mathfrak{n}_+(\mathfrak{so}(n - 1))$ of $\mathfrak{so}(n - 1, \mathbb{C})$ than the basis \mathcal{N}_+ , defining

$$\begin{aligned} \mathcal{N}_+(\mathfrak{so}(n - 1)): = & \quad \{E_{e_i \pm e_j} \mid 1 \leq i < j < k\} \\ & \cup \{E_{e_i + e_k} + E_{e_i - e_k} \mid 1 \leq i < k\} \\ & \cup \{E_{e_i + e_k} - E_{e_i - e_k} \mid 1 \leq i < k\}. \end{aligned}$$

We order $\mathcal{N}_+(\mathfrak{so}(n - 1))$ such that

$$E_{e_i + e_k} - E_{e_i - e_k} < E_{e_j + e_k} + E_{e_j - e_k} < E_{e_l \pm e_p}$$

for any i, j, l , and p such that $1 \leq i, j, l, p < k$ and $l < p$, while $(E_{e_i + e_k} - E_{e_i - e_k}) < (E_{e_{i-1} + e_k} - E_{e_{i-1} - e_k})$ for any i with $1 \leq i < k$. Next, we define \mathcal{B} to be the union of a basis for \mathfrak{h} and the negative root vectors. We extend $\mathcal{N}_+(\mathfrak{so}(n - 1))$ to a basis $\mathcal{N}_+(\mathfrak{so}(n - 1)) \cup \mathcal{B}(\mathfrak{so}(n - 1))$ for $\mathfrak{so}(n - 1, \mathbb{C})$ and impose an ordering on this basis such that $v < b$ for any v in $\mathcal{N}_+(\mathfrak{so}(n - 1))$ and any b in $\mathcal{B}(\mathfrak{so}(n - 1))$. This ordering determines a PBW basis for the universal enveloping algebra $\mathfrak{U}(\mathfrak{so}(n - 1, \mathbb{C}))$. We let $\mathfrak{b}(\mathfrak{so}(n - 1))$ be the Borel subalgebra of $\mathfrak{so}(n - 1, \mathbb{C})$.

Lemma 6.2 *We have*

$$\begin{aligned} & (E_{e_{k-1} + e_k} - E_{e_{k-1} - e_k})^{a_{k-1}} (E_{e_{k-2} + e_k} - E_{e_{k-2} - e_k})^{a_{k-2}} \cdots \\ & \cdots (E_{e_1 + e_k} - E_{e_1 - e_k})^{a_1} \nu_{n-2}(\gamma_{n-1}, \gamma_{n-2}) = d\nu_{n-1}(\gamma_{n-1}) \end{aligned}$$

for some scalar d and where $(\gamma_{n-1})_j - (\gamma_{n-2})_j = a_j$ for each j such that $1 \leq j \leq k-1$.

Proof The reasoning used to prove this lemma follows more or less precisely the reasoning used to prove Lemma 6.1. Since $\nu_{n-1}(\gamma_{n-1})$ is a highest weight for $mu_{\gamma_{n-1}}$, we know

$$E_1 E_2 \cdots E_p \nu_{n-2}(\gamma_{n-2}, \gamma_{n-1}) = c \nu_{n-1}(\gamma_{n-1})$$

for some natural number p , some scalar c , and some E_1, E_2, \dots, E_p such that E_i is an element of $\mathcal{N}_+(\mathfrak{so}(n-1))$ for each i such that $1 \leq i \leq p$. By the Poincaré-Birkhoff-Witt theorem ([3] p.92), we know $\mathfrak{U}(\mathfrak{so}(n-1, \mathbb{C}))$ is a free $\mathfrak{n}_+(\mathfrak{so}(n-1))$ -module with basis

$$\{1\} \cup \bigcup_{\substack{j \in \mathbb{N} \\ j \geq 1}} \{b_1 b_2 \cdots b_j \mid b_i \in \mathcal{B} \text{ for } 1 \leq i \leq j \text{ and } b_1 \leq b_2 \leq \cdots \leq b_j\}.$$

Hence, we can express $E_1 E_2 \cdots E_p$ as

$$\sum_{i=1}^m c(i) E_{i,1}^{a_{i,1}} E_{i,2}^{a_{i,2}} \cdots E_{i,p(i)}^{a_{i,p(i)}} \quad (22)$$

for some natural number m and an m -tuple of scalars $(c(1), \dots, c(m))$ where for $1 \leq i \leq m$ and $1 \leq j \leq p(i)$ the expression $E_{i,j}$ is an element of \mathcal{N}_+ and where $E_{i,j} \leq E_{i,j+1}$ if $j < p(i)$. Here, $a_{i,j}$ is some natural number for each i and j with $1 \leq i \leq m$ and $1 \leq j \leq k-1$. For i and j with $1 \leq i < j < k$, the positive root vector $E_{e_i \pm e_j}$ for $\mathfrak{so}(n-1, \mathbb{C})$ is the image of a positive root vector for $\mathfrak{so}(n-3, \mathbb{C})$ under the differential of $\iota_{n-1} \circ \iota_{n-2}$ mapping $\mathfrak{so}(n-3, \mathbb{C})$ to $\mathfrak{so}(n-1, \mathbb{C})$. Somewhat less obviously, for i such that $1 \leq i < k$, the vector $E_{e_i + e_k} + E_{e_i - e_k}$ in \mathcal{N}_+ is the image of a vector in the span of the positive root spaces for $\mathfrak{so}(n-2, \mathbb{C})$ under the differential of ι mapping $\mathfrak{so}(n-2, \mathbb{C})$ to $\mathfrak{so}(n-1, \mathbb{C})$. In fact, we see $E_{e_i + e_k} + E_{e_i - e_k}$ is the image of $2E_{e_i}(\mathfrak{so}(n-2))$ where $E_{e_i}(\mathfrak{so}(n-2))$ is the positive root vector for $\mathfrak{so}(n-2, \mathbb{C})$ corresponding to the root e_i as described in Equation 6. Since $\nu_{n-2}(\gamma_{n-1}, \gamma_{n-2})$ is a

highest weight vector for $\mathfrak{so}(n-2, \mathbb{C})$, we know

$$E_{e_i \pm e_j} \nu_{n-2}(\gamma_{n-1}, \gamma_{n-2}) = E_{e_i+e_k} + E_{e_i-e_k} \nu_{n-2}(\gamma_{n-1}, \gamma_{n-2}) = 0$$

for i and j such that $1 \leq i < j < k$. As such, in the sum from Expression 22, we may exclude any term such that $E_{i_{p(i)}} > (E_{e_1+e_k} - E_{e_1-e_k})$. We have shown

$$\begin{aligned} & \sum_{i=1}^m c(i) (E_{e_{k-1}+e_k} - E_{e_{k-1}-e_k})^{a_{i,k-1}} (E_{e_{k-2}+e_k} - E_{e_{k-2}-e_k})^{a_{i,k-2}} \dots \\ & \dots (E_{e_1+e_k} - E_{e_1-e_k})^{a_{i,1}} \nu_{n-2}(\gamma_{n-1}, \gamma_{n-2}) = d \nu_{n-1}(\gamma_{n-1}). \end{aligned}$$

For any i such that $1 \leq i \leq m$, we see

$$\begin{aligned} & (E_{e_{k-1}+e_k} - E_{e_{k-1}-e_k})^{a_{i,k-1}} (E_{e_{k-2}+e_k} - E_{e_{k-2}-e_k})^{a_{i,k-2}} \dots \\ & \dots (E_{e_1+e_k} - E_{e_1-e_k})^{a_{i,1}} \nu_{n-2}(\gamma_{n-1}, \gamma_{n-2}) \end{aligned}$$

is the sum of weight vectors, each one having weight

$$\gamma_{n-2} + \sum_{j=1}^{k-1} a_{i,j}(e_j) + z(e_k)$$

for some integer z . For only one $k-1$ -tuple $(a'_{k-1}, a'_{k-2}, \dots, a'_1)$ does

$$\gamma_{n-2} + \sum_{j=1}^{k-1} a'_j(e_j) + z(e_k) = \gamma_{n-1}$$

for some integer z . Hence, we may further exclude from the sum in Expression 22 any term such that $a_{i,j} \neq a'_j$ for each j such that $1 \leq j \leq k-1$. There exists exactly

one i with $1 \leq i \leq m$ such that

$$c(i)(E_{e_{k-1}+e_k} - E_{e_{k-1}-e_k})^{a_{i,k-1}}(E_{e_{k-2}+e_k} - E_{e_{k-2}-e_k})^{a_{i,k-2}} \cdots \\ \cdots (E_{e_1+e_k} - E_{e_1-e_k})^{a_{i,1}} \nu_{n-2}(\gamma_{n-1}, \gamma_{n-2}) = c \nu_{n-1}(\gamma_{n-1}).$$

Setting $a'_j = a_{i,j}$ for each j with $1 \leq j \leq k-1$, and letting $c/c(i) = d$, we arrive at the desired result. ■

Lemmas 6.1 and 6.2 give us a description of the vectors in \mathcal{L} precise enough to make calculating the action of M on these vectors a tractable problem for odd n . When n is even, we will rely on Lemma 4.3. We will use these lemmas within an inductive framework built to accomodate either parity for n . From section 3, we recall our notation m^j for the element of M having j^{th} and $j+1^{\text{st}}$ diagonal elements equal to -1 and all other diagonal elements equal to 1 . According to the arguments in Lemma 3.1, $\{m^j \mid 1 \leq j < n-1\}$ generates M , so we focus on the action of elements from this set in determining the action of M on each of the elements in \mathcal{L} . If $n=2$, we understand completely the action of m^1 on $\nu_n(\lambda) = \nu_2(\bar{\Omega})$, and $\mathcal{L} = \{\nu_n(\lambda)\}$. By our definition for $\nu_2(\bar{\Omega})$, we notice $\nu_2(\bar{\Omega}) = c\nu_2(\bar{\pi}\bar{\Omega})$ for some scalar c where $\bar{\pi}$ is the map defined in Theorem 5.4 and $\bar{\pi}\bar{\Omega}$ is a decorated semistandard Young tableau admissible for $SO(n-1, \mathbb{R})$. We may, of course, choose $\nu_2(\bar{\pi}\bar{\Omega})$ such that $\nu_2(\bar{\Omega}) = \nu_2(\bar{\pi}(\bar{\Omega}))$. For j such that $1 \leq j < n-1$, the element m^j of $M = M(SO(n)) \subset SO(n, \mathbb{R})$ is the image of an element $m^j_{SO(n-1)}$ in $M(SO(n-1)) \subset SO(n-1, \mathbb{R})$ under the map ι . The action of $m^j_{SO(n-1)}$ on the vector $\nu_2(\bar{\pi}\bar{\Omega})$ in the representation $\mu_{\gamma_{n-1}}|_{M(SO(n-1))}$ is exactly the action of m^j on the vector $\nu_2(\bar{\Omega})$ in the representation $\mu_\lambda|_M$ whenever $1 \leq j < n-1$. Hence, in the context of any inductive proof, the burden will shift to discovering the action of m^{n-1} on each of the vectors in \mathcal{L} .

With an eye toward making use of Lemmas 6.1 and 6.2, we calculate

$$m^{n-1}(E_{e_k}^{a_k})\nu, \quad (23)$$

$$m^{n-1}(E_{e_{k-1}}^{a_{k-1}} \cdots E_{e_1}^{a_1})\nu, \text{ and} \quad (24)$$

$$m^{n-1}((E_{e_{k-1}+e_k} - E_{e_{k-1}-e_k})^{a_{k-1}}(E_{e_{k-2}+e_k} - E_{e_{k-2}-e_k})^{a_{k-2}} \cdots \cdots (E_{e_1+e_k} - E_{e_1-e_k})^{a_1})\nu \quad (25)$$

where ν is any vector in a representation for $SO(n, \mathbb{R})$. If E is any element of $\mathfrak{so}(n, \mathbb{C})$ and ν is some vector in an $SO(n, \mathbb{R})$ -representation, we know $m^{n-1} \cdot (E\nu) = (\text{Ad}(m^{n-1})E)m^{n-1}\nu$. We know $\text{Ad}(m^{n-1})E$ is conjugation of the matrix E by the matrix m^{n-1} . Conjugation of E_{e_k} by m^{n-1} is $-E_{-e_k}$, and, for j such that $1 \leq j < k$, conjugation of E_{e_j} by m^{n-1} is $-E_{e_j}$. For any j such that $1 \leq j \leq k-1$, conjugation of $E_{e_j+e_k}$ by m^{n-1} is $E_{e_j-e_k}$ and conjugation of $E_{e_j-e_k}$ by m^{n-1} is $E_{e_j+e_k}$. We see Equation23 is

$$((-1)^{a_k} E_{-e_k})m^{n-1}\nu, \quad (26)$$

while Equation24 is

$$((-1)^{a_{k-1}+\cdots+a_1} E_{e_{k-1}} \cdots E_{e_1})m^{n-1}\nu. \quad (27)$$

Also, Equation24 is

$$(-1)^{a_{k-1}+a_{k-1}+\cdots+a_1}((E_{e_{k-1}+e_k} - E_{e_{k-1}-e_k})^{a_{k-1}}(E_{e_{k-2}+e_k} - E_{e_{k-2}-e_k})^{a_{k-2}} \cdots \cdots (E_{e_1+e_k} - E_{e_1-e_k})^{a_1})m^{n-1}\nu. \quad (28)$$

For the most part, describing the action of m^{n-1} reduces to performing computations in $\mathfrak{so}(3, \mathbb{C})$ and $SO(3, \mathbb{R})$. In order to study representations of $SO(3, \mathbb{R})$, we turn to the simply connected covering group for $SO(3, \mathbb{R})$ and study explicit descriptions

of representations for that covering group. The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ has basis

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and these basis elements obey the relations

$$[Z, X] = 2X, [Z, Y] = -2Y, \text{ and } [X, Y] = Z.$$

We determine a map $\phi: \mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{so}(3, \mathbb{C})$ by defining ϕ on basis vectors for $\mathfrak{sl}(2, \mathbb{C})$ as follows:

$$X \mapsto E_{e_1}, Y \mapsto -E_{-e_1}, \text{ and } Z \mapsto H: = \begin{pmatrix} 0 & 2i & 0 \\ -2i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly, E_{e_1}, E_{-e_1} , and H form a basis for $\mathfrak{so}(3, \mathbb{C})$, and a quick calculation shows

$$[E_{e_1}, -E_{-e_1}] = H, \quad [Y, E_{e_1}] = 2E_{e_1}, \text{ and } [Y, -E_{-e_1}] = 2E_{e_1},$$

so we see $\mathfrak{sl}(2, \mathbb{C})$ is isomorphic to $\mathfrak{so}(3, \mathbb{C})$. Now $SL(2, \mathbb{C})$ is simply connected, hence a covering group for $SO(3, \mathbb{C})$, and there exists a unique surjective homomorphism of Lie groups Γ such that the diagram

$$\begin{array}{ccc} \mathfrak{sl}(2, \mathbb{C}) & \xrightarrow{\phi} & \mathfrak{so}(3, \mathbb{C}) \\ \exp \downarrow & & \downarrow \exp \\ SL(2, \mathbb{C}) & \xrightarrow{\Gamma} & SO(3, \mathbb{C}) \end{array}$$

commutes. We construct Γ explicitly. First, we choose a new basis

$$X' = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, Y' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ and } Z' = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

for $\mathfrak{sl}(2, \mathbb{C})$, and we let \langle, \rangle be the symmetric inner product on $\mathfrak{sl}(2, \mathbb{C})$ given by $\langle A, B \rangle = \text{tr}(AB)$. With respect to this inner product, the basis X', Y' , and Z' is orthogonal, and each basis vector has length -2 . The adjoint action of $SL(2, \mathbb{C})$ on $\mathfrak{sl}(2, \mathbb{C})$ is orthogonal with respect to this inner product since conjugation leaves trace invariant. We get a map

$$\Gamma': SL(2, \mathbb{C}) \rightarrow O(3, \mathbb{C}) \subset \text{Aut}(\mathfrak{sl}(2, \mathbb{C})),$$

determined completely by the adjoint action of $SL(2, \mathbb{C})$ on the basis vectors X', Y' , and Z' . Since $SL(2, \mathbb{C})$ is connected, $\Gamma'(SL(2, \mathbb{C}))$ is connected, and $1_{SL(2, \mathbb{C})} \mapsto 1_{O(3, \mathbb{C})}$, so $\Gamma'(SL(2, \mathbb{C})) \subset SO(3, \mathbb{C})$. In writing $\Gamma'(g)$ as a matrix for any g in $SL(2, \mathbb{C})$, we need to order X', Y' , and Z' in such a way as to ensure agreement between the Cartan subalgebra $\mathfrak{h}(\mathfrak{sl}(2))$ for $\mathfrak{sl}(2, \mathbb{C})$ spanned by Z' and our choice for a Cartan subalgebra \mathfrak{h} of $\mathfrak{so}(3, \mathbb{C})$. We notice

$$\text{Ad}(\exp Z')Z' = Z', \text{Ad}(\exp Z')X' = \left(\frac{1}{2}e^{2i}(X' + iY')\right) + \left(\frac{1}{2}e^{-2i}(X' - iY')\right), \text{ and} \tag{29}$$

$$\text{Ad}(\exp Z')Y' = \left(\frac{1}{2}e^{-2i}(Y' + iX')\right) + \left(\frac{1}{2}e^{2i}(Y' - iX')\right).$$

Hence, we determine the matrix form of $\Gamma'(g)$ for any g in $SL(2, \mathbb{C})$ such that, if $\text{Ad}(g)(X') = a_1X' + b_1Y' + c_1Z'$, $\text{Ad}(g)(Y') = a_2X' + b_2Y' + c_2Z'$, and $\text{Ad}(g)(Z') =$

$a_3X' + b_3Y' + c_3Z'$, then

$$\Gamma'(g) = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

By Equations 29, Γ' maps $\exp Z'$ to the torus for $SO(3, \mathbb{C})$ corresponding to our choice for \mathfrak{h} . By calculating Γ' on $\exp X'$, $\exp Y'$, and $\exp Z'$, we see Γ' is the surjective map Γ . The irreducible representations of $SO(3, \mathbb{R})$ correspond exactly to the dominant analytically integral forms of $\mathfrak{so}(3, \mathbb{C})$. Since $SL(2, \mathbb{C})$ is simply connected, any dominant weight for $\mathfrak{sl}(2, \mathbb{C})$ is analytically integral for $SL(2, \mathbb{C})$. Every irreducible representation for $SO(3, \mathbb{C})$ descends from a representation of $SL(2, \mathbb{C})$ via the map Γ . (This fact follows from the arguments in [5] 5.110.) We denote by I_n the n by n diagonal matrix with each diagonal entry equal to 1. The kernel of Γ is $\{\pm 1_{SL(2, \mathbb{C})}\}$, so $\widehat{SO(3, \mathbb{C})}$ is the subset of $\widehat{SL(2, \mathbb{C})}$ consisting of all representations such that $-I_2$ acts as the identity. We take a description of $\widehat{SL(2, \mathbb{C})}$ from [1] p. 117. We define V_a to be the space of homogeneous polynomials of degree a in two variables, z_1 and z_2 . The space V_a has basis

$$\{x_j = \binom{m}{j} z_1^{a-j} z_2^j \mid 0 \leq j \leq a\},$$

and $SL(2, \mathbb{C})$ acts on any polynomial $P(z_1, z_2)$ in V_a by

$$\begin{pmatrix} b & c \\ d & e \end{pmatrix} \cdot P(z_1, z_2) = P(bz_1 + dz_2, cz_1 + ez_2) \text{ for } \begin{pmatrix} b & c \\ d & e \end{pmatrix} \in SL(2, \mathbb{C}).$$

Any irreducible representation for $SL(2, \mathbb{C})$ is, in particular, a representation for $\mathfrak{sl}(2, \mathbb{C})$. We calculate the action of the basis X, Y , and Z for $\mathfrak{sl}(2, \mathbb{C})$ on V_a as in [1]

10.7 to find

$$Zx_j = (a - 2j)x_j, Xx_j = (a - j + 1)x_{j-1}, \text{ and } Yx_j = -(j + 1)x_{j+1}. \quad (30)$$

We see x_0 is a highest weight of weight a . Clearly, V_a is isomorphic to the representation of highest weight $a/2e_1$ for $\mathfrak{so}(3, \mathbb{C})$. For any j such that $0 \leq j \leq a$, we know

$$-I_2 \cdot \binom{a}{j} z_1^{a-j} z_2^j = \binom{a}{j} (-1)^a z_1^{a-j} z_2^j.$$

Hence, the representation V_a descends to $SO(3, \mathbb{C})$ if and only if a is even. The element m^2 of $M(SO(3, \mathbb{C}))$ has preimage $\{\pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\}$ under Γ because

$$\text{Ad}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)X' = X', \text{ Ad}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)Y' = -Y', \text{ and } \text{Ad}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)Z' = -Z'.$$

Since $SO(3, \mathbb{R})$, the group of real points in $SO(3, \mathbb{C})$, is compact, we know $\widehat{SO(3, \mathbb{R})}$ is exactly the set of restrictions of $\widehat{SO(3, \mathbb{C})}$ to $SO(3, \mathbb{R})$.

For the following two theorems, we choose λ to be an analytically integral form for $SO(3, \mathbb{R})$, and by M we mean $M(SO(3, \mathbb{R}))$. We fix some $\bar{\Omega}$ in $\bar{\Psi}_\lambda$, and we get $\nu_2(\bar{\Omega}) = \nu_2(\gamma_2)$ in \mathcal{L} .

Theorem 6.3 *If $\gamma_2 = 0$ and $\lambda = a(e_1)$ for some natural number a , then*

$$m^2(\nu_2(\bar{\Omega})) = (-1)^a(\nu_2(\bar{\Omega})).$$

Proof We work with the $\mathfrak{sl}(2, \mathbb{C})$ representation V_{2a} isomorphic to the representation of $\mathfrak{so}(3, \mathbb{C})$ of highest weight $a(e_1)$. Choosing $x_0 = \nu_3(\lambda)$, we find $\nu_2(\bar{\Omega})$ is some scalar multiple c of the basis vector x_a in V_{2a} with weight 0. Now $m^2x_0 = (-z_2)^{2a} = z_2^{2a} =$

x_{2a} . Using the action of $\mathfrak{sl}(2, \mathbb{C})$ on V_{2a} given by Equations 30, we find

$$\begin{aligned} E_{e_1}^a x_m &= E_{-e_1}^a x_0 = \left(\sum_{i=1}^a i \right) x_a, \text{ so} \\ E_{e_1}^a m^2 \nu_3(\lambda) &= E_{-e_1}^a \nu_3(\lambda) \end{aligned} \quad (31)$$

From Lemma 6.1 we know we can choose the scalar c such that $\nu_3(\lambda) = E_{e_1}^a \nu_2(\bar{\Omega})$. The following equation holds:

$$\begin{aligned} E_{e_1}^a m^2 \nu_3(\lambda) &= E_{e_1}^a m^2 E_{e_1}^a \nu_2(\bar{\Omega}) \\ &= E_{e_1}^a (-1)^a E_{-e_1}^a m^2 \nu_2(\bar{\Omega}) \text{ (by Equation 26)} \\ &= (-1)^a E_{e_1}^a E_{-e_1}^a m^2 \nu_2(\bar{\Omega}) \\ &= E_{-e_1}^a E_{e_1}^a \nu_2(\bar{\Omega}) \text{ (by Equation 31)}. \end{aligned} \quad (*)$$

We want to show

$$E_{-e_1}^a E_{e_1}^a \nu_2(\bar{\Omega}) = E_{e_1}^a E_{-e_1}^a \nu_2(\bar{\Omega}). \quad (**)$$

Considering the expression

$$E_{\epsilon_1 e_1} E_{\epsilon_2 e_1} \cdots E_{\epsilon_{2a} e_1} \nu_2(\bar{\Omega}), \quad (32)$$

where, for each i such that $1 \leq i \leq 2a$, we have $\epsilon_i = \pm 1$ and $\epsilon_i = 1$ for exactly half of the natural numbers i with $1 \leq i \leq 2a$, we attempt to find the highest value for j such that $\epsilon_j = -1$ and $\epsilon_{j+1} = 1$. If no such j exists, Expression 32 is $E_{e_1}^a E_{-e_1}^a \nu_2(\bar{\Omega})$. Otherwise, Expression 32 is

$$\begin{aligned} &E_{\epsilon_1 e_1} \cdots E_{\epsilon_{j+1} e_1} E_{\epsilon_j e_1} \cdots E_{\epsilon_{2a} e_1} \nu_2(\bar{\Omega}) \\ &+ E_{\epsilon_1 e_1} \cdots [E_{\epsilon_j e_1}, E_{\epsilon_{j+1} e_1}] \cdots E_{\epsilon_{2a} e_1} \nu_2(\bar{\Omega}). \end{aligned}$$

We induct on $2a - 1 - j$ to show the second term in Expression 33 is 0. If $j = 2a - 1$, the second term in Expression 33 is 0 since $\nu_2(\bar{\Omega})$ has weight 0 and $[E_{\epsilon_j e_1}, E_{\epsilon_{j+1} e_1}]$ is in the subspace of $\mathfrak{sl}(2, \mathbb{C})$ spanned by Z . Otherwise, the second term in Expression 33 is

$$\begin{aligned} & E_{\epsilon_1 e_1} \cdots E_{\epsilon_{j+2} e_1} [E_{\epsilon_j e_1}, E_{\epsilon_{j+1} e_1}] E_{\epsilon_{j+3} e_1} \cdots E_{\epsilon_{2a} e_1} \nu_2(\bar{\Omega}) \\ & + E_{\epsilon_1 e_1} \cdots E_{\epsilon_{j-1}} [[E_{\epsilon_j e_1}, E_{\epsilon_{j+1} e_1}], E_{\epsilon_{j+2} e_1}] E_{\epsilon_{j+3} e_1} \cdots E_{\epsilon_{2a} e_1} \nu_2(\bar{\Omega}). \end{aligned}$$

The second term in Expression 33 has weight not equal to 0, so, since Expression 32 has weight 0, the second term in expression 33 must be 0. By inductive hypothesis, the first term in Expression 33 is 0. Hence, the second term in Expression 33 is 0, and we may freely interchange $E_{\epsilon_l e_1}$ and $E_{\epsilon_{l+1} e_1}$ in Expression 32 for any l such that $1 \leq l < 2a$. Thus, Equation** holds.

To conclude, we combine Equations * and ** to see

$$(-1)^a E_{e_1}^a E_{-e_1}^a m^2 \nu_2(\bar{\Omega}) = E_{e_1}^a E_{-e_1}^a \nu_2(\bar{\Omega}),$$

so $m^2 \nu_2(\bar{\Omega}) = (-1)^a \nu_2(\bar{\Omega})$, as desired. ■

For the next theorem, we specify Ω , the admissible semistandard Young tableau associated to $\bar{\Omega} = (\Omega, \epsilon)$, where we use the term ‘‘associated’’ in the sense described by section 5

Theorem 6.4 *If $\lambda = a(e_1)$ and $\nu_2(\Omega, \epsilon)$ has weight $b(e_i)$ with $1 \leq b \leq a$, we may choose $\nu_2(\Omega, \epsilon)$ and $\nu_2(\Omega, -\epsilon)$ such that $m^2 \nu_2(\Omega, \epsilon) = \nu_2(\Omega, -\epsilon)$.*

Proof As in Theorem 6.3, we choose $\nu_3(\lambda) = x_0$, identifying μ_λ with the $SL(2, \mathbb{R})$ representation V_{2a} . We know $m^2 \nu_3(\lambda) = x_m$. Also, by Lemma 6.1, we may choose

$\nu_2(\Omega, \epsilon)$ such that $\nu_3(\lambda) = E_{\epsilon_1}^{a-b} \nu_2(\Omega, \epsilon)$. Hence, we may use Equation 26 to see

$$(-1)^{a-b} E_{-\epsilon_1}^{a-b} m^2 \nu_2(\Omega, \epsilon) = x_m.$$

This equation shows the weight of $m^2 \nu_2(\Omega, \epsilon)$ is $-2b$ (working even still within the representation V_{2a}). There exists only one weight space of weight $-2b$ in V_{2a} , and $\nu_2(\Omega, -\epsilon)$ must have weight $-2b$, so we can choose $m^2 \nu_2(\Omega, \epsilon) = \nu_2(\Omega, -\epsilon)$. ■

Beyond the case $n = 3$, a complete understanding of the action of M on \mathcal{L} proves somewhat too complicated to pursue all at once. To continue, we narrow our focus to a small portion of the larger project. For each Ω in Ψ_λ , we consider a subset $\mathcal{L}_\Omega \subset \mathcal{L}$ defined as

$$\mathcal{L}_\Omega := \{\nu_2(\bar{\Omega}) \in \mathcal{L} \mid \Omega \text{ is associated to } \bar{\Omega}\}.$$

In addition, we consider a subset $M_\Omega \subset \{m^j \mid 1 \leq j < n\}$ defined as

$$M_\Omega := \{m^j \mid j \in \mathcal{C}_\Omega\}, \tag{33}$$

where j is an element of $\mathcal{C}_\Omega \subset \{1, \dots, n-1\}$ if and only if j is even and $\Omega(1, -j/2) = j$. If $|\mathcal{C}_\Omega| = l$, then $|\mathcal{L}_\Omega| = 2^l$. We fix some $\bar{\Omega} = (\Omega, (\epsilon_1, \dots, \epsilon_{n/2}))$.

Theorem 6.5 *For each j in \mathcal{C}_Ω , the element m^j in M_Ω acts on $\nu_2(\Omega, (\epsilon_1, \dots, \epsilon_{n/2}))$ according to the equation*

$$m^j \nu_2(\Omega, (\epsilon_1, \dots, \epsilon_{[n/2]})) = \nu_2(\Omega, (\epsilon_1, \dots, \epsilon_{j/2-1}, -\epsilon_{j/2}, \epsilon_{j/2+1}, \dots, \epsilon_{[n/2]})).$$

for some choice of elements in the set \mathcal{L}_Ω .

Proof For $n = 2$ the statement of the theorem is empty, and for $n = 3$ the statement amounts to Theorem 6.4. We suppose \mathcal{C}_Ω is nonempty, as, otherwise, the statement of the theorem is empty. We prove the result for general n by induction. If $j < n - 1$,

our inductive hypothesis shows the preimage of m^j under ι acts on $\nu_2(\bar{\pi}(\bar{\Omega}))$ according to the equation

$$\begin{aligned} & m^j \nu_2 \left(\pi(\Omega), (\epsilon_1, \dots, \epsilon_{\lfloor n-1/2 \rfloor}) \right) \\ &= \nu_2 \left(\pi(\Omega), (\epsilon_1, \dots, \epsilon_{j/2-1}, -\epsilon_{j/2}, \epsilon_{j/2+1}, \dots, \epsilon_{\lfloor n-1/2 \rfloor}) \right). \end{aligned}$$

Hence, m^j acts on $\nu_2(\bar{\Omega})$ as desired.

It remains to prove the statement for $j = n - 1$, in which case $n = 2k + 1$ is odd. In this case, we consider the subgroup $SO(n - 3, \mathbb{R}) \times SO(3, \mathbb{R})$ of $SO(n, \mathbb{R})$ where we identify $SO(n - 3, \mathbb{R})$ with the upper left $n - 3$ by $n - 3$ block diagonal part of $SO(n, \mathbb{R})$ and $SO(3, \mathbb{R})$ with the lower right 3 by 3 block diagonal part of $SO(n, \mathbb{R})$. We focus on the action of the subgroup $SO(3, \mathbb{R})$ of $SO(n - 3, \mathbb{R}) \times SO(3, \mathbb{R})$ on $\nu_n(\lambda)$ where we identify $SO(3, \mathbb{R})$ with the image of $SO(3, \mathbb{R})$ under the inclusion $\tilde{\iota}: SO(3, \mathbb{R}) \rightarrow SO(n - 3, \mathbb{R}) \times SO(3, \mathbb{R})$ given by the mapping $g \mapsto (1_{SO(n-3, \mathbb{R})}, g)$. We examine the irreducible $SO(3, \mathbb{R})$ -representation $\tilde{\mu}_\lambda$ generated by $\nu_n(\lambda)$. By Lemma 6.1, $\nu_n(\lambda) = E_k^{a_k} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})$. The weight vector $E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})$ is a highest weight vector for $SO(2, \mathbb{R})$ where we identify $SO(2, \mathbb{R})$ with the image of $SO(2, \mathbb{R})$ under the usual inclusion ι . Moreover, since $\Omega(1, -j/2) = j$, we know $E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})$ has weight $b(e_1)$ not equal to 0. By Theorem 6.4, the element $m^2(SO(3))$ of $M(SO(3))$ maps $E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})$ to a highest weight vector for $SO(2, \mathbb{R})$ of weight $-b(e_1)$. Now, $m^2(SO(3))$ maps under $\tilde{\iota}$ to the element $(1, m^2(SO(3)))$ and $(1, m^2(SO(3)))$ maps to m^j under the natural inclusion of $SO(n - 3, \mathbb{R}) \times SO(3, \mathbb{R})$ in $SO(n, \mathbb{R})$. We now consider the entirety of the $SO(n, \mathbb{R})$ representation μ_λ . In this representation, $E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})$ has weight $b_1(e_1) + \cdots + b_k(e_k)$ where $b_k = b$. According to our understanding of the action of $m^2(SO(3))$ on $\tilde{\mu}_\lambda$, the element m^j of M maps $E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})$ to a vector of weight $b_1(e_1) + \cdots + b_{k-1}(e_{k-1}) - b_k(e_k)$. By our

definition for $\nu_2(\bar{\Omega})$, and by Lemmas 6.1 and 6.2, we know

$$\begin{aligned} E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1}) &= E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \cdots \\ \cdots (E_{e_{k-1}+e_k} - E_{e_{k-1}-e_k})^{b_{k-1}} \cdots (E_{e_1+e_k} - E_{e_1-e_k})^{b_1} U \nu_2(\bar{\Omega}) \end{aligned}$$

for some U in the universal enveloping algebra of the image of $(\mathfrak{so}(n-2, \mathbb{C}))$, the complexification of the Lie algebra for $\iota_{n-1} \circ \iota_{n-2}(SO(n-2, \mathbb{R}))$, under the differential of $\iota_{n-1} \circ \iota_{n-2}$. We know m^j acts as the identity on $\iota_{n-1} \circ \iota_{n-2}(SO(n-2, \mathbb{R}))$, so, combining Equations 27 and 28, we see

$$\begin{aligned} m^j(E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})) &= \pm E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \cdots \\ \cdots (E_{e_{k-1}+e_k} - E_{e_{k-1}-e_k})^{b_{k-1}} \cdots (E_{e_1+e_k} - E_{e_1-e_k})^{b_1} U m^{j-1} \nu_2(\bar{\Omega}) \end{aligned}$$

Considering the weight of $m^j(E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1}))$, we know

$$m^{j-1} \nu_2(\Omega, (\epsilon_1, \dots, \epsilon_k)) = \pm \nu_2(\Omega, (\epsilon_1, \dots, -\epsilon_k)).$$

■

Suppose G is a finite abelian group generated by g_1, g_2, \dots, g_n where $g_i^2 = 1$ for each i such that $1 \leq i \leq n$. For any subset S of $\{1, \dots, n\}$, we define g_S to be $\prod_{i \in S} g_i$. We consider V , a 2^n -dimensional representation of G with basis

$$\beta: = \{g_S \nu \mid S \subset \{1, \dots, n\}\}$$

for some ν in V .

Lemma 6.6 *The representation V decomposes into the sum $\bigoplus_{S \subset \{1, \dots, n\}} V_S$ where, if ν_S is an element of V_S , we have $g_i \nu_S = \nu_S$ when i is an element of S and $g_i \nu_S = -\nu_S$*

when n is an element of S^C .

Proof The representation V is the regular representation of G , so the lemma is a special case of the Peter-Weyl theorem. We produce a simple proof so as to keep this section self-contained. We induct on n . If $n = 1$, we have $\{\nu, g_1\nu\}$ as a basis of V . Then $\{\nu + g_1\nu, \nu - g_1\nu\}$ is also a basis of V . Moreover, $g_1(\nu + g_1\nu) = g_1\nu + \nu$, and $g_1(\nu - g_1\nu) = g_1\nu - \nu$. Hence, $\nu + g_1\nu$ generates the irreducible representation V_\emptyset , and $\nu - g_1\nu$ generates the irreducible representation V_1 .

In general, V has $\{g_S\nu \mid n \notin S\} \cup \{g_S\nu \mid n \in S\}$ as a basis. We consider the basis $\beta(n) \cup \beta(n)'$ where $\beta(n) := \{g_S\nu + g_{S \setminus \{n\}}\nu \mid n \in S\}$ and $\beta(n)' := \{g_S\nu - g_{S \setminus \{n\}}\nu \mid n \in S\}$. If \tilde{G} is the subgroup of G generated by $\{g_1, \dots, g_{n-1}\}$, then consider the representations $V(n)$ and $V(n)'$ of \tilde{G} generated by $\beta(n)$ and $\beta(n)'$ respectively. For any subset S_0 of $\{1, \dots, n\}$ such that n is an element of S , we let \tilde{S}_0 be the set $S_0 \setminus \{n\}$. Then $g_{\tilde{S}_0}(g_n\nu + \nu) = g_{S_0}\nu + g_{S_0 \setminus \{n\}}\nu$. Similarly, $g_{\tilde{S}_0}(g_n\nu - \nu) = g_{S_0}\nu - g_{S_0 \setminus \{n\}}\nu$. Hence, both $V(n)$ and $V(n)'$ are 2^{n-1} -dimensional representations of \tilde{G} , and

$$\begin{aligned} \beta(n) &= \left\{ g_{\tilde{S}}(g_n\nu + \nu) \mid \tilde{S} \subset \{1, \dots, n-1\} \right\}, \text{ whereas} \\ \beta(n)' &= \left\{ g_{\tilde{S}}(g_n\nu - \nu) \mid \tilde{S} \subset \{1, \dots, n-1\} \right\}. \end{aligned}$$

We may apply our inductive hypothesis to $V(n)$ and $V(n)'$ to show

$$\begin{aligned} V(n) &= \bigoplus_{\tilde{S} \subset \{1, \dots, n-1\}} V(n)_{\tilde{S}}, \text{ and} \\ V(n)' &= \bigoplus_{\tilde{S} \subset \{1, \dots, n-1\}} V(n)'_{\tilde{S}} \end{aligned}$$

where $g_i\nu_{\tilde{S}} = \nu_{\tilde{S}}$ when i is an element of \tilde{S} and $g_i\nu_{\tilde{S}} = -\nu_{\tilde{S}}$ when n is an element of \tilde{S}^C for any $\nu_{\tilde{S}}$ in $V(n)_{\tilde{S}} \cup V(n)'_{\tilde{S}}$. If we let $V_{S_0} = V(n)_{\tilde{S}_0}$ and $V_{S_0 \setminus \{n\}} = V(n)'_{\tilde{S}_0}$ for any

subset S_0 of $\{1, \dots, n\}$, the decomposition

$$V = \bigoplus_{S \subset \{1, \dots, n\}} V_S$$

has the desired property. ■

Remark 6.7 *We suppose $\mathcal{C}_\Omega \subset \{1, \dots, n-1\} = \{i_1, \dots, i_l\}$, and we fix a bijection $\theta: \{1, \dots, 2^l\} \rightarrow \mathcal{E}$ where \mathcal{E} is the set of subsets of \mathcal{C}_Ω . Applying Lemma 6.6 to Theorem 6.5, we can, for each Ω in Ψ_λ , find a basis $\{\nu(\Omega)_1, \dots, \nu(\Omega)_{2^l}\}$ for the subspace of μ_λ spanned by \mathcal{L}_Ω with the following property. For each j and each p such that $1 \leq j \leq l$ and $1 \leq p \leq 2^l$, we have*

$$m^{i_j} \nu(\Omega)_p = \begin{cases} \nu(\Omega)_p & i_j \in \theta(p), \\ -\nu(\Omega)_p & \text{otherwise.} \end{cases}$$

Fixing any Ω in Ψ_λ , we must now consider the action of m^j on each vector $\nu_2(\bar{\Omega})$ where Ω is associated to $\bar{\Omega}$ and for each j such that j is an element of the complement \mathcal{C}_Ω^c of \mathcal{C}_Ω in $\{1, \dots, n-1\}$. We start with the even elements of \mathcal{C}_Ω^c .

Theorem 6.8 *If j is an even element of \mathcal{C}_Ω^c , then*

$$m^j \nu_2(\bar{\Omega}) = \epsilon(\nu_2(\bar{\Omega}))$$

where $\epsilon = (-1)^{a_1 + \dots + a_{[j+1/2]}}$ for $\gamma_{j+1} - \gamma_{j-1} = a_1(e_1) + \dots + a_{[j+1/2]}(e_{[j+1/2]})$.

Proof It suffices to consider the action of $m^j(SO(j+1))$ on the weight vector

$$\nu_2(\bar{\pi}_{j+2} \circ \bar{\pi}_{j+3} \circ \dots \circ \bar{\pi}_n(\bar{\Omega}))$$

in the representation $\mu_{\gamma_{j+1}}$ for

$$\iota_{n-1} \circ \iota_{n-2} \circ \cdots \circ \iota_{j+1}(SO(j+1, \mathbb{R})),$$

so we may as well assume $j = n - 1$. We also write $n = 2k + 1$. As in Theorem 6.5, we identify $SO(3, \mathbb{R})$ with the image of $SO(3, \mathbb{R})$ under the map $\tilde{\iota}$ followed by the natural inclusion of $SO(n-3, \mathbb{R}) \times SO(3, \mathbb{R})$ in $SO(n, \mathbb{R})$. We study the irreducible $SO(3, \mathbb{R})$ representation $\tilde{\mu}_\lambda$ generated by $\nu_n(\lambda)$. By Lemma 6.1, $\nu_n(\lambda) = E_k^{a_k} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})$. The weight vector $E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})$ is a highest weight vector for $SO(2, \mathbb{R})$ where we identify $SO(2, \mathbb{R})$ with the image of $SO(2, \mathbb{R})$ under the usual inclusion ι . Since j is an element of C_Ω^c , we know $E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})$ has weight equal to 0. By Theorem 6.3,

$$m^2(SO(3))(E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})) = (-1)^{a_k} E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1}). \quad (34)$$

Now, $m^2(SO(3))$ maps under $\tilde{\iota}$ to the element $(1, m^2(SO(3)))$, and $(1, m^2(SO(3)))$ maps to m^j under the natural inclusion of $SO(n-3, \mathbb{R}) \times SO(3, \mathbb{R})$ in $SO(n, \mathbb{R})$. We consider the entirety of the $SO(n, \mathbb{R})$ representation μ_λ . By our definition for $\nu_2(\bar{\Omega})$, and by Lemmas 6.1 and 6.2, we know

$$\begin{aligned} E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1}) &= E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \cdots \\ &\cdots (E_{e_{k-1}+e_k} - E_{e_{k-1}-e_k})^{b_{k-1}} \cdots (E_{e_1+e_k} - E_{e_1-e_k})^{b_1} U \nu_2(\bar{\Omega}) \end{aligned}$$

for some U in the universal enveloping algebra of the image of $(\mathfrak{so}(n-2, \mathbb{C}))$, the complexification of the Lie algebra for $\iota_{n-1} \circ \iota_{n-2}(SO(n-2, \mathbb{R}))$, under the differential of $\iota_{n-1} \circ \iota_{n-2}$. We know m^j acts as the identity on $\iota_{n-1} \circ \iota_{n-2}(SO(n-2, \mathbb{R}))$, so,

combining Equations 27 and 28, we see

$$m^j(E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \nu_{n-1}(\gamma_{n-1})) = (-1)^{a_{k-1} + \cdots + a_1 + b_{k-1} + \cdots + b_1} E_{k-1}^{a_{k-1}} \cdots E_1^{a_1} \cdots \quad (35)$$

$$\cdots (E_{e_{k-1}+e_k} - E_{e_{k-1}-e_k})^{b_{k-1}} \cdots (E_{e_1+e_k} - E_{e_1-e_k})^{b_1} U m^{j-1} \nu_2(\bar{\Omega}).$$

Combining Equations 34 and 35, we see

$$m^j \nu_2(\bar{\Omega}) = (-1)^{a_k + \cdots + a_1 + b_{k-1} + \cdots + b_1} \nu_2(\bar{\Omega}).$$

Clearly, $\lambda - \gamma_{n-2} = b_1 + a_1(e_1) + \cdots + b_{k-1}a_{k-1}(e_{k-1}) + a_k(e_k)$. ■

By Theorem 6.8, m^j acts by a scalar on the space spanned by \mathcal{L}_Ω for any even j in \mathcal{C}_Ω^c . Combining this observation with remark 6.7, we see the vector $\nu(\Omega)_p$ for any p with $1 \leq p \leq 2^l$ spans a representation for M_{even} , the subgroup of M generated by $\{m^j \mid j \text{ is even}\}$. We want to show, for each Ω in Ψ_λ and each p such that $1 \leq p \leq 2^l$, the vector $\nu(\Omega)_p$ spans a representation for all of M , and we want eventually to describe this action completely. We will do so by showing m^j acts by a scalar on the space spanned by \mathcal{L}_Ω for any odd j such that $1 \leq j \leq n-1$. For even n , we do already know the action of one element of M on any vector in μ_λ . Namely, we know the action of $-I_n$ on any vector in $\nu_n(\lambda)$ by the arguments in Lemma 4.3. The element $-I_n$ acts by ± 1 as determined by the weight of $\nu_n(\lambda)$: if $\lambda = a_1(e_1) + \cdots + a_{n/2}(e_{n/2})$, then $-I_n v = (-1)^{a_1 + \cdots + a_{n/2}} v$ for any vector v in μ_λ . We define

$$\epsilon(\lambda) := (-1)^{a_1 + \cdots + a_{n/2}},$$

and we consider any two decorated admissible tableaux $\bar{\Omega}$ and $\bar{\Omega}'$ such that Ω is associated to both $\bar{\Omega}$ and $\bar{\Omega}'$.

Theorem 6.9 *For any odd j with $1 \leq j \leq n-1$, the element m^j of M acts on $\nu_2(\bar{\Omega})$*

by $\epsilon(\Omega)_j$ where

$$\epsilon(\Omega)_j = \epsilon(\gamma_{j+1})\epsilon(\gamma_{j-1}).$$

Moreover, for each odd j with $1 \leq j \leq n-1$, the element m^j acts on $\nu_2(\bar{\Omega}')$ by $\epsilon(\Omega)_j$ as well.

Proof For any odd j such that $1 \leq j \leq n-1$, we may calculate the action of m^j on $\nu_2(\bar{\Omega})$ by determining the action of $m^j(SO(j+1))$ on the weight vector

$$\nu_2(\bar{\pi}_{j+2} \circ \bar{\pi}_{j+3} \circ \cdots \circ \bar{\pi}_n(\bar{\Omega}))$$

in the representation $\mu_{\gamma_{j+1}}$ for

$$\iota_{n-1} \circ \iota_{n-2} \circ \cdots \circ \iota_{j+1}(SO(j+1, \mathbb{R})),$$

so we may as well assume $j = n-1$. Now,

$$-I_n(\iota_{n-1} \circ \iota_{n-2}(-I_{n-2})) = m^j,$$

so the first statement follows.

As for the second statement, we compare γ_{j-1} , defined with respect to $\bar{\Omega}$, and γ'_{j-1} , defined with respect to $\bar{\Omega}'$. The weights γ_{j-1} and γ'_{j-1} can differ only in the sign of the coefficients for $e_{j-1/2}$. According to the definitions for $\epsilon(\gamma_{j-1})$ and $\epsilon(\gamma'_{j-1})$, we have

$$\epsilon(\gamma_{j-1}) = \epsilon(\gamma'_{j-1}).$$

The second statement follows. ■

Remark 6.10 *Theorem 6.9 provides a simple way to calculate the action of m^j on $\nu_2(\bar{\Omega})$ for any odd j and any Ω in Ψ_λ . If $j = n-1$, then, for $\lambda - \gamma_{n-2} = c_1(e_1) +$*

$\cdots + c_{n/2}(e_{n/2})$, we have

$$m^j(\nu_2(\bar{\Omega})) = (-1)^{c_1 + \cdots + c_{n/2}}.$$

Likewise, if $1 \leq j < n - 1$, then, for $\gamma_{j+1} - \gamma_{j1} = c_1(e_1) + \cdots + c_{j+1/2}(e_{j+1/2})$, we have

$$m^j(\nu_2(\bar{\Omega})) = (-1)^{c_1 + \cdots + c_{j+1/2}}.$$

We have shown

$$\mu_\lambda|_M = \bigoplus_{\Omega \in \Psi_\lambda} \bigoplus_{p=1}^{2^l} \mathbb{C}\nu(\Omega)_p,$$

and we have described entirely the action of $\{m^j \mid 1 \leq j \leq n - 1\}$, hence the action of M , on each vector $\nu(\Omega)_p$ with Ω in Ψ_λ and with p such that $1 \leq p \leq 2^l$. Using this information, we now introduce a map $\xi_n: \bar{\Psi} \rightarrow F_n$ where Equation3 defines the set F_n .

We define ξ_n inductively as follows. For $n = 2$, we have $\bar{\Omega} \mapsto \{2\}$ if $m^1\nu(\bar{\Omega}) = 1\nu(\bar{\Omega})$, and $\bar{\Omega} \mapsto \emptyset$ otherwise. For general n , we may assume $\xi_{n-1}(\bar{\pi}\bar{\Omega}) = \hat{S}$ where \hat{S} is some element of \mathcal{S}_{n-1} . For even n , if $m^{n-1}\nu(\bar{\Omega}) = -\nu(\bar{\Omega})$, and if $n - 1$ is an element of \hat{S} , then we define ξ_n such that $\xi_n(\bar{\Omega}) = \hat{S}$. If $n - 1$ is not an element of \hat{S} , we define ξ_n such that $\xi_n(\bar{\Omega}) = \hat{S} \cup \{n\}$. On the other hand, if $m^{n-1}\nu(\bar{\Omega}) = \nu(\bar{\Omega})$, we define ξ_n such that $\xi_n(\bar{\Omega}) = \hat{S} \cup \{n\}$ when $n - 1$ is an element of \hat{S} and such that $\xi_n(\bar{\Omega}) = \hat{S}$ when $n - 1$ is not an element of \hat{S} . Now we suppose n is odd. We define S' to be $\hat{S} \cup \{n\}$ if $\hat{S} \cup \{n\}$ is an element of \mathcal{S}_n . Otherwise, we define S' to be $\{1, \dots, n\} \setminus \hat{S} \cup \{n\}$. If $n - 1$ is an element of \mathcal{C}_Ω^c , we define ξ_n such that $\xi_n(\bar{\Omega}) = \hat{S}$ if $m^{n-1}\nu(\bar{\Omega}) = -\nu(\bar{\Omega})$ and $n - 1$ is an element of \hat{S} or if $m^{n-1}\nu(\bar{\Omega}) = \nu(\bar{\Omega})$ and $n - 1$ is not an element of \hat{S} . We define ξ_n such that $\xi_n(\bar{\Omega}) = S'$ if $m^{n-1}\nu(\bar{\Omega}) = -\nu(\bar{\Omega})$ and $n - 1$ is not an element of \hat{S} or if $m^{n-1}\nu(\bar{\Omega}) = \nu(\bar{\Omega})$ and $n - 1$ is an element of \hat{S} . When n is odd and $n - 1$ is an element of \mathcal{C}_Ω , we make an arbitrary choice. If $\bar{\Omega} = (\Omega, (\epsilon_1, \dots, \epsilon_{[n/2]}))$, we define ξ_n such that $\xi_n(\bar{\Omega}) = S'$ if $\epsilon_{[n/2]} = -1$ and $n - 1$ is not an element of \hat{S} or if $\epsilon_{[n/2]} = 1$

and $n - 1$ is an element of \hat{S} . we define ξ_n such that $\xi_n(\bar{\Omega}) = \hat{S}$ if $\epsilon_{[n/2]} = -1$ and $n - 1$ is an element of \hat{S} or if $\epsilon_{[n/2]} = 1$ and $n - 1$ is not an element of \hat{S} .

Proposition 6.11 *The order of the preimage of S under the map ξ_n is exactly $m(\chi_{n,S}, \mu_\lambda)$.*

Proof We recall the Definition 33 of M_Ω . For each Ω in Ψ_λ and for each j in \mathcal{C}_Ω^e , we have shown m^j acts uniformly on every vector in \mathcal{L}_Ω . Hence, for any i and q such that $1 \leq i < q \leq 2^l$, the M_Ω -representation spanned by $\nu(\Omega)_i$ is isomorphic to the M_Ω -representation spanned by $\nu(\Omega)_q$. We write $\nu(\Omega, M_\Omega)$ to denote this representation. If $\bar{\Omega} \mapsto S$, then we conclude $\chi_{n,S}|_{M_\Omega}$ is isomorphic to $\nu(\Omega, M_\Omega)$ from the definition of ξ_n . For any l -tuple $(\epsilon_{h_1}, \dots, \epsilon_{h_l})$ indexed by the elements of \mathcal{C}_Ω where, for each i such that $1 \leq i \leq l$, we know $\epsilon_i = \pm 1$, the following equation holds:

$$|\{p \in \{1, \dots, 2^l\} \mid m^j \nu(\Omega)_p = \epsilon_j \nu(\Omega)_p \text{ for each } j \in \mathcal{C}_\Omega\}| = 1.$$

We know

$$\xi_n(\Omega, (\epsilon_1, \dots, \epsilon_{[n/2]})) = S$$

where $\nu(\Omega)_p$ spans the M -representation $\chi_{n,S}$ and p with $1 \leq p \leq 2^l$ is such that $m^j \nu(\Omega)_p = \epsilon_j \nu(\Omega)_p$ for each j in \mathcal{C}_Ω . There exists only one $[n/2]$ -tuple $(\epsilon'_1, \dots, \epsilon'_{[n/2]})$ such that Ω is associated to the element

$$(\Omega, (\epsilon'_1, \dots, \epsilon'_{[n/2]}))$$

and $\epsilon_j = \epsilon'_j$ for each j in \mathcal{C}_Ω . The statement of the proposition follows. \blacksquare

The method given by proposition 6.11 to determine $m(\chi_{n,S}, \mu_\lambda)$ allows for computation with much less overhead than the method given in section 4, especially for large n . Although proposition 6.11 does call for induction in determining the action

of M on a chosen decomposition for μ_λ into irreducible representations for M , this proposition then allows for the calculation of $m(\chi_{n,S}, \mu_\lambda)$ for any S in F_n without having to resort to induction.

We give a synopsis of the results in this section, focusing on the visual representation of decorated admissible semistandard Young tableaux. The set $\bar{\Psi}_\lambda$ corresponds to a basis

$$\{\nu_2(\bar{\Omega}) \mid \bar{\Omega} \in \bar{\Psi}\}$$

for the underlying space of the representation μ_λ . Using this correspondence, we see each tableau in $\bar{\Psi}$ as a basis vector. The set \mathcal{C}_Ω comprises all even natural numbers $2k \leq n-1$ such that the number $2k$ actually appears in row k of the admissible semistandard Young tableau Ω . The cardinality of \mathcal{C}_Ω is l , so $l \leq \lfloor n/2 \rfloor$. According to Theorem 6.8, if j is an even natural number such that $1 \leq j \leq n-1$ and such that the number j does not appear in the $j/2^{\text{th}}$ row of $\bar{\Omega}$, then m^j acts on $\bar{\Omega}$ by -1^q where q is the quantity of the numbers $j+1$ and j in $\bar{\Omega}$. According to Theorem 6.9, if j is an odd natural number such that $1 \leq j \leq n-1$, then m^j acts on $\bar{\Omega}$ by -1^q where q is the quantity of the numbers $\pm j+1$ and j in $\bar{\Omega}$. On the other hand, by Theorem 6.5, if j is an element of \mathcal{C}_Ω , then m^j maps $\bar{\Omega}$ to $\bar{\Omega}_j$ where every occurrence of $\pm j$ in row $j/2$ of $\bar{\Omega}$ is $\mp j$ in row $j/2$ of $\bar{\Omega}$ and all other entries are equal. The set \mathcal{L}_Ω comprises each decorated tableau $\bar{\Omega}$ such that taking the absolute values of every number in $\bar{\Omega}$ yields Ω . The order of \mathcal{L}_Ω is 2^l , one element for each choice of sign given to the numbers j in row $j/2$ of Ω for every j in \mathcal{C}_Ω . We define M_Ω to be the subgroup of M generated by $\{m^j \mid j \in \mathcal{C}_\Omega\}$. The group M_Ω has 2^l distinct irreducible representations, one for each choice of the action for m^j where j is an element of \mathcal{C}_Ω . (The element m^j must act by ± 1 .) By the Peter-Weyl theorem, the subspace spanned by \mathcal{L}_Ω has a basis such that each of the 2^l basis vectors spans a unique irreducible M_Ω -representation.

Since

$$\cup_{\Omega \in \Psi_\lambda} \mathcal{L}_\Omega$$

spans the underlying space for μ_λ , the Peter-Weyl theorem give a basis for this underlying space such that each basis vector spans an irreducible representation for M . Moreover, we understand completely the action of M on this basis because $\{m^j \mid 1 \leq j \leq n-1\}$ generates M .

This information allows for us to find $m(\chi_{n,S}, \mu_\lambda)$ for any S in F_n . We express this information in terms of an algorithm for associating each $\bar{\Omega}$ in $\bar{\Psi}_\lambda$ with some subset S in F_n such that the number of decorated tableaux associated to S is exactly $m(\chi_{n,S}, \mu_\lambda)$. The algorithm used to associate each $\bar{\Omega}$ with some subset S is our map ξ_n , and Proposition 6.11 proves our map ξ_n does in fact map exactly $m(\chi_{n,S}, \mu_\lambda)$ -many decorated tableaux to the subset S . To more cleanly express the definition of the map ξ_n , we introduce a map

$$* : \{1, \dots, n-1\} \times \bar{\Psi} \rightarrow \pm 1.$$

We define $*(j, \bar{\Omega}) = \epsilon$ where $m^j(\bar{\Omega}) = \epsilon(\bar{\Omega})$ for j in \mathcal{C}_Ω^e . For j in \mathcal{C}_Ω , we define $*(j, \bar{\Omega}) = \epsilon$ where $\epsilon = -1^q$ for q equal to the quantity of the numbers $j+1$ and $\pm j$ in $\bar{\Omega}$ if each occurrence of the number $\pm j$ in row $j/2$ of $\bar{\Omega}$ is positive and where $\epsilon = -(-1^q)$ if each occurrence of the number $\pm j$ in row $j/2$ of $\bar{\Omega}$ is negative. In terms of tableaux, the map ξ_n acts recursively as follows. If $n=2$, then $\bar{\Psi}_\lambda$ contains only one element, a tableau having one row such that each entry is equal to 2 or such that each entry is equal to -2 . In this case, $\xi_2(\bar{\Omega}) = \{1\}$ if the number of entries is odd, while $\xi_2(\bar{\Omega}) = \emptyset$ if the number of entries is even. For $n > 2$, we suppose $\xi_{n-1}(\bar{\pi}(\bar{\Omega})) = \tilde{S}$ for some element \tilde{S} of F_{n-1} . If $*(n-1, \bar{\Omega}) = -1$ and \tilde{S} contains $\{n-1\}$, then we define S' to be \tilde{S} . If $*(n-1, \bar{\Omega}) = -1$ and \tilde{S} does not contain

$\{n-1\}$, then we define S' to be $\tilde{S} \cup \{n\}$. If $*(n-1, \bar{\Omega}) = 1$ and \tilde{S} contains $\{n-1\}$, then we define S' to be $\tilde{S} \cup \{n\}$. If $*(n-1, \bar{\Omega}) = 1$ and \tilde{S} does not contain $\{n-1\}$, then we define S' to be \tilde{S} . Either S' is an element of F_n or the complement of S' in $\{1, \dots, n\}$ is an element of F_n . If we define S to be the element of F_n equal either to S' or the complement of S' in $\{1, \dots, n\}$, then $\xi_n(\bar{\Omega}) = S$.

7 Application to Split Real Reductive Groups of Type B_n and D_n

We have given a neat description of the branching law from K to M for $SL(n, \mathbb{R})$, the split real group of type A_n . As far as possible, we would like to use that result to determine similarly complete descriptions of the branching laws from K_e to $M \cap K_e$ for the split real groups of type B_n and D_n . We start by identifying these groups.

We consider the group

$$SO(n, n) = \{g \in SL(2n, \mathbb{R}) \mid {}^t g I_{n,n} g = I_{n,n}\}$$

where $I_{n,n}$ is the $2n$ by $2n$ diagonal matrix with j^{th} diagonal entry equal to 1 for j such that $1 \leq j \leq n$ and j^{th} diagonal entry equal to -1 for j such that $n+1 \leq j \leq 2n$. Differentiating the defining relations for $SO(n, n)$, we see $SO(n, n)$ has Lie algebra

$$\mathfrak{so}(n, n) = \{X \in \mathfrak{sl}(2n, \mathbb{R}) \mid {}^t X I_{n,n} + I_{n,n} X = 0\}.$$

Any two quadratic forms over \mathbb{C} are equivalent. In particular, $I_{n,n}$ is equivalent to I_{2n} via the transformation ${}^t g I_{n,n} g$ where g is the diagonal $2n$ by $2n$ matrix with j^{th} diagonal entry equal to 1 for j such that $1 \leq i \leq n$ and j^{th} diagonal entry equal to i for j such that $n+1 \leq j \leq 2n$. We see the complexification of $\mathfrak{so}(n, n)$ is $\mathfrak{so}(2n, \mathbb{C})$,

the set $M(\widehat{SO(n)})$ as $\{\chi_{n,S} \mid S \in F_n\}$ where F_n takes its definition from Equation 3. We can determine a branching law from K_e to $M \cap K_e$ using proposition 6.11 so long as we can determine a branching law from $M(SO(n)) \times M(SO(n))$ to $M \cap K_e$. For any subset S of $\{1, \dots, n\}$ we refer to the complement of S in $\{1, \dots, n\}$ by S^c . We choose an irreducible representation $\chi_{n,S'} \otimes \chi_{n,S''}$ of $M(SO(n)) \times M(SO(n))$ and an irreducible representation $\chi_{n,S}$ of $M \cap K_e$.

Theorem 7.1 *We have $m(\chi_{n,S}, \chi_{n,S'} \otimes \chi_{n,S''}) = 1$ if and only if $S = S' \ominus S''$ or $S = (S' \ominus S'')^c$. Otherwise, $m(\chi_{n,S}, \chi_{n,S'} \otimes \chi_{n,S''}) = 0$.*

Proof Each irreducible representation for $M(SO(n)) \times M(SO(n))$ has dimension equal to 1, so $(\chi_{n,S'} \otimes \chi_{n,S''})|_{M \cap K_e}$ is itself irreducible. We need only find the subset S in F_n such that $\chi_{n,S} = (\chi_{n,S'} \otimes \chi_{n,S''})|_{M \cap K_e}$. With m^j defined as in Lemma 3.1, we consider the action of (m^j, m^j) on some vector (v', v'') in $(\chi_{n,S'} \otimes \chi_{n,S''})$ for each j such that $1 \leq j < n$. We have

$$(m^j, m^j)(v', v'') = (m^j(v'), m^j(v'')),$$

so $(m^j, m^j)(v', v'') = -1(v', v'')$ if

$$\begin{aligned} \{j, j+1\} \cup S' &= \{j, j+1\} \text{ or } \{j, j+1\} \cup S' = \emptyset \text{ and} \\ \{j, j+1\} \cup S'' &= \{j\} \text{ or } \{j, j+1\} \cup S'' = \{j+1\}. \end{aligned}$$

Also, $(m^j, m^j)(v', v'') = -1(v', v'')$ if

$$\begin{aligned} \{j, j+1\} \cup S' &= \{j\} \text{ or } \{j, j+1\} \cup S' = \{j+1\} \text{ and} \\ \{j, j+1\} \cup S'' &= \{j+j+1\} \text{ or } \{j, j+1\} \cup S'' = \emptyset. \end{aligned}$$

Otherwise, $(m^j, m^j)(v', v'') = (v', v'')$. From this description of the action of $M \cap K_e$ on $(\chi_{n,S'} \otimes \chi_{n,S''})$, we know $\chi_{n,S} = (\chi_{n,S'} \otimes \chi_{n,S''})|_{M \cap K_e}$ if and only if $S = S' \ominus S''$ or

$$S = (S' \ominus S'')^c. \quad \blacksquare$$

We now choose an irreducible representation $\chi_{n,S}$ for $M \cap K_e$ and an irreducible representation $\mu_{\lambda'} \otimes \mu_{\lambda''}$.

Proposition 7.2 *The branching law from K_e to $M \cap K_e$ for the split real reductive group of type D_n determines $m(\chi_{n,S}, \mu_{\lambda'} \otimes \mu_{\lambda''})$ to be*

$$\begin{aligned} & \sum_{\substack{S', S'' \in F_n \\ S' \ominus S'' = S}} [m(\chi_{n,S'}, \mu_{\lambda'})m(\chi_{n,S''}, \mu_{\lambda''}) \\ & \quad + m(\chi_{n,S''}, \mu_{\lambda'})m(\chi_{n,S'}, \mu_{\lambda''})] \\ & \sum_{\substack{S', S'' \in F_n \\ (S' \ominus S'')^c = S}} [m(\chi_{n,S'}, \mu_{\lambda'})m(\chi_{n,S''}, \mu_{\lambda''}) \\ & \quad + m(\chi_{n,S''}, \mu_{\lambda'})m(\chi_{n,S'}, \mu_{\lambda''})] \end{aligned}$$

Proof This result follows directly from proposition 6.11 and Theorem 7.1. \blacksquare

A branching law for the split real reductive group of type B_n develops along very similar lines. We focus on the group

$$SO(n+1, n) = \{g \in SL(2n+1, \mathbb{R}) \mid {}^t g I_{n+1, n} g = I_{n+1, n}\}$$

where $I_{n+1, n}$ is the $2n+1$ by $2n+1$ diagonal matrix with i^{th} diagonal entry equal to 1 for i such that $1 \leq i \leq n+1$ and with i^{th} diagonal entry equal to -1 for i such that $n+2 \leq i \leq 2n+1$. Differentiating the defining relations for $SO(n+1, n)$, we see $SO(n+1, n)$ has Lie algebra

$$\mathfrak{so}(n+1, n) = \{X \in \mathfrak{sl}(2n+1, \mathbb{R}) \mid {}^t X I_{n+1, n} + I_{n+1, n} X = 0\}.$$

Any two quadratic forms over \mathbb{C} are equivalent. In particular, $I_{n, n, 1}$ is equivalent to

Component 37 is the identity component K_e of K . Clearly, K_e is $SO(n+1, \mathbb{R}) \times SO(n, \mathbb{R})$. The closed subgroup $M \cap K_e$ of K_e is simply the image of $M(SO(n))$ under the diagonal map $\delta: SO(n, \mathbb{R}) \rightarrow SO(n, \mathbb{R}) \times SO(n, \mathbb{R})$ followed by the map $\iota': SO(2n, \mathbb{R}) \hookrightarrow SO(2n+1, \mathbb{R})$ where $M(SO(n))$ is the Langlands subgroup for $SO(n)$ and where ι' is the embedding

$$A \mapsto \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix}.$$

We know $M \cap K_e$ is a subgroup of $M(SO(n)) \times M(SO(n)) \subset M(SO(n+1)) \times M(SO(n))$. The map δ isomorphically identifies $M(SO(n))$ with $M \cap K_e$, and Theorem 3.2 describes the set $\widehat{M(SO(n))}$ as $\{\chi_{n,S} \mid S \in F_n\}$ where F_n takes its definition from Equation 3.

We can determine a branching law from K_e to $M \cap K_e$ using proposition 6.11, so long as we can determine a branching law from $M(SO(n+1)) \times M(SO(n))$ to $M \cap K_e$. We choose an irreducible representation $\chi_{n,S''} \otimes \chi_{n,S'}$ of $M(SO(n+1)) \times M(SO(n))$ and an irreducible representation $\chi_{n,S}$ of $M \cap K_e$. For any subset S'' of $\{1, \dots, n+1\}$ we denote by \tilde{S}'' the subset $S'' \setminus \{n+1\}$.

Theorem 7.3 *We have $m(\chi_{n,S}, \chi_{n,S''} \otimes \chi_{n,S'}) = 1$ if and only if $S = S' \ominus \tilde{S}''$ or $S = (S' \ominus \tilde{S}'')^c$. Otherwise, $m(\chi_{n,S}, \chi_{n,S''} \otimes \chi_{n,S'}) = 0$.*

Proof Branching from $M(SO(n+1)) \times M(SO(n))$ to $M(SO(n)) \times M(SO(n))$, we find

$$m(\chi_{n,\tilde{S}''} \otimes \chi_{n,S'}, \chi_{n+1,S''} \otimes \chi_{n,S'}) = 1$$

where we realize $M(SO(n)) \times M(SO(n))$ as a subgroup of $M(SO(n+1)) \times M(SO(n))$

via the embedding ι' and where $\chi_{n+1,S''} \otimes \chi_{n,S'}$ is any irreducible representation of $M(SO(n+1)) \times M(SO(n))$. Since any irreducible representation of $M(SO(n+1)) \times M(SO(n))$ has dimension equal to 1, any irreducible representation of $M(SO(n)) \times M(SO(n))$ not isomorphic to $\chi_{n,\tilde{S}''} \otimes \chi_{n,S'}$ has multiplicity in $\chi_{n+1,S''} \otimes \chi_{n,S'}$ equal to 0. Combining this result with the branching law from $M(SO(n)) \times M(SO(n))$ to $M \cap K_e$ given in Theorem 7.1, the statement of this theorem follows. \blacksquare

We choose an irreducible representation $\mu_{\lambda''} \otimes \mu_{\lambda'}$ of $SO(n+1, \mathbb{R}) \times SO(n, \mathbb{R})$ and an irreducible representation $\chi_{n,S}$ of $M \cap K_e$

Proposition 7.4 *The branching law from K_e to $M \cap K_e$ for the split real reductive group of type B_n determines $m(\chi_{n,S}, \mu_{\lambda''} \otimes \mu_{\lambda'})$ to be*

$$\begin{aligned} & \sum_{\substack{S' \in F_n, S'' \in F_{n+1} \\ S' \ominus \tilde{S}'' = S}} [m(\chi_{n,S'}, \mu_{\lambda'})m(\chi_{n+1,S''}, \mu_{\lambda''}) \\ & \quad + m(\chi_{n,\tilde{S}''}, \mu_{\lambda'})m(\chi_{n+1,S'}, \mu_{\lambda''})] \\ & \sum_{\substack{S', S'' \in F_n \\ (S' \ominus S'')^c = S}} [m(\chi_{n,S'}, \mu_{\lambda'})m(\chi_{n+1,S''}, \mu_{\lambda''}) \\ & \quad + m(\chi_{n,\tilde{S}''}, \mu_{\lambda'})m(\chi_{n+1,S'}, \mu_{\lambda''})] \end{aligned}$$

Proof This result follows directly from proposition 6.11 and Theorem 7.3. \blacksquare

For the split classical groups, then, we have determined to extensive detail the branching law from K_e to $M \cap K_e$ with respect to these classical groups. We have done so in a manner making the multiplicities of irreducible representations for $M \cap K_e$ in irreducible representations for K_e fairly easy to calculate. All the benefits these branching laws yield for the study of principal series representations attach.

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