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Complexity of stability and controllability of elementary hybrid systems *

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In this paper, we consider simple classes of nonlinear systems and prove that basic questions related to their stability and controllability are either undecidable or computationally intractable (NP-hard). As a special case, we consider a class of hybrid systems in which the state space is partitioned into two halfspaces, and the dynamics in each halfspace correspond to a different linear system.

Keywords: Hybrid systems, nonlinear systems, control, decidability, computability, computational complexity, NP-hardness.

1 Introduction

In recent years, much research has focused on *hybrid* systems. These are systems that involve a combination of continuous dynamics (e.g., differential equations or linear evolution equations) and discrete dynamics. The motivation lies in the fact that most complex systems involve a physical layer described by continuous variables, together with higher level layers involving symbolic manipulations and discrete supervisory decisions. Applications range from intelligent traffic systems to industrial process control.

Hybrid systems can be usually described by state space models, using a suitably defined state space (often the Cartesian product of continuous and discrete sets). Classical systems theory provides us with efficient methods for analyzing and controlling certain classes of continuous-variable systems (e.g., linear sys-

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tems) and certain classes of discrete-variable systems (e.g., finite state Markov chains). However, equally efficient generalizations are not available even for the simplest classes of hybrid systems. This is thought to be a reflection of the inherent complexity of such systems. The research reported in this paper is aimed at elucidating this complexity.

As an illustration, consider a hybrid system with state $(x_t, q_t) \in \mathbf{R}^n \times \{1, \ldots, m\}$ where x_t and q_t are, respectively, the continuous and discrete parts of the state. Let A_i $(i = 1, \ldots, m)$ be square matrices and let the dynamics of x_t depend on the discrete state by

$$x_{t+1} = A_i x_t \text{ when } q_t = i.$$

In addition, let a finite partition of $\mathbf{R}^n$ be given , $\mathbf{R}^n = H_1 \cup H_2 \cup \cdots \cup H_m$, and suppose that the discrete state q_t depends only on the location of the continuous state x_t in the partition, i.e.,

$$q_t = i \text{ when } x_t \in H_i.$$

Then, the overall hybrid system can be written in the form of a nonlinear system

$$x_{t+1} = A_i x_t \text{ when } x_t \in H_i. \tag{1}$$

In the case where the partition consists of two regions separated by a hyperplane, the system becomes

$$x_{t+1} = \begin{cases} A_1 x_t & \text{when} \quad c^T x_t \geq 0 \\ A_2 x_t & \text{when} \quad c^T x_t < 0 \end{cases} \tag{2}$$

A system is *stable* if its state vector always converges to zero. Deciding stability for hybrid systems as simple as (2) is already a nontrivial task. Let us illustrate this with an example. We build a state space model for a system described by a state vector (v_t, y_t, z_t), where v_t and y_t are scalars and z_t is a vector in $\mathbf{R}^n$. The dynamics of the system is of the form

$$\begin{pmatrix} v_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & A_+ \end{pmatrix} \begin{pmatrix} v_t \\ y_t \\ z_t \end{pmatrix} \text{ when } y_t \geq 0,$$

and

$$\begin{pmatrix} v_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & A_- \end{pmatrix} \begin{pmatrix} v_t \\ y_t \\ z_t \end{pmatrix} \text{ when } y_t < 0.$$

This hybrid system consists of two linear systems, each of which is enabled in one of two halfspaces, as determined by the sign of y_t.

Let us now look at the evolution of an initial state vector (v_0, y_0, z_0). Suppose that $v_0 = 1$ in which case we have $v_t = 2^{-t}$ for all t. Suppose in addition, that y_0 can take any value in $[-1, 1]$. Then, it is easily seen that y_1 can take any value in $[-1/2, 1/2]$, no matter what was the sign of y_0. Continuing inductively, we see that y_t can take any value in $[-2^{-t}, 2^{-t}]$, can have either sign, and this is independent of the signs of y_s for $s < t$. This shows that every possible sign sequence can be generated by suitable choice of y_0. Hence, the dynamics of the state subvector z_t are of the form $z_{t+1} = A_t z_t$, where A_t is an arbitrary matrix from $\{A_-, A_+\}$. We conclude that the state vector converges to zero, for all possible initial states, if and only if all sequences of products of the matrices A_- and A_+ (taken in an arbitrary order) converge to zero.

Unfortunately, a test for the stability of all possible sequences of products of two matrices is not available. The decidability of this problem is a major open question and is intimately related to the so-called "finiteness conjecture" (see, e.g., Daubechies and Lagarias [5], Lagarias and Wang [11], and Gurvits [7,8]). If the stability of all possible sequences of products of two matrices turns out to be undecidable, it will immediately follow that the stability of the class of hybrid systems of form (2) is also undecidable. Given the present state of knowledge, we are unable to prove such an undecidability result. On the other hand, NP-hardness of the stability problem for systems of the form (2) is obtained with a straightforward adaptation of the arguments in [23].

In Section 2 we build on this last observation and prove NP-hardness of the stability problem for many more classes of systems. Let us note that systems of the form (2) can also be written

$$x_{t+1} = (B_0 + \nu(c^T x_t) B_1) x_t \tag{3}$$

with $B_0 = A_1$, $B_1 = A_2 - A_1$, and with the function ν defined by $\nu(\alpha) = 0$ for $\alpha \geq 0$, and by $\nu(\alpha) = 1$ for $\alpha < 0$. In Theorem 1, we consider nonlinear systems of the form (3) where ν is an arbitrary scalar function. We show that for a large class of nonconstant functions ν, the stability of these systems is NP-hard to decide. In particular, our result applies to the function defined above, and so the stability of systems of the form (2) is NP-hard to decide.

In Section 3 we consider classes of elementary hybrid systems similar to (2) but with an additional control variable. The nth-dimensional *sign system* associated with $A_+, A_0, A_- \in \mathbf{R}^{n \times n}$ and $b, c \in \mathbf{R}^n$ is the system

$$x_{t+1} = A_{\mathrm{sgn}(c^T x_t)} x_t + b u_t, \qquad t = 0, 1, \ldots,$$

where sgn$(\cdot)$ is the sign function defined by

$$\mathrm{sgn}(x) = \begin{cases} +, & \text{when } x > 0, \\ 0, & \text{when } x = 0, \\ -, & \text{when } x < 0. \end{cases}$$

In Theorem 2, we establish that null-controllability and complete reachability are both undecidable for such systems. A related result is given by Toker [22] who considers a class of systems similar to sign systems. He shows that the question of deciding whether all possible control actions drive a given initial state to the origin is undecidable. Theorem 2 is also related to our earlier work on the complexity of certain questions involving products of matrices coming from a given finite family [2], [23]. In our earlier work, matrices could be multiplied in an arbitrary order. The present work is different in that the choice of the next matrix in the product is determined by a feedback mechanism involving the state of the system.

Systems of the form (1) are the *piecewise linear* systems introduced by Sontag in [18], and for which some complexity results are already available, see [19] for a survey of these results together with results for other type of nonlinear systems. The systems (1) are also similar to the *piecewise constant derivative* systems analyzed by Arasin, Maler and Pnueli. A piecewise constant derivative system is given by a finite partition of $\mathbf{R}^n$, $\mathbf{R}^n = H_1 \cup H_2 \cup \cdots \cup H_m$, and by slope vectors b_i for every region H_i of the partition. On any given region of the partition, the state $x(t)$ of a piecewise constant derivative system has a fixed constant derivative,

$$\frac{d\,x(t)}{dt} = b_i \text{ when } x \in H_i.$$

The trajectories of such systems are continuous broken lines, with breaking points occurring on the boundaries of the regions. In [1] Asarin et al. show that, for given states x_b and x_e, the problem of deciding whether x_b is reached by a trajectory starting from x_b, is decidable for systems of dimension two, but is undecidable for systems of dimension three or more. This undecidability result is obtained by simulating Turing machines. Similar Turing machine simulations by hybrid systems are possible with other type of hybrid systems; see, e.g., Bournez and Cosnard [3] for simulation by analog automata, Siegelmann and Sontag [17] for simulation by saturated systems, and Branicky [4] for simulation by differential equations. (See also the general reference, Henzinger et al. [9].) In all these constructions, the "partition" of the state space is used to encode the configuration of a Turing machine. Simulation of arbitrary Turing machines is therefore possible only if there is no apriori limit on the "size" of the partition. An original aspect of our results, when compared with those mentioned above, is that they are valid for hybrid systems with very few regions.

2 Autonomous systems

A *discrete-time autonomous system* $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ is said to be *globally asymptotically stable* (or, for short, *asymptotically stable*) if the sequences defined by

$$x_{t+1} = f(x_t), \qquad t = 0, 1, \ldots,$$

converge to the origin for all initial states $x_0 \in \mathbf{R}^n$.

Let A be an $n \times n$ real matrix. It is well-known that the linear system $x_{t+1} = Ax_t$ is asymptotically stable if and only if all eigenvalues of A have magnitude strictly less than one. Furthermore, asymptotic stability can be decided efficiently, e.g., by solving a Lyapunov equation. No such simple and computationally efficient test exists for general nonlinear systems.

In this section, we define particular classes of systems involving a single scalar nonlinearity, and we prove that algorithms for deciding asymptotic stability of systems in any one of our classes are inherently inefficient. Unless P=NP, the running time of any such algorithm must increase faster than any polynomial in the size of the description of the system. (See, e.g., Garey and Johnson [6] or Papadimitriou [14] for the definitions of P, NP, and the implications of NP-hardness.) Some of our classes are elementary and can be viewed as the "least nonlinear" systems. In particular, one of our classes corresponds to the class of systems that are linear on each side of a hyperplane that divides the state space in two.

Systems with a single scalar nonlinearity. Let us fix a scalar function $\nu : \mathbf{R} \mapsto \mathbf{R}$. The *$\nu$-system* associated with $n \geq 1$, $A_0, A_1 \in \mathbf{R}^{n \times n}$, and $c \in \mathbf{R}^n$, is defined by

$$x_{t+1} = \big(A_0 + \nu(c^T x_t) A_1\big) x_t, \qquad t = 0, 1, \ldots.$$

(Here, the superscript T denotes matrix transposition.) When ν is a constant function, ν-systems are linear and their stability can be decided easily. We show in Theorem 1 below that for a broad variety of nonconstant functions ν, the stability of ν-systems is NP-hard to decide.

Let us note that stability can be difficult to check for the simple reason that ν may be difficult to compute. For this reason, the result that we present below is of interest primarily for the case where ν is an easily computable function.

Theorem 1. Let us fix a nonconstant scalar function $\nu : \mathbf{R} \mapsto \mathbf{R}$ such that

$$\lim_{x \to -\infty} \nu(x) \leq \nu(x) \leq \lim_{x \to +\infty} \nu(x)$$

for all $x \in \mathbf{R}$. Then, the asymptotic stability of ν-systems is NP-hard to decide.

Proof. Our proof relies on a construction developed in [23], which in turn is based on a reduction technique introduced in [15]. Rather than repeating here the construction in [23], we simply state its conclusions, in the form of the lemma that follows. The lemma makes reference to 3SAT, which is the Boolean satisfiability problem with three literals per clause, and is a well-known NP-complete problem.

Lemma 1. Given an instance of 3SAT with n variables and m clauses, we can construct (in polynomial time) two matrices A_0 and A_1, of dimensions $r \times r$, where $r = (n+1)(m+1)$, whose entries belong to $\{0,1\}$, and with the following properties:

(a) If we have a "yes" instance of 3SAT, there exist indices $k_1, k_2, \ldots, k_{n+2} \in \{0,1\}$, and a nonnegative nonzero integer vector x such that $A_{k_{n+2}} \cdots A_{k_2} A_{k_1} x = mx$.

(b) If we have a "no" instance of 3SAT, then $\|A_{k_{n+2}} \cdots A_{k_2} A_{k_1} x\| \leq (m-1)\|x\|$, for every vector x, and for every choice of indices $k_1, k_2, \ldots, k_{n+2} \in \{0,1\}$. Here, and throughout the paper, $\|\cdot\|$ stands for the maximum (ℓ_∞) norm.

Let us now fix a nonconstant function $\nu(\cdot)$ with

$$\lim_{x \to -\infty} \nu(x) \leq \nu(x) \leq \lim_{x \to +\infty} \nu(x),$$

for all $x \in \mathbf{R}$, and let $a_- = \lim_{x \to -\infty} \nu(x)$ and $a_+ = \lim_{x \to +\infty}$. For simplicity we assume that a_- and a_+ are rational numbers. This restriction is not essential and can be removed with a slightly more complicated proof.

Since we have assumed that $\nu(\cdot)$ is not constant, we have $a_- < a_+$. Given an instance of 3SAT, we construct the matrices A_0 and A_1 as in Lemma 1. We then let

$$B_0 = \frac{a_+ A_0 - a_- A_1}{a_+ - a_-}, \qquad B_1 = \frac{A_1 - A_0}{a_+ - a_-}.$$

It is seen that for any $a \in \mathbf{R}$, we have

$$B_0 + aB_1 = \frac{a_+ - a}{a_+ - a_-} A_0 + \frac{a - a_-}{a_+ - a_-} A_1, \tag{4}$$

and that for any $a \in [a_-, a_+]$, $B_0 + aB_1$ is a convex combination of A_0, A_1.

We will now define the dynamics of a ν-system. The system we construct has a state vector $x_t = (z_t, y_t)$, consisting of a subvector $z_t \in \mathbf{R}^r$ and a subvector $y_t \in \mathbf{R}^{n+2}$. Let y_t^i and z_t^i stand for the ith component of y_t and z_t, respectively, and let the vector c in the definition of a ν-system be such that $c^T x_t = y_t^1$. Next, we describe the dynamics of the state vector.

Regarding z_t, we let

$$z_{t+1} = g\big(B_0 + \nu(y_t^1)B_1\big)z_t. \tag{5}$$

Here, g is a rational number such that

$$\left(m - \frac{1}{3}\right)^{-1} \leq g^{n+2} \leq \left(m - \frac{2}{3}\right)^{-1}. \tag{6}$$

Such a rational number exists whose size (number of bits in a binary encoding) is polynomial in m and n, and can be constructed in polynomial time. Regarding y_t, we have the following equations:

$$y_{t+1}^i = y_t^{i+1}, \qquad i = 1, \ldots, n+1, \tag{7}$$

and

$$y_{t+1}^{n+2} = \left(\nu(y_t^1) - \frac{a_- + a_+}{2}\right)\sum_{i=1}^{r} z_t^i. \tag{8}$$

We will show that the resulting ν-system is asymptotically stable if and only if the instance of 3SAT that we started with is a "no" instance.

Suppose that we have a "no" instance of 3SAT. By the construction of Lemma 1, we have $||A_{k_{n+2}} \cdots A_{k_2}A_{k_1}z|| \leq (m-1)||z||$, for any vector z, and any choice of indices $k_1, \ldots, k_{n+2}$. Because of Eq. (4), we see that for every value of y^1, $B_0 + \nu(y^1)B_1$ is a convex combination of the matrices A_0, A_1, i.e., $B_0 + \nu(y^1)B_1 = \gamma A_0 + (1-\gamma)A_1$, for some $\gamma \in [0,1]$. Hence, using Eq. (5),

$$\begin{aligned} ||z_{n+2}|| &\leq g^{n+2} \max_{\gamma_1,\ldots,\gamma_{n+2}} \Big\| \big(\gamma_{n+2}A_0 + (1-\gamma_{n+2})A_1\big) \cdots \big(\gamma_1 A_0 + (1-\gamma_1)A_1\big)z_0 \Big\| \\ &= g^{n+2} \max_{k_1,\ldots,k_{n+2}} ||A_{k_{n+2}} \cdots A_{k_2}A_{k_1}z_0|| \\ &\leq g^{n+2}(m-1)||z_0||. \end{aligned}$$

The first maximum is subject to the constraints $0 \leq \gamma_i \leq 1$. It is easily shown that the maximum is attained with each γ_i equal to either zero or one, which explains the equality. Since $g^{n+2} \leq (m - (2/3))^{-1}$, we conclude that $||z_{n+2}|| \leq \alpha||z_0||$, for some constant $\alpha < 1$, from which it easily follows that z_t converges to zero. In particular, $\sum_{i=1}^r z_t^i$ converges to zero, and by inspecting Eqs. (7)-(8), we conclude that y_t also converges to zero. Since this argument was carried out for arbitrary initial conditions, we conclude that the ν-system is asymptotically stable.

We now consider the case where we start with a "yes" instance of 3SAT. By the construction of Lemma 1, there exists a nonnegative nonzero integer vector $\overline{z}$, and some choice of indices $k_1, \ldots, k_{n+2}$, such that $A_{k_{n+2}} \cdots A_{k_2}A_{k_1}\overline{z} = m\overline{z}$. Using scaling, we can assume that the components of $\overline{z}$ are nonnegative integer multiples of a positive integer constant K, whose value will be determined shortly. We choose the initial subvector z_0 to be any vector that satisfies

$$z_0 \geq \overline{z}.$$

Let M be another positive integer constant to be determined shortly. Let us say that a vector $y \in \mathbf{R}^{n+2}$ *encodes* $k_1, \ldots, k_{n+2}$ if the following two conditions hold for $i = 1, \ldots, n+2$:

$$y^i \geq M, \qquad \text{if } k_i = 1,$$

$$y^i \leq -M, \qquad \text{if } k_i = 0.$$

We let the initial subvector y_0 be such that it encodes $k_1, \ldots, k_{n+2}$. We will show that with a suitably large choice of K and M, we have $z_{n+2} \geq \overline{z}$ and y_{n+2} also encodes $k_1, \ldots, k_{n+2}$. It will then follow (by induction) that $z_t \geq \overline{z}$ for all times t that are integer multiples of $n+2$, and we will have completed the proof that the ν-system is not asymptotically stable.

We now set the values of the constants K and M. We first choose some $\epsilon > 0$ such that

$$\left(1 - \frac{\epsilon}{a_+ - a_-}\right)^{n+2} \frac{m}{m - \frac{1}{3}} \geq 1.$$

We then choose M so that

$$\nu(b) \geq a^+ - \epsilon, \qquad \text{if } b \geq M,$$

$$\nu(b) \leq a^- + \epsilon, \qquad \text{if } b \leq -M.$$

Finally, we choose K so that

$$g^{n+2}\left(a_+ - \epsilon - \frac{a_- + a_+}{2}\right)\left(1 - \frac{\epsilon}{a_+ - a_-}\right)^{n+2} K \geq M.$$

For $t = 1, \ldots, n+2$, Eq. (7) yields $y_{t-1}^1 = y_0^t$, which implies $\nu(y_{t-1}^1) = \nu(y_0^t)$. Since y_0 encodes $k_1, \ldots, k_{n+2}$, it follows that $\nu(y_{t-1}^1)$ is within ϵ of a^+ or a_-, depending on whether k_t is 1 or 0, respectively. Suppose that $k_t = 1$. In that case, $\nu(y_{t-1}^1) \geq a^+ - \epsilon$, and Eq. (4) yields

$$B_0 + \nu(y_{t-1}^1)B_1 \geq \frac{\nu(y_{t-1}^1) - a_-}{a_+ - a_-}A_1 \geq \frac{a_+ - \epsilon - a_-}{a_+ - a_-}A_1 = \left(1 - \frac{\epsilon}{a_+ - a_-}\right)A_{k_t}.$$

(The inequality between matrices is to be understood componentwise.) A symmetric argument also shows that if $k_t = 0$, we again have

$$B_0 + \nu(y_{t-1}^1)B_1 \geq \left(1 - \frac{\epsilon}{a_+ - a_-}\right)A_{k_t}.$$

This shows that we have

$$z_t \geq g\left(1 - \frac{\epsilon}{a_+ - a_-}\right)A_{k_t}z_{t-1}, \qquad t = 1, \ldots, n+2. \tag{9}$$

In particular,

$$\begin{aligned} z_{n+2} &\geq g^{n+2}\left(1-\frac{\epsilon}{a_+ - a_-}\right)^{n+2} A_{k_{n+2}}\cdots A_{k_1} z_0 \\ &\geq \frac{1}{m-\frac{1}{3}}\left(1-\frac{\epsilon}{a_+ - a_-}\right)^{n+2} A_{k_{n+2}}\cdots A_{k_1}\overline{z} \\ &= \frac{1}{m-\frac{1}{3}}\left(1-\frac{\epsilon}{a_+ - a_-}\right)^{n+2} m\overline{z} \\ &\geq \overline{z}. \end{aligned}$$

The second inequality made use of the definition of g [cf. Eq. (6)]. The equality was based on the definition of $\overline{z}$. Finally, the last inequality relied on the definition of ϵ.

Recall that the matrices A_0, A_1 have nonnegative integer entries. Since the entries of $\overline{z}$ are nonnegative integer multiples of K, we see that the entries of $A_{k_t}\cdots A_{k_1}\overline{z}$ have the same property, for $t = 1,\ldots,n+2$. Furthermore, for t in that range, the vector $A_{k_t}\cdots A_{k_1}\overline{z}$ must be nonzero; otherwise, we would have $m\overline{z} = A_{k_{n+2}}\cdots A_{k_1}\overline{z} = 0$, contradicting the fact that $\overline{z}$ is nonzero. Using Eq. (9), and the fact $g < 1$, we conclude that

$$\sum_{i=1}^{r} z_t^i \geq g^{n+2}\left(1-\frac{\epsilon}{a_+ - a_-}\right)^{n+2} K, \qquad t = 1,\ldots,n+2. \tag{10}$$

Suppose that $y_t^1 \geq M$. Then, $\nu(y_t^1) \geq a^+ - \epsilon$. Using this inequality in Eq. (8), and using also Eq. (10), we obtain

$$y_{t+1}^{n+2} \geq g^{n+2}\left(a_+ - \epsilon - \frac{a_- + a_+}{2}\right)\left(1-\frac{\epsilon}{a_+ - a_-}\right)^{n+2} K \geq M,$$

due to the choice of K. By a symmetrical argument, if $y_t^1 \leq -M$, we obtain $y_{t+1}^{n+2} \leq -M$.

We have shown that starting with $z_0 \geq \overline{z}$, and for $t = 1,\ldots,n+2$, the dynamics of y_t amount to a cyclic shift of its sign pattern, while the magnitude of each component of y_t stays above M. After $n+2$ time steps, and since y has dimension $n+2$, the same sign pattern is repeated, and y_{n+2} is again an encoding of $k_1,\ldots,k_{n+2}$. Furthermore, $z_{n+2} \geq \overline{z}$, and the same argument can be repeated. As argued earlier, this establishes that the ν-system is not asymptotically stable.

We have therefore completed a reduction of the 3SAT problem to the problem of interest. The first step in the reduction, as described by Lemma 1, takes polynomial time. The remaining steps (the definition of the matrices A_0, A_1 and the constant g) also take polynomial time. Thus, the overall reduction takes polynomial time and the NP-hardness proof is complete. □

Remarks:

1. Particular choices of nonconstant functions ν lead to particular classes of systems for which asymptotic stability is NP-hard to decide. Consider for example the function

$$\nu(\alpha) = \begin{cases} +1 & \text{when} \quad \alpha \geq 0, \\ -1 & \text{when} \quad \alpha < 0. \end{cases}$$

   This function satisfies the hypothesis of the theorem. After elementary algebraic manipulations we easily obtain:

   Corollary. The problem of deciding, for given matrices $A_+, A_- \in \mathbf{Q}^{n \times n}$ and vector $c \in \mathbf{Q}^n$, whether the system

$$x_{t+1} = \begin{cases} A_+ x_t & \text{when} \quad c^T x_t \geq 0, \\ A_- x_t & \text{when} \quad c^T x_t < 0, \end{cases}$$

   is asymptotically stable, is NP-hard.

2. An interesting corollary of Theorem 1 is obtained by letting ν be a "sigmoidal nonlinearity" of the type used in artificial neural networks. Theorem 1 implies that the stability of recurrent neural networks involving just one sigmoidal nonlinearity is NP-hard to decide.

3. Another interesting corollary is obtained for linear systems controlled by bang-bang controllers. A linear system $x_{t+1} = Ax_t + Bu_t$ controlled by a bang-bang controller of the type

$$u_k = \begin{cases} K_0 x_t & \text{when} \quad y_t \geq 0, \\ K_1 x_t & \text{when} \quad y_t < 0, \end{cases}$$

   leads to a closed-loop system

$$x_k = \begin{cases} (A + BK_0)x_t & \text{when} \quad y_t \geq 0, \\ (A + BK_1)x_t & \text{when} \quad y_t < 0. \end{cases}$$

   From Theorem 1 we see that the stability of such systems is NP-hard to decide.

4. A discrete-time autonomous system $f : \mathbf{R}^n \mapsto \mathbf{R}^n$ is *marginally stable* if the sequences defined by $x_{k+1} = f(x_k)$, $k = 0, 1, \ldots$, remain bounded for all initial states $x_0 \in \mathbf{R}^n$ and it is *locally stable* (asymptotically or marginally) if it is stable (asymptotically or marginally) in some neighborhood of the origin. The proof of NP-hardness of asymptotic global stability can be adapted so as to cover the other three cases in the four possible combinations of local/global asymptotic/marginal stability.

5. Note that we do not know whether the asymptotic stability of ν-systems is *decidable* for any or for some nonconstant function ν. As mentioned earlier, this is related to the decidability of the stability of all possible sequences of products of two matrices, which is an open problem.

3 Controlled systems

A *discrete-time system* is a map $f : \mathbf{R}^n \times \mathbf{R}^m \mapsto \mathbf{R}^n : (x_t, u_t) \mapsto x_{t+1} = f(x_t, u_t)$. Let $x_b, x_e \in \mathbf{R}^n$ (the subscripts b and e stand for $b$eginning and end). The state x_b *can be controlled to* x_e, or, equivalently, x_e *is reachable from* x_b, if there exists some $p \geq 1$ and $u_i \in \mathbf{R}^m$ $(i = 0, \ldots, p-1)$ such that the iterates

$$x_{t+1} = f(x_t, u_t), \qquad t = 0, \ldots, p-1,$$

drive $x_0 = x_b$ to $x_p = x_e$.

A system is *controllable to* x_e if all states can be controlled to x_e, it is *reachable from* x_b if all states can be reached from x_b. In particular, the system is *null-controllable* if all states can be controlled to the origin and it is *null-reachable* if all states can be reached from the origin.

A system is *completely controllable* (or, simply, *controllable*) if all states can be controlled to all states. This notion being symmetric with respect to time, it coincides with the notion of *complete reachability*.

Asymptotic versions of these definitions are also possible by requiring the sequences to converge to the given state rather than reaching it exactly.

For linear systems the notions of complete controllability, null-reachability, and reachability from a state, are all equivalent and can be proved equivalent to the condition that the matrices A and B form a controllable pair (see, e.g., Sontag [21]). When the matrix A is invertible, these notions furthermore coincide with those of null-controllability and of controllability to a state. Controllability of a pair of matrices can be decided in polynomial time using elementary linear algebra algorithms. For general nonlinear systems no such algorithms exist.

We define below a particular family of nonlinear systems which we consider to be the simplest possible controlled nonlinear systems, and also the simplest possible controlled hybrid systems. In Theorem 2, we analyze controllability and reachability of these systems from a computational complexity point of view.

The nth-dimensional *sign system* associated with $A_+, A_0, A_- \in \mathbf{R}^{n \times n}$ and $b, c \in \mathbf{R}^n$ is the system

$$x_{t+1} = A_{\mathrm{sgn}(c^T x_t)} x_t + b u_t, \qquad t = 0, 1, \ldots,$$

where sgn($\cdot$) is the sign function defined by

$$\operatorname{sgn}(x) = \begin{cases} +, & \text{when } x > 0, \\ 0, & \text{when } x = 0, \\ -, & \text{when } x < 0. \end{cases}$$

When the control variables u_i are all zero or when $b = 0$, sign systems degenerate into autonomous systems of the form described in the previous section and for which we have shown that it is NP-hard to check asymptotic stability. It is therefore clear that asymptotic null-controllability is NP-hard to decide for sign systems. We show in Theorem 2 below that null-controllability and reachability are *undecidable* for sign systems. For proving this, we need preliminary results on Post's correspondence problem and on mortality of sets of matrices.

POST'S CORRESPONDENCE PROBLEM.

Instance: A set of pairs of words $\{(U_i, V_i) : i = 1, \ldots, n\}$ over a finite alphabet.

Question: Does there exist a non-empty sequence of indices $i_1, i_2, \ldots, i_k$ where $1 \leq i_j \leq n$, such that $U_{i_1} U_{i_2} \cdots U_{i_k} = V_{i_1} V_{i_2} \cdots V_{i_k}$?

Post's correspondence problem is trivially decidable for one letter alphabets. Furthermore, it is easy to see that the solvability of the problem does not depend on the size of the alphabet, as long as the alphabet contains more than one letter. Post proved that the correspondence problem for an alphabet with more than one letter is undecidable (for a proof of this classical result see, e.g., Hopcroft and Ullman [10]). In a recent contribution Matiyasevich and Sénizergues [12] have improved this result by showing that the problem remains undecidable in the case where there are only seven pairs of words. On the other hand, the problem is known to be decidable for two pairs of words. The limit between decidability/undecidability is somewhere between three and seven pairs; according to Matiyasevich [13] this limit is likely to be equal to three.

Post's correspondence problem can be used to prove a result on mortality of matrices. Let $k \geq 1$. A set $\mathcal{A}$ of square real matrices of the same dimension is *k-mortal* if there exist $A_i \in \mathcal{A}$ $(i = 1, \ldots, k)$ such that $A_k \cdots A_2 A_1 = 0$. The set is *mortal* if it is k-mortal for some finite k. In [16] Paterson uses Post's correspondence problem to show that mortality of integer matrices is undecidable. This result is improved slightly in [2] where the following can be found:

Proposition 1. Mortality of two integer matrices of size $n \times n$ is undecidable for $n = 6(n_p + 1)$ where n_p is any number of pairs of words for which Post's correspondence problem is undecidable.

As mentioned earlier we can take $n_p = 7$, and thus mortality of pairs of 48×48

integer matrices is undecidable. We are now able to prove our theorem.

Theorem 2. Let n_p be any number of pairs of words for which Post's correspondence problem is undecidable (we can take $n_p = 7$).

(a) The problem of deciding, for a given nth-dimensional sign system, whether the system is null-controllable is undecidable when $n \geq 6n_p + 7$.

(b) The problem of deciding, for a given nth-dimensional system and for given states $x_e, x_b \in \mathbf{Q}^n$, whether x_e can be reached from x_b, is undecidable when $n \geq 3n_p + 1$.

Proof.

(a) Let $B_0, B_1 \in \mathbf{Z}^{n \times n}$ be two arbitrary matrices of size $n \times n$. The sign system we construct has a state vector $x_t = (z_t, y_t)$ where z_t is a scalar and y_t is a vector in $\mathbf{R}^n$. Let the vector c in the definition of a sign system be such that $c^T x_t = z_t$ and let $A_- = A_0 = B_0$ and $A_+ = B_1$. We define the dynamics of the sign system by $z_{t+1} = u_t$ and $y_{t+1} = A_{\mathrm{sgn}(c^T x_t)} y_t = A_{\mathrm{sgn}(z_t)} y_t$.

For a given initial state $x_0 \in \mathbf{R}^{n+1}$ and $p \geq 1$, the state x_t is obtained by $x_t = (z_t, y_t)$ with $z_t = u_{t-1}$ and

$$y_t = A_{\mathrm{sgn}(u_{t-1})} \cdots A_{\mathrm{sgn}(u_1)} A_{\mathrm{sgn}(u_0)} A_{\mathrm{sgn}(c^T x_0)} y_0.$$

We claim that the sign system is null-controllable if and only if the matrices B_0, B_1 are mortal.

If the matrices B_0, B_1 are mortal, then the sign system is clearly null-controllable, and so this part is trivial. For the other direction, assume that the sign system is null-controllable and let e_r be the rth unit vector of $\mathbf{R}^n$. Since the system is null controllable, there exists a $k_1 \geq 0$ and a sequence $j_i \in \{-, 0, +\}$, for $i = 1, \ldots, k_1$ such that $A_{j_{k_1}} \cdots A_{j_2} A_{j_1} e_1 = 0$. Let $x_2 = A_{j_{k_1}} \cdots A_{j_2} A_{j_1} e_2$. By using the null-controllability assumption again, we find some $k_2 \geq 0$ and a sequence $j'_i \in \{-, 0, +\}$ for $i = 1, \ldots, k_2$ such that $A_{j'_{k_2}} \cdots A_{j'_2} A_{j'_1} x_2 = 0$. The product $A = A_{j'_{k_2}} \cdots A_{j'_2} A_{j'_1} A_{j_{k_1}} \cdots A_{j_2} A_{j_1}$ is such that $Ae_1 = 0$ and $Ae_2 = 0$. Continuing in the same way for all unit vectors, we eventually obtain a product A of matrices in $\{A_-, A_0, A_+\}$ such that $Ae_r = 0$ for $r = 1, \ldots, n$. This implies that the set $\{A_-, A_0, A_+\}$ is mortal and thus so is the set $\{B_0, B_1\}$.

We have shown that null-controllability of the $(n+1)$th-dimensional sign system is equivalent to mortality of the set $\{B_0, B_1\}$. According to Proposition 1, the latter problem is undecidable when $n \geq 6(n_p + 1)$, hence the result.

(b) Let an instance of Post's correspondence problem be given by the pairs of words $\{(U_i, V_i) : i = 1, \ldots, n\}$ over the alphabet $\{1, 2\}$. We construct a sign system of dimension $(3n + 1)$ and states x_b and x_e such that x_e can be reached from x_b if and only if the correspondence problem has a solution. Our construction is similar to the one given by Paterson in [16].

Let $|a|$ denote the length of the word a. Note that every word U_i or V_i over the alphabet $\{1,2\}$ can also be viewed as a nonnegative integer u_i or v_i, respectively. For each pair (U_i, V_i) we construct a matrix

$$W_i = \begin{pmatrix} q_i & 0 & 0 \\ 0 & s_i & 0 \\ u_i & v_i & 1 \end{pmatrix}$$

were u_i and v_i are as described above, $q_i = 10^{|U_i|}$, and $s_i = 10^{|V_i|}$. The product of the matrices W_i and W_j is given by

$$W_i W_j = \begin{pmatrix} q_i q_j & 0 & 0 \\ 0 & s_i s_j & 0 \\ u_i \oplus u_j & v_i \oplus v_j & 1 \end{pmatrix}$$

were $a \oplus b$ denotes the positive integer resulting from the concatenation of the positive integers a and b. It is therefore clear that the correspondence problem admits a solution if and only if there exist a product $B_k \cdots B_1$ with $B_j \in \mathcal{W} := \{W_i : i = 1, \dots, n\}$ such that

$$10^{-p} B_k \cdots B_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

for some $p \geq 1$ (the integer p is equal to the length of the word resulting from the correspondence). We transform this problem into a reachability problem for sign systems.

Let I_m denote the identity matrix of size m and define

$$V_1 = \text{diag}(W_1, W_2, \dots, W_n),$$

(The reason for the notation V_1 will appear shortly.)

$$S = 10^{-1} I_{3n},$$

and

$$T = \begin{pmatrix} 0 & I_{3(n-1)} \\ I_3 & 0 \end{pmatrix}.$$

All these matrices have size $3n \times 3n$. We define a sign system of dimension $(3n+1)$ by $A_+ = \text{diag}(0, V_1), A_0 = \text{diag}(0, S), A_- = \text{diag}(0, T)$ and $b = c =$

$\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T$. Finally, we define the beginning and end states by

$$x_b = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } x_e = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

and claim that the sign system

$$x_{t+1} = A_{\mathrm{sgn}(c^T x_t)} x_t + b u_t$$

can be driven from x_b to x_e if and only if the correspondence problem has a solution.

For notational convenience, let us partition the state vector x_t by $x_t = (z_t, y_t)$ where z_t is a scalar and y_t is a subvector of dimension $3n$. We use the corresponding decompositions of the beginning and end states $x_b = (z_b, y_b)$ and $x_e = (z_e, y_e)$. The dynamics of z_t is given by $z_0 = 1$ and $z_{t+1} = u_t$. The dynamics of y_t is given by $y_1 = V_1 y_0$ and

$$y_{t+1} = \begin{cases} V_1 y_t & \text{when} \quad u_{t-1} > 0, \\ S y_t & \text{when} \quad u_{t-1} = 0, \\ T y_t & \text{when} \quad u_{t-1} < 0. \end{cases}$$

The matrix S commutes with T and V_1 and so we obtain

$$y_t = S^s V_1^{w_q} T^{t_q} \cdots V_1^{w_1} T^{t_1} V_1 y_0$$

for some $s, t_i, w_i \geq 0$. Notice that $T^n = I_{3n}$ and define

$$V_k = T^{k-1} V_1 T^{n-(k-1)}.$$

We have then

$$V_k = \mathrm{diag}(W_k, W_{k+1}, \ldots, W_n, W_1, \ldots, W_{k-1})$$

for $k = 1 \ldots, n$. Using the property $T^n = I_{3n}$ we arrive, after elementary manipulations, at

$$y_t = S^s T^{t_*} V y_0$$

where V is a nonempty product of matrices V_i and $s, t_* \geq 0$. The matrices V_i are block-diagonal and so the blocks of V are obtained by forming non-empty products of matrices from the set $\mathcal{W}$. We can now conclude. If the Post correspondence problem has a solution, then x_e can be reached from x_b by choosing the control u_i such that $y_t = S^s V y_0$ where the last block in V is constructed from the solution of the correspondence problem and s is equal to the length of the word resulting from the correspondence. Conversely, if $y_e = S^s T^{t_*} V y_b$ for some nonempty product V and $s, t_* \geq 0$ then, since all $3(n-1)$ first components of y_b are equal to zero, and V is block-diagonal, we must have $t_* = kn$ for some $k \in \mathbf{Z}$. But then $y_e = S^s V y_b$ and the correspondence problem has a solution. $\square$

Remarks:

1. In the proof of the first part of the theorem we use matrices and vectors that have integer entries. Therefore null-controllability remains undecidable when matrices and vectors are constrained to have integer entries. For an integer valued sequence, convergence to zero is equivalent to equality with zero after finitely many steps. From this it follows that the asymptotic version of null-controllability is undecidable for sign systems.

2. The class of piecewise linear systems is arguably the smallest possible class of systems that contains the classical linear systems, the finite automata, and that is closed under interconnection of such systems, see Sontag [20]. A sign systems is a piecewise linear system with elementary partitions $c^T x > 0$, $c^T x = 0$ and $c^T x < 0$, and the results stated in Theorem 2 therefore apply to the class of piecewise linear systems.

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