Contributions to the Theory of Ehrhart Polynomials

by

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Abstract

In this thesis, we study the Ehrhart polynomials of different polytopes. In the 1960's Eugène Ehrhart discovered that for any rational d-polytope P, the number of lattice points, $i(P,m)$, in the mth dilated polytope mP is always a quasi-polynomial of degree d in m, whose period divides the least common multiple of the denominators of the coordinates of the vertices of P. In particular, if P is an integral polytope, $i(P,m)$ is a polynomial. Thus, we call $i(P,m)$ the Ehrhart (quasi-)polynomial of P.

In the first part of my thesis, motivated by a conjecture given by De Loera, which gives a simple formula of the Ehrhart polynomial of an integral cyclic polytope, we define a more general family of polytopes, lattice-face polytopes, and show that these polytopes have the same simple form of Ehrhart polynomials. we also give a conjecture which connects our theorem to a well-known fact that the constant term of the Ehrhart polynomial of an integral polytope is 1. In the second part (joint work with Brian Osserman), we use Mochizuki's work in algebraic geometry to obtain identities for the number of lattice points in different polytopes. We also prove that Mochizuki's objects are counted by polynomials in the characteristic of the base field.

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Contents

Chapter I

Introduction

I.1 Outline of the thesis

In the 1960's Eugène Ehrhart [9] discovered that for any rational d-polytope P, the number of lattice points, $i(P, m)$, in the mth dilated polytope mP is always a quasi-polynomial of degree d in m. In particular, $i(P, m)$ is a polynomial when P is an integral polytope. Therefore, we call $i(P, m)$ the Ehrhart (quasi-)polynomial of P. Much work on the theory of Ehrhart polynomials has been done since then. In this thesis, we try two approaches to study the Ehrhart (quasi-)polynomials.

Chapter II is based on the two papers [18, 17]. Motivated by a conjecture given by De Loera [4, Conjecture 1.5] on the Ehrhart polynomial of an integral cyclic polytope, we look for integral polytopes with the same simple form of Ehrhart polynomials. We generalize the family of integral cyclic polytopes to a new family of polytopes, *lattice-face polytopes*, and show that they have the same form of Ehrhart polynomials. The method we use is to first reduce the problem to the simplex case, and then develop a way of decomposing a d-dimensional simplex into $d!$ signed sets. By summing the number of lattice points in these sets (with signs) and using a property of Bernoulli polynomials, we are able to show that the Ehrhart polynomial of a lattice-face polytope has the desired form.

Chapter III is joint work with Brian Osserman [19]. Mochizuki's work on torally indigenous bundles [23] yields combinatorial identities by degenerating to different curves of the same genus. We rephrase these identities in combinatorial language and strengthen them, giving relations between Ehrhart quasi-polynomials of different polytopes. We then apply the theory of Ehrhart quasi-polynomials to conclude that the number of dormant torally indigenous bundles on a general curve of a given type is expressed as a polynomial in the characteristic of the base field. In particular, we conclude the same for the number vector bundles of rank two and trivial determinant whose Frobenius-pullbacks are maximally unstable, as well as self-maps of the projective line with prescribed ramification.

In the next two sections of this chapter, we will give the basic terminology we need for polytopes, and discuss the theory of Ehrhart polynomials in more details.

I.2 Basic definitions related to polytopes

Throughout this thesis, the notation for polytopes mostly follows [32].

Definition I.2.1 ($\mathcal{V}$-representation)**.** A *convex polytope* P in the d-dimensional Euclidean space $\mathbb{R}^d$ is the convex hull of finitely many points $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^d$. In other words,

$$P = \mathrm{conv}(V) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \;:\; \text{all } \lambda_i \geq 0, \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}.$$

We often omit convex and just say polytope. There is an alternative definition of polytopes in terms of halfspaces.

Definition I.2.2 ($\mathcal{H}$-representation)**.** A *convex polytope* $P \subset \mathbb{R}^d$ is a bounded intersection of halfspaces:

$$P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{z}\},$$

for some $A \in \mathbb{R}^{m \times d}, \mathbf{z} \in \mathbb{R}^m$.

The proof [32, Theorem 1.1] of the equivalence between this two definitions is nontrivial. We will not include it here.

The set of all affine combinations of points in some set $S \subset \mathbb{R}^d$ is called the *affine hull* of S, and denoted $\text{aff}(S)$:

$$\text{aff}(S) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n \ : \ v_1, v_2, \ldots, v_n \in S, \text{ all } \lambda_i \in \mathbb{R}, \text{ and } \sum_{i=1}^{n} \lambda_i = 1\}.$$

The *dimension* of a polytope is the dimension of its affine hull. A *d-polytope* is a polytope of dimension d in some $\mathbb{R}^e$ ($e \geq d$).

Definition I.2.3 (Definition 2.1 [32]). Let $P \subset \mathbb{R}^d$ be a convex polytope. A linear inequality $\mathbf{cx} \leq c_0$ is *valid* for P if it is satisfied for all points $\mathbf{x} \in P$. A *face* of P is any set of the form

$$F = P \cap \{\mathbf{x} \in \mathbb{R}^d \ : \ \mathbf{cx} = c_0\},$$

where $\mathbf{cx} \leq c_0$ is a valid inequality for P. The *dimension* of a face is the dimension of its affine hull: $\dim(F) := \dim(\text{aff}(F))$.

The faces of dimension $0, 1, \dim(P) - 2$, and $\dim(P) - 1$ are called *vertices, edges, ridges,* and *facets,* respectively.

It's easy to see that the convex hull of all of the vertices of a convex polytope P is P itself.

Definition I.2.4. Let $P \subset \mathbb{R}^d$ be a convex polytope. The *boundary* of P, denoted by ∂P, is the union of all of the facets of P. The *interior* of P, denoted as $I(P)$, is $P \setminus \partial P$.

A d-dimensional *lattice* $\mathbb{Z}^d = \{\mathbf{x} = (x_1, \ldots, x_d) \mid \forall x_i \in \mathbb{Z}\}$ is the collection of all points with integer coordinates in $\mathbb{R}^d$. Any point in a lattice is called a *lattice point.*

Now we are ready to define two important functions of polytopes.

Definition I.2.5. For any polytope $P \subset \mathbb{R}^d$ and some positive integer $m \in \mathbb{N}$, the *mth dilated polytope* of P is $mP = \{m\mathbf{x} \ : \ \mathbf{x} \in P\}$. We denote by

$$i(m, P) = |mP \cap \mathbb{Z}^d|,$$

and

$$\widehat{i}(P,m) = |I(mP) \cap \mathbb{Z}^d|,$$

the number of lattice points in mP, and the number of lattice points in the interior of mP, respectively.

A *rational* polytope is a convex polytope, the coordinates of whose vertices are all rational and an *integral* polytope is a convex polytope, the coordinates of whose vertices are all integers. In next section, we will discuss Ehrhart's work on $i(P,m)$ and $\widehat{i}(P,m)$ when P is a rational polytope, or an integral polytope.

I.3 Theory of Ehrhart (quasi-)polynomials

We first look at some examples of $i(P,m)$ for different polytopes P.

Example I.3.1 (Example of integral polytopes). (i) When $d = 1$, P is an interval $[a,b]$, where $a,b \in \mathbb{Z}$. Then $i(P,m) = (b-a)m + 1$.

(ii) When $d = 2$, P is an integral polygon. Recall that Pick's theorem states that for any integral polygon P':

$$\text{area}(P') = |I(P') \cap \mathbb{Z}^2| + \frac{1}{2}|\partial(P') \cap \mathbb{Z}^d| - 1.$$

Thus,

$$\begin{aligned} i(P,m) &= \text{area}(mP) + \frac{1}{2}|\partial(mP) \cap \mathbb{Z}^d| + 1 \\ &= \text{area}(P)m^2 + \frac{1}{2}|\partial(P) \cap \mathbb{Z}^d|m + 1 \end{aligned}$$

(iii) For any d, let P be the convex hull of the set $\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_i = 0 \text{ or } 1\}$, i.e. P is the *unit cube* in $\mathbb{R}^d$. Then it is obvious that

$$i(P,m) = (m+1)^d.$$

In the above three examples, we can see that $i(P,m)$ are all polynomials in m. Next, we look at an example of a rational polytope.

Example I.3.2 (An example of a rational polytope). When $P = [\frac{1}{3}, \frac{3}{2}]$,

$$i(P,m) = \begin{cases} \frac{7}{6}m + 1, & \text{if } m \equiv 0 \mod 6 \\ \frac{7}{6}m - \frac{1}{6}, & \text{if } m \equiv 1 \mod 6 \\ \frac{7}{6}m + \frac{2}{3}, & \text{if } m \equiv 2 \mod 6 \\ \frac{7}{6}m + \frac{1}{2}, & \text{if } m \equiv 3 \mod 6 \\ \frac{7}{6}m + \frac{1}{3}, & \text{if } m \equiv 4 \mod 6 \\ \frac{7}{6}m + \frac{1}{6}, & \text{if } m \equiv 5 \mod 6 \end{cases}$$

In this example, $i(P,m)$ are polynomials depending on ($m \mod 6$). We call this kind of function *quasi-polynomial.*

Definition I.3.3. A function $f : \mathbb{N} \to \mathbb{C}$ (or $f : \mathbb{Z} \to \mathbb{C}$) is a *quasi-polynomial* if there exists an integer $N > 0$ and polynomials $f_0, f_1, \ldots, f_{N-1}$ such that

$$f(n) = f_i(n), \text{ if } n \equiv i \mod N.$$

The integer N (which is not unique) will be called a *quasi-period* of f.

The observation we have in examples I.3.1, I.3.2 is not a coincidence. In 1962, Eugène Ehrhart gave his famous theorem on $i(P,m)$ in [9].

Theorem I.3.4. *Given P a rational d-polytope, $i(P,m)$ is always a quasi-polynomial of degree n in m, whose period divides the least common multiple of the denominators of the coordinates of the vertices of P.*

In particular, if P is an integral polytope, $i(P,m)$ is a polynomial.

Therefore, we call $i(P,m)$ the Ehrhart (quasi-)polynomial of P.

Moreover, Ehrhart discovered there is a beautiful reciprocity relation between $i(P,m)$ and $\widehat{i}(P,m)$ and proved it for several special cases [10]. Ian Macdonald found a general proof in 1971 [21].

Theorem I.3.5 (Ehrhart-Macdonald reciprocity). *Let P be a rational d-polytope. Then*

$$i(P,-m) = (-1)^d \hat{i}(P,m).$$

There has been much study on the Ehrhart quasi-polynomials via different approaches. One of them is to determine the coefficients of the Ehrhart polynomial $i(P,m)$, when P is an integral polytope. In fact, the coefficients of the Ehrhart polynomial play an essential role in combinatorics, discrete geometry and geometry of numbers (cf., e.g., [2, 6, 11, 12, 14, 16, 29]). However, although it has been well known for a long time that the leading, second and last coefficients of $i(P,m)$ are the normalized volume of P, one half of the normalized volume of the boundary of P, and 1, there is no known explicit method of describing all the coefficients of Ehrhart polynomials of general integral d-polytopes, when $d \geq 3$.

Another approach people tried is to compute the generating function of the Ehrhart quasi-polynomial of a rational polytope. Barvinok [3] proved that the generating function can be efficiently computed for fixed dimension. Base on his method, De Loera et al. developed an algorithm to calculate the coefficients of the Ehrhart quasi-polynomials (cf. [20]).

Some recent work has also been done on the roots of the Ehrhart polynomials [4, 13] and the quasi-period of the Ehrhart quasi-polynomials [22, 31].

Despite the work mentioned above, basic questions on Ehrhart polynomials remain poorly understood. Questions like whether there are general geometric descriptions of coefficients of Ehrhart polynomials and when two polytopes have the same Ehrhart polynomials are still mysteries.

Chapter II

Ehrhart polynomials of lattice-face polytopes

II.1 Introduction

As we mentioned in the last section, the leading, second and last coefficients of the Ehrhart polynomial of an integral polytope have a geometric interpretation. However, no interpretation of this kind is known for the other coefficients for general polytopes, except for certain special classes of polytopes (e.g. [5, 7, 8, 15, 24, 25, 28]). The purpose of this chapter is to produce a new class of integral polytopes, *lattice-face polytopes*, all coefficients of whose Ehrhart polynomials have geometric meaning.

The results of this chapter are based on the two papers [18, 17]. Instead of presenting results separately, we will combine them together to avoid repetitions of definitions and construction. The motivation for study of this topic is a conjecture given by De Loera in [4, Conjecture 1.5]. It states that given P a d-dimensional integral cyclic polytope, we have

$$i(P, m) = \mathrm{Vol}(mP) + i(\pi(P), m) = \sum_{k=0}^{d} \mathrm{Vol}_k(\pi^{d-k}(P)) m^k, \tag{II.1.1}$$

where π^k is the map which ignores the last k coordinates of a point and $\mathrm{Vol}_k(P)$ is the volume of P in k-dimensional Euclidean space $\mathbb{R}^k$. In other words, the coefficient of m^k in the Ehrhart polynomial $i(P, m)$ is the volume of $\pi^{n-k}(P)$.

We examine cyclic polytopes and find special properties which we suspect is the important criteria that make the Ehrhart polynomial of a cyclic polytope have such a simple form. Therefore, we define a more general family of integral polytopes, *lattice-face polytopes*, and our goal becomes to prove that the Ehrhart polynomial of a lattice-face polytope has the form of (II.1.1). We use a standard triangulation decomposition of polytopes, and careful counting of lattice points to reduce problem to the simplex case.

We then develop a way of decomposing any d-dimensional simplex in *general position* into $d!$ signed sets, each of which corresponds to a permutation in the symmetric group $\mathfrak{S}_d$. When applying this decomposition to cyclic polytopes (which recovers the decomposition we built in [18]), the sign of each of these $d!$ signed sets has the same sign as the corresponding permutation. By summing the number of lattice points in these sets (with signs), we show (Theorem II.3.2) that the Ehrhart polynomial of a cyclic polytope is in the form of (II.1.1).

However, when we apply the decomposition to an arbitrary lattice-face polytope, the sign of each signed set is not necessarily the same as the corresponding permutation. Thus, the situation is more complicated than the case for cyclic polytopes. We show that the number of lattice points is given by a formula (II.7.13) involving Bernoulli polynomials and signs of permutations, and then we analyze this formula further ((II.8.2), whose proof is relegated to Appendix A) to derive Theorem II.3.10, relating the lattice points to the volume. Theorem II.3.10, together with some simple observations in section II.3 and II.4, implies Theorem II.3.8.

In the last section, we give several examples to illustrate the methods we describe in the proofs. We also give a conjecture which connects the main theorem to the well-known fact that the constant term of the Ehrhart polynomial of an integral polytope is 1.

II.2 Preliminaries

We first recall some definitions we defined in Chapter I and give some more notation.

All polytopes we will consider are full-dimensional, so for any convex polytope P, we use d to denote both the dimension of the ambient space $\mathbb{R}^d$ and the dimension of P. Recall that we call a d-dimensional polytope a d-polytope. Also, we use ∂P and $I(P)$ to denote the boundary and the interior of P, respectively.

A *d-simplex* is a polytope given as the convex hull of $d+1$ affinely independent points in $\mathbb{R}^d$.

For any set S, we use $\text{conv}(S)$ to denote the convex hull of all of points in S.

The projection $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is the map that forgets the last coordinate. For any set $S \subset \mathbb{R}^d$ and any point $y \in \mathbb{R}^{d-1}$, let $\rho(y, S) = \pi^{-1}(y) \cap S$ be the intersection of S with the inverse image of y under π. Let $p(y, S)$ and $n(y, S)$ be the point in $\rho(y, S)$ with the largest and smallest last coordinate, respectively. If $\rho(y, S)$ is the empty set, i.e., $y \notin \pi(S)$, then let $p(y, S)$ and $n(y, S)$ be empty sets as well. Clearly, if S is a d-polytope, $p(y, S)$ and $n(y, S)$ are on the boundary of S. Also, we let $\rho^+(y, S) = \rho(y, S) \setminus n(y, S)$, and for any $T \subset \mathbb{R}^{d-1}$, $\rho^+(T, S) = \cup_{y \in T} \rho^+(y, S)$.

Definition II.2.1. Define $PB(P) = \bigcup_{y \in \pi(P)} p(y, P)$ to be the *positive boundary* of P; $NB(P) = \cup_{y \in \pi(P)} n(y, P)$ to be the *negative boundary* of P and $\Omega(P) = P \setminus NB(P) = \rho^+(\pi(P), P) = \cup_{y \in \pi(P)} \rho^+(y, P)$ to be the *nonnegative part* of P.

Definition II.2.2. For any facet F of P, if F has an interior point in the positive boundary of P, then we call F a *positive facet* of P and define the sign of F as $+1$: $\text{sign}(F) = +1$. Similarly, we can define the *negative facets* of P with associated sign -1. For the facets that are neither positive nor negative, we call them *neutral facets* and define the sign as 0.

It's easy to see that $F \subset PB(P)$ if F is a positive facet and $F \subset NB(P)$ if F is a negative facet.

We write $P = \bigsqcup_{i=1}^k P_i$ if $P = \bigcup_{i=1}^k P_i$ and for any $i \neq j$, $P_i \cap P_j$ is contained in their boundaries. If $F_1, F_2, \ldots, F_\ell$ are all the positive facets of P and $F_{\ell+1}, \ldots, F_k$ are all the negative facets of P, then

$$\pi(P) = \bigsqcup_{i=1}^{\ell} \pi(F_i) = \bigsqcup_{i=\ell+1}^{k} \pi(F_i).$$

Because the usual set union and set minus operation do not count the number of occurrences of an element, which is important in our paper, from now on we will consider any polytopes or sets as *multisets* which allow *negative multiplicities.* In other words, we consider any element of a multiset as a pair $(\mathbf{x}, m)$, where m is the multiplicity of element $\mathbf{x}$. Then for any multisets M_1, M_2 and any integers m, n and i, we define the following operators:

a) Scalar product: $iM_1 = i \cdot M_1 = \{(\mathbf{x}, im) \mid (\mathbf{x}, m) \in M_1\}$.

b) Addition: $M_1 \oplus M_2 = \{(\mathbf{x}, m+n) \mid (\mathbf{x}, m) \in M_1, (\mathbf{x}, n) \in M_2\}$.

c) Subtraction: $M_1 \ominus M_2 = M_1 \oplus ((-1) \cdot M_2)$.

It's clear that the following holds:

Lemma II.2.3. *For any polytope $P \subset \mathbb{R}^d$, $\forall R_1, \dots, R_k \subset \mathbb{R}^{d-1}$, $\forall i_1, \dots, i_k \in \mathbb{Z}$:*

$$\rho^+\left(\bigoplus_{j=1}^{k} i_j R_j,\ P\right) = \bigoplus_{j=1}^{k} i_j \rho^+(R_j,\ P).$$

Definition II.2.4. We say a set S has *weight* w, if each of its elements has multiplicity either 0 or w. And S is a *signed* set if it has weight 1 or -1.

Let P be a convex polytope. For any y an interior point of $\pi(P)$, since π is a continuous open map, the inverse image of y contains an interior point of P. Thus $\pi^{-1}(y)$ intersects the boundary of P exactly twice. For any y a boundary point of $\pi(P)$, again because π is an open map, we have that $\rho(y, P) \subset \partial P$, so $\rho(y, P) = \pi^{-1}(y) \cap \partial P$ is either one point or a line segment. We are only interested in polytopes P where $\rho(y, P)$ always has only one point for a boundary point y.

Lemma II.2.5. *If a polytope P satisfies:*

$$|\rho(y, P)| = 1, \forall y \in \partial\pi(P), \tag{II.2.6}$$

then P has the following properties:

(i) For any $y \in I(\pi(P))$, $\pi^{-1}(y) \cap \partial P = \{p(y, P), n(y, P)\}$.

(ii) For any $y \in \partial\pi(P)$, $\pi^{-1}(y) \cap \partial P = \rho(y, P) = p(y, P) = n(y, P)$, so $\rho^+(y, P) = \emptyset$.

(iii) Let R be a region containing $I(\pi(P))$. Then

$$\Omega(P) = \rho^+(R, P) = \bigoplus_{y \in R} \rho^+(y, P).$$

(iv) If $P = \bigsqcup_{i=1}^k P_i$, where the P_i's all satisfy (II.2.6), then $\Omega(P) = \bigoplus_{i=1}^k \Omega(P_i)$.

(v) The set of facets of P are partitioned into the set of positive facets and the set of negative facets, i.e., there is no neutral facets.

(vi) π gives a bijection between $PB(P) \cap NB(P)$ and $\partial\pi(P)$.

The proof of this lemma is straightforward, so we won't include it here.

The main purpose of this chapter is to discuss the number of lattice points in a polytope. Therefore, for simplicity, for any set $S \in \mathbb{R}^d$, we denote by

$$\mathcal{L}(S) = S \cap \mathbb{Z}^d$$

the set of lattice points in S. It's not hard to see that $\mathcal{L}$ commutes with some of the operations we defined earlier, e.g. ρ, ρ^+, Ω.

II.3 Cyclic polytopes vs. lattice-face polytopes

In this section, we will introduce the definitions of cyclic polytopes and lattice-face polytopes, and also describe the main theorems of this chapter.

Definition II.3.1. The *moment curve* in $\mathbb{R}^d$ is defined by

$$\nu_d : \mathbb{R} \to \mathbb{R}^d, t \mapsto \nu_d(t) = \left(t, t^2, \ldots, t^d\right).$$

Let $T = \{t_1, \ldots, t_n\}_<$ be a linearly ordered set. Then the *cyclic polytope* $C_d(T) = C_d(t_1, \ldots, t_n)$ is the convex hull $\mathrm{conv}\{v_d(t_1), v_d(t_2), \ldots, v_d(t_n)\}$ of $n > d$ distinct points $\nu_d(t_i), 1 \leq i \leq n$, on the moment curve.

The first important theorem in this chapter is the one conjectured in [4, Conjecture 1.5]:

Theorem II.3.2. *For any integral cyclic polytope $C_d(T)$, (i.e., when T is an integral linearly ordered set)*

$$i(C_d(T), m) = \mathrm{Vol}(mC_d(T)) + i(C_{d-1}(T), m).$$

Hence,

$$i(C_d(T), m) = \sum_{k=0}^{d} \mathrm{Vol}_k(mC_k(T)) = \sum_{k=0}^{d} \mathrm{Vol}_k(C_k(T))m^k,$$

where $\mathrm{Vol}_k(mC_k(T))$ *is the volume of $mC_k(T)$ in k-dimensional space, and we let* $\mathrm{Vol}_0(mC_0(T)) = 1$.

Noting that $C_k(T) = \pi^{d-k}(C_d(T))$, this theorem is equivalent to saying that the Ehrhart polynomial of an integral cyclic polytope is in the form of (II.1.1).

In [4, Lemma 5.1], the authors showed that the inverse image under π of a lattice point $y \in \mathcal{L}(C_{d-1}(T))$ is a line that intersects the boundary of $C_d(T)$ at integral points, and by using this lemma, they proved Theorem II.3.2 when $d \leq 2$. In fact, their proof of the lemma says more than what was stated. We restate their lemma and include one additional fact:

Lemma II.3.3. *Let $T = \{t_1, t_2, \ldots, t_n\}_<$ be an integral linearly ordered set. When $d = 1$, $C_d(T)$ is just an integral 1-polytope.*

For $d \geq 2$, let V be the vertex set of $C_d(T)$. For any d-subset U of V, let H_U be the affine space spanned by U. Then

a) $\pi(\mathrm{conv}(U))$ *is an integral cyclic polytope, and*

b) $\pi(\mathcal{L}(H_U)) = \mathbb{Z}^{d-1}$. *In other words, after dropping the last coordinate of the lattice of* H_U, *we get the* $(d-1)$*-dimensional lattice.*

Proof. When $d = 1$, $C_d = [t_1, t_n]$ is an integral interval.

a) is clearly true and b) follows the proof of [4, Lemma 5.1]. □

We suspect that a) and especially b), are the essential properties to make the Ehrhart polynomial of an integral cyclic polytope have such a simple form. Therefore, we define the following new family of polytopes.

Definition II.3.4. We define *lattice-face* polytopes recursively. We call a one dimensional polytope a *lattice-face* polytope if it is integral.

For $d \geq 2$, we call a d-dimensional polytope P with vertex set V a *lattice-face* polytope if for any d-subset $U \subset V$,

a) $\pi(\mathrm{conv}(U))$ is a lattice-face polytope, and

b) $\pi(\mathcal{L}(H_U)) = \mathbb{Z}^{d-1}$, where H_U is the affine space spanned by U.

By Lemma II.3.3, any integral cyclic polytope is a lattice-face polytope. Hence, we consider the family of lattice-face polytopes as a generalization of the family of cyclic polytopes.

To understand the definition, let's look at examples of 2-polytopes.

Example II.3.5. Let P_1 be the polytope with vertices $v_1 = (0,0), v_2 = (2,0)$ and $v_3 = (2,1)$. Clearly, for any 2-subset U, condition *a)* is always satisfied. When $U = \{v_1, v_2\}$, H_U is $\{(x,0) \mid x \in \mathbb{R}\}$. So $\pi(\mathcal{L}(H_U)) = \mathbb{Z}$, i.e., *b)* holds. When $U = \{v_1, v_3\}$, H_U is $\{(x,y) \mid x = 2y\}$. Then $\mathcal{L}(H_U) = \{(2y,y) \mid y \in \mathbb{Z}\} \Rightarrow \pi(\mathcal{L}(H_U)) = 2\mathbb{Z} \neq \mathbb{Z}$. When $U = \{v_2, v_3\}$, H_U is $\{(2,y) \mid y \in \mathbb{R}\}$. Then $\pi(\mathcal{L}(H_U)) = \{2\} \neq \mathbb{Z}$. Therefore, P_1 is not a lattice-face polytope.

Let P_2 be the polytope with vertices $(0,0), (1,1)$ and $(2,0)$. One can check that P_2 is a lattice-face polytope.

The following lemma gives some properties of a lattice-face polytope.

Lemma II.3.6. *Let P be a lattice-face d-polytope with vertex set V, then we have:*

(i) $\pi(P)$ is a lattice-face $(d-1)$-polytope.

(ii) mP is a lattice-face d-polytope, for any positive integer m.

(iii) π induces a bijection between $\mathcal{L}(NB(P))$ (or $\mathcal{L}(PB(P))$) and $\mathcal{L}(\pi(P))$.

(iv) $\pi(\mathcal{L}(P)) = \mathcal{L}(\pi(P))$.

(v) Any d-subset U of V forms a $(d-1)$-simplex. Thus $\pi(\text{conv}(U))$ is a $(d-1)$-simplex.

(vi) Let H be the affine space spanned by some d-subset of V. Then for any lattice point $y \in \mathbb{Z}^{d-1}$, we have that $\rho(y, H)$ is a lattice point.

(vii) P is an integral polytope.

Proof. (i), (ii), (v) and (vi) can be checked directly from the conditions *a)* and *b)* of the definition. (iii) and (iv) both follow from (vi). We prove (vii) by induction on d.

Any 1-dimensional lattice-face polytope is integral by definition.

For $d \geq 2$, suppose any $(d-1)$ dimensional lattice-face polytope is an integral polytope. Let P be a d dimensional lattice-face polytope with vertex set V. For any vertex $v_0 \in V$, let U be a subset of V that contains v_0. Let $U = \{v_0, v_1, \ldots, v_{d-1}\}$. We know that $P' = \pi(\text{conv}(U))$ is a lattice-face $(d-1)$-simplex with vertices $\{\pi(v_0), \ldots, \pi(v_{d-1})\}$. Thus, by the induction hypothesis, P' is an integral polytope. In particular, $\pi(v_0)$ is a lattice point. Therefore, $v_0 = \rho(\pi(v_0), H_U)$ is a lattice point. □

Remark II.3.7. One sees that condition b) in the definition of lattice-face polytopes is equivalent to (vi).

It turns out that our guess regarding the importance of the properties of Lemma II.3.3 is correct. The Ehrhart polynomial of a lattice-face polytope is indeed in the form of (II.1.1). This is the main theorem of this chapter.

Theorem II.3.8. *Let P be a lattice-face d-polytope, then*

$$i(P,m) = \mathrm{Vol}(mP) + i(\pi(P),m) = \sum_{k=0}^{d} \mathrm{Vol}_k(\pi^{(d-k)}(P))m^k. \tag{II.3.9}$$

Although Theorem II.3.2 follows from Theorem II.3.8, we will prove Theorem II.3.2 first in Section II.6, since the proof for Theorem II.3.2 is simpler and more elegant. However, we will continue making definitions for both cases before applying them to cyclic polytopes separately.

By Lemma II.3.6/(iii), we have that

$$i(P,m) = |\mathcal{L}(\Omega(mP))| + i(\pi(P),m).$$

Therefore, by Lemma II.3.6/(i),(ii), to prove Theorem II.3.8, it is sufficient to prove the following theorem:

Theorem II.3.10. *For any P a lattice-face polytope,*

$$|\mathcal{L}(\Omega(P))| = \mathrm{Vol}(P).$$

Note that when P is an integral cyclic polytope, although Lemma II.3.6/(i), (ii), (iii) are all satisfied, mP is not a cyclic polytope. Thus, Theorem II.3.2 is equivalent to the following:

Theorem II.3.11. *For any integral cyclic polytope $C_d(T)$, and any positive integer m,*

$$\mathcal{L}(\Omega(mC_d(T))) = |\mathrm{Vol}(mC_d(T))|.$$

Remark II.3.12. We have an alternative definition of lattice-face polytopes, which is equivalent to Definition II.3.4. Indeed, a d-polytope on a vertex set V is a lattice-face polytope if and only if for all k with $0 \leq k \leq d-1$,

$$\text{for any } (k+1)\text{-subset } U \subset V,\ \pi^{d-k}(\mathcal{L}(H_U)) = \mathbb{Z}^k, \tag{II.3.13}$$

where H_U is the affine space spanned by U. In other words, after dropping the last $d-k$ coordinates of the lattice of H_U, we get the k-dimensional lattice.

II.4 A signed decomposition of the nonnegative part of a simplex in general position

The volume of a polytope is not very hard to characterize. So our main problem is to find a way to describe the number of lattice points in the nonnegative part of a lattice-face polytope. We are going to do this via a signed decomposition.

II.4.1 Polytopes in general position

For the decomposition, we will work with a more general type of polytope (which contains the family of lattice-face polytopes).

Definition II.4.1. We say that a d-polytope P with vertex set V is in *general position* if for any $k : 0 \leq k \leq d-1$, and any $(k+1)$-subset $U \subset V$, $\pi^{d-k}(\text{conv}(U))$ is a k-simplex, where $\text{conv}(U)$ is the convex hull of all of points in U.

By the alternative definition of lattice polytopes in Remark II.3.12, it's easy to see that a lattice-face polytope is a polytope in general position. Therefore, the following discussion can be applied to lattice-face polytopes.

The following lemma states some properties of a polytope in general position. The proof is omitted.

Lemma II.4.2. *Given a d-polytope P in general position with vertex set V, then*

(i) P satisfies (II.2.6).

(ii) $\pi(P)$ is a $(d-1)$-polytope in general position.

(iii) For any nonempty subset U of V, let $Q = \text{conv}(U)$. If U is has dimension $k (0 \leq k \leq d)$, then $\pi^{d-k}(Q)$ is a k-polytope in general position. In particular, for any facet F of P, $\pi(F)$ is a $(d-1)$-polytope in general position.

(iv) *For any triangulation of* $P = \bigsqcup_{i=1}^{k} P_i$ *without introducing new vertices,* $\Omega(P) = \bigoplus_{i=1}^{k} \Omega(P_i)$. *Thus,* $\mathcal{L}(\Omega(P)) = \bigoplus_{i=1}^{k} \mathcal{L}(\Omega(P_i))$.

(v) *If* $F_1, F_2, \ldots, F_\ell$ *are all the positive facets of* P *and* $F_{\ell+1}, \ldots, F_k$ *are all the negative facets of* P*, then* $\Omega(\pi(P)) = \bigoplus_{i=1}^{\ell} \Omega(\pi(F_i)) = \bigoplus_{i=\ell+1}^{k} \Omega(\pi(F_i))$.

(vi) *For any hyperplane* H *determined by one facet of* P *and any* $y \in \mathbb{R}^{d-1}$, $\rho(y, H)$ *is one point.*

(vii) *For any* $k : 0 \leq k \leq d-1$*, any* $(k+1)$*-subset* U *of* V*, any* $y_1, \ldots, y_k \in \mathbb{R}$*, there exists a unique point* $w \in \mathbb{R}^d$*, such that the first* k *coordinates of* w *are* $y_1, \ldots, y_k$ *and* w *is affinely dependent with the points in* U*.*

Remark II.4.3. By (iv), the problem of counting number of lattice points in a polytope in general position is reduced to that of counting lattice points in a simplex in general position. In particular, together with the fact that $\mathrm{Vol}(\bigsqcup_{i=1}^{k} P_i) = \sum_{i=1}^{k} \mathrm{Vol}(P_i)$, to prove Theorem II.3.10 and Theorem II.3.11, it is sufficient to prove the case when P is a lattice-face simplex and an integral cyclic polytope, respectively.

Therefore, we will only construct our decomposition in the case of simplices in general position. However, before the construction, we need one more proposition about the nonnegative part of a polytope in general position.

Proposition II.4.4. *Let* P *be a d-polytope in general position with facets* $F_1, F_2 \ldots F_k$*. Let* H *be the hyperplane determined by* F_k*. For* $i : 1 \leq i \leq k$*, let* $F_i' = \pi^{-1}(\pi(F_i)) \cap H$ *and* $Q_i = \mathrm{conv}(F_i \cup F_i')$*. Then*

$$\Omega(P) = -\,\mathrm{sign}(F_k) \bigoplus_{i=1}^{k-1} \mathrm{sign}(F_i)\rho^+(\Omega(\pi(F_i)), Q_i). \tag{II.4.5}$$

Proof. We are going to just prove the case when F_k is a negative facet; for the other case we can prove it analogously. Suppose $F_1, F_2, \ldots, F_\ell$ are positive facets and $F_{\ell+1}, \ldots, F_k$ are negative facets.

A special case of Lemma II.2.5/(iii) is when $R = \Omega(\pi(P))$, so we have

$$\Omega(P) = \rho^+(\Omega(\pi(P)), P) = \bigoplus_{y \in \Omega(\pi(P))} \rho^+(y, P).$$

Now for any points a and b, we use $(a, b]$ to denote the half-open line segment between a and b. Then, $\rho^+(y, P) = (n(y, P), p(y, P)] = (\rho(y, H), p(y, P)] \ominus (\rho(y, H), n(y, P)]$. Therefore,

$$\begin{aligned}\Omega(P) &= \bigoplus_{y \in \Omega(\pi(P))} ((\rho(y, H), p(y, P)] \ominus (\rho(y, H), n(y, P)]) \\ &= \left(\bigoplus_{y \in \Omega(\pi(P))} (\rho(y, H), p(y, P)] \right) \oplus \left(\bigoplus_{y \in \Omega(\pi(P))} (-1) \cdot (\rho(y, H), n(y, P)] \right).\end{aligned}$$

By Lemma II.4.2/(v), we have $\Omega(\pi(P)) = \bigoplus_{i=1}^{\ell} \Omega(\pi(F_i))$. Therefore,

$$\begin{aligned}\bigoplus_{y \in \Omega(\pi(P))} (\rho(y, H), p(y, P)] &= \bigoplus_{i=1}^{\ell} \bigoplus_{y \in \Omega(\pi(F_i))} (\rho(y, H), p(y, P)] \\ &= \bigoplus_{i=1}^{\ell} \bigoplus_{y \in \Omega(\pi(F_i))} (\rho(y, F_i'), \rho(y, F_i)] \\ &= \bigoplus_{i=1}^{\ell} \rho^+(\Omega(\pi(F_i)), Q_i).\end{aligned}$$

Similarly, we will have

$$\bigoplus_{y \in \Omega(\pi(P))} (-1) \cdot (\rho(y, H), n(y, P)] = \bigoplus_{i=\ell+1}^{k} (-1)\rho^+(\Omega(\pi(F_i)), Q_i).$$

Note that $\rho^+(\Omega(\pi(F_k)), Q_k)$ is the empty set. Thus, putting everything together, we get (II.4.5). □

Now, we can use this proposition to inductively construct a decomposition of the nonnegative part $\Omega(P)$ of a d-simplex P in general position into $d!$ signed sets.

Decomposition of $\Omega(P)$:

- If $d = 1$, we do nothing: $\Omega(P) = \Omega(P)$.

- If $d \geq 2$, then by applying Proposition II.4.4 to P and letting $k = d + 1$, we have

$$\Omega(P) = -\operatorname{sign}(F_{d+1}) \bigoplus_{i=1}^{d} \operatorname{sign}(F_i)\rho^+(\Omega(\pi(F_i)), Q_i). \qquad \text{(II.4.6)}$$

  However, by Lemma II.4.2/(iii), each $\pi(F_i)$ is a $(d-1)$-simplex in general position. By the induction hypothesis, $\Omega(\pi(F_i)) = \bigoplus_{j=1}^{(d-1)!} S_{i,j}$, where $S_{i,j}$'s are signed sets.

$$\rho^+(\Omega(\pi(F_i)), Q_i) = \rho^+(\bigoplus_{j=1}^{(d-1)!} S_{i,j}, Q_i) = \bigoplus_{j=1}^{(d-1)!} \rho^+(S_{i,j}, Q_i).$$

  Since each $\rho^+(S_{i,j}, Q_i)$ is a signed set, we have decomposed $\Omega(P)$ into $d!$ signed sets.

Now we know that we can decompose $\Omega(P)$ into $d!$ signed sets. But we still need to figure out what these sets are and which signs they have. In the next subsection, we are going to discuss the sign of a facet of a d-simplex, which is going to help us determine the signs in our decomposition.

II.4.2 The sign of a facet of a d-simplex

From now on, we will always use the following setup for a d-simplex unless otherwise stated:

Suppose P is a d-simplex in general position with vertex set $V = \{v_1, v_2, \ldots, v_{d+1}\}$, where the coordinates of v_i are $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,d})$.

For any i, we denote by F_i the facet determined by vertices in $V \setminus \{v_i\}$ and H_i the hyperplane determined by F_i.

For any $\sigma \in \mathfrak{S}_d$ and $k : 1 \leq k \leq d$, we define matrices $X_V(\sigma, k)$ and $Y_V(\sigma, k)$ to be the matrices

$$X_V(\sigma, k) = \begin{pmatrix} 1 & x_{\sigma(1),1} & x_{\sigma(1),2} & \cdots & x_{\sigma(1),k} \\ 1 & x_{\sigma(2),1} & x_{\sigma(2),2} & \cdots & x_{\sigma(2),k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{\sigma(k),1} & x_{\sigma(k),2} & \cdots & x_{\sigma(k),k} \\ 1 & x_{d+1,1} & x_{d+1,2} & \cdots & x_{d+1,k} \end{pmatrix}_{(k+1)\times(k+1)},$$

$$Y_V(\sigma, k) = \begin{pmatrix} 1 & x_{\sigma(1),1} & x_{\sigma(1),2} & \cdots & x_{\sigma(1),k-1} \\ 1 & x_{\sigma(2),1} & x_{\sigma(2),2} & \cdots & x_{\sigma(2),k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{\sigma(k),1} & x_{\sigma(k),2} & \cdots & x_{\sigma(k),k-1} \end{pmatrix}_{k\times k}.$$

We also define $z_V(\sigma, k)$ to be

$$z_V(\sigma, k) = \det(X_V(\sigma, k)) / \det(Y_V(\sigma, k)),$$

where $\det(M)$ is the determinant of a matrix M.

We often omit the subscript V for $X_V(\sigma, k)$, $Y_V(\sigma, k)$ and $z_V(\sigma, k)$ if there is no confusion.

Now we can determine the sign of a facet F_i of P by looking at the determinants of these matrices, denoting by $\text{sign}(x)$ the usual definition of sign of a real number x.

Lemma II.4.7. *We have*

(i) $\forall i : 1 \leq i \leq d$ *and* $\forall \sigma \in \mathfrak{S}_d$ *with* $\sigma(d) = i$,

$$\text{sign}(F_i) = \text{sign}(\det(X(\sigma, d)) / \det(X(\sigma, d-1))). \tag{II.4.8}$$

(ii) When $i = d + 1$ *and for* $\forall \sigma \in \mathfrak{S}_d$,

$$\text{sign}(F_{d+1}) = -\,\text{sign}(\det(X(\sigma, d)) / \det(Y(\sigma, d))) = -\,\text{sign}(z(\sigma, d)). \tag{II.4.9}$$

Proof. For any $i : 1 \leq i \leq d+1$, let $v_i' = \rho(\pi(v_i), H_i)$, i.e. v_i' is the unique point of the hyperplane spanned by F_i which has the same coordinates as v_i except for the last one. Suppose the coordinates of v_i' are $(x_{i,1}, \ldots, x_{i,d-1}, x'_{i,d})$. Then F_i is a positive facet if and only if $x_{i,d} < x'_{i,d}$. Therefore,

$$\operatorname{sign}(F_i) = -\operatorname{sign}(x_{i,d} - x'_{i,d}).$$

$\forall i : 1 \leq i \leq d$ and $\forall \sigma \in \mathfrak{S}_d$ with $\sigma(d) = i$, because v_i' is in the hyperplane determined by F_i, we have that

$$\det\left(\begin{pmatrix} 1 & x_{\sigma(1),1} & \cdots & x_{\sigma(1),d-1} & x_{\sigma(1),d} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_{\sigma(d-1),1} & \cdots & x_{\sigma(d-1),d-1} & x_{\sigma(d-1),d} \\ 1 & x_{\sigma(d),1} & \cdots & x_{\sigma(d),d-1} & x'_{\sigma(d),d} \\ 1 & x_{d+1,1} & \cdots & x_{d+1,d-1} & x_{d+1,d} \end{pmatrix}\right) = 0.$$

Therefore,

$$\det(X(\sigma, d)) = (-1)^{2d+1}(x_{i,d} - x'_{i,d}) \det(X(\sigma, d-1)).$$

Thus,

$$\operatorname{sign}(\det(X(\sigma, d))/\det(X(\sigma, d-1))) = -\operatorname{sign}(x_{i,d} - x'_{i,d}) = \operatorname{sign}(F_i).$$

We can similarly prove the formula for $i = d + 1$. □

II.4.3 Decomposition formulas

The following theorem describes the signed sets in our decomposition.

Theorem II.4.10. *Let P be a d-simplex in general position with vertex set $V = \{v_1, v_2, \ldots, v_{d+1}\}$, where the coordinates of v_i are $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,d})$. For any $\sigma \in \mathfrak{S}_d$, and $k : 0 \leq k \leq d-1$, let $v_{\sigma,k}$ be the point with first k coordinates the same as v_{d+1} and affinely dependent with $v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(k)}, v_{\sigma(k+1)}$. (By Lemma II.4.2/*(vii)*, we know that there exists one and only one such point.) We also let*

$v_{\sigma,d} = v_{d+1}$. *Then*

$$\Omega(P) = \bigoplus_{\sigma \in \mathfrak{S}_d} \operatorname{sign}(\sigma, P) S_\sigma, \tag{II.4.11}$$

where

$$\operatorname{sign}(\sigma, P) = \operatorname{sign}(\det(X(\sigma, d))) \operatorname{sign}\left(\prod_{i=1}^{d} z(\sigma, i)\right), \tag{II.4.12}$$

and

$$S_\sigma = \{\mathbf{s} \in \mathbb{R}^d \mid \pi^{d-k}(\mathbf{s}) \in \Omega(\pi^{d-k}(\operatorname{conv}(\{v_{\sigma,0}, \ldots, v_{\sigma,k}\})))\forall 1 \leq k \leq d\} \tag{II.4.13}$$

is a set of weight 1, *i.e. a regular set.*

Hence,

$$\mathcal{L}(\Omega(P)) = \bigoplus_{\sigma \in \mathfrak{S}_d} \operatorname{sign}(\sigma, P)\mathcal{L}(S_\sigma).$$

Proof. We prove it by induction on d.

When $d = 1$, the only permutation $\sigma \in \mathfrak{S}_1$ is the identity permutation $\mathbf{1}$. One can check that $\operatorname{sign}(\mathbf{1}, P) = 1$ and $S_\mathbf{1} = \Omega(\operatorname{conv}(v_1, v_2))$. Thus (II.4.11) holds.

Assuming (II.4.11) holds for $d = d_0 \geq 1$, we consider for $d = d_0 + 1$. For any $i : 1 \leq i \leq d$, $\pi(F_i)$ is a $(d-1)$-simplex in general position with vertex set $W = \{w_1, \ldots, w_d\}$, where $w_j = \begin{cases} \pi(v_j), & j < i, \\ \pi(v_{j+1}), & j \geq i. \end{cases}$ Therefore, by the induction hypothesis,

$$\Omega(\pi(F_i)) = \bigoplus_{\varsigma \in \mathfrak{S}_{d-1}} \operatorname{sign}(\varsigma, \pi(F_i)) S'_\varsigma, \tag{II.4.14}$$

where

$$\operatorname{sign}(\varsigma, \pi(F_i)) = \operatorname{sign}(\det(X_W(\varsigma, d-1))) \prod_{i=1}^{d-1} \operatorname{sign}(z_W(\varsigma, i)),$$

$$S'_\varsigma = \{\mathbf{s} \in \mathbb{R}^{d-1} \mid \pi^{d-1-k}(\mathbf{s}) \in \Omega(\pi^{d-1-k}(\operatorname{conv}(\{w_{\varsigma,0}, \ldots, w_{\varsigma,k}\})))\forall 1 \leq k \leq d-1\}.$$

For any $\varsigma \in \mathfrak{S}_{d-1}$, if we let $\sigma \in \mathfrak{S}_d$ with $\sigma(j) = \begin{cases} i, & j = d, \\ \varsigma(j), & \varsigma(j) < i, \\ \varsigma(j) + 1, & \varsigma(j) \geq i, \end{cases}$ then this gives a bijection between $\varsigma \in \mathfrak{S}_{d-1}$ and $\sigma \in \mathfrak{S}_d$ with $\sigma(d) = i$. In particular, for any $j : 1 \leq j \leq d-1$, $w_{\varsigma(j)} = \pi(v_{\sigma(j)})$. Hence,

$$\text{sign}(\varsigma, \pi(F_i)) = \text{sign}(\det(X(\sigma, d-1))) \prod_{i=1}^{d-1} \text{sign}(z(\sigma, i)).$$

Note that $w_{\varsigma,d-1} = w_d = \pi(v_{d+1}) = \pi(v_{\sigma,d-1})$, so

$$S'_\varsigma = \{\mathbf{s} \in \mathbb{R}^{d-1} \mid \pi^{d-1-k}(\mathbf{s}) \in \Omega(\pi^{d-k}(\text{conv}(\{v_{\sigma,0}, \dots, v_{\sigma,k}\})))\forall 1 \leq k \leq d-1\}.$$

One can check that $F'_i = \pi^{-1}(\pi(F_i) \cap H_{d+1} = \text{conv}(\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_d, v_{\sigma,d-1}\})$ and $Q_i = \text{conv}(F_i \cup F'_i) = \text{conv}(V \cup \{v_{\sigma,d-1}\} \setminus \{v_i\})$. Hence,

$$\rho^+(S'_\varsigma, Q_i) = \{\mathbf{s} \in \mathbb{R}^d \mid \pi^{d-k}(\mathbf{s}) \in \Omega(\pi^{d-k}(\text{conv}(\{v_{\sigma,0}, \dots, v_{\sigma,k}\})))\forall 1 \leq k \leq d\}.$$

By letting $S_\sigma = \rho^+(S'_\varsigma, Q_i)$ and $\text{sign}(\sigma, P) = -\text{sign}(F_{d+1})\,\text{sign}(F_i)\,\text{sign}(\varsigma, \pi(F_i))$ and using Lemma II.4.7, we get

$$-\text{sign}(F_{d+1})\,\text{sign}(F_i)\rho^+(\pi(F_i), Q_i) = \bigoplus_{\sigma \in \mathfrak{S}_d, \sigma(d)=i} \text{sign}(\sigma, P) S_\sigma.$$

Thus, together with (II.4.6), summing over all $i : 1 \leq i \leq d$ gives (II.4.11). □

Corollary II.4.15. *If P is a d-simplex in general position, then*

$$|\mathcal{L}(\Omega(P))| = \sum_{\sigma \in \mathfrak{S}_d} \text{sign}(\sigma, P) |\mathcal{L}(S_\sigma)|. \qquad \text{(II.4.16)}$$

Therefore, if we can calculate the number of lattice points in S_σ's, then we can calculate the number of lattice points in the nonnegative part of a d-simplex in general position. Although it's not so easy to find $|\mathcal{L}(S_\sigma)|$'s for an arbitrary polytope, we can do it for any lattice-face d-simplex.

II.5 Lattice enumeration in S_σ

In this section, we will count the number of lattice points in S_σ's when P is a lattice-face d-simplex.

We say a map from $\mathbb{R}^d \to \mathbb{R}^d$ is *lattice preserving* if it is invertible and it maps lattice points to lattice points. Clearly, given a lattice preserving map f, for any set $S \in \mathbb{R}^d$ we have that $|\mathcal{L}(S)| = |\mathcal{L}(f(S))|$.

Let P be a lattice face d-simplex with vertex set $V = \{v_1, \dots, v_{d+1}\}$, where we use the same setup as before for d-simplices.

Given any $\sigma \in \mathfrak{S}_d$, recall that S_σ is defined as in (II.4.13). To count the number of lattice points in S_σ, we want to find a lattice preserving affine transformation which simplifies the form of S_σ.

Before trying to find such a transformation, we will define more notation.

For any $\sigma \in \mathfrak{S}_d$, $k : 1 \le k \le d$ and $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, we define matrix $\tilde{X}(\sigma, k; \mathbf{x})$ as

$$\tilde{X}(\sigma, k; \mathbf{x}) = \begin{pmatrix} 1 & x_{\sigma(1),1} & x_{\sigma(1),2} & \cdots & x_{\sigma(1),k} \\ 1 & x_{\sigma(2),1} & x_{\sigma(2),2} & \cdots & x_{\sigma(2),k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{\sigma(k),1} & x_{\sigma(k),2} & \cdots & x_{\sigma(k),k} \\ 1 & x_1 & x_2 & \cdots & x_k \end{pmatrix}_{(k+1)\times(k+1)},$$

and for $j : 0 \le j \le k$, let $\mathfrak{m}(\sigma, k; j)$ be the minor of the matrix $\tilde{X}(\sigma, k; \mathbf{x})$ obtained by omitting the last row and the $(j+1)$th column. Then

$$\det(\tilde{X}(\sigma, k; \mathbf{x})) = (-1)^k \left(\mathfrak{m}(\sigma, k; 0) + \sum_{j=1}^{k} (-1)^j \mathfrak{m}(\sigma, k; j) x_j \right). \tag{II.5.1}$$

Note that $\mathfrak{m}(\sigma, k; k) = \det(Y(\sigma, k))$. Therefore,

$$\frac{\det(\tilde{X}(\sigma, k; \mathbf{x}))}{\det(Y(\sigma, k))} = (-1)^k \frac{\mathfrak{m}(\sigma, k; 0)}{\det(Y(\sigma, k))} + \sum_{j=1}^{k-1} (-1)^{k+j} \frac{\mathfrak{m}(\sigma, k; j)}{\det(Y(\sigma, k))} x_j + x_k. \tag{II.5.2}$$

We will construct our transformation based on (II.5.2). Before that, we give the following lemma which discusses the coefficients in the right hand side of (II.5.2).

Lemma II.5.3. *Suppose P is a lattice-face d-simplex. For any $\sigma \in \mathfrak{S}_d$, for any $k : 1 \leq k \leq d$, and for any $j : 0 \leq j \leq k-1$, we have that*

$$\frac{\mathfrak{m}(\sigma, k; j)}{\det(Y(\sigma, k))} \in \mathbb{Z}.$$

Proof. By the definition of lattice-face polytope and Lemma II.3.6/(i), one can see that $\pi^{d-k}(\operatorname{conv}(v_{\sigma(1)}, \ldots, v_{\sigma(k)}, v_{d+1})) = \operatorname{conv}(\pi^{d-k}(v_{\sigma(1)}), \ldots, \pi^{d-k}(v_{\sigma(k)}), \pi^{d-k}(v_{d+1}))$ is a lattice-face k-polytope. Choose $U = \{\pi^{d-k}(v_{\sigma(1)}), \ldots, \pi^{d-k}(v_{\sigma(k)})\}$, then $\pi(\mathcal{L}(H_U)) = \mathbb{Z}^{k-1}$, where H_U is the affine space spanned by U. However,

$$H_U = \{\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k \mid \det(\tilde{X}(\sigma, k; \mathbf{x})) = 0\}.$$

Therefore, we must have that

$$\det(\tilde{X}(\sigma, k; \mathbf{x})) = 0, \quad x_1, \ldots, x_{k-1} \in \mathbb{Z} \Rightarrow x_k \in Z.$$

Let $x_1 = \cdots = x_{k-1} = 0$, then $\det(\tilde{X}(\sigma, k; \mathbf{x})) = 0$ implies that

$$(-1)^k \frac{\mathfrak{m}(\sigma, k; 0)}{\det(Y(\sigma, k))} + x_k = 0 \Rightarrow \frac{\mathfrak{m}(\sigma, k; 0)}{\det(Y(\sigma, k))} = (-1)^{k+1} x_k \in Z.$$

For any $j : 1 \leq j \leq k-1$, let $x_i = x_{\sigma(j),i} + \delta_{i,j}$ for $1 \leq i \leq k-1$, where $\delta_{i,j}$ is the Kronecker delta function. Then, $\det(\tilde{X}(\sigma, k; x)) = 0$ implies that

$$\begin{aligned} 0 &= (-1)^k \frac{\mathfrak{m}(\sigma, k; 0)}{\det(Y(\sigma, k))} + \sum_{i=1}^{k-1} (-1)^{k+i} \frac{\mathfrak{m}(\sigma, k; i)}{\det(Y(\sigma, k))} x_{\sigma(j),i} + x_k + (-1)^{k+j} \frac{\mathfrak{m}(\sigma, k; j)}{\det(Y(\sigma, k))} \\ &= x_k - x_{\sigma(j),k} + (-1)^{k+j} \frac{\mathfrak{m}(\sigma, k; j)}{\det(Y(\sigma, k))}, \end{aligned}$$

where the second equality follows from the fact that $(x_{\sigma(j),1}, \ldots, x_{\sigma(j),k})$ is in H_U.

Thus,

$$\frac{\mathfrak{m}(\sigma,k;j)}{\det(Y(\sigma,k))} = (-1)^{k+j+1}(x_k - x_{\sigma(j),k}) \in \mathbb{Z}.$$

□

Given this lemma, we have the following proposition.

Proposition II.5.4. *There exist a lattice-preserving affine transformation T_σ which maps $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ to*

$$\left(\frac{\det(\tilde{X}(\sigma,1;\mathbf{x}))}{\det(Y(\sigma,1))}, \frac{\det(\tilde{X}(\sigma,2;\mathbf{x}))}{\det(Y(\sigma,2))}, \ldots, \frac{\det(\tilde{X}(\sigma,d;\mathbf{x}))}{\det(Y(\sigma,d))}\right).$$

Proof. Let $\alpha_\sigma = \left(-\frac{\mathfrak{m}(\sigma,1;0)}{\det(Y(\sigma,1))}, \frac{\mathfrak{m}(\sigma,2;0)}{\det(Y(\sigma,2))}, \ldots, (-1)^d \frac{\mathfrak{m}(\sigma,d;0)}{\det(Y(\sigma,d))}\right)$ and $M_\sigma = (m_{\sigma,j,k})_{d\times d}$, where

$$m_{\sigma,j,k} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j > k, \\ (-1)^{k+j} \frac{\mathfrak{m}(\sigma,k;j)}{\det(Y(\sigma,k))}, & \text{if } j < k. \end{cases}$$

We define $T_\sigma : \mathbb{R}^d \to \mathbb{R}^d$ by mapping $\mathbf{x}$ to $\alpha_\sigma + \mathbf{x}M_\sigma$. By (II.5.2),

$$\alpha_\sigma + \mathbf{x}M_\sigma = \left(\frac{\det(\tilde{X}(\sigma,1;\mathbf{x}))}{\det(Y(\sigma,1))}, \frac{\det(\tilde{X}(\sigma,2;\mathbf{x}))}{\det(Y(\sigma,2))}, \ldots, \frac{\det(\tilde{X}(\sigma,d;\mathbf{x}))}{\det(Y(\sigma,d))}\right).$$

Also, because all of the entries in M_σ and α_σ are integers and the determinant of M_σ is 1, T_σ is lattice preserving. □

Corollary II.5.5. *Give P a lattice-face polytope with vertex set $V = \{v_1, v_2, \ldots, v_{d+1}\}$, we have that*

(i) $\forall i : 1 \le i \le d$, the last $d+1-i$ coordinates of $T_\sigma(v_{\sigma(i)})$ are all zero.

(ii) $T_\sigma(v_{d+1}) = (z(\sigma,1), z(\sigma,2), \ldots, z(\sigma,d))$.

(iii) Recall that for $k : 0 \le k \le d-1$, $v_{\sigma,k}$ is the unique point with first k coordinates the same as v_{d+1} and affinely dependent with $v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(k)}, v_{\sigma(k+1)}$. Then

the first k coordinates of $T_\sigma(v_{\sigma,k})$ are the same as $T_\sigma(v_{d+1})$ and the rest of the coordinates are zero. In other words, $T_\sigma(v_{\sigma,k}) = (z(\sigma,1),\ldots,z(\sigma,k),0,\ldots,0)$.

Proof. (i) This follows from that fact that $\det(\tilde{X}(\sigma,k;\mathbf{x}_{\sigma(i)})) = 0$ if $1 \le i \le k \le d$.

(ii) This follows from the fact that $\tilde{X}(\sigma,k;\mathbf{x}_{d+1}) = X(\sigma,k)$ and $z(\sigma,k) = \det(X(\sigma,k)) / \det(Y(\sigma,k))$.

(iii) Because for any $\mathbf{x} \in \mathbb{R}^d$, the kth coordinate of $T_\sigma(\mathbf{x})$ only depends on the first k coordinates of $\mathbf{x}$, $T_\sigma(v_{\sigma,k})$ has the same first k coordinates as $T_\sigma(v_{d+1})$. T_σ is an affine transformation. So $T_\sigma(v_{\sigma,k})$ is affinely dependent with $T_\sigma(v_{\sigma(1)})$, $T_\sigma(v_{\sigma(2)})$, $\ldots$, $T_\sigma(v_{\sigma(k)})$, $T_\sigma(v_{\sigma(k+1)})$, the last $d-k$ coordinates of which are all zero. Therefore the last $d-k$ coordinates of $T_\sigma(v_{\sigma,k})$ are all zero as well. □

Recalling that $v_{\sigma,d} = v_{d+1}$, we are able to describe $T_\sigma(S_\sigma)$ now.

Proposition II.5.6. *Let $\widehat{S}_\sigma = T_\sigma(S_\sigma)$. Then*

$$\mathbf{s} = (s_1, s_2, \ldots, s_d) \in \widehat{S}_\sigma \Leftrightarrow \forall 1 \le k \le d, s_k \in \Omega(\mathrm{conv}(0, \frac{z(\sigma,k)}{z(\sigma,k-1)} s_{k-1})), \quad \text{(II.5.7)}$$

where by convention we let $z(\sigma,0) = 1$ and $s_0 = 1$.

Proof. T_σ is an affine transformation whose corresponding matrix M_σ is upper triangular. So T_σ commutes with Ω, π and conv. Therefore,

$$\widehat{S}_\sigma = \{\mathbf{s} \in \mathbb{R}^d \mid \pi^{d-k}(\mathbf{s}) \in \Omega(\pi^{d-k}(\mathrm{conv}(\{\widehat{v}_{\sigma,0}, \ldots, \widehat{v}_{\sigma,k}\})))\forall 1 \le k \le d\},$$

where $\widehat{v}_{\sigma,i} = T_\sigma(v_{\sigma,i}) = (z(\sigma,1),\ldots,z(\sigma,i),0,\ldots,0)$, for $0 \le i \le d$.

(II.5.7) follows. □

Because T_σ is a lattice preserving map, $|\mathcal{L}(S_\sigma)| = |\mathcal{L}(\widehat{S}_\sigma)|$. Hence, our problem becomes to find the number of lattice points in $\widehat{S}_\sigma$. However, $\widehat{S}_\sigma$ is much nicer than S_σ. Actually, we can give a formula to calculate all of the sets having the same shape as $\widehat{S}_\sigma$.

Lemma II.5.8. *Given real nonzero numbers* $b_0 = 1, b_1, b_2, \ldots, b_d$, *let* $a'_k = b_k/b_{k-1}$ *and* $a_k = b_k/|b_{k+1}|, \forall k : 1 \leq k \leq d$. *Let* S *be the set defined by the following:*

$$\mathbf{s} = (s_1, s_2, \ldots, s_d) \in S \Leftrightarrow \forall 1 \leq k \leq d, s_k \in \Omega(\mathrm{conv}(0, a'_k s_{k-1})),$$

where s_0 *is set to* 1. *Then*

$$|\mathcal{L}(S)| = \sum_{s_1 \in \mathcal{L}(\Omega(\mathrm{conv}(0,a'_1)))} \sum_{s_2 \in \mathcal{L}(\Omega(\mathrm{conv}(0,a'_2 s_1)))} \cdots \sum_{s_d \in \mathcal{L}(\Omega(\mathrm{conv}(0,a'_d s_{d-1})))} 1. \tag{II.5.9}$$

In particular, if $b_d > 0$, *then*

$$|\mathcal{L}(S)| = \sum_{s_1=1}^{\overline{\lfloor a_1 \rfloor}} \sum_{s_2=1}^{\overline{\lfloor a_2 s_1 \rfloor}} \cdots \sum_{s_d=1}^{\overline{\lfloor a_d s_{d-1} \rfloor}} 1, \tag{II.5.10}$$

where for any real number x, $\lfloor x \rfloor$ *is the largest integer no greater than* x *and* $\overline{x}$ *is defined as*

$$\overline{x} = \begin{cases} x, & \text{if } x \geq 0, \\ -x-1, & \text{if } x < 0. \end{cases}$$

Note that $\overline{\lfloor x \rfloor} \in \mathbb{Z}_{\geq 0}$, and if any of the sums in (II.5.10) have upper bound equal to 0, we consider the sum to be 0.

Proof. (II.5.9) is straightforward. (II.5.10) follows from the facts that for any real numbers x,

$$\mathcal{L}(\Omega(\mathrm{conv}(0,x))) = \begin{cases} \{z \in \mathbb{Z} \mid 1 \leq z \leq \overline{\lfloor x \rfloor}\}, & \text{if } x \geq 0, \\ \{z \in \mathbb{Z} \mid -\overline{\lfloor x \rfloor} \leq z \leq 0\}, & \text{if } x < 0, \end{cases}$$

the sign of s_i is the same as the sign of b_i, and, because $b_d > 0$, all the s_i's are non-zero. □

We want to give a formula for the number of lattice points in $\widehat{S}_\sigma$ in the form of (II.5.10). We first need the condition "$b_d > 0$", which in our case is that "$z(\sigma, d) > 0$".

However, for any d-simplex P in general position, we can always find a way to order its vertices into $V = \{v_1, v_2, \dots, v_{d+1}\}$, so that the corresponding $\det(X(\mathbf{1}, d))$ and $\det(Y(\mathbf{1}, d))$ are positive, where $\mathbf{1}$ stands for the identity permutation in $\mathfrak{S}_d$. Note $z(\sigma, d)$ is independent of σ. So it is positive.

Moreover, for lattice-polytopes, we have another good property of the $z(\sigma, k)$'s which allows us to remove the $\lfloor\ \rfloor$ operation in (II.5.10).

Lemma II.5.11. *If P is a lattice-polytope d-simplex, then*

$$z(\sigma, k)/z(\sigma, k-1) \in \mathbb{Z},$$

where by convention $z(\sigma, 0)$ is set to 1.

Proof. Let $P' = T_\sigma(P)$ with vertex set $V' = \{v'_1, \dots, v'_{d+1}\}$, where $v'_i = T_\sigma(v_i)$ with coordinates $\mathbf{x}'_i = (x'_{i,1}, \dots, x'_{i,d})$. Because T_σ is an upper triangular lattice preserving map, P' is a lattice-face d-simplex as well. Similar to the proof of Lemma II.5.3, $\mathrm{conv}(\pi^{d-k}(v'_{\sigma(1)}), \dots, \pi^{d-k}(v'_{\sigma(k)}), \pi^{d-k}(v'_{d+1}))$ is a lattice-face k-polytope. We choose $U = \{\pi^{d-k}(v'_{\sigma(1)}), \dots, \pi^{d-k}(v'_{\sigma(k-1)}), \pi^{d-k}(v'_{d+1})\}$, then $\pi(\mathcal{L}(H_U)) = \mathbb{Z}^{k-1}$. Note that by Corollary II.5.5/(i),(ii), we have that

a) the last 2 coordinates of $\pi^{d-k}(v'_{\sigma(j)})$ are both zero, for any $j : 1 \leq j \leq k-1$.

b) $\pi^{d-k}(v'_{d+1}) = (z(\sigma, 1), \dots, z(\sigma, k-1), z(\sigma, k))$.

Hence, $(x_1, \dots, x_k) \in H_U$ if and only if $\det\left(\left(\begin{array}{cc} z(\sigma, k-1) & z(\sigma, k) \\ x_{k-1} & x_k \end{array}\right)\right) = 0$, where we set $x_0 = 1$.

We have that for any $(x_1, \dots, x_k) \in H_U$, if $x_1, \dots, x_{k-1} \in \mathbb{Z}$, then $x_k \in \mathbb{Z}$. Thus, by setting $x_{k-1} = 1$, we get $z(\sigma, k)/z(\sigma, k-1) = x_k \in \mathbb{Z}$. □

Therefore, by Lemma II.5.8 and Lemma II.5.11, we have the following result.

Proposition II.5.12. *Let P be a lattice-face d-simplex with vertex set V, where the order of vertices makes both* $\det(X(\mathbf{1}, d))$ *and* $\det(Y(\mathbf{1}, d))$ *positive. Define*

$$a(\sigma, k) = \frac{z(\sigma, k)}{|z(\sigma, k-1)|}, \forall k : 1 \leq k \leq d.$$

Then

$$|\mathcal{L}(S_\sigma)| = \sum_{s_1=1}^{\overline{a(\sigma,1)}} \sum_{s_2=1}^{\overline{a(\sigma,2)s_1}} \cdots \sum_{s_d=1}^{\overline{a(\sigma,d)s_{d-1}}} 1. \tag{II.5.13}$$

Although now we have a formula to describe the number of lattice points in S_σ, those bars on the top of $a(\sigma, i)$'s make our calculation hard. However, when P is a dilation of an integral cyclic polytope, (II.5.13) becomes a formula without bars. In the next section, we will discuss this case and complete the proof for Theorem II.3.11 and thus Theorem II.3.2.

II.6 The case when P is a dilation of an integral simplex cyclic polytope

Given positive integer m, let $T = \{t_1, t_2, \ldots, t_{d+1}\}_<$ be an integral linearly ordered set and $P = mC_d(T)$ be the simplex polytope with vertex set $V = \{v_1, v_2, \ldots, v_{d+1}\}$, where the coordinates of v_i are $\mathbf{x}_i = (mt_i, mt_i^2, \ldots, mt_i^d)$.

One can calculate that for any $\sigma \in \mathfrak{S}_d$ and $1 \leq k \leq d$:

$$\begin{aligned}
\det(X(\sigma,k)) &= m^k \prod_{1\leq i<j\leq k} (t_{\sigma(j)} - t_{\sigma(i)}) \prod_{i=1}^{k} (t_{d+1} - t_{\sigma(i)}), \\
\det(Y(\sigma,k)) &= m^{k-1} \prod_{1\leq i<j\leq k} (t_{\sigma(j)} - t_{\sigma(i)}), \\
z(\sigma,k) &= m \prod_{i=1}^{k} (t_{d+1} - t_{\sigma(i)}).
\end{aligned}$$

II.6.1 Decomposition formula for $\Omega(P)$

We first want to restate the terms $\operatorname{sign}(\sigma, P)$ and S_σ in the decomposition formula in Theorem II.4.10. Because $t_1 < t_2 < \cdots < t_{d+1}$, both $\det(X(\mathbf{1}, d))$ and $\det(Y(\mathbf{1}, d))$

are positive. Also, all the $z(\sigma, k)$'s are positive. Thus,

$$\begin{aligned} \operatorname{sign}(\sigma, P) &= \operatorname{sign}(\det(X(\sigma, d))) \operatorname{sign}\left(\prod_{i=1}^{d} z(\sigma, i)\right) \\ &= \operatorname{sign}(\sigma) \operatorname{sign}(X(\sigma, \mathbf{1})) \\ &= \operatorname{sign}(\sigma). \end{aligned}$$

However, S_σ is still not easy to describe. Recall $\widehat{S}_\sigma = T_\sigma(S_\sigma)$, so we have $S_\sigma = T_\sigma^{-1}(\widehat{S}_\sigma)$. We will use T_σ and $\widehat{S}_\sigma$ to describe S_σ.

Recall $T_\sigma : \mathbb{R}^d \to \mathbb{R}^d$ is defined in Proposition II.5.4 by mapping $\mathbf{x}$ to $\alpha_\sigma + \mathbf{x}M_\sigma$, where α_σ and M_σ are both involved with entries $\frac{\mathfrak{m}(\sigma,k;j)}{\det(Y(\sigma,k))}$, for $1 \le k \le d, 0 \le j \le k-1$. Fortunately, there is a well-known result [30, Theorem 7.15.1] which simplifies $\frac{\mathfrak{m}(\sigma,k;j)}{\det(Y(\sigma,k))}$ in terms of symmetric functions. Namely,

$$\frac{\mathfrak{m}(\sigma, k; j)}{\det(Y(\sigma, k))} = \begin{cases} e_{k-j}(t_{\sigma(i)} : 1 \le i \le k, i \ne j), & 1 \le j \le k-1, \\ m\prod_{i=1}^{k} t_{\sigma(i)}, & j = 0, \end{cases}$$

where e_{k-j} is the elementary symmetric function (see [30, (7.2)] for the definition).

By the proof of Proposition II.5.6, we have

$$\widehat{S}_\sigma = \{\mathbf{s} \in \mathbb{R}^d \mid \pi^{d-k}(\mathbf{s}) \in \Omega(\pi^{d-k}(\operatorname{conv}(\{\widehat{v}_{\sigma,0}, \dots, \widehat{v}_{\sigma,k}\}))), \forall 1 \le k \le d\},$$

where

$$\begin{aligned} \widehat{v}_{\sigma,i} &= (z(\sigma,1), \dots, z(\sigma,i), 0, \dots, 0) \\ &= \left(m(t_{d+1} - t_{\sigma(1)}), \dots, m\prod_{j=1}^{i}(t_{d+1} - t_{\sigma(j)}), 0, \dots, 0\right), \end{aligned}$$

for $0 \le i \le d$.

Let $R_\sigma = \operatorname{conv}\{\widehat{v}_{\sigma,i}\}_{i=0}^d$ be the convex polytope with vertices $\{\widehat{v}_{\sigma,i}\}_{i=0}^d$. We know that $\widehat{S}_\sigma \subset R_\sigma$. However, we can say even more than that.

Lemma II.6.1.

$$\widehat{S}_\sigma = \Omega(R_\sigma).$$

Proof. For any $\mathbf{s} \in \widehat{S}_\sigma$, when $k = d$, the condition $\pi^{d-k}(\mathbf{s}) \in \Omega(\pi^{d-k}(\text{conv}(\{\widehat{v}_{\sigma,0}, \ldots, \widehat{v}_{\sigma,k}\})))$ just says $s \in \Omega(R_\sigma)$. Thus, $\widehat{S}_\sigma \subset \Omega(R_\sigma)$.

Because $t_{d+1} > t_i$, for any $1 \leq i \leq d$, the coordinates of the $\widehat{v}_{\sigma,i}$'s are all nonnegative.

For any $1 \leq k \leq d$, if we let $P_k = \pi^{d-k}(\text{conv}(\{\widehat{v}_{\sigma,0}, \widehat{v}_{\sigma,1}, \ldots, \widehat{v}_{\sigma,k}\}))$, then $\pi(P_k) = \text{conv}(\pi^{d-k+1}(\{\widehat{v}_{\sigma,0}, \widehat{v}_{\sigma,1}, \ldots, \widehat{v}_{\sigma,k-1}\}))$. Together with the fact that the last coordinate of $\pi^{d-k}(\widehat{v}_{\sigma,k})$ is $m\prod_{i=1}^{k}(t_{d+1} - t_{\sigma(i)}) > 0$, we have

$$\Omega(P_k) = P_k \cap \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_k > 0\}.$$

For any $\mathbf{s} = (s_1, \ldots, s_d) \in \Omega(R_\sigma)$, $\mathbf{s}$ can be written as $\sum_{i=0}^{d} \lambda_i \widehat{v}_{\sigma,i}$, where $\sum_{i=0}^{d} \lambda_i = 1$ and all $\lambda_i \geq 0$. Note that $R_\sigma = P_d$, so $s_d > 0$. However, since $\widehat{v}(\sigma, d)$ is the only vertex whose last coordinate is positive, $\lambda_d > 0$. Because all of the coordinates of $\widehat{v}(\sigma, d)$ are positive, the s_i's are all positive. Thus the last coordinate of $\pi^{d-k}(\mathbf{s})$ is positive. So

$$\pi^{d-k}(\mathbf{s}) \in \Omega(P_k) = \Omega(\pi^{d-k}(\text{conv}(\{\widehat{v}_{\sigma,0}, \ldots, \widehat{v}_{\sigma,k}\}))), \forall 1 \leq k \leq d.$$

Therefore, $\mathbf{s} \in \widehat{S}_\sigma$ and $\Omega(R_\sigma) \subset \widehat{S}_\sigma$. □

We now have everything to restate the decomposition formula.

Theorem II.6.2. *Let $T = \{t_1, t_2, \ldots, t_{d+1}\}_<$ be an integral linearly ordered set and m be a positive integer. Let $P = mC_d(T)$. Then,*

$$\Omega(P) = \bigoplus_{\sigma \in \mathfrak{S}_d} \text{sign}(\sigma) T_\sigma^{-1}(\Omega(R_\sigma)), \tag{II.6.3}$$

where $R_\sigma = \text{conv}(\{\widehat{v}_{\sigma,i} = (m(t_{d+1} - t_{\sigma(1)}), \ldots, m\prod_{j=1}^{i}(t_{d+1} - t_{\sigma(j)}), 0, \ldots, 0)\}_{i=0}^{d})$, and $T_\sigma(\mathbf{x}) = \alpha_\sigma + \mathbf{x}M_\sigma$, with $\alpha_\sigma = (-mt_{\sigma(1)}, mt_{\sigma(1)}t_{\sigma(2)}, \ldots, (-1)^d m\prod_{i=1}^{d} t_{\sigma(i)})$ and $M_\sigma = (m_{\sigma,j,k})_{d\times d}$, with

$$m_{\sigma,j,k} = \begin{cases} 1, & \textit{if } j = k, \\ 0, & \textit{if } j > k, \\ (-1)^{k+j} e_{k-j}(t_{\sigma(i)} : 1 \leq i \leq k, i \neq j), & \textit{if } j < k. \end{cases}$$

II.6.2 The number of lattice points in $\Omega(P)$

For our dilated cyclic polytope P, the hypothesis of Proposition II.5.12 is satisfied, so we can either use (II.5.13) or calculate the number of lattice points in $\widehat{S}_\sigma = \Omega(R_\sigma)$ directly. We will take the former approach. Since $\forall k : 1 \leq k \leq d$,

$$a(\sigma, k) = \frac{z(\sigma, k)}{|z(\sigma, k-1)|} = \begin{cases} t_{d+1} - t_{\sigma(k)}, & \text{if } 2 \leq k \leq d, \\ m(t_{d+1} - t_{\sigma(k)}), & \text{if } k = 1 \end{cases}$$

is a positive integer, we can remove the bars in (II.5.13):

$$|\mathcal{L}(S_\sigma)| = |\mathcal{L}(\widehat{S}_\sigma)| = |\mathcal{L}(\Omega(R_\sigma))| = \sum_{s_1=1}^{a(\sigma,1)} \sum_{s_2=1}^{a(\sigma,2)s_1} \cdots \sum_{s_d=1}^{a(\sigma,d)s_{d-1}} 1. \tag{II.6.4}$$

Therefore, we can write the number of lattice points in $\Omega(P) = \Omega(mC_d(T))$ using the following formula.

Corollary II.6.5.

$$|\mathcal{L}(\Omega(mC_d(T)))| = \sum_{\sigma \in \mathfrak{S}_d} \text{sign}(\sigma) \sum_{s_1=1}^{m(t_{d+1}-t_{\sigma(1)})} \sum_{s_2=1}^{(t_{d+1}-t_{\sigma(2)})s_1} \cdots \sum_{s_d=1}^{(t_{d+1}-t_{\sigma(d)})s_{d-1}} 1. \tag{II.6.6}$$

Because of (II.6.4) and (II.6.6), it's natural for us to define

$$f_d(a_1, a_2, \ldots, a_d) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} \cdots \sum_{s_d=1}^{a_d s_{d-1}} 1, \tag{II.6.7}$$

and

$$\mathcal{H}_{m,d}(a_1, a_2, \ldots, a_d) = \sum_{\sigma \in S_n} \text{sign}(\sigma) f_d(m a_{\sigma(1)}, a_{\sigma(2)} \ldots, a_{\sigma(d)}),$$

for any positive integers $a_1, a_2, \ldots, a_d$.

Thus, (II.6.6) can be rewritten as

$$|\mathcal{L}(\Omega(mC_d(T)))| = \mathcal{H}_{m,d}(t_{d+1} - t_{\sigma(1)}, t_{d+1} - t_{\sigma(2)}, \cdots, t_{d+1} - t_{\sigma(d)}). \qquad \text{(II.6.8)}$$

We want to analyze f_d and $\mathcal{H}_{m,d}$ so that we can simplify the right hand side of (II.6.8). However, f_d is closely related to power sums, so we will first discuss some properties of power sums and use them to give lemmas on f_d.

Given any x a positive integer, we define

$$P_k(x) = \sum_{i=0}^{x} i^k = \begin{cases} \sum_{i=1}^{x} i^k, & \text{if } k \geq 1, \\ x+1, & \text{if } k = 0. \end{cases}$$

It's well known that for $k \geq 1$,

$$P_k(x) \text{ is a polynomial in } x \text{ of degree } k+1, \qquad \text{(II.6.9)}$$

$$\text{the constant term of } P_k(x) \text{ is } 0, \text{ i.e., } x \text{ is a factor of } P_k(x), \qquad \text{(II.6.10)}$$

$$\text{the leading coefficient of } P_k(x) \text{ is } \tfrac{1}{k+1}. \qquad \text{(II.6.11)}$$

Therefore, we can extend the domain of $P_k(x)$ from $\mathbb{Z}_{>0}$ to $\mathbb{R}$ and thus we call $P_k(x)$ the *kth power sum polynomial.*

Extension of the sum operation

Given $h = h(s) = \sum_{k\geq 0} h_k s^k$ a polynomial in s, the upper bound u of a sum $\sum_{s=1}^{u} h$ should be a positive integer in the usual definition. We extend this definition to allow u (as well as the h_k's) to be in any polynomial ring over $\mathbb{R}$ using the formula

$$\sum_{s=1}^{u} h = h_0 u + \sum_{k\geq 1} h_k P_k(u). \qquad \text{(II.6.12)}$$

One can check that this extension agrees with the case when u is a positive integer.

Since f_d is defined by (II.6.7), which recursively uses the sum operation, we can use (II.6.12) to extend the domain of f_d from $(\mathbb{Z}_{>0})^d$ to $\mathbb{Z}^d$ or even $\mathbb{R}^d$. Hence the domain of $\mathcal{H}_{m,d}$ is extended to $\mathbb{R}^d$ as well.

Lemma II.6.13. *The only highest degree term of f_d is $\frac{1}{d!}a_1^d a_2^{d-1} a_3^{d-2} \dots a_d$. This is also true when we consider f_d as a polynomial just in the variable a_1.*

Proof. We will prove it by induction on d.

When $d = 1$, $f_1(a_1) = \sum_{s_1=1}^{a_1} 1 = P_0(a_1) - 1 = a_1$. Thus the lemma holds.

Assume the lemma is true for $d(\geq 1)$, and note that

$$f_{d+1}(a_1, a_2, \dots, a_{d+1}) = \sum_{s_1=1}^{a_1} f_d(a_2 s_1, a_3, \dots, a_{d+1}).$$

By assumption, $\frac{1}{d!}a_2^d a_3^{d-1} \dots a_{d+1} s_1^d$ is the only highest degree term of $f_d(a_2 s_1, a_3, \dots, a_{d+1})$ when we consider it as polynomial both in $y = a_2 s_1, a_3, \dots, a_{d+1}$ and in y. This implies that $\frac{1}{d!}a_2^d a_3^{d-1} \dots a_{d+1} s_1^d$ is the only highest degree term of $f_d(a_2 s_1, a_3, \dots, a_{d+1})$ when we consider it both in $a_2, a_3, \dots, a_{d+1}$ and in s_1. Then our lemma immediately follows from the fact that the highest degree term of $\sum_{s_1=1}^{a_1} s_1^d = P_d(a_1)$ is $\frac{1}{d+1}a_1^{d+1}$. $\square$

Lemma II.6.14. *$f_d(a_1, \dots, a_d)$ is a polynomial in a_1 of degree d, having a factor of $\prod_{i=1}^d a_i$. In particular, f_d can be written as*

$$f_d(a_1, \dots, a_d) = \sum_{k=1}^{d} f_{d,k}(a_2, \dots, a_d) a_1^k, \tag{II.6.15}$$

where $f_{d,k}(a_2, \dots, a_d)$ is a polynomial in $a_2, \dots, a_d$ with a factor of $\prod_{i=2}^d a_i$.

Proof. This can be proved by induction on d, using (II.6.9) and (II.6.10). $\square$

Proposition II.6.16.

$$\mathcal{H}_{m,d}(a_1, a_2, \dots, a_d) = \frac{m^d}{d!} \prod_{i=1}^{d} a_i \prod_{1 \leq i < j \leq d} (a_i - a_j).$$

Proof. By Lemma II.6.14, $\mathcal{H}_{m,d}(a_1, \dots, a_d)$ has a factor of $\prod_{i=1}^d a_i$.

For $1 \leq i < j \leq d$, it's easy to check that $\mathcal{H}_{m,d}$ changes sign when we switch a_i and a_j, i.e.,

$$\mathcal{H}_{m,d}(\dots, a_i, \dots, a_j, \dots) = -\mathcal{H}_{m,d}(\dots, a_j, \dots, a_i, \dots).$$

Therefore, $\mathcal{H}_{m,d}(a_1, \dots, a_d)$ must be a multiple of

$$\prod_{i=1}^{d} a_i \prod_{1 \leq i < j \leq d} (a_i - a_j),$$

which has degree $\frac{1}{2}d(d+1)$.

So now it's enough to show that $\mathcal{H}_{m,d}(a_1, \dots, a_d)$ is of degree $\frac{1}{2}d(d+1)$ and the coefficient of $a_1^d a_2^{d-1} a_3^{d-2} \dots a_d$ in $\mathcal{H}_{m,d}(a_1, \dots, a_d)$ is $\frac{m^d}{d!}$, which follows from Lemma II.6.13. $\square$

Proof of Theorem II.3.11. By remark II.4.3, it is enough to prove the case that $C_d(T)$ is a simplex. But

$$\begin{aligned}
|\mathcal{L}(\Omega(mC_d(T)))| &= \sum_{\sigma \in \mathfrak{S}_d} \operatorname{sign}(\sigma) \sum_{s_1=1}^{m(t_{d+1}-t_{\sigma(1)})} \sum_{s_2=1}^{(t_{d+1}-t_{\sigma(2)})s_1} \cdots \sum_{s_d=1}^{(t_{d+1}-t_{\sigma(d)})s_{d-1}} 1 \\
&= \mathcal{H}_{m,d}(t_{d+1} - t_{\sigma(1)}, t_{d+1} - t_{\sigma(2)}, \dots, t_{d+1} - t_{\sigma(d)}) \\
&= \frac{m^d}{d!} \prod_{i=1}^{d} (t_{d+1} - t_i) \prod_{1 \leq i < j \leq d} (t_i - t_j) \\
&= \frac{m^d}{d!} \prod_{1 \leq i < j \leq d+1} (t_i - t_j) = \operatorname{Vol}(mC_d(T)).
\end{aligned}$$

$\square$

As we argued earlier, the proof of Theorem II.3.11 completes the proof of Theorem II.3.2.

II.7 Back to lattice-face polytopes

In the previous section, we applied the decomposition to a dilation of integral cyclic polytopes and proved the theorem on the Ehrhart polynomial of an integral cyclic polytope. One reason why we were able to carry out the proof was that the $a(\sigma, k)$'s were all positive and thus the formula (II.5.13) for the number of lattice points in S_σ became a simple formula without bars on the top of $a(\sigma, k)$. However, this is not true for general lattice-face polytopes. We have to find some other way to remove the bars in (II.5.13).

Recall we have defined f_d as

$$f_d(a_1, a_2, \ldots, a_d) = \sum_{s_1=1}^{a_1} \sum_{s_2=1}^{a_2 s_1} \cdots \sum_{s_d=1}^{a_d s_{d-1}} 1, \tag{II.7.1}$$

for any positive integers $a_1, a_2, \ldots, a_d$, and then we extend the domain from $(\mathbb{Z}_{>0})^d$ to $\mathbb{R}^d$.

We define another function g_d in terms of f_d, with which we will rewrite (II.5.13). Fixing $b_0 = 1$, we define

$$g_d(b_1, b_2, \ldots, b_d) = f_d(b_1/b_0, b_2/b_1, \ldots, b_d/b_{d-1}), \tag{II.7.2}$$

for any $(b_1, b_2, \ldots, b_d) \in (\mathbb{Z}_{>0})^d$ such that b_i is a multiple of $b_{i-1} (\forall 1 \leq i \leq d)$. As we extend the domain of f_d, the domain of g_d can be extended to $(\mathbb{R} \setminus \{0\})^d$.

In last section, we discussed how the properties of f_d follow from those of power sum polynomials $P_k(x)$. In this section, we will discuss the relationship between Bernoulli polynomials and power sums, and then use a property of Bernoulli polynomials to rewrite (II.5.13) in terms of g_d. Please refer to [6, Section 2.4] for other examples discussing Bernoulli polynomials and their relation to integral polytopes.

The *kth Bernoulli polynomial*, $B_k(x)$, is defined as [1, p. 264]

$$\frac{te^{tx}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.$$

The Bernoulli polynomials satisfy [1, Theorem 12.19]

$$B_k(1-x) = (-1)^k B_k(x), \forall k \geq 0, \tag{II.7.3}$$

as well as the relation [1, Theorem 12.14]

$$B_k(x+1) - B_k(x) = kx^{k-1}, \forall k \geq 1. \tag{II.7.4}$$

We call $B_k = B_k(0)$ a *Bernoulli number.* It satisfies [1, Theorem 12.16] that

$$B_k(0) = 0, \text{for any odd number } k \geq 3. \tag{II.7.5}$$

By (II.7.4), it is easy to see that for $k \geq 0$,

$$P_k(x) = \frac{B_{k+1}(x+1) - B_{k+1}}{k+1}.$$

Lemma II.7.6. *For any* $k \geq 1$,

$$P_k(x) = (-1)^{k+1} P_k(-x-1). \tag{II.7.7}$$

Proof. It follows from (II.7.3) and (II.7.5). □

Lemma II.7.8. *Given* $(a_1, a_2, \ldots, a_d) \in \mathbb{R}^d$,

$$f_d(a_1, a_2, \ldots, a_d) = -\sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} \sum_{s_3=1}^{a_3 s_2} \cdots \sum_{s_d=1}^{a_d s_{d-1}} 1.$$

Proof.

$$-\sum_{s_1=1}^{-a_1-1} \sum_{s_2=1}^{-a_2 s_1} \sum_{s_3=1}^{a_3 s_2} \cdots \sum_{s_d=1}^{a_d s_{d-1}} 1 = -\sum_{s_1=1}^{-a_1-1} f_{d-1}(-a_2 s_1, a_3, \ldots, a_d).$$

By (II.6.15) and (II.7.7), we have

$$f_{d-1}(-a_2 s_1, a_3, \ldots, a_d) = \sum_{k=1}^{d-1} f_{d-1,k}(a_3, \ldots, a_d)(-a_2 s_1)^k,$$

and
$$\sum_{s_1=1}^{-a_1-1} s_1^k = P_k(-a_1-1) = (-1)^{k+1}P_k(a_1) = (-1)^{k+1}\sum_{s_1=1}^{a_1} s_1^k.$$

Therefore,

$$-\sum_{s_1=1}^{-a_1-1}\sum_{s_2=1}^{-a_2 s_1}\sum_{s_3=1}^{a_3 s_2}\cdots\sum_{s_d=1}^{a_d s_{d-1}} 1 = \sum_{s_1}^{a_1}\sum_{k=1}^{d-1} f_{d-1,k}(a_3,\dots,a_d)(a_2 s_1)^k$$
$$= \sum_{s_1}^{a_1} f_{d-1}(a_2 s_1, a_3, \dots, a_d) = f_d(a_1, a_2, \dots, a_d).$$

□

Proposition II.7.9. *Given $b_0 = 1$, $b_1, b_2, \dots, b_d \in (\mathbb{R}\setminus\{0\})$ with $b_d > 0$, let $a_k = b_k/|b_{k-1}|$, then*

$$g_d(b_1, b_2, \dots, b_d) = f_d(b_1, \frac{b_2}{b_1}, \dots, \frac{b_d}{b_{d-1}}) = \operatorname{sign}\left(\prod_{i=1}^{d} b_i\right)\sum_{s_1=1}^{\overline{a_1}}\sum_{s_2=1}^{\overline{a_2 s_1}}\cdots\sum_{s_d=1}^{\overline{a_d s_{d-1}}} 1, \quad \text{(II.7.10)}$$

where we always treat s_i as positive when determining the meaning of $\overline{a_{i+1}s_i}$. That is, for $a_{i+1} > 0$, we set $\overline{a_{i+1}s_i} = a_{i+1}s_i$, and for $a_{i+1} < 0$, we set $\overline{a_{i+1}s_i} = -a_{i+1}s_i - 1$. Note that this agrees with the original definition when the a_i's are all positive integers.

Proof. We prove the proposition by induction on d. When $d = 1$, it's trivial.

Assume (II.7.10) holds for $d = d_0 \geq 1$. s_1 is positive. Thus, by the induction hypothesis,

$$g_d(\frac{b_2}{|b_1|}s_1, \frac{b_3}{|b_1|}s_1, \dots, \frac{b_d}{|b_1|}s_1) = f_d(\frac{b_2}{|b_1|}s_1, \frac{b_3}{b_2}, \dots, \frac{b_d}{b_{d-1}}) = \operatorname{sign}\left(\prod_{i=2}^{d} b_i\right)\sum_{s_2=1}^{\overline{a_2 s_1}}\cdots\sum_{s_d=1}^{\overline{a_d s_{d-1}}} 1.$$

It's clear that (II.7.10) holds when $b_1 > 0$. In the case that $b_1 < 0$, (II.7.10) follows from the above equation and Lemma II.7.8. □

Proposition II.7.11. *Let P be a lattice-face d-simplex with vertex set V, where the* *order of vertices makes both* $\det(X(\mathbf{1},d))$ *and* $\det(Y(\mathbf{1},d))$ *positive. Then*

$$|\mathcal{L}(S_\sigma)| = \operatorname{sign}\left(\prod_{i=1}^{d} z(\sigma,i)\right) g_d(z(\sigma,1), z(\sigma,2), \dots, z(\sigma,d)). \tag{II.7.12}$$

Therefore,

$$|\mathcal{L}(\Omega(P))| = \sum_{\sigma\in\mathfrak{S}_d} \operatorname{sign}(\sigma) g_d(z(\sigma,1), z(\sigma,2), \dots, z(\sigma,d)). \tag{II.7.13}$$

Proof. We can get (II.7.12) by comparing (II.5.13) and (II.7.10). And (II.7.13) follows from (II.4.16), (II.4.12), (II.7.12) and the fact that $\det(X(\sigma,d)) = \operatorname{sign}(\sigma)\det(X(\mathbf{1},d))$. □

II.8 Proof of the Main Theorems

We now have all the ingredients but one to prove our main theorems: Theorem II.3.8 and Theorem II.3.10. The missing one is stated as the following proposition; it is proved in Appendix A, because the proof is self-contained and different from the rest of the chapter.

Proposition II.8.1. *Let* $V = \{v_1, v_2, \dots, v_{d+1}\}$ *be the vertex set of a d-simplex in general position, where the coordinates of* v_i *are* $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \dots, x_{i,d})$. *Recall that* $X(\sigma,k)$, $Y(\sigma,k)$ *and* $z(\sigma,k)$ *are defined in §II.4.2 and* g_d *is defined in (II.7.2). Then*

$$\sum_{\sigma\in\mathfrak{S}_d} \operatorname{sign}(\sigma) g_d(z(\sigma,1), z(\sigma,2), \dots, z(\sigma,d)) = \frac{1}{d!}\det(X(\mathbf{1},d)), \tag{II.8.2}$$

where $\mathbf{1}$ *is the identity in* $\mathfrak{S}_d$.

Given this proposition, we can prove the theorems.

Proof of Theorem II.3.8 and Theorem II.3.10. As we mentioned in Remark II.4.3, to prove Theorem II.3.10, it is sufficient to prove the case when P is a lattice-face simplex.

When P is a lattice-face d-simplex, we still assume that the order of the vertices of P makes both $\det(X(\mathbf{1},d))$ and $\det(Y(\mathbf{1},d))$ positive. Thus, (II.7.13), (II.8.2) and the fact that the volume of P is $\frac{1}{d!}|\det(X(\mathbf{1},d))|$ imply Theorem II.3.10, and Theorem II.3.8 follows. □

Recall that we use $I(P)$ to denote the interior of a d-polytope P. We denote by $\widehat{i}(P,m) = |I(mP) \cap \mathbb{Z}^d|$ the number of lattice points in the interior of mP.

Corollary II.8.3. *For any lattice-face d-polytope P, we have that*

$$\widehat{i}(P,m) = \mathrm{Vol}(mP) - \widehat{i}(\pi(P),m) = \sum_{k=0}^{d}(-1)^{d-k}\mathrm{Vol}_k(\pi^{(d-k)}(P))m^k. \tag{II.8.4}$$

Thus,

$$i(P,-m) = (-1)^d\widehat{i}(P,m). \tag{II.8.5}$$

Proof. Since P satisfies (II.2.6), by Lemma II.2.5/(vi) and Lemma II.3.6/(vi), π induces a bijection between $\mathcal{L}(PB(P) \cap NB(P))$ and $\mathcal{L}(\partial\pi(P))$. Together with Lemma II.3.6/(ii), (iii), this implies

$$\widehat{i}(P,m) = i(P,m) - i(\pi(P),m) - \widehat{i}(\pi(P),m).$$

Therefore, (II.8.4) and (II.8.5) follow from Theorem II.3.8. □

Note that (II.8.5) recovers the Ehrhart-Macdonald reciprocity law (Theorem I.3.5).

II.9 Examples and Further discussion

II.9.1 Examples of lattice-face polytopes

In this subsection, we use a fixed family of lattice-face polytopes to illustrate our results. Let $d = 3$, and for any positive integer k, let P_k be the polytope with the vertex set $V = \{v_1 = (0,0,0), v_2 = (4,0,0), v_3 = (3,6,0), v_4 = (2,2,10k)\}$. One can check that P_k is a lattice-face polytope.

Example II.9.1 (Example of Theorem II.3.8). The volume of P_k is $40k$, and

$$i(P_k, m) = 40km^3 + 12m^2 + 4m + 1.$$

$\pi(P_k) = \text{conv}\{(0,0),(4,0),(3,6)\}$, where

$$i(\pi(P_k), m) = 12m^2 + 4m + 1.$$

So

$$i(P_k, m) = 40km^3 + i(\pi(P_k), m),$$

which agrees with Theorem II.3.8.

Example II.9.2 (Example of Formula (II.4.5)). $F_4 = \text{conv}(v_1, v_2, v_3)$ is a negative facet. The hyperplane determined by F_4 is $H = \{(x_1, x_2, x_3) \mid x_3 = 0\}$. Thus, $v_4' = \pi^{-1}(\pi(v_4)) \cap H = (2,2,0)$.

$F_3 = \text{conv}(v_1, v_2, v_4)$ is a positive facet. $\pi(F_3) = \text{conv}((0,0),(4,0),(2,2))$. $\Omega(\pi(F_3)) = \pi(F_3) \setminus \text{conv}((0,0),(4,0))$. $F_3' = \pi^{-1}(\pi(F_3)) \cap H = \text{conv}(v_1, v_2, v_4')$. So

$$Q_3 = \text{conv}(F_3 \cup F_3') = \text{conv}(v_1, v_2, v_4, v_4'),$$
$$\rho^+(\Omega(\pi(F_3)), Q_3) = Q_3 \setminus F_3'.$$

$F_2 = \text{conv}(v_1, v_3, v_4)$ is a positive facet. $\pi(F_2) = \text{conv}((0,0),(3,6),(2,2))$. $\Omega(\pi(F_2)) = \pi(F_2) \setminus (\text{conv}((0,0),(2,2)) \cup \text{conv}((2,2),(3,6)))$. $F_2' = \pi^{-1}(\pi(F_2)) \cap H = \text{conv}(v_1, v_3, v_4')$. So

$$Q_2 = \text{conv}(F_2 \cup F_2') = \text{conv}(v_1, v_3, v_4, v_4'),$$
$$\rho^+(\Omega(\pi(F_2)), Q_2) = Q_2 \setminus (F_2' \cup \text{conv}(v_1, v_4, v_4') \cup \text{conv}(v_3, v_4, v_4')).$$

$F_1 = \text{conv}(v_2, v_3, v_4)$ is a positive facet. $\pi(F_1) = \text{conv}((4,0),(3,6),(2,2))$. $\Omega(\pi(F_1)) =$

$\pi(F_1) \setminus \operatorname{conv}((4,0),(2,2))$. $F_1' = \pi^{-1}(\pi(F_1)) \cap H = \operatorname{conv}(v_2, v_3, v_4')$. So

$$Q_1 = \operatorname{conv}(F_1 \cup F_1') = \operatorname{conv}(v_2, v_3, v_4, v_4'),$$
$$\rho^+(\Omega(\pi(F_1)), Q_1) = Q_1 \setminus (F_1' \cup \operatorname{conv}(v_2, v_4, v_4')).$$

Therefore,

$$\Omega(P_k) = P_k \setminus F_4 = -\operatorname{sign}(F_4) \bigoplus_{i=1}^{3} \operatorname{sign}(F_i)\rho^+(\Omega(\pi(F_i)), Q_i),$$

which agrees with Proposition II.4.4.

Example II.9.3 (Example of decomposition). In this example, we decompose P_k into 3! sets, where 5 of them have positive signs and one has negative sign, which is different from the cases for cyclic polytopes, where half of the sets have positive signs and the other half have negative signs.

Recall that $v_{\sigma,3} = v_4 = (2, 2, 10k)$, for any $\sigma \in \mathfrak{S}_3$.

When $\sigma = 123 \in \mathfrak{S}_3$, $v_{123,2} = v_4' = (2,2,0)$, $v_{123,1} = (2,0,0)$ and $v_{123,0} = v_1 = (0,0,0)$. Then

$$S_{123} = \operatorname{conv}(\{v_{123,i}\}_{0 \le i \le 3}) \setminus \operatorname{conv}(\{v_{123,i}\}_{0 \le i \le 2}),$$

with $\operatorname{sign}(123, P_k) = +1$.

When $\sigma = 213 \in \mathfrak{S}_3$, $v_{213,2} = v_4' = (2,2,0)$, $v_{213,1} = (2,0,0)$ and $v_{213,0} = v_2 = (4,0,0)$. Then

$$S_{213} = \operatorname{conv}(\{v_{213,i}\}_{0 \le i \le 3}) \setminus (\operatorname{conv}(\{v_{213,i}\}_{0 \le i \le 2}) \cup \operatorname{conv}(\{v_{213,i}\}_{1 \le i \le 3})),$$

with $\operatorname{sign}(213, P_k) = +1$.

One can check that

$$S_{123} \oplus S_{213} = \rho^+(\Omega(\pi(F_3)), Q_3).$$

When $\sigma = 231 \in \mathfrak{S}_3$, $v_{231,2} = v_4' = (2,2,0)$, $v_{231,1} = (2,12,0)$ and $v_{231,0} = v_2 =$

$(4, 0, 0)$. Then

$$S_{231} = \text{conv}(\{v_{231,i}\}_{0 \le i \le 3}) \backslash (\text{conv}(\{v_{231,i}\}_{0 \le i \le 2}) \cup \text{conv}(\{v_{231,i}\}_{i=0,2,3} \cup \text{conv}(\{v_{231,i}\}_{1 \le i \le 3})),$$

with $\text{sign}(231, P_k) = +1$.

When $\sigma = 321 \in \mathfrak{S}_3$, $v_{321,2} = v_4' = (2, 2, 0)$, $v_{321,1} = (2, 12, 0)$ and $v_{321,0} = v_3 = (3, 6, 0)$. Then

$$S_{321} = \text{conv}(\{v_{321,i}\}_{0 \le i \le 3}) \backslash (\text{conv}(\{v_{321,i}\}_{0 \le i \le 2}) \cup \text{conv}(\{v_{321,i}\}_{i=0,2,3} \cup \text{conv}(\{v_{321,i}\}_{1 \le i \le 3})),$$

with $\text{sign}(321, P_k) = -1$.

One can check that

$$S_{231} \ominus S_{321} = \rho^+(\Omega(\pi(F_1)), Q_1).$$

Similarly, we have that

$$S_{132} \oplus S_{312} = \rho^+(\Omega(\pi(F_2)), Q_2).$$

Therefore, $\Omega(P_k) = \bigoplus_{\sigma \in \mathfrak{S}_3} \text{sign}(\sigma, P_k) S_\sigma$, which coincides with Theorem II.4.10.

II.9.2 Further discussion

Recall that Remark II.3.12 gives an alternative definition for lattice-face polytopes. Note that in this definition, when $k = 0$, satisfying (II.3.13) is equivalent to saying that P is an integral polytope, which implies that the last coefficient of the Ehrhart polynomial of P is 1. Therefore, one may ask

Question II.9.4. If P is a polytope that satisfies (II.3.13) for all $k \in K$, where K is a fixed subset of $\{0, 1, \ldots, d-1\}$, can we say something about the Ehrhart polynomials of P?

A special set K can be chosen as the set of consecutive integers from 0 to d', where d' is an integer no greater than $d-1$. Based on some examples in this case, the Ehrhart polynomials seems to follow a certain pattern, so we conjecture the following:

Conjecture II.9.5. *Given $d' \leq d-1$, if P is a d-polytope with vertex set V such that for any $k : 0 \leq k \leq d'$, (II.3.13) is satisfied, then for $0 \leq k \leq d'$, the coefficient of m^k in $i(P,m)$ is the same as in $i(\pi^{d-d'}(P), m)$. In other words,*

$$i(P,m) = i(\pi^{d-d'}(P), m) + \sum_{i=d'+1}^{d} c_i m^i.$$

When $d' = 0$, the condition on P is simply that it is integral. And when $d' = d-1$, we are in the case that P is a lattice-face polytope. Therefore, for these two cases, this conjecture is true.

Chapter III

Mochizuki's Indigenous Bundles and Ehrhart Polynomials

This chapter is joint work with Brian Osserman [19].

III.1 Introduction

In this chapter, we bring together work of Mochizuki in algebraic geometry and the theory of Ehrhart quasi-polynomials in combinatorics, obtaining results in both fields. There is already a combinatorial result implicit in [23]; we strengthen it and state it explicitly in terms of familiar combinatorial objects: namely, we obtain in Theorem III.2.4 below an infinite family of polytopes with Ehrhart quasi-polynomials agreeing at all odd values. As we said in Chapter I, Ehrhart quasi-polynomials are not in general well-understood. For instance, criteria for two polytopes to have the same Ehrhart quasi-polynomial have not been well studied. Thus, one might hope that a family of non-trivial identities such as these could help to shed light on the situation.

Additionally, we use the theory Ehrhart quasi-polynomials to conclude in Theorem III.2.1 below that the number of dormant torally indigenous bundles of Mochizuki's theory (see the following section for references and discussion, including relationships to certain rational functions and Frobenius-destabilized vector bundles) may be expressed as a polynomial in the characteristic of the base field. Both these phenomena are observed (but not pursued) in [23, p. 46] in a slightly different case, so this work may be considered a more complete exploration of these phenomena in the case of dormant torally indigenous bundles.

In fact, the ideas of this chapter make no use of the precise definition of dormant torally indigenous bundles, but rather of formal properties which one might expect to find in a number of other settings in the geometry of algebraic curves. The basic idea of Mochizuki's work is that he counts the dormant torally indigenous bundles on general curves of a given type (g, r) (that is, having genus g and r marked points) by degenerating to totally degenerate curves (Definition III.3.9); such curves are determined entirely by the combinatorial data of their dual graphs (Definition III.3.10). He shows that the number of dormant torally indigenous bundles on a totally degenerate curve can be described combinatorially, essentially as the number of lattice points inside a polytope whose dimensions depend on the characteristic of the base field, and he also shows that the number of dormant torally indigenous bundles on a general curve of type (g, r) is equal to the number on any totally degenerate curve of the same type. In particular, the number agrees for any two totally degenerate curves, which is how we obtain our combinatorial formulas. In addition, the relationship to lattice points of polytopes allows the application of the theory of Ehrhart quasi-polynomial to conclude that these numbers are given by polynomials in the characteristic of the base field.

We remark that for any situation where enumerative invariants are associated to curves in such a way that the invariant for a general curve can be computed at totally degenerate curves, one can expect the computations at totally degenerate curves to be of a combinatorial nature, and then one can hope to obtain non-trivial combinatorial identities by comparing these formulas at different totally degenerate curves, as is done here. Thus, the easiest generalizations of our combinatorial results are likely to arise not from attempting to generalize dormant torally indigenous bundles (although that possibility is discussed briefly at the end of the final section below), but from finding examples of completely unrelated algebro-geometric objects which are associated to curves and satisfy the same formal properties with respect to degeneration.

III.2 Statements

We first state our theorem in algebraic geometry. Dormant torally indigenous bundles are algebro-geometric objects associated to curves with marked points, and their precise definition, which is rather technical and given in [23, Def. I.1.2, p. 89, Def. I.4.1, p. 113, Def. II.1.1, p. 127], is not relevant to this chapter. However, for the purposes of motivating our Theorem III.2.1, we remark on two important cases in which dormant torally indigenous bundles correspond to more concrete and widely-studied objects. In the case of a smooth curve of genus $g \geq 2$ with no marked points, dormant torally indigenous bundles are equivalent up to a factor of 2^{2g} to semistable vector bundles of rank 2 with trivial determinant whose pullbacks under the relative Frobenius morphism are maximally unstable – specifically, contain a line-bundle of degree $g-1$. In the case that C is smooth with $g=0$ and r general marked points, dormant torally indigenous bundles are equivalent up to a factor of 2^{r-1} to rational functions on $\mathbb{P}^1$ ramified to order less than p at the marked points, and unramified elsewhere, up to linear fractional transformation. For both these assertions, see [26]. Theorem III.2.1 is new and no easier to prove in both these special cases, and we state the general case partly because it is the most natural level of generality given the arguments, and partly because the phenomenon of invariants in algebraic geometry being expressible by polynomials in the characteristic of the base field is ubiquitous and poorly understood, and rather than simply providing two apparently unrelated examples of this phenomenon, it seems preferable to have a single example which simultaneously generalizes the special cases.

We now state our first theorem, denoting by F the relative Frobenius morphism:

Theorem III.2.1. *Fix $g, r \geq 0$ with $2g-2+r>0$. Then there exists a polynomial $f_{g,r}(n) \in \mathbb{Q}[n]$ such that if k is an algebraically closed field of characteristic $p>2$, and C a general smooth curve over k of genus g with r general marked points, then the number of dormant torally indigenous bundles on C is given by $f_{g,r}(p)$. Furthermore, $f_{g,r}(n)$ has degree $3g-3+2r$, is even or odd as determined by its degree, and is always strictly positive for $n \geq 2$.*

In particular, if $r = 0$ we have that the number of semistable vector bundles $\mathscr{E}$ of rank two and trivial determinant on C such that $F^*\mathscr{E}$ contains a line bundle of degree $g-1$ is given by $2^{2g} f_{g,0}(p)$. If $g = 0$ we have that the number of maps $f : C = \mathbb{P}^1 \to \mathbb{P}^1$ ramified to order less than p at the marked points and unramified elsewhere, counted modulo automorphism of the image, is given by $2^{r-1} f_{0,r}(p)$.

The combinatorial result will require some preliminary definitions. We have:

Definition III.2.2. Let V, E be sets, and suppose that we are given φ a map from E to $V \cup \left(\binom{V}{2}\right)$. We then call $G = (V, E, \varphi)$ a *quasi-graph.* The standard notions of *edges*, *vertices*, and edges being *adjacent* to vertices generalize immediately to quasi-graphs. The set of edges E is naturally subdivided into *free* edges, which are $\varphi^{-1}(V)$, and *fixed* edges, given by $\varphi^{-1}\left(\binom{V}{2}\right)$.

Thus, a quasi-graph may be thought of simply as a graph where some edges – the free edges – are allowed to be attached to only a single vertex. A quasi-graph which consists of only fixed edges is simply a standard graph. Quasi-graphs arise naturally as the dual objects to nodal curves with marked points, where the marked points correspond to free edges of the dual quasi-graph; see Definition III.3.10. The usual notions of connectedness, regularity, loops, and simplicity for graphs immediately make sense in the context of quasi-graphs as well. When there is no ambiguity, we will denote by V and E the vertex and edge sets of a quasi-graph G.

We next associate a polytope to certain special quasi-graphs, denoting by $A(v)$ the set of edges adjacent to a vertex v:

Definition III.2.3. Let G be a quasi-graph which is regular of degree 3. The convex polytope $\mathscr{P}_G$ associated to G is defined to be the space of real-valued weight functions $w : E \to \mathbb{R}$ on the edge set of G satisfying the following inequalities:

(i) for each $e \in E$, $w(e) \geq 0$;

(ii) for each $v \in V$, $\sum_{e \in A(v)} w(e) \leq 1$;

(iii) for each $v \in V$ and $e \in A(v)$, $w(e) \leq \sum_{e' \in A(v) \smallsetminus \{e\}} w(e')$.

Note that condition (iii) is just the triangle inequality for the edges adjacent to any given vertex. Note also that (i) and (ii) bound all the $w(e)$ between 0 and 1, so in particular $\mathscr{P}_G$ is in fact a polytope.

With these definitions, we can now state the combinatorial result:

Theorem III.2.4. *Let G, G' be any two quasi-graphs, connected, regular of degree three, and having the same number of vertices and edges. Then the Ehrhart quasi-polynomials for $\mathscr{P}_G$ and $\mathscr{P}_{G'}$ agree at all odd values.*

When presented with cross-disciplinary results such as these, one naturally wonders whether they can be obtained more directly by further exploration of the situation. In algebraic geometry, it is a general phenomenon that answers to enumerative questions are given as polynomials in the characteristic, but as often as not, as in the case here, the only way to show this is to compute the answer and show *a posteriori* that it is a polynomial. We are thus motivated to ask:

Question III.2.5. Can one show *a priori* by methods of algebraic geometry that the number of dormant indigenous bundles must be given by a polynomial in p? Can such an argument be given covering a wider range of enumerative problems?

Correspondingly, we wonder:

Question III.2.6. Can one demonstrate directly a combinatorial relationship between the polytopes $\mathscr{P}_G$ for G as in Theorem III.2.4 which implies agreement of their Ehrhart quasi-polynomials?

We discuss additional, more concrete combinatorial questions in the final section.

III.3 Proofs

We start by associating a second polytope to any quasi-graph which is regular of degree 3, which is affinely isomorphic to $\mathscr{P}_G$, but imbedded in a larger-dimensional space:

Definition III.3.1. Let G be a quasi-graph which is regular of degree 3. We describe a second polytope $\mathscr{P}'_G$ associated to G, defined to be the space of real-valued weight functions $w : E \cup V \to \mathbb{R}$ on the edges and vertices of G satisfying the following inequalities:

(i) for each $e \in E$, $w(e) \geq 0$;

(ii) for each $v \in V$, $\sum_{e \in A(v)} w(e) = 2w(v)$;

(iii) for each $v \in V$ and $e \in A(v)$, $w(e) \leq w(v)$;

(iv) for each $v \in V$, $w(v) \leq 1$.

Indeed, one checks that points of $\mathscr{P}'_G$ correspond to points of $2\mathscr{P}_G$, by leaving the $w(e)$ unchanged and setting $w(v)$ as determined by (iii) above. The $w(v)$ act as 'dummy variables' to insure that lattice points of $n\mathscr{P}'_G$ are merely lattice points of $2n\mathscr{P}_G$ with $w(e)$'s having even sum at any v.

We also specify:

Definition III.3.2. Let G be a quasi-graph. A *sub-quasi-graph* H of G is a quasi-graph obtained by restricting the adjacency function φ for G to subsets of the vertex and edge sets on which φ remains well-defined.

In particular, a sub-quasi-graph may not change a fixed edge to a free edge.

Lemma III.3.3. *Let G be a quasi-graph which is connected and regular of degree 3. Then the odd values of the Ehrhart quasi-polynomials for $\mathscr{P}_G$ and $\mathscr{P}'_G$ differ by an integer multiple determined by G.*

Proof. Let $e_1, \dots, e_d$ be the edges of G, and $v_1, \dots, v_m$ be the vertices. Then the nth value of the Ehrhart quasi-polynomial of $\mathscr{P}_G$ (respectively, $\mathscr{P}_{G'}$) is by definition the number of possible integer values for the $w(e_i)$ (respectively, the $w(e_i)$ and $w(v_j)$) lying inside the (closed) polytope $n\mathscr{P}_G$ (respectively, $n\mathscr{P}'_G$), which is obtained by replacing the 1 in the definition of $\mathscr{P}_G$ (respectively, $\mathscr{P}_{G'}$) by n. We claim that for n odd, both of these are equivalent (up to constant integer multiple) to:

$$\#\{(\lambda_1, \ldots, \lambda_d, d_1, \ldots, d_m) \in \mathbb{Z}^{d+m} : \forall i, 0 < \lambda_i < n+2; \quad \forall j, d_j < n+2;$$
$$\forall i, j \text{ such that } e_i \in A(v_j), \lambda_i \leq d_j; \quad \forall j, 2d_j + 1 = \sum_{i: e_i \in A(v_j)} \lambda_i\} \quad \text{(III.3.4)}$$

First, note that the conditions $\lambda_i < n+2$ are superfluous. If we set $\lambda_i = w(e_i) + 1$ for each i and $d_j = w(v_j) + 1$ for each j, we recover the description of the nth value of the Ehrhart quasi-polynomial of $\mathscr{P}'_G$. Next, it is easily checked that if we set $\lambda_i = 2w(e_i) + 1$ (at which point the d_j are all uniquely determined) as the $w(e_i)$ range over all possibilities for the nth value of the Ehrhart quasi-polynomial of $\mathscr{P}_G$, we recover all possible assignments of the λ_i, d_j for which λ_i are all odd. The key observation is that for n odd, the number of possibilities with all λ_i odd is $\frac{1}{N_G}$ times the total number of possibilities, where N_G is the number of (not necessarily spanning) sub-quasi-graphs of G which are regular of degree 2. Indeed, if one starts with an arbitrary assignment of λ_i, d_j, the edges e_i for which λ_i are even gives such a sub-quasi-graph, and if all λ_i which are even are replaced by $n + 2 - \lambda_i$ (and the d_j adjusted accordingly), one can check that all conditions are preserved, and one obtains an assignment with all λ_i odd. This sets up a natural, visibly invertible correspondence between arbitrary assignments of λ_i and d_j satisfying the required inequalities, and pairs of assignments with all λ_i odd together with an arbitrary sub-quasi-graph of G which is regular of degree 2. This completes the proof of the claim, and the lemma. □

We now proceed to describe the vertices of the polytopes we have constructed. We start with:

Lemma III.3.5. *Let G be a regular quasi-graph of degree* 3*. Then any vertex of $\mathscr{P}_G$ whose coordinates are all non-zero has coordinates equal to $\frac{1}{4}$ or $\frac{1}{2}$. More precisely, for any vertex of $\mathscr{P}_G$, the weights associated to the three edges adjacent to any vertex of G are $\{\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\}$. Moreover, given such a vertex w of $\mathscr{P}_G$, let H be the sub-quasi-graph of G with edge set $E(H) = \{e \mid w(e) = \frac{1}{4}\}$. Then H consists of cycles of odd length.*

Proof. Denote by E_2 and E_1 the sets of fixed and free edges of G respectively. Then one has $3(\#V) = 2(\#E_2) + \#E_1$. A vertex of $\mathscr{P}_G$ must satisfy all of the inequalities listed in Definition III.2.3; moreover, by replacing the inequalities with equalities, one obtains a collection of linear constraints, and among these, the vertex must satisfy some $\#E$ independent constraints. By hypothesis, none of these constraints are of the form $w(e) = 0$, so they are chosen from the $4(\#V)$ constraints of the form $w(e) + w(e') + w(e'') = 1$ or $w(e) = w(e') + w(e'')$ where e, e', e'' are the three edges (possibly with multiplicity) adjacent to some vertex of G. We note that for any given vertex of G, we cannot have two constraints of the second form, since that would force one of the weights to be 0. Therefore, each vertex of G can supply at most two constraints, and in the case of two, one of the two is necessarily $w(e)+w(e')+w(e'') = 1$. Moreover, since we need $\#E = \#E_1 + \#E_2$ constraints, we must have at least $\frac{1}{3}(\#E_2) + \frac{2}{3}(\#E_1)$ vertices of G with two constraints; denote these vertices by V^2. For any given such vertex in V^2, we may write the two constraints as $w(e) + w(e') + w(e'') = 1$ and $w(e) = w(e') + w(e'')$, which yields $w(e) = \frac{1}{2}$. Let E^2 denote the set of edges e forced to have weight $\frac{1}{2}$ in this manner.

Now, by the hypothesis that all weights are non-zero, for any given vertex of G we cannot have two edges with weight $\frac{1}{2}$, so every vertex in V (and in particular in V^2) is adjacent to a unique edge of E^2. The next observation is that conversely, every edge $e \in E^2$ is adjacent to a unique vertex of V^2. Indeed, if both vertices adjacent to e were in V^2, the two constraints at each would both necessarily force e to have weight $\frac{1}{2}$, which would imply that they were linearly dependent. We claim that in fact every vertex of G is adjacent to an edge in E^2; that is, E^2 gives a perfect matching of G. We subdivide V^2 into V^{21} and V^{22}, according to whether the corresponding edge of E^2 is free or fixed, respectively. We therefore want to show that $\#V^{21} + 2(\#V^{22}) \geq \#V$. But $2(\#V^{21}) + 2(\#V^{22}) = 2(\#V^2) \geq \frac{2}{3}(\#E_2) + \frac{4}{3}(\#E_1)$, and by definition $\#V^{21} \leq \#E_1$, so subtracting we find that $\#V^{21} + 2(\#V^{22}) \geq \frac{2}{3}(\#E_2) + \frac{1}{3}(\#E_1) = \#V$, as claimed. Finally, we can conclude that $\#V^{21} + 2(\#V^{22}) = \#V$, and it follows that we must have had $\#V^{21} = \#E_1$, and no vertices of G without any associated constraints.

The next step is to show that given our description so far, if one assigns $\frac{1}{4}$ to all edges not in E^2, then this assignment satisfies every constraint which is permissible based on the hypotheses that all weights are non-zero and that the weights of E^2 are predetermined as $\frac{1}{2}$. Indeed, let v be any vertex of G, and e, e', e'' its adjacent edges. Suppose without loss of generality that $e \in E^2$. Then under the assignment $w(e') = w(e'') = \frac{1}{4}$, both constraints $w(e)+w(e')+w(e'') = 1$ and $w(e) = w(e')+w(e'')$ will be satisfied. All that remains is to note that with $w(e) = \frac{1}{2}$, no valid assignment of $w(e')$ and $w(e'')$ can achieve $w(e') = w(e) + w(e'')$ or $w(e'') = w(e) + w(e')$, since with all three weights positive, their sum would have to be greater than 1. Thus, our assignment satisfies any possible choice of constraints associated to the vertices of G not in V^2, and we conclude that our chosen vertex of $\mathscr{P}_G$ has coordinates of the desired form.

We prove the last assertion by contradiction. Suppose the statement is false for a given vertex w of $\mathscr{P}_G$, and H is the corresponding sub-quasi-graph. Since each vertex of G must have assignments $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ to the three edges attached to it, H is a regular quasi-sub-graph of G of degree 2. Thus, H consists of cycles or paths. By assumption, H either has a cycle of even length or a path. In either case, we can replace the weight on the path or the even length cycle with alternating values of $\frac{1}{8}, \frac{3}{8}$. Then the new assignment still satisfies every constraint satisfied by the old values. Therefore, this contradicts the fact that in order to be a vertex, w must be uniquely determined by those constraints. So H must only consists of odd cycles. ☐

We can now conclude:

Proposition III.3.6. *Let G be a regular quasi-graph of degree 3. Then any vertex of $\mathscr{P}_G$ has each of its coordinates equal to 0, $\frac{1}{4}$ or $\frac{1}{2}$, with the only possible weights associated to the edges adjacent to a given vertex of G being $\{0,0,0\}, \{0, \frac{1}{2}, \frac{1}{2}\}, \{0, \frac{1}{4}, \frac{1}{4}\}$, and $\{\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\}$. Any vertex of $\mathscr{P}'_G$ has each of its coordinates equal to 0, $\frac{1}{2}$ or 1.*

Proof. We begin with the assertion for $\mathscr{P}_G$, working by induction on the number of coordinates of a given vertex which are equal to 0. The base case is that all coordinates are non-zero, which we handled in the previous lemma. Now, suppose we have an edge e of G whose weight is zero for our chosen vertex of $\mathscr{P}_G$. Suppose e is a fixed edge, and let v_1, v_2 be the two adjacent vertices; note that it suffices to consider the case that v_1 and v_2 are distinct, since if e is a loop, the triangle inequalities in the definition of $\mathscr{P}_G$ will force the other edge adjacent to $v_1 = v_2$ to have weight 0 as well, and we could instead choose this edge to be e. Next, note that if e_1, e_1' and e_2, e_2' are the adjacent edges other than e at v_1, v_2 respectively, then the triangle inequalities at v_1 and v_2 force $w(e_1) = w(e_1')$ and $w(e_2) = w(e_2')$. We define a new graph G' obtained by removing e, v_1, and v_2, and replacing each pair e_1, e_1' and e_2, e_2' by single edges e_1'' and e_2'' in the obvious way: that is, if e_i, e_i' are each adjacent to vertices other than v_1 or v_2, replace them with an edge adjacent to those two vertices; if only one is adjacent to a vertex other than v_1 or v_2, replace them by a single free edge; if neither is adjacent to a vertex other than v_1 or v_2, remove them entirely.

Now, we claim that we obtain a vertex of $\mathscr{P}_{G'}$ by assigning weights to the edges of G' which are the same as G where the graphs are the same, and which assign the common weight of e_i, e_i' to the new edges e_i'' for $i = 1, 2$. It suffices to show that we can provide constraints from the definition of $\mathscr{P}_{G'}$ to replace any constraints that were lost when v_1 and v_2 were removed. The constraints coming from triangle inequalities at v_i are easily replaced: they can either require $w(e_i) = w(e_i')$, or $w(e_i) = w(e_i) = 0$. The first condition is superfluous, while the second can be replaced by the constraint $w(e_i'') = 0$. So we need only show that we can effectively replace the condition that the sum of the weights at a v_i be equal to 1, which gives $w(e_i) = w(e_i') = \frac{1}{2}$, so is equivalent to requiring $w(e_i'') = \frac{1}{2}$. Choose v_3 to be a vertex adjacent to e_i''; without loss of generality, suppose this was the vertex adjacent to e_i in G. Since we had $w(e_i) = \frac{1}{2}$ in G, the inequality requiring the sum of the three weights at v_3 to be at most 1, together with the triangle inequality for $w(e_i)$, implies that in fact both of these inequalities are sharp, giving corresponding constraints satisfied in G, so we can then require them also in G' in order to force the weight of e_i'' to be $\frac{1}{2}$, as desired.

By the induction hypothesis, we can assume that the vertex of $\mathscr{P}_{G'}$ we have constructed has weights only equal to 0, $\frac{1}{4}$ or $\frac{1}{2}$, with the weights of the asserted form for edges adjacent to a particular vertex. One checks easily that our description of weights of the edges adjacent to a given vertex is preserved by the construction, as long as we verify that if in constructing G' we removed an edge of G other than e, its weight must also have been one of $0, \frac{1}{4}, \frac{1}{2}$. Such a removal only occurred if both e_i and e_i' were both adjacent only to v_1, v_2 for $i = 1$ or 2. There are only two possibilities: either $e_i = e_i'$ is a loop, or e_i and e_i' are both adjacent to both v_1 and v_2, in which case these three edges and two vertices are necessarily all of G. In the first case, one checks that in order for the weight of the loop to be uniquely determined, given that $w(e) = 0$, its weight is necessarily either 0 or $\frac{1}{2}$. In the second case, one can check the assertion of the Proposition directly for G (see also Example III.4.1). Finally, note that although we carried out this process in the case that e was fixed, the argument works equally well (and is in fact simpler) in the case that e is free. This completes the induction argument for $\mathscr{P}_G$.

We may now conclude the desired statement for $\mathscr{P}'_G$: for finding vertices of our polytopes, we work over $\mathbb{R}$, and in this setting, as mentioned above, the introduction of the $w(v)$ coordinates are irrelevant, and if we ignore these coordinates, the polytope $\mathscr{P}'_G$ is the same as $2\mathscr{P}_G$. In particular, with the possible exception of the $w(v)$, all coordinates of vertices of $\mathscr{P}'_G$ are equal to 0, $\frac{1}{2}$ or 1. But the same follows for $w(v)$ from our sharp description of the possible weights associated to edges adjacent to a vertex of G in the case of a vertex of $\mathscr{P}_G$. □

This gives us our key result:

Corollary III.3.7. *The odd values of the Ehrhart quasi-polynomial of $\mathscr{P}'_G$, and hence $\mathscr{P}_G$, are given by a single polynomial, of degree equal to $\#E$.*

Proof. We note that the dimension of $\mathscr{P}_G$ (and hence $\mathscr{P}'_G$) is equal to $\#E$: indeed, it is easily verified that the $(\#E)$-cube with all weights between $\frac{1}{6}$ and $\frac{1}{3}$ lies inside $\mathscr{P}_G$. The assertion for $\mathscr{P}'_G$ is then immediate from Proposition III.3.6 and the existence theorem for Ehrhart-quasi-polynomials. The assertion for $\mathscr{P}_G$ then follows by Lemma III.3.3. □

In Proposition III.3.6, we state that the coordinates of any vertex of $\mathscr{P}_G$ can be $0, \frac{1}{4}$ or $\frac{1}{2}$. Because the appearance of $\frac{1}{4}$ allows the quasi-period to be 4, we wonder in which cases $\frac{1}{4}$ does not appear in the coordinates. The following lemma answers this question.

Lemma III.3.8. *Let G be a connected regular quasi-graph of degree* 3, *associated to a curve of type* (g, r). *If $g = 0$, $\mathscr{P}_G$ has no vertices with quarter-integer coordinates. If $g > 0$, we have following two cases.*

(i) If $r > 0$ or G has a "bridge" edge e (i.e., removing e makes G disconnected), then $\mathscr{P}_G$ has a vertex with quarter-integer coordinates.

(ii) If $r = 0$ and G does not have a "bridge", when $g = 2, 3$, or 4, there is only one graph G such that $\mathscr{P}_G$ does not have vertices with quarter-integer coordinates; when $g \geq 5$ (or equivalently $\#V \geq 8$), $\mathscr{P}_G$ always has a vertex with quarter-integer coordinates.

Proof. By Lemma III.3.5 and the construction in Proposition III.3.6, it is clear that vertices with quarter-integer coordinates appear only if there are cycles in G, which is equivalent to $g > 0$. Thus, when $g = 0$, $\mathscr{P}_G$ does not have vertices with quarter-integer coordinates.

Now we assume $g > 0$.

We prove (i) by induction on the number of vertices in G. The base case is when $\#V = 1$: the only graph G that satisfies the conditions is the one with a single vertex attached to a free edge and a loop. One can check that $\mathscr{P}_G$ has a vertex $(\frac{1}{2}, \frac{1}{4})$. We now assume $\#V \geq 2$, and (i) holds for any G' with $\#V(G') < \#V$.

If G has a bridge e, we can cut e into two free edges, and obtain two graphs G_1 and G_2. If both of $g(G_1)$ and $g(G_2)$ are positive, let w_1 and w_2 be vertices in $\mathscr{P}_{G_1}$ and $\mathscr{P}_{G_2}$ with quarter-integer coordinates, respectively. If one of $w_1(e)$ and $w_2(e)$ is 0, without loss of generality we may assume $w_1(e) = 0$. Then if we assign weights w to the edges of G by $w(e) = w_1(e)$ for e in G_1 and $w(e) = 0$ for e in G_2, we obtain a vertex in $\mathscr{P}_G$ with quarter-integer coordinates. If neither of $w_1(e)$ and $w_2(e)$ is 0, then $w_1(e) = w_2(e) = \frac{1}{2}$. Combining w_1 and w_2 gives the desired vertex in $\mathscr{P}_G$. If one of $g(G_1)$ and $g(G_2)$ is 0, without loss of generality we may assume $g(G_1) > 0$ and

$g(G_2) = 0$. Let w_1 be a vertex in $\mathscr{P}_{G_1}$ with quarter-integer coordinates. We know $w_1(e) = 0$ or $\frac{1}{2}$. Since $g(G_2) = 0$, we have that G_2 is a tree. One can check that $\mathscr{P}_{G_2}$ has a vertex w_2 such that $w_2(e) = w_1(e)$. Once again, combining w_1 and w_2 gives the desired vertex in $\mathscr{P}_G$.

If G does not have a bridge and $r > 0$, let e be a free edge adjacent to a vertex v, and let e_1 and e_2 be the other two edges connecting to v. There exists a cycle C (without repeated edges) in G starting from e_1 and ending at e_2. $\mathscr{P}_G$ has a vertex with $w(e) = \frac{1}{2}$, with $w(e') = \frac{1}{4}$ for all e' in C, and with $w(e') = 0$ for all remaining edges e'. This completes the proof for (i).

For (ii), suppose $r = 0$ and G does not have a bridge. When $g = 2$, there are only two graphs associated to this type (see example III.4.1). Only G_1, the one with two vertices connected by three edges, does not have vertices with quarter-integer coordinates in $\mathscr{P}_{G_1}$. When $g = 3$, suppose G is a graph such that none of the vertices of $\mathscr{P}_G$ have quarter-integer coordinates. Then each vertex of $\mathscr{P}_G$ must have an edge e with weight 0. Suppose e is attached to vertices v_1 and v_2. We do the same thing as in Proposition III.3.6. We define a new graph G' obtained by removing e, v_1 and v_2 and merging the other two edges connecting to v_1 and v_2. Since e is not a bridge, G' is still connected. Because $w(e) = 0$, the weights of the two other edges adjacent to v_1 or v_2 are the same. Therefore, we have a natural way to give a weight assignment to the edges of G' and clearly it gives a vertex of $\mathscr{P}_{G'}$. Thus, we conclude that G' has to be G_1. One checks that the only G which can be changed to G_1 in this way is the graph with four vertices, each of which connects to the other three vertices. When $g = 4$, by a similar argument, we find that the only graph G not having vertices with quarter-integer coordinates in $\mathscr{P}_G$ is the one with $V = \{v_1, v_2, \dots, v_6\}$ and $E = \{v_i v_{i+1} (1 \le i \le 5), v_6 v_1, v_j v_{j+3} (1 \le j \le 3)\}$. Finally, similar arguments help us to confirm that when $g = 5$, all of the graphs have a vertex with quarter-integer coordinate in $\mathscr{P}_G$, and the case that $g > 5$ follows. □

Recall the following:

Definition III.3.9. A *nodal curve* C is a curve obtained from a (not necessarily connected) smooth curve by gluing together pairs of points transversely, creating *nodes* at these points. The smooth curve from which C is obtained is unique, and called the *normalization* of C, and will be denoted $\tilde{C}$. When considering curves with marked points, we set the marked points of $\tilde{C}$ to be the points lying above marked points or nodes of C. Finally, a *totally degenerate curve* is a nodal curve such that $\tilde{C}$ consists of disjoint copies of $\mathbb{P}^1$ with three marked points each.

Because any three points on $\mathbb{P}^1$ are equivalent up to automorphism, a totally degenerate curve is determined by combinatorial data, and specifically by the dual quasi-graph:

Definition III.3.10. Let C be a nodal curve with marked points. Then the *dual quasi-graph* associated to C is defined to be the quasi-graph whose vertices are the components of C, whose fixed edges correspond to nodes of C and are adjacent to the components intersecting at a given node, and whose free edges correspond to marked points of C, and are adjacent to the component on which the marked point lies.

One checks directly that a C such that $\tilde{C}$ is a disjoint union of $\mathbb{P}^1$'s is totally degenerate if and only if its dual quasi-graph is regular of degree 3, and that conversely given a quasi-graph which is regular of degree 3, there is a unique totally degenerate curve having the chosen dual quasi-graph. Finally, one checks that the type (g, r) of the curve is related to the number of vertices and edges of the dual quasi-graph by the formulas $\#V = 2g - 2 + r, \#E = 3g - 3 + 2r$.

The theorem we will use which is implicit in Mochizuki's work may be stated as:

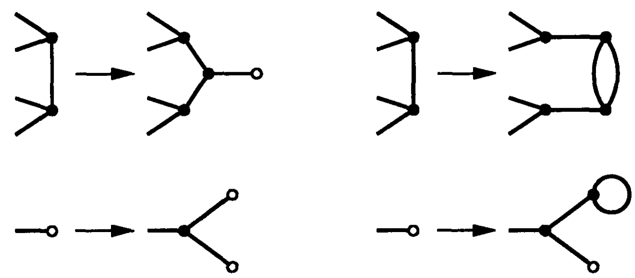
Theorem III.3.11. *(Mochizuki) Fix $g, r \geq 0$ with $2g - 2 + r > 0$, and p an odd prime. Then the number of dormant torally indigenous bundles on a general curve of type (g, r) over an algebraically closed field of characteristic p is given as the $(p-2)$nd value of the Ehrhart quasi-polynomial of $\mathscr{P}_G$, where G is any connected regular quasi-graph of degree 3 satisfying $\#V = 2g - 2 + r, \#E = 3g - 3 + 2r$; in particular, these values depend only on $\#V$ and $\#E$.*

Proof. The first relevant statement is that the number of dormant torally indigenous bundles on a general curve of type (g, r) over an algebraically closed field may be computed at any totally degenerate curve of type (g, r). This follows immediately from the assertions of [23, Thm 2.8, p. 153] in the $n = 0$ case that the stack of dormant torally indigenous bundles is finite and flat over $\overline{\mathcal{M}}_{g,r}$ and is étale over points corresponding to totally degenerate curves. Next, one needs to know that a dormant torally indigenous bundle on a totally degenerate curve C is equivalent to dormant torally indigenous bundles on each component of $\tilde{C}$ having radii which agree at any two points which are glued together; this is immediate from [23, $ I.4.4, p. 118] when one takes into account that the dormancy condition is simply a condition of vanishing p-curvature, and will not be affected by gluing.

The final ingredient is the description of dormant torally indigenous bundles on $\mathbb{P}^1$ with three marked points, given as the $n = 0$ case of [23, Thm. IV.2.3, p. 211]. If we are given λ_i as in this last theorem (these always exist, since the radii are only defined up to ± 1, so we could choose all the λ_i to be odd), we have to check that the existence of a finite separable morphism from $\mathbb{P}^1$ to itself ramified to orders λ_i at 0, 1, ∞ and unramified elsewhere is equivalent to the conditions that the degree d, which by the Riemann-Hurwitz formula is determined by $2d + 1 = \sum \lambda_i$, must also satisfy $d < p$ and $d \geq \lambda_i$ for all i. This is shown in [27], but could also be deduced directly from [23, (2), p. 232]. Given all of this, one can verify directly that the dormant torally indigenous bundles on a totally degenerate curve C are nearly counted by setting $n = p - 2$ in Equation III.3.4 as applied to the dual quasi-graph G of C. The only discrepancy is a factor of the N_G of the proof of Lemma III.3.3, since radii are only defined up to ± 1 and therefore assignments of λ_i which differ by ± 1 mod p give the same dormant torally indigenous bundle. Since we saw in the proof of Lemma III.3.3 that $\frac{1}{N_G}$ times the value of Equation III.3.4 computed the odd values of the Ehrhart quasi-polynomial of $\mathscr{P}_G$, we thus conclude the desired result. □

We can now easily give the proofs of our main theorems:

Proof of Theorem III.2.1. We note that given any specified (g, r) with $2g - 2 + r > 0$, we can find a connected regular quasi-graph of degree 3 having the number of vertices and edges required by Theorem III.3.11. This is equivalent to the standard algebraic



Increment r, fixing g Increment g, fixing r

geometry statement that there exist totally degenerate curves of any hyperbolic type, but one can easily verify it directly. Indeed, the figure demonstrates how to increase either g or r by 1 while keeping the other fixed, and given this it suffices to check the base cases of $(g, r) = (0, 3), (1, 1), (2, 0)$, which is easily accomplished. Putting together Corollary III.3.7 with Theorem III.3.11 then gives the existence and degree of the desired polynomial. The positivity follows from the fact that for any $n \geq 0$, $\mathscr{P}_G$ contains the lattice point with all weights equal to 0. Lastly, to see that the polynomial is always odd or even, we note that simply by translating all coordinates by 1 one sees that the number of lattice points in the interior of $n\mathscr{P}_G$ is equal to the number in the the closed polytope $(n-4)\mathscr{P}_G$. Applying the reciprocity theorem for Ehrhart polynomials then easily gives the desired result. □

Proof of Theorem III.2.4. This is almost the same as Theorem III.3.11, except that it asserts agreement of the nth value of the Ehrhart quasi-polynomial for all odd n rather than those for which $n + 2$ is a prime. The stronger statement then follows from Corollary III.3.7, although in fact for this application one need not consider $\mathscr{P}'_G$ at all: it is enough to use the existence of Ehrhart quasi-polynomials once one knows that the Ehrhart quasi-polynomial of $\mathscr{P}_G$ has quasi-period 4, since Mochizuki's values then give infinitely many values for n congruent to either 1 or 3 mod 4. Thus, to prove Theorem III.2.4 it suffices to know the statement of Proposition III.3.6 for $\mathscr{P}_G$ only. □

Remark III.3.12. Translating from self-maps of $\mathbb{P}^1$ to indigenous bundles and back in order to obtain the statement for $g = 0$ of Theorem III.2.1 may seem superfluous, and indeed one could argue directly using the results of [27] that the number of such maps is counted by the Ehrhart polynomial of $\mathscr{P}'_G$. However, there is something to be said for concluding the statement as a special case of a more general result.

In fact, an induction argument similar to that carried out for Proposition III.3.6 can be used to show that the number N_G of Lemma III.3.3 also depends only on the number of vertices and edges of G:

Lemma III.3.13. *Let G be a quasi-graph which is connected and regular of degree 3. N_G, the number sub-quasi-graphs of G which are regular of degree 2, is 2^g for $r = 0$ and 2^{g+r-1} for $r > 0$.*

Proof. For any sub-quasi-graph H of G of degree 2, we can consider its vertex set $V(H)$ to be the vertex set V of G. Thus, we only care about its edge set $E(H)$. Let A_G be the set of all of sub-quasi-graphs of G of degree 2 with a product operation defined by setting $H \cdot H'$ to be the graph whose edge set is $\{e \mid e \in E(H) \text{ or } e \in E(H'), \text{ but } e \notin E(H) \cap E(H')\}$. One can check that this product operation is well-defined, i.e., $H \cdot H' \in A_G$, for any $H, H' \in A_G$, and is commutative. Therefore, A_G is an abelian group with identity given by the empty graph I with vertex set V. For any $H \in A_G$, $H \cdot H = I$. Hence, the order N_G of A_G is 2^s, where s is the size of a minimal set of generators of A_G. Another observation we have for $H \in A_G$ is that H consists of loops and path, where every loop consists of fixed edges, and every path starts with a free edge, ends with another free edge, and has fixed edges in between.

We prove the lemma by induction on the number of edges $\#E$ of G. The base cases are $(\#V, \#E) = (1,3), (1,2), (2,3)$, (equivalently, $(g,r) = (0,3), (1,1), (2,0)$,) which are easy to check. Now, suppose $\#E \geq 4$ and the lemma holds for all the graphs whose edge set size is less than $\#E$. Denote by E_1 and E_2 the sets of free and fixed edges of G respectively. Recall that $\#V = 2g - 2 + r$ and $\#E = 3g - 3 + 2r$. Hence, $g = 2\#V - \#E + 1$ and $r = 2\#E - 3\#V$. Also, it's easy to calculate that $\#E_1 = r$.

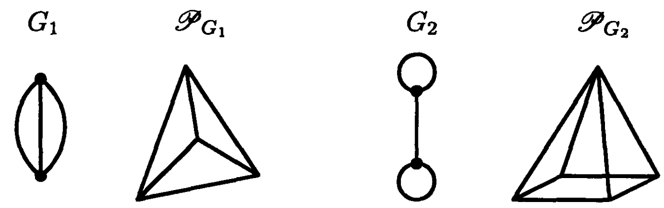
If $r = \#E_1 > 0$, let e be a free edge connecting to a vertex v; then the other two edges e_1, e_2 connected to v cannot be both free edges since $\#E \geq 4$. We construct a new graph G' by removing e and v and merging e_1 and e_2 into one edge. One can check that G' is a connected regular quasi-graph of degree 3 with $\#E(G') = \#E - 2$ and $\#V(G') = \#V - 1$. Thus, $g(G') = g$ and $r(G') = r - 1$. By the induction hypothesis, $N_{G'}$ is 2^g for $r = 1$ and 2^{g+r-2} for $r \geq 2$. However, when $r = 1$, there is no $H \in A_G$ containing e as an edge. Hence, there is a bijection between A_G and $A_{G'}$. Thus, $N_G = N_{G'} = 2^g = 2^{g+r-1}$. When $r \geq 2$, let e' be another free edge in G. Let H be a path from e to e' (without using same edges more than once). Then $H \in A_G$. One can see that H together with a minimal set of generators of $A_{G'}$ gives a minimal set of generators for A_G. Therefore, $N_G = 2N_{G'} = 2^{g+r-1}$.

If $r = \#E_1 = 0$, G is a graph without free edges. Because $\#E \geq 4$, there exist two vertices v_1 and v_2, such that there is one and only one edge e connecting them. If e is a "bridge", i.e., by removing e, G becomes disconnected, then we cut e into two free edges and disconnect G into two graphs G_1 and G_2. Clearly $r(G_1) = r(G_2) = 1$. One can calculate that $g(G_1) + g(G_2) = g$. Since, e is a bridge, there are no loops containing e. Thus, $\nexists H \in A_G$ such that $e \in E(H)$. Therefore $N_G = N_{G_1} \cdot N_{G_2} = 2^g$. If e is not a bridge, let H be a loop containing e (without repeating edges). Then $H \in A_G$. We obtain G' be removing e, v_1 and v_2 and merging the two other edges connecting to v_i, for $i = 1, 2$. Using the similar argument as for the case $r \geq 2$, we can get that $N_G = 2N_{G'} = 2^g$. □

It then follows that one also obtains identities for the odd values of the Ehrhart quasi-polynomials of $\mathscr{P}'_G$.

Corollary III.3.14. *Let G, G' be any two quasi-graphs, connected, regular of degree three, and having the same number of vertices and edges. Then the odd values of the Ehrhart quasi-polynomials for $\mathscr{P}'_G$ and $\mathscr{P}'_{G'}$ agree to a single polynomial, of degree $\#E$.*

We have chosen to phrase our main result in terms of $\mathscr{P}_G$ partly for the sake of simplicity, and partly because it seems like the more natural object, in that its Ehrhart quasi-polynomial computes the number of dormant torally indigenous bundles directly, and it is imbedded in a space of its own dimension.



The case $(g, r) = (2, 0)$

III.4 Further Remarks and Questions

In this section, we discuss some explicit examples and possible directions of further investigation, with a particular focus towards the combinatorial side. We begin by describing the simplest example of our results, seeing that the combinatorial identities obtaining in Theorem III.2.4 do in fact appear to be non-trivial.

Example III.4.1. Consider the case of $(g, r) = (2, 0)$, or equivalently graphs with three edges and two vertices. One checks that we get only two graphs: the G_1 and G_2 of the figure. The corresponding polytopes $\mathscr{P}_{G_1}$ and $\mathscr{P}_{G_2}$ are, respectively: a regular tetrahedron with vertices at $(0,0,0)$, $(\frac{1}{2}, \frac{1}{2}, 0)$, $(0, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2})$; and a square pyramid with vertices $(0,0,0)$, $(\frac{1}{2}, 0, 0)$, $(0, \frac{1}{2}, 0)$, $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. One finds that in fact not only the odd values, but the entire Ehrhart quasi-polynomials of $\mathscr{P}_{G_1}$ and $\mathscr{P}_{G_2}$ agree, and are given by $\frac{1}{24}(n^3 + 6n^2 + 20n + 24)$ for even n and $\frac{1}{24}(n^3 + 6n^2 + 11n + 6)$ for odd n. The number of dormant torally indigenous bundles in this case is thus given by $\frac{1}{24}(p^3 - p)$.

While we have not explicitly presented $\mathscr{P}_G$ for different G in further cases, one can compute that in the case of $(g, r) = (3, 0)$, there are five different graphs, for which the corresponding $\mathscr{P}_G$ have 8, 10 or 14 vertices depending on G. Thus, in this situation there are at least three polytopes which are not combinatorially equivalent for which we obtain relations. It seems reasonable to expect that the number of non-trivial identities obtained will grow with (g, r).

In computing further examples, there are two phenomena which stand out. The first is that in all examples computed so far, for any two G, G' as in Theorem III.2.4 we find that not only the odd values of the Ehrhart quasi-polynomial agree, but the even values agree as well. This holds for examples with (g, r) up to $(4, 0)$, as well $(0, 6)$ and $(1, 2)$. The data for $(5, 0)$ is also consistent with this conclusion, although computation of the entire Ehrhart polynomial for even a single graph in this case appears unfeasible. We therefore conjecture:

Conjecture III.4.2. *In Theorem III.2.4, the restriction to odd values of the Ehrhart quasi-polynomials is unnecessary.*

We also remark that the same seems to hold for $\mathscr{P}'_G$ in the few examples we have computed thus far. This is interesting in its own right, as there is no obvious relation between the even values of the Ehrhart quasi-polynomials of $\mathscr{P}_G$ and $\mathscr{P}'_G$.

We also make some observations on the period of the Ehrhart quasi-polynomial of $\mathscr{P}_G$. First, in certain cases with $r > 0$ (for instance, when $(g, r) = (1, 1)$, $(1, 2)$, or $(3, 1)$) one can compute that the Ehrhart quasi-polynomial in fact has quasi-period 4. This is as expected based on the fact that for any graph G of the corresponding type, some vertices of $\mathscr{P}_G$ have quarter-integer coordinates, but it is interesting in that it means that Theorem III.2.4 is producing potentially infinitely many examples of rational polytopes for which the Ehrhart quasi-polynomial has different quasi-periods when restricting attention to even or odd values. In contract, we have observed that when $r = 0$ the Ehrhart quasi-polynomials of $\mathscr{P}_G$ and $\mathscr{P}_{G'}$ always appear to have quasi-period 2 and 1 respectively, rather than the *a priori* expected quasi-periods of 4 and 2. This would follow from the above conjecture for $g \leq 4$, since in this case by Lemma III.3.8, there is always a G with all vertices of $\mathscr{P}_G$ lying on half-integers (and correspondingly, all vertices of $\mathscr{P}'_G$ lying on integers). However, Lemma III.3.8 also says that for $g = 5$ and above every $\mathscr{P}_G$ must have some vertices with quarter-integer coordinates. Yet, the data we have for $g = 5$ is consistent with the Ehrhart quasi-polynomial of $\mathscr{P}_G$ having quasi-period 2. Thus, we seem to have a separate pattern not explained by our previous conjecture, and we ask:

Question III.4.3. Is it true that for $r = 0$, the quasi-period of the Ehrhart quasi-polynomial of $\mathscr{P}_G$ is always 2? Are there other cases where the quasi-period is smaller than expected based on the denominators of the vertices of $\mathscr{P}_G$ as G ranges over all quasi-graphs corresponding to (g, r)? Is the quasi-period of the Ehrhart quasi-polynomial for $\mathscr{P}'_G$ always half the quasi-period for $\mathscr{P}_G$?

Finally, we would like to add that Mochizuki had already remarked on the existence of apparently non-trivial combinatorial identities implicit in his work (see [23, p. 238-239]), but in a more general setting than we have treated here. Additional families of combinatorial identities are obtainable by considering more general nilpotent torally indigenous bundles than the dormant ones we have examined. For instance, Mochizuki treats a few cases of these identities for $g = 0, 1$ in the situation of ordinary torally indigenous bundles; see [23, p. 24] and [23, Cor. V.1.3, p. 237]. He also develops the basic combinatorial algorithms necessary to describe the intermediate cases, called spiked torally indigenous bundles; see [23, Def. II.3.1, p. 160], [23, Thm. V.1.1, p. 236], as well as the diagram [23, p. 270]. Translating these identities into combinatorial language is likely to be more complicated than for the dormant case, but may yield correspondingly more interesting identities.

Appendix A

Proof of Proposition II.8.1

The purpose of this appendix is to prove Proposition II.8.1 by showing both sides of (II.8.1) are equal to

$$\frac{1}{d!}\sum_{\sigma\in\mathfrak{S}_d}\operatorname{sign}(\sigma)\prod_{j=1}^{d} z(\sigma,j).$$

We always assume that $V=\{v_1,v_2,\ldots,v_{d+1}\}$ is the vertex set of a d-simplex in general position, where the coordinates of v_i are $\mathbf{x}_i=(x_{i,1},x_{i,2},\ldots,x_{i,d})$. We first give some new notation and definitions.

For all $k: 1\le k\le d$, let $\widehat{\mathbf{x}}_k=(\widehat{x}_{k,1},\ldots,\widehat{x}_{k,d})=(x_{k,1}-x_{d+1,1},\ldots,x_{k,d}-x_{d+1,d})=\mathbf{x}_k-\mathbf{x}_{d+1}$. Define

$$\widehat{X}_V(\sigma,k)=\begin{pmatrix}\widehat{x}_{\sigma(1),1} & \widehat{x}_{\sigma(1),2} & \cdots & \widehat{x}_{\sigma(1),k}\\ \widehat{x}_{\sigma(2),1} & \widehat{x}_{\sigma(2),2} & \cdots & \widehat{x}_{\sigma(2),k}\\ \vdots & \vdots & \ddots & \vdots\\ \widehat{x}_{\sigma(k),1} & \widehat{x}_{\sigma(k),2} & \cdots & \widehat{x}_{\sigma(k),k}\end{pmatrix},$$

$$\widehat{Y}_V(\sigma,k)=\begin{pmatrix}1 & \widehat{x}_{\sigma(1),1} & \cdots & \widehat{x}_{\sigma(1),k-1}\\ 1 & \widehat{x}_{\sigma(2),1} & \cdots & \widehat{x}_{\sigma(2),k-1}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & \widehat{x}_{\sigma(k),1} & \cdots & \widehat{x}_{\sigma(k),k-1}\end{pmatrix},$$

and

$$\widehat{z}_V(\sigma, k) = \det(\widehat{X}_V(\sigma, k)) / \det(\widehat{Y}_V(\sigma, k)).$$

Then

$$\widehat{z}_V(\sigma, k) = (-1)^k z_V(\sigma, k). \tag{A.0.1}$$

Again, when there is no confusion, we omit the subscript V from $\widehat{X}_V(\sigma, k)$, $\widehat{Y}_V(\sigma, k)$ and $\widehat{z}_V(\sigma, k)$.

We define certain subsets of the symmetric group $\mathfrak{S}_d$, which we will use in our later proofs. We denote by $\mathfrak{S}_T$ the set of permutations on some set T and use one-line notation for all permutations.

Definition A.0.2. a) Let $(\Lambda, \Gamma, \Delta)$ be a partition of $[d]$ with the sizes of Λ and Γ to be ℓ and i, respectively. For any $\lambda \in \mathfrak{S}_\Lambda$, $\gamma \in S_\Gamma$ and $\delta \in \mathfrak{S}_\Delta$, we denote by $(\delta, \gamma, \lambda)$ the permutation $(\lambda(1), \ldots, \lambda(\ell), \gamma(1), \ldots, \gamma(i), \delta(1), \ldots, \delta(d-\ell-i))$. For fixed λ and δ, we denote by $\tilde{\mathfrak{S}}_{\lambda,d,\delta}$ the set of all of the permutations in the form of $(\lambda, \gamma, \delta)$.

b) In particular, when Δ is the empty set, i.e., (Λ, Γ) is a partition of $[d]$, we simply write $\tilde{\mathfrak{S}}_{\lambda,d,\delta}$ as $\tilde{\mathfrak{S}}_{\lambda,d}$ which is the set of all of permutations in the form of (λ, γ), for some fixed $\lambda \in S_\Lambda$.

c) We analogously define $\tilde{\mathfrak{S}}_{d,\delta}$ in the case that Λ is the empty set, i.e., (Γ, Δ) is a partition of $[d]$.

A.1 Right side of (II.8.1)

Because $\widehat{z}(\sigma, j) = (-1)^j z(\sigma, j)$ and $\det(\widehat{X}(1, d)) = (-1)^d \det(X(1, d))$ to prove that the right side of (II.8.1) is equal to $\frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \operatorname{sign}(\sigma) \prod_{j=1}^{d} z(\sigma, j)$ is equivalent to showing that

$$\sum_{\sigma \in \mathfrak{S}_d} \operatorname{sign}(\sigma) \prod_{j=1}^{d} \widehat{z}(\sigma, j) = (-1)^{\frac{d(d-1)}{2}} \det(\widehat{X}(\mathbf{1}, d)). \tag{A.1.1}$$

The following lemma gives a stronger statement. It involves $\tilde{\mathfrak{S}}_{d,\delta}$. For any $\sigma =$

$(\gamma, \delta) \in \tilde{\mathfrak{S}}_{d,\delta}$, and $\forall 1 \leq j \leq i$, $\det(\widehat{X}(\sigma, j))$ and $\det(\widehat{Y}(\sigma, j))$ do not depend on δ. So we simply write them as $\det(\widehat{X}(\gamma, j))$ and $\det(\widehat{Y}(\gamma, j))$.

Lemma A.1.2. *For any $1 \leq i \leq d$, let (Γ, Δ) be a partition of $[d]$ with the size of Γ equal to i. For any $\delta \in \mathfrak{S}_\Delta$ and $\gamma \in \mathfrak{S}_\Gamma$, we have that*

$$\sum_{\sigma \in \tilde{\mathfrak{S}}_{d,\delta}} \operatorname{sign}(\sigma) \prod_{j=1}^{i} \widehat{z}(\sigma, j) = (-1)^{\frac{i(i-1)}{2}} \operatorname{sign}((\gamma, \delta)) \det(\widehat{X}(\gamma, i)). \tag{A.1.3}$$

In particular, (A.1.1) holds.

Proof. We prove (A.1.3) by induction on i. The base case is $i = 1$: there is only one σ in $\tilde{\mathfrak{S}}_{d,\delta}$ and $\operatorname{sign}(\sigma) = \operatorname{sign}((\gamma, \delta))$. Together with the fact that $\det(\widehat{Y}(\gamma, 1)) = 1$, (A.1.3) holds.

Now assuming that (A.1.3) holds when $i = i_0 \geq 1$, we consider $i = i_0 + 1$. For any $m : 1 \leq m \leq i$, let $\Gamma^{(m)} = \Gamma \setminus \{\gamma(m)\}$ and $\Delta^{(m)} = \Delta \cup \{\gamma(m)\}$. Then $(\Gamma^{(m)}, \Delta^{(m)})$ is a partition of $[d]$, where the size of $\Gamma^{(m)}$ is $i - 1 = i_0$. Let $\gamma^{(m)} = (\gamma(1), \ldots, \gamma(m-1), \gamma(m+1), \ldots, \gamma(i))$ and $\delta^{(m)} = (\gamma(m), \delta(1), \ldots, \delta(d-i))$. We know that $\operatorname{sign}((\gamma^{(m)}, \delta^{(m)})) = (-1)^{i+m} \operatorname{sign}((\gamma, \delta))$. Then by the induction hypothesis,

$$\begin{aligned} \sum_{\sigma \in \tilde{\mathfrak{S}}_{d,\delta^{(m)}}} \operatorname{sign}(\sigma) \prod_{j=1}^{i-1} \widehat{z}(\sigma, j) &= (-1)^{\frac{(i-1)(i-2)}{2}} \operatorname{sign}((\gamma^{(m)}, \delta^{(m)})) \det(\widehat{X}(\gamma^{(m)}, i-1)) \\ &= (-1)^{\frac{(i-1)(i-2)}{2}+i+m} \operatorname{sign}((\gamma, \delta)) \det(\widehat{X}(\gamma^{(m)}, i-1)) \end{aligned}$$

However, $(\tilde{\mathfrak{S}}_{d,\delta^{(m)}})_{1 \leq m \leq i}$ gives a partition for $\tilde{\mathfrak{S}}_{d,\delta}$, and for any $\sigma \in \tilde{\mathfrak{S}}_{d,\delta}$, $\widehat{z}(\sigma, i)$ is an invariant. In particular, $\widehat{z}(\sigma, i) = \widehat{z}((\gamma, \delta), i) = \det(\widehat{X}(\gamma, i)) / \det(\widehat{Y}(\gamma, i))$. Therefore,

$$\begin{aligned} \sum_{\sigma \in \tilde{\mathfrak{S}}_{d,\delta}} \operatorname{sign}(\sigma) \prod_{j=1}^{i} \widehat{z}(\sigma, j) &= \sum_{m=1}^{i} \widehat{z}((\gamma, \delta), i) \sum_{\sigma \in \tilde{\mathfrak{S}}_{d,\delta(j)}} \operatorname{sign}(\sigma) \prod_{j=1}^{i-1} \widehat{z}(\sigma, j) \\ &= (-1)^{\frac{(i-1)(i-2)}{2}+i-1} \operatorname{sign}((\gamma, \delta)) \widehat{z}((\gamma, \delta), i) \sum_{m=1}^{i} (-1)^{m+1} \det(\widehat{X}(\gamma^{(m)}, i-1)) \\ &= (-1)^{\frac{i(i-1)}{2}} \operatorname{sign}((\gamma, \delta)) \widehat{z}((\gamma, \delta), i) \det(\widehat{Y}(\gamma, i)) \\ &= (-1)^{\frac{i(i-1)}{2}} \operatorname{sign}((\gamma, \delta)) \det(\widehat{X}(\gamma, i)). \end{aligned}$$

Therefore, (A.1.3) holds. If we set $i = d$, then $\Delta = \emptyset$, and $\Gamma = [d]$. Letting $\gamma = \mathbf{1}$ be the identity in $\mathfrak{S}_d$, we obtain (A.1.1). □

A.2 Left side of (II.8.1)

The proof that

$$\sum_{\sigma\in\mathfrak{S}_d} \operatorname{sign}(\sigma) g_d(z(\sigma,1),\ldots,z(\sigma,d)) = \frac{1}{d!}\sum_{\sigma\in\mathfrak{S}_d} \operatorname{sign}(\sigma)\prod_{j=1}^{d} z(\sigma,j) \tag{A.2.1}$$

is relatively harder than what we did in the previous section. We need to use the following lemma.

Lemma A.2.2. *For any $0 \le \ell + k \le d-2$, given $p(y_1,\ldots,y_\ell)$ a function on ℓ variables, let $q(\sigma) = p(z(\sigma,1),\ldots,z(\sigma,\ell)), \forall\sigma \in \mathfrak{S}_d$. Then*

$$\sum_{\sigma\in\mathfrak{S}_d} \operatorname{sign}(\sigma) q(\sigma)\frac{\prod_{j=\ell+1}^{d} z(\sigma,j)}{(z(\sigma,\ell+1))^{k+1}} = 0. \tag{A.2.3}$$

Given this lemma, we are able to prove the following proposition which implies (A.2.1) when we set $\ell = 0$.

Proposition A.2.4. *Define $s_0 = 1, z(\sigma,0) = 1$. For any $\ell : 0 \le \ell \le d$, we have that*

$$\begin{aligned}&\sum_{\sigma\in\mathfrak{S}_d} \operatorname{sign}(\sigma) g_d(z(\sigma,1),\ldots,z(\sigma,d)) \\ =\ & \frac{1}{(d-\ell)!}\sum_{\sigma\in\mathfrak{S}_d}\operatorname{sign}(\sigma)\sum_{s_1=1}^{\frac{z(\sigma,1)}{z(\sigma,0)}s_0}\cdots\sum_{s_\ell=1}^{\frac{z(\sigma,\ell)}{z(\sigma,\ell-1)}s_{\ell-1}}\frac{\prod_{j=\ell+1}^{d} z(\sigma,j)}{(z(\sigma,\ell))^{d-\ell}}s_\ell^{d-\ell}.\end{aligned} \tag{A.2.5}$$

Proof. We proceed by descending induction on ℓ.

When $\ell = d$, (A.2.5) holds by the definition of g_d.

When $\ell = d-1$, it's easy to check that (A.2.5) holds.

Assuming (A.2.5) holds for $\ell = \ell_0 + 1 \le d-1$, we consider $\ell = \ell_0 \le d-2$. By the induction hypothesis,

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_d} \mathrm{sign}(\sigma) g_d(z(\sigma,1),\ldots,z(\sigma,d)) \\
=\ & \frac{1}{(d-\ell-1)!}\sum_{\sigma\in\mathfrak{S}_d}\mathrm{sign}(\sigma)\sum_{s_1=1}^{\frac{z(\sigma,1)}{z(\sigma,0)}s_0}\cdots\sum_{s_{\ell+1}=1}^{\frac{z(\sigma,\ell+1)}{z(\sigma,\ell)}s_\ell}\frac{\prod_{j=\ell+2}^d z(\sigma,j)}{(z(\sigma,\ell+1))^{d-\ell-1}}s_{\ell+1}^{d-\ell-1} \\
=\ & \frac{1}{(d-\ell-1)!}\sum_{\sigma\in\mathfrak{S}_d}\mathrm{sign}(\sigma)\sum_{s_1=1}^{\frac{z(\sigma,1)}{z(\sigma,0)}s_0}\cdots\sum_{s_\ell=1}^{\frac{z(\sigma,\ell)}{z(\sigma,\ell-1)}s_{\ell-1}}\frac{\prod_{j=\ell+2}^d z(\sigma,j)}{(z(\sigma,\ell+1))^{d-\ell-1}}P_{d-\ell-1}\left(\frac{z(\sigma,\ell+1)}{z(\sigma,\ell)}s_\ell\right).
\end{aligned}$$

Recall that $P_{d-\ell-1}(x)$ is the power sum polynomial. Note that $d-\ell-1\geq 1$. By (II.6.9), (II.6.10) and (II.6.11), we can assume

$$P_{d-\ell-1}(x) = \frac{1}{d-\ell}x^{d-\ell} + \sum_{m=1}^{d-\ell-1} c_m x^m,$$

where $c_m \in \mathbb{R}$.

For $\forall m : 1 \leq m \leq d-\ell-1$, defining $x_0 = 1$, let

$$p_m(x_1,\ldots,x_\ell) = \sum_{s_1=1}^{\frac{x_1}{x_0}s_0}\cdots\sum_{s_\ell=1}^{\frac{x_\ell}{x_{\ell-1}}s_{\ell-1}}\left(\frac{s_\ell}{x_\ell}\right)^m.$$

Then p_m is a function on ℓ variables. Let

$$q_m(\sigma) = p_m(z(\sigma,1),\ldots,z(\sigma,\ell)).$$

Then

$$\begin{aligned}
&\sum_{\sigma\in\mathfrak{S}_d}\mathrm{sign}(\sigma)\sum_{s_1=1}^{\frac{z(\sigma,1)}{z(\sigma,0)}s_0}\cdots\sum_{s_\ell=1}^{\frac{z(\sigma,\ell)}{z(\sigma,\ell-1)}s_{\ell-1}}\frac{\prod_{j=\ell+2}^d z(\sigma,j)}{(z(\sigma,\ell+1))^{d-\ell-1}}\left(\frac{z(\sigma,\ell+1)}{z(\sigma,\ell)}s_\ell\right)^m \\
=\ & \sum_{\sigma\in\mathfrak{S}_d}\mathrm{sign}(\sigma)q_m(\sigma)\frac{\prod_{j=\ell+2}^d z(\sigma,j)}{(z(\sigma,\ell+1))^{d-\ell-1-m}} \\
=\ & \sum_{\sigma\in\mathfrak{S}_d}\mathrm{sign}(\sigma)q_m(\sigma)\frac{\prod_{j=\ell+1}^d z(\sigma,j)}{(z(\sigma,\ell+1))^{d-\ell-m}} = 0.
\end{aligned}$$

The last equality is by (A.2.3). Therefore,

$$\sum_{\sigma\in\mathfrak{S}_d} \operatorname{sign}(\sigma) g_d(z(\sigma,1),\ldots,z(\sigma,d))$$

$$= \frac{1}{(d-\ell-1)!}\sum_{\sigma\in\mathfrak{S}_d}\operatorname{sign}(\sigma)\sum_{s_1=1}^{\frac{z(\sigma,1)}{z(\sigma,0)}s_0}\cdots\sum_{s_\ell=1}^{\frac{z(\sigma,\ell)}{z(\sigma,\ell-1)}s_{\ell-1}}\frac{\prod_{j=\ell+2}^{d} z(\sigma,j)}{(z(\sigma,\ell+1))^{d-\ell-1}}\frac{1}{d-\ell}\left(\frac{z(\sigma,\ell+1)}{z(\sigma,\ell)}s_\ell\right)^{d-\ell}$$

$$= \frac{1}{(d-\ell)!}\sum_{\sigma\in\mathfrak{S}_d}\operatorname{sign}(\sigma)\sum_{s_1=1}^{\frac{z(\sigma,1)}{z(\sigma,0)}s_0}\cdots\sum_{s_\ell=1}^{\frac{z(\sigma,\ell)}{z(\sigma,\ell-1)}s_{\ell-1}}\frac{\prod_{j=\ell+1}^{d} z(\sigma,j)}{(z(\sigma,\ell))^{d-\ell}}s_\ell^{d-\ell}.$$

□

Now we have everything we need to prove Proposition II.8.1.

Proof of Proposition II.8.1

The proposition follows from (A.2.1), (A.1.1) and the facts that $\widehat{z}(\sigma,j) = (-1)^j z(\sigma,j)$ and $\det(\widehat{X}(1,d)) = (-1)^d \det(X(1,d))$.

A.3 Proof of Lemma A.2.2

It remains to prove Lemma A.2.2, which is most complicated part of this section. We will break the proof into several steps. The first lemma we need involves symmetric polynomials.

A *symmetric polynomial* on d variables $y_1, ..., y_d$ is a polynomial that is unchanged by any permutation of its variables.

Lemma A.3.1. *For any $k \geq 0$, there exist symmetric polynomials $\phi_i^k(y_1,y_2,\ldots,y_i)$ on variables $y_1,y_2,\ldots,y_i$ for any $i : 1 \leq i \leq k+2$ and symmetric polynomials $\varphi_j^k(y_1,y_2,\ldots,y_j)$ on variables $y_1,y_2,\ldots,y_j$ for any $j : 2 \leq j \leq k+2$, so that*

$$\phi_1^k = 1, \quad \phi_{k+1}^k = 1, \quad \phi_{k+2}^k = 0, \tag{A.3.2}$$

$$\varphi_2^k(y_1, y_2) = \sum_{i=0}^{k} y_1^i y_2^{k-i}, \quad \varphi_{k+2}^k = 1, \tag{A.3.3}$$

$$\begin{aligned} \forall 1 \le i \le k+1, \qquad & \phi_i^k(y_1, \ldots, y_i) y_{i+1}^{k+2-i} - \varphi_{i+1}^k(y_1, \ldots, y_{i+1}) y_{i+1} \\ = \; & -\phi_{i+1}^k(y_1, \ldots, y_{i+1}) y_1 \cdots y_{i+1}. \end{aligned} \tag{A.3.4}$$

Proof. Proof by induction on k.

When $k = 0$, $\phi_1^0 = 1, \phi_2^0 = 0, \varphi_2^0 = 1 \Rightarrow \phi_1^0 y_2 - \varphi_2^0 y_2 = -\phi_2^0 y_1 y_2$.

Assume that (A.3.2),(A.3.3) and (A.3.4) hold for $k = k_0 \ge 0$. We check the case $k = k_0 + 1$.

We set

$$\phi_i^k(y_1, \ldots, y_i) = \begin{cases} 1, & \text{if } i = 1, \\ \varphi_i^{k-1}(y_1, \ldots, y_i), & \text{if } 2 \le i \le k+1, \\ 0, & \text{if } i = k+2. \end{cases}$$

Note that $\phi_{k+1}^k = \varphi_{k+1}^{k-1} = 1$ by the induction hypothesis. Thus, (A.3.2) holds.

Now all of the ϕ_i^k's are given. In order to satisfy (A.3.4), for $\forall 1 \le i \le k+1$, we set

$$\varphi_{i+1}^k(y_1, \ldots, y_{i+1}) = \phi_i^k(y_1, \ldots, y_i) y_{i+1}^{k+1-i} + \phi_{i+1}^k(y_1, \ldots, y_{i+1}) y_1 \cdots y_i. \tag{A.3.5}$$

Hence, it is left to show that φ_{i+1}^k's are symmetric polynomials and satisfy (A.3.3).

When $i = 1$, $\varphi_2^k(y_1, y_2) = \phi_1^k(y_1) y_2^k + \phi_2^k(y_1, y_2) y_1 = y_2^k + \varphi_2^{k-1}(y_1, y_2) y_1 = y_2^k + \left(\sum_{i=0}^{k-1} y_1^i y_2^{k-1-i}\right) y_1 = \sum_{i=0}^{k} y_1^i y_2^{k-i}$.

When $2 \le i \le k$, because the right hand side of (A.3.5) is symmetric in $y_1, y_2, \ldots, y_i$, it's enough to show that it is symmetric in y_1 and y_{i+1}. However,

$$\begin{aligned} \phi_i^k(y_1, \ldots, y_i) &= \varphi_i^{k-1}(y_1, \ldots, y_i) \\ &= \phi_{i-1}^{k-1}(y_1, \ldots, y_{i-1}) y_i^{k+1-i} + \phi_i^{k-1}(y_1, \ldots, y_i) y_1 \cdots y_{i-1}. \end{aligned}$$

Because ϕ_i^k is symmetric, we can switch y_1 and y_i. So

$$\phi_i^k(y_1,\ldots,y_i) = \phi_{i-1}^{k-1}(y_2,\ldots,y_i)y_1^{k+1-i} + \phi_i^{k-1}(y_1,\ldots,y_i)y_2\cdots y_i.$$

Similarly,

$$\phi_{i+1}^k(y_1,\ldots,y_{i+1}) = \phi_i^{k-1}(y_2,\ldots,y_{i+1})y_1^{k-i} + \phi_{i+1}^{k-1}(y_1,\ldots,y_{i+1})y_2\cdots y_{i+1}.$$

Therefore,

$$\begin{aligned}&\varphi_{i+1}^k(y_1,\ldots,y_{i+1})\\ =\;& \phi_{i-1}^{k-1}(y_2,\ldots,y_i)y_1^{k+1-i}y_{i+1}^{k+1-i} + \phi_i^{k-1}(y_1,\ldots,y_i)y_2\cdots y_i y_{i+1}^{k+1-i} + \\ & \phi_i^{k-1}(y_2,\ldots,y_{i+1})y_1^{k+1-i}y_2\cdots y_i + \phi_{i+1}^{k-1}(y_1,\ldots,y_{i+1})y_1y_2^2\cdots y_i^2 y_{i+1}\end{aligned}$$

is symmetric in y_1 and y_{i+1}.

When $i = k+1$, $\varphi_{k+2}^k(y_1,\ldots,y_{i+1}) = \phi_{k+1}^k + \phi_{k+2}^k y_1\cdots y_{k+1} = 1$. □

Lemma A.3.6. *For any $0 \le k \le d-2$, $1 \le i \le k+2$, let (Γ,Δ) be a partition of $[d]$ with the size of Γ equal to i. For any $\delta \in \mathfrak{S}_\Delta$ and $\gamma \in \mathfrak{S}_\Gamma$, we have that*

$$\sum_{\sigma\in\tilde{\mathfrak{S}}_{d,\delta}} \operatorname{sign}(\sigma)\frac{\prod_{j=1}^{i-1}\widehat{z}(\sigma,j)}{(\widehat{z}(\sigma,1))^{k+1}} = (-1)^{\frac{i(i+1)}{2}-1}\operatorname{sign}((\gamma,\delta))\frac{\phi_i^k(\widehat{x}_{\gamma(1),1},\ldots,\widehat{x}_{\gamma(i),1})}{\prod_{j=1}^{i}(\widehat{x}_{\gamma(j),1})^{k+2-i}}\det(\widehat{Y}(\gamma,i)). \tag{A.3.7}$$

Proof. We prove (A.3.7) by induction on i.

When $i=1$, there is only one σ in $\tilde{\mathfrak{S}}_{d,\delta}$ and $\operatorname{sign}(\sigma) = \operatorname{sign}((\gamma,\delta))$. Together with the facts that $\phi_1^k = 1$, $\widehat{z}(\sigma,1) = \widehat{x}_{\sigma(1),1} = \widehat{x}_{\gamma(1),1}$ and $\det(\widehat{Y}(\gamma,1)) = 1$, we conclude (A.3.7).

Assuming that (A.3.7) holds when $i = i_0 \ge 1$, consider $i = i_0+1$.

For any $m : 1 \le m \le i$, let $\Gamma^{(m)} = \Gamma \setminus \{\gamma(m)\}$ and $\Delta^{(m)} = \Delta \cup \{\gamma(m)\}$. Then $(\Gamma^{(m)},\Delta^{(m)})$ is a partition of $[d]$, where the size of $\Gamma^{(m)}$ is $i-1 = i_0$. Let $\gamma^{(m)} = (\gamma(1),\ldots,\gamma(m-1),\gamma(m+1),\ldots,\gamma(i))$ and $\delta^{(m)} = (\gamma(m),\delta(1),\ldots,\delta(d-i))$.

Then by the induction hypothesis, we have that

$$\begin{aligned}
&\sum_{\sigma\in\tilde{\mathfrak{S}}_{d,\delta^{(m)}}} \operatorname{sign}(\sigma)\frac{\prod_{j=1}^{i-2}\widehat{z}(\sigma,j)}{(\widehat{z}(\sigma,1))^{k+1}} \\
= &\ (-1)^{\frac{i(i-1)}{2}-1}\operatorname{sign}((\gamma^{(m)},\delta^{(m)}))\frac{\phi_{i-1}^{k}(\widehat{x}_{\gamma^{(m)}(1),1},\ldots,\widehat{x}_{\gamma^{(m)}(i-1),1})}{\prod_{j=1}^{i-1}(\widehat{x}_{\gamma^{(m)}(j),1})^{k+3-i}}\det(\widehat{Y}(\gamma^{(m)},i-1)).
\end{aligned}$$

However, $(\tilde{\mathfrak{S}}_{d,\delta^{(m)}})_{1\le m\le i}$ gives a partition for $\tilde{\mathfrak{S}}_{d,\delta}$. Therefore,

$$\sum_{\sigma\in\tilde{\mathfrak{S}}_{d,\delta}} \operatorname{sign}(\sigma)\frac{\prod_{j=1}^{i-1}\widehat{z}(\sigma,j)}{(\widehat{z}(\sigma,1))^{k+1}} = \sum_{m=1}^{i}\sum_{\sigma\in\tilde{\mathfrak{S}}_{d,\delta^{(j)}}} \operatorname{sign}(\sigma)\frac{\prod_{j=1}^{i-2}\widehat{z}(\sigma,j)}{(\widehat{z}(\sigma,1))^{k+1}}\widehat{z}(\sigma,i-1).$$

But for any $\sigma\in\tilde{\mathfrak{S}}_{d,\delta^{(m)}}$, $\widehat{z}(\sigma,i-1)$ is an invariant. In particular, $\widehat{z}(\sigma,i-1) = \widehat{z}((\gamma^{(m)},\delta^{(m)}),i-1) = \det(\widehat{X}(\gamma^{(m)},i-1))/\det(\widehat{Y}(\gamma^{(m)},i-1))$. Hence,

$$\begin{aligned}
&\sum_{\sigma\in\tilde{\mathfrak{S}}_{d,\delta}} \operatorname{sign}(\sigma)\frac{\prod_{j=1}^{i-1}\widehat{z}(\sigma,j)}{(\widehat{z}(\sigma,1))^{k+1}} \\
= &\ \sum_{m=1}^{i}(-1)^{\frac{i(i-1)}{2}-1}\operatorname{sign}((\gamma^{(m)},\delta^{(m)}))\frac{\phi_{i-1}^{k}(\widehat{x}_{\gamma^{(m)}(1),1},\ldots,\widehat{x}_{\gamma^{(m)}(i-1),1})}{\prod_{j=1}^{i-1}(\widehat{x}_{\gamma^{(m)}(j),1})^{k+3-i}}\det(\widehat{X}(\gamma^{(m)},i-1)).
\end{aligned}$$

Note that $(\widehat{x}_{\gamma(m),1})^{k+3-i}\prod_{j=1}^{i-1}(\widehat{x}_{\gamma^{(m)}(j),1})^{k+3-i} = \prod_{j=1}^{i}(\widehat{x}_{\gamma(j),1})^{k+3-i}$, and $\operatorname{sign}((\gamma^{(m)},\delta^{(m)})) = (-1)^{i+m}\operatorname{sign}((\gamma,\delta))$. Therefore,

$$\sum_{\sigma\in\tilde{\mathfrak{S}}_{d,\delta}} \operatorname{sign}(\sigma)\frac{\prod_{j=1}^{i-1}\widehat{z}(\sigma,j)}{(\widehat{z}(\sigma,1))^{k+1}} = (-1)^{\frac{i(i-1)}{2}-1+i}\frac{\operatorname{sign}((\gamma,\delta))}{\prod_{j=1}^{i}(\widehat{x}_{\gamma(j),1})^{k+3-i}}A,$$

where

$$\begin{aligned}
A &= \sum_{m=1}^{i}(-1)^{m}\phi_{i-1}^{k}(\widehat{x}_{\gamma^{(m)}(1),1},\ldots,\widehat{x}_{\gamma^{(m)}(i-1),1})(\widehat{x}_{\gamma(m),1})^{k+3-i}\det(\widehat{X}(\gamma^{(m)},i-1)) \\
&= -\det\left(\begin{pmatrix}
\phi_{i-1}^{k}(\widehat{x}_{\gamma^{(1)}(1),1},\ldots,\widehat{x}_{\gamma^{(1)}(i-1),1})(\widehat{x}_{\gamma(1),1})^{k+3-i} & \widehat{x}_{\gamma(1),1} & \cdots & \widehat{x}_{\gamma(1),i-1} \\
\phi_{i-1}^{k}(\widehat{x}_{\gamma^{(2)}(1),1},\ldots,\widehat{x}_{\gamma^{(2)}(i-1),1})(\widehat{x}_{\gamma(2),1})^{k+3-i} & \widehat{x}_{\gamma(2),1} & \cdots & \widehat{x}_{\gamma(2),i-1} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{i-1}^{k}(\widehat{x}_{\gamma^{(i)}(1),1},\ldots,\widehat{x}_{\gamma^{(i)}(i-1),1})(\widehat{x}_{\gamma(i),1})^{k+3-i} & \widehat{x}_{\gamma(i),1} & \cdots & \widehat{x}_{\gamma(i),i-1}
\end{pmatrix}\right)
\end{aligned}$$

By (A.3.4), if we subtract the second column times $\varphi_i^k(\widehat{x}_{\gamma(1),1}, \ldots, \widehat{x}_{\gamma(i),1})$ from the first column, then

$$\begin{aligned} A &= -\det\left(\begin{pmatrix} -\phi_i^k(\widehat{x}_{\gamma(1),1}, \ldots, \widehat{x}_{\gamma(i),1})\widehat{x}_{\gamma(1),1}\cdots\widehat{x}_{\gamma(i),1} & \widehat{x}_{\gamma(1),1} & \cdots & \widehat{x}_{\gamma(1),i-1} \\ -\phi_i^k(\widehat{x}_{\gamma(1),1}, \ldots, \widehat{x}_{\gamma(i),1})\widehat{x}_{\gamma(1),1}\cdots\widehat{x}_{\gamma(i),1} & \widehat{x}_{\gamma(2),1} & \cdots & \widehat{x}_{\gamma(2),i-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\phi_i^k(\widehat{x}_{\gamma(1),1}, \ldots, \widehat{x}_{\gamma(i),1})\widehat{x}_{\gamma(1),1}\cdots\widehat{x}_{\gamma(i),1} & \widehat{x}_{\gamma(i),1} & \cdots & \widehat{x}_{\gamma(i),i-1} \end{pmatrix}\right) \\ &= \phi_i^k(\widehat{x}_{\gamma(1),1}, \ldots, \widehat{x}_{\gamma(i),1})\widehat{x}_{\gamma(1),1}\cdots\widehat{x}_{\gamma(i),1}\det(\widehat{Y}(\gamma, i)). \end{aligned}$$

Therefore,

$$\sum_{\sigma\in\tilde{\mathfrak{S}}_{d,\delta}} \mathrm{sign}(\sigma)\frac{\prod_{j=1}^{i-1}\widehat{z}(\sigma,j)}{(\widehat{z}(\sigma,1))^{k+1}} = (-1)^{\frac{i(i+1)}{2}-1}\,\mathrm{sign}((\gamma,\delta))\frac{\phi_i^k(\widehat{x}_{\gamma(1),1},\ldots,\widehat{x}_{\gamma(i),1})}{\prod_{j=1}^{i}(\widehat{x}_{\gamma(j),1})^{k+2-i}}\det(\widehat{Y}(\gamma,i)).$$

□

Corollary A.3.8. *For any $0 \le \ell + k \le d-2$, $i = k+2$, let $(\Lambda, \Gamma, \Delta)$ be a partition of $[d]$ with the sizes of λ and Γ equal to ℓ and i, respectively. For any $\delta \in \mathfrak{S}_\Delta$ and any $\lambda \in \mathfrak{S}_\Lambda$, we have that*

$$\sum_{\sigma\in\tilde{\mathfrak{S}}_{\lambda,d,\delta}} \mathrm{sign}(\sigma)\frac{\prod_{j=\ell+1}^{\ell+k+1}\widehat{z}(\sigma,j)}{(\widehat{z}(\sigma,\ell+1))^{k+1}} = 0, \tag{A.3.9}$$

and

$$\sum_{\sigma\in\tilde{\mathfrak{S}}_{\lambda,d,\delta}} \mathrm{sign}(\sigma)\frac{\prod_{j=\ell+1}^{\ell+k+1} z(\sigma,j)}{(z(\sigma,\ell+1))^{k+1}} = 0. \tag{A.3.10}$$

Proof. Proof by induction on ℓ.

When $\ell = 0$, (A.3.9) follows from Lemma A.3.6 and the fact that $\phi_{k+2}^k = 0$. Thus, (A.3.10) holds by (A.0.1).

We assume for $\ell = \ell_0 \ge 0$, (A.3.9) and (A.3.10) holds. We check the case $\ell = \ell_0+1$. Because (A.3.9) and (A.3.10) are equivalent by (A.0.1), it's enough to show (A.3.9).

Without loss of generality, we assume that $d \in \Lambda$ and $\lambda(1) = d$. For $1 \le q \le d-1$, define

$$y_{p,q} = \begin{cases} (\widehat{x}_{p,q+1} - \widehat{x}_{d,q+1})/(\widehat{x}_{p,1} - \widehat{x}_{d,1}), & \text{if } 1 \le p \le d-1, \\ \widehat{x}_{p,q+1}/\widehat{x}_{p,1}, & \text{if } p = d. \end{cases}$$

Let W be the vertex set $\{w_1, w_2, \dots, w_{d-1}\}$, where the coordinates of w_p are $(y_{p,1}, y_{p,2}, \dots, y_{p,d-1})$. For any $\sigma \in \tilde{\mathfrak{S}}_{\lambda,\delta}$, let $\varsigma = (\sigma(2), \sigma(3), \dots, \sigma(d))$. Because $\lambda(1) = d$, $\varsigma \in \mathfrak{S}_{d-1}$. Clearly, $(\Lambda \setminus \{d\}, \Gamma, \Delta)$ is a partition for $[d-1]$ and $\varsigma \in \tilde{\mathfrak{S}}_{\lambda',d-1,\gamma}$, where $\lambda' = (\lambda(2), \dots, \lambda(\ell))$. Therefore, for $j \ge 2$,

$$\begin{aligned} &\det(\widehat{X}(\sigma, j)) \\ = \;& (-1)^{j-1} \det\left(\begin{pmatrix} \widehat{x}_{\sigma(2),1} & \widehat{x}_{\sigma(2),2} & \cdots & \widehat{x}_{\sigma(2),j} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{x}_{\sigma(j),1} & \widehat{x}_{\sigma(j),2} & \cdots & \widehat{x}_{\sigma(j),j} \\ \widehat{x}_{d,1} & \widehat{x}_{d,2} & \cdots & \widehat{x}_{d,j} \end{pmatrix}\right) \\ = \;& (-1)^{j-1} \det\left(\begin{pmatrix} \widehat{x}_{\sigma(2),1} - \widehat{x}_{d,1} & \widehat{x}_{\sigma(2),2} - \widehat{x}_{d,2} & \cdots & \widehat{x}_{\sigma(2),j} - \widehat{x}_{d,j} \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{x}_{\sigma(j),1} - \widehat{x}_{d,1} & \widehat{x}_{\sigma(j),2} - \widehat{x}_{d,2} & \cdots & \widehat{x}_{\sigma(j),j} - \widehat{x}_{d,j} \\ \widehat{x}_{d,1} & \widehat{x}_{d,2} & \cdots & \widehat{x}_{d,j} \end{pmatrix}\right) \\ = \;& (-1)^{j-1} \widehat{x}_{d,1} \prod_{p=2}^{j} (\widehat{x}_{\sigma(p),1} - \widehat{x}_{d,1}) \det\left(\begin{pmatrix} 1 & y_{\varsigma(1),1} & \cdots & y_{\varsigma(1),j-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & y_{\varsigma(j-1),1} & \cdots & y_{\varsigma(j-1),j-1} \\ 1 & y_{d-1,2} & \cdots & y_{d-1,j-1} \end{pmatrix}\right) \\ = \;& (-1)^{j-1} \widehat{x}_{d,1} \prod_{p=2}^{j} (\widehat{x}_{\sigma(p),1} - \widehat{x}_{d,1}) \det(X_W(\varsigma, j-1)). \end{aligned}$$

Similarly,

$$\det(\widehat{Y}(\sigma, j)) = \prod_{p=2}^{j} (\widehat{x}_{\sigma(p),1} - \widehat{x}_{d,1}) \det(Y_W(\varsigma, j-1)).$$

Hence,

$$\widehat{z}(\sigma, j) = \det(\widehat{X}(\sigma, j)) / \det(\widehat{Y}(\sigma, j)) = (-1)^{j-1} \widehat{x}_{d,1} z_W(\varsigma, j-1).$$

Note that $\operatorname{sign}(\sigma) = (-1)^{d-1} \operatorname{sign}(\varsigma)$. Hence, by the induction hypothesis,

$$\begin{aligned}
&\sum_{\sigma \in \tilde{\mathfrak{S}}_{\lambda,d,\delta}} \operatorname{sign}(\sigma) \frac{\prod_{j=\ell+1}^{\ell+k+1} \widehat{z}(\sigma, j)}{(\widehat{z}(\sigma, \ell+1))^{k+1}} \\
= &\sum_{\varsigma \in \tilde{\mathfrak{S}}_{\lambda',d-1,\delta}} (-1)^{d-1} \operatorname{sign}(\varsigma) \frac{\prod_{j=\ell+1}^{\ell+k+1} (-1)^{j-1} z_W(\varsigma, j-1)}{(z_W(\varsigma, \ell))^{k+1}} = 0.
\end{aligned}$$

□

Proof of Lemma A.2.2. Consider any partition $(\Lambda, \Gamma, \Delta)$ of $[d]$, where the size of Λ is ℓ and the size of Γ is $i = k+2$. If we fix $\lambda \in \mathfrak{S}_\Lambda$ and $\delta \in \mathfrak{S}_\Delta$, then $\forall \sigma \in \tilde{\mathfrak{S}}_{\lambda,d,\delta}$, $z(\sigma, j)$ is an invariant when $1 \leq j \leq \ell$ or $\ell + i \leq j \leq d$. Therefore, by (A.3.10),

$$\sum_{\sigma \in \tilde{\mathfrak{S}}_{\lambda,d,\delta}} \operatorname{sign}(\sigma) q(\sigma) \frac{\prod_{j=\ell+1}^{d} z(\sigma, j)}{(z(\sigma, \ell+1))^{k+1}} = 0.$$

But all of the $\tilde{\mathfrak{S}}_{\lambda,d,\delta}$'s give a partition for $\mathfrak{S}_d$. Thus, (A.2.3) holds. □

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