The Combinatorics of Reduced Decompositions

by

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Abstract

This thesis examines several aspects of reduced decompositions in finite Coxeter groups. Effort is primarily concentrated on the symmetric group, although some discussions are subsequently expanded to finite Coxeter groups of types B and D.

In the symmetric group, the combined frameworks of permutation patterns and reduced decompositions are used to prove a new characterization of vexillary permutations. This characterization and the methods used yield a variety of new results about the structure of several objects relating to a permutation. These include its commutation classes, the corresponding graph of the classes, the zonotopal tilings of a particular polygon, and a poset defined in terms of these tilings. The class of freely braided permutations behaves particularly well, and its graphs and posets are explicitly determined.

The Bruhat order for the symmetric group is examined, and the permutations with boolean principal order ideals are completely characterized. These form an order ideal which is a simplicial poset, and its rank generating function is computed. Moreover, it is determined when the set of permutations avoiding a particular set of patterns is an order ideal, and the rank generating functions of these ideals are computed. The structure of the intervals and order ideals in this poset is elucidated via patterns, including progress towards understanding the relationship between pattern containment and subintervals in principal order ideals.

The final discussions of the thesis are on reduced decompositions in the finite Coxeter groups of types B and D. Reduced decompositions of the longest element in the hyperoctahedral group are studied, and expected values are calculated, expanding on previous work for the symmetric group. These expected values give a quantitative interpretation of the effects of the Coxeter relations on reduced decompositions of the longest element in this group. Finally, the Bruhat order in types B and D is studied, and the elements in these groups with boolean principal order ideals are characterized and enumerated by length.

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Chapter 1

Introduction

This thesis studies the combinatorics of reduced decompositions, also known as reduced words or reduced expressions, of Coxeter group elements. The interplay between reduced decompositions, patterns, and the Bruhat order is investigated, with particular emphasis on these relationships in the symmetric group (the finite Coxeter group of type A). Chapters 3 and 4 thoroughly examine the combinatorics of reduced decompositions in the symmetric group from the two viewpoints of patterns and the Bruhat order, while Chapter 5 considers reduced decompositions of a distinguished element in the hyperoctahedral group, and Chapter 6 discusses aspects of the Bruhat order for the finite Coxeter groups of types B and D.

Coxeter groups have been studied from several mathematical perspectives, including algebra, combinatorics, and geometry. The finite Coxeter groups of types A, B, and D have combinatorial interpretations as permutations, signed permutations, and signed permutations with certain restrictions. The combinatorial aspects of all Coxeter groups are treated in depth in [4]. These groups are classical objects with a bountiful literature, although there are still many open questions, particularly in reference to patterns and the Bruhat order, as discussed in this thesis. Reduced decompositions of Coxeter group elements are a similarly classical topic in mathematics, appearing throughout the literature. As evidenced in this thesis, their close relationships with patterns and the Bruhat order give rise to very interesting combinatorics.

In the last two decades, subsequent to the work of Simion and Schmidt in [31], there has been a surge of interest in permutation patterns. This field has seen extensive research and yielded many intriguing results. Unfortunately, some of the most basic questions, such as how many permutations avoid a given pattern, remain unanswered. However, recent work (see Chapter 3) has uncovered connections between reduced decompositions and permutation patterns that may prove useful to resolving some of these issues.

The Bruhat order is a partial ordering, related to reduced decompositions, of the elements in a Coxeter group. This order plays a remarkably significant role in the study of Coxeter groups. Somewhat surprisingly, very little is known about the structure of this order, particularly in terms of its order ideals and intervals. Results of Lakshmibai and Sandhya in [19], and more recently those presented in Chapters 4 and 6, elucidate some pattern-related facts about this structure. These results, combined with the relationship between reduced decompositions and patterns in Chapter 3, are significant steps towards understanding the more general structural aspects of this partial order.

After providing necessary background material in Chapter 2, this thesis examines reduced decompositions in four different ways. The first two examine reduced decompositions in the finite Coxeter group of type A, and the latter two consider types B and D. Two of these discussions are based on [42] and [43]. The individual chapters include more detailed introductory material pertinent to their respective topics.

Many properties of reduced decompositions are already well known, particularly in the case of the symmetric group. For example, in [35], Stanley uses symmetric functions to present a formula for the number of reduced decompositions of an element in the symmetric group. Perhaps the earliest link between reduced decompositions and permutation patterns in the symmetric group occurs in [2]. There Billey, Jockusch, and Stanley show that 321-avoiding permutations are exactly those permutations whose reduced decompositions never contain a factor $i(i \pm 1)i$. Relatedly, Reiner shows in [26] that the number of $i(i \pm 1)i$ factors in a reduced decomposition of the longest element in the symmetric group, which has the most occurrences of the pattern 321, is equal to the number of such reduced decompositions. These results suggest an underlying relationship in the symmetric group between reduced decompositions and permutation patterns, which is examined extensively in Chapter 3.

The primary result of Chapter 3 is a new definition of vexillary permutations in terms of principal dual order ideals in a particular poset (Theorem 3.2.8). In broad terms, the result states that *any* permutation containing a vexillary p-pattern has a reduced decomposition with a factor that is a reduced decomposition of p (a constant may be added to each letter in this factor). The converse of this statement is also true: if p is not vexillary, then there exists a permutation containing p that has no such reduced decomposition. The structure of the graph of the commutation classes of a permutation is also described, as in Theorem 3.4.10, which shows that the number of commutation classes of a permutation is monotonically increasing with respect to pattern containment. The work of Elnitsky in [9] is expanded upon to describe the zonotopal tilings of a particular polygon associated with a permutation. For example, a tiling of Elnitsky's polygon can include a $2k$-gon if and only if the permutation has a decreasing sequence of length k (Theorem 3.5.4). Corollary 3.5.7 and Theorem 3.5.8 discuss when Elnitsky's polygon can be tiled entirely by $2k$-gons. The latter of these results states that a centrally symmetric $2n$-gon with unit sides can be tiled by centrally symmetric $2k$-gons with unit sides if and only if $k \in \{2, n\}$. The poset of zonotopal tilings of Elnitsky's polygon is also discussed, and the permutations whose posets have a maximal element are shown in Theorem 3.5.14 to be exactly those permutations that avoid the patterns 4231, 4312, and 3421.

In Chapter 4, the combinatorics of reduced decompositions is studied from the perspective of the Bruhat order for the symmetric group. The Bruhat ordering can be defined in terms of reduced decompositions, and this chapter examines the structure of order ideals and intervals in this poset. Several of the results further relate the concepts of reduced decompositions and permutation patterns. For example, Therem 4.3.2 characterizes those permutations with boolean principal order ideals as exactly those which avoid the patterns 321 and 3412. The discussion is generalized in a natural way to consider permutations whose principal order ideals are isomorphic to other classes of posets, and these too can be characterized by patterns. For a fixed $k \geq 3$, Theorem 4.4.3 shows that the permutations whose principal order ideals are isomorphic to a power of the principal order ideal for the longest element in $\mathfrak{S}_k$ are those permutations in which every inversion is in exactly one decreasing sequence of length k. Additionally, the question of when the set of permutations that avoid a pattern p or two patterns p and q will form a nonempty order ideal in this poset is completely answered. Somewhat surprisingly, the set of permutations that avoid a single pattern $p \in \mathfrak{S}_k$, for $k \geq 3$, is never an order ideal (Theorem 4.5.1). On the other hand, as discussed in Theorem 4.5.2, the set of permutations avoiding two patterns $p \in \mathfrak{S}_k$ and $q \in \mathfrak{S}_l$, for $k, l \geq 3$, is an order ideal in exactly three situations: $\{p,q\} \in \big\{\{321,3412\},\{321,231\},\{321,312\}\big\}$. Finally, Theorem 4.6.10 shows that if a permutation p avoids sixteen subpatterns, then every permutation containing a p-pattern has a principal order ideal in which an interval is isomorphic to the principal order ideal for p. The reverse implication is only proved in a special case (Theorem 4.6.11), although it is conjectured always to be true.

Chapter 5 consists of calculations for the longest element in the finite Coxeter group of type B, analogous to Reiner's work for type A in [26]. In this chapter, the expected number of occurrences of each of the "interesting" Coxeter relations in an arbitrary reduced decomposition of the longest element in the hyperoctahedral group is computed. Unlike Reiner's result, where the expected number of $i(i \pm 1)i$ factors was always 1, each of the expectations for this group are dependent on the cardinality of the group. The expected number of $i(i+1)i$ or $(i+1)i(i+1)$ factors, for $i \in [n-2]$, is computed to be $2 - 4/n$ in Theorem 5.3.1, while the expected number of 0101 or 1010 factors is computed to be $2/(n^2-2)$ in Theorem 5.4.1. The former result is perhaps not surprising, as the length of the longest element in the hyperoctahedral group is n^2, which is approximately twice the length of the longest element in the symmetric group (which is $\binom{n}{2}$).

Expanding on some of the results in Chapter 4, Chapter 6 examines the Bruhat order for the finite Coxeter groups of types B and D. In particular, those elements with boolean principal order ideals are precisely defined in Theorems 6.2.1 and 6.3.1. Once again, permutation patterns emerge, although now for signed permutations, and the avoidance of certain patterns is equivalent to having a boolean principal order ideal. While the case for type A required avoiding only two patterns, it is necessary to avoid ten patterns in type B, and twenty patterns must be avoided to have a boolean principal order ideal in type D. For both types B and D, the elements avoiding these patterns are enumerated by length in Corollaries 6.2.3 and 6.3.3.

Chapter 2

Definitions and tools

In preparation for the examination of reduced decompositions and their combinatorics, some background material is necessary. To define a reduced decomposition, it is first necessary to introduce Coxeter groups. In this thesis, the primary group considered is the finite Coxeter group of type A. Additionally, the finite Coxeter groups of types B and D will be discussed, although in somewhat less detail. These groups can all be defined in terms of permutations (unsigned for type A and signed for types B and D), thus emphasizing their combinatorial nature. Much of this thesis works with permutation patterns, either unsigned or signed, and with a partial order structure on a Coxeter group known as the Bruhat order. Each of these concepts is introduced in this chapter.

2.1 Permutations

This section establishes basic facts and notation for permutations. Additional information can be found in [21].

Let $\mathfrak{S}_n$ be the group of permutations on n elements, and let $[n]$ denote the set of integers $\{1, \ldots, n\}$. An element $w \in \mathfrak{S}_n$ is the bijection on $[n]$ mapping $i \mapsto w(i)$. A permutation will be written in one-line notation as $w = w(1)w(2)\cdots w(n)$.

Example 2.1.1. The permutation $4213 \in \mathfrak{S}_4$ maps $1 \mapsto 4$, $2 \mapsto 2$, $3 \mapsto 1$, and $4 \mapsto 3$.

Definition 2.1.2. An *inversion* in w is a pair (i, j) such that $i < j$ and $w(i) > w(j)$. The *inversion set* is $I(w) = \{(i, j) : (i, j) \text{ is an inversion in } w\}$.

Since $I(w) \subset [n]^2$, the inversion set can also be viewed as an array.

Definition 2.1.3. The number of inversions in a permutation w is equal to the *length* of w, denoted $\ell(w)$. The permutation $w_0 \stackrel{\text{def}}{=} n \cdots 21 \in \mathfrak{S}_n$, is aptly named the *longest element* in $\mathfrak{S}_n$, and $\ell(w_0) = \binom{n}{2}$. If the value of n is unclear from the context, this longest element may be denoted $w_0^{(n)}$.

Let $[\pm n]$ denote the set of integers $\{\pm 1, \ldots, \pm n\}$. For ease of notation, a negative sign may be written beneath an integer: $\underset{-}{i} \stackrel{\text{def}}{=} -i$.

Definition 2.1.4. A *signed permutation* of the set $[\pm n]$ is a bijection $w : [\pm n] \to [\pm n]$ with the requirement that

$$w(\underline{i}) = \underline{w(i)}. \tag{2.1}$$

Let $\mathfrak{S}_n^B$ be the set of signed permutations on $[\pm n]$.

Equation (2.1) indicates that the signed permutation w is entirely defined by $w(1), \ldots, w(n)$. Therefore one-line notation will be used, although some values may now be negative.

Example 2.1.5. The signed permutation $\underline{4}21\underline{3} \in \mathfrak{S}_4^B$ maps $\pm 1 \mapsto \mp 4$, $\pm 2 \mapsto \pm 2$, $\pm 3 \mapsto \pm 1$, and $\pm 4 \mapsto \mp 3$.

As with unsigned permutations, there is a notion of length for signed permutations. The precise definition is not necessary to this discussion, and the reader is referred to [4] for more details. Analogous to $w_0 \in \mathfrak{S}_n$, the *longest element* in type B is $w_0^B \stackrel{\text{def}}{=} \underline{12} \cdots \underline{n}$, and $\ell(w_0^B) = n^2$.

2.2 Finite Coxeter groups of types A, B, and D

The only Coxeter groups considered in this thesis are the finite Coxeter groups of types A, B, and D. Consequently, these are the only groups defined in this section. For more thorough discussions of general Coxeter groups, the reader is encouraged to read [4] and [16].

Define bijections, called *simple reflections*, on $[\pm n]$ as follows:

$$\begin{aligned} s_i &: [\pm n] \to [\pm n] \text{ transposes } i \leftrightarrow i+1 \text{ (and } \underline{i} \leftrightarrow \underline{i+1}) \text{ for } i \in [n-1]; && (2.2) \\ s_0 &: [\pm n] \to [\pm n] \text{ transposes } 1 \leftrightarrow \underline{1}; \text{ and} && (2.3) \\ s_{1'} &\stackrel{\text{def}}{=} s_0 s_1 s_0, \text{ thus this map transposes } 1 \leftrightarrow \underline{2} \text{ (and } \underline{1} \leftrightarrow 2). && (2.4) \end{aligned}$$

All other elements are fixed by these maps. For $i > 0$, the map s_i may be restricted to $[n]$ and considered as the bijection transposing $i \leftrightarrow i+1$ and fixing all other elements. The maps $\{s_1, \ldots, s_{n-1}\}$ are sometimes called adjacent transpositions. Let $\mathcal{S}_n = \{s_{1'}, s_0, s_1, \ldots, s_{n-1}\}$.

It is not hard to see from the definitions in equations (2.2)-(2.4) that the following relations hold for the maps in $\mathcal{S}_n$:

$$\begin{aligned} s^2 &= e \text{ for all } s \in \mathcal{S}_n, \text{ where } e \text{ is the identity permutation;} && (2.5) \\ s_i s_j &= s_j s_i \text{ for } i, j \in [0, n-1] \text{ and } |i-j| > 1; && (2.6) \\ s_{1'} s_i &= s_i s_{1'} \text{ for } i \in \big([n-1] \setminus \{2\}\big); && (2.7) \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \text{ for } i \in [n-2]; && (2.8) \\ s_0 s_1 s_0 s_1 &= s_1 s_0 s_1 s_0; \text{ and} && (2.9) \\ s_{1'} s_2 s_{1'} &= s_2 s_{1'} s_2. && (2.10) \end{aligned}$$

Definition 2.2.1. The finite Coxeter group of *type* A is the set of permutations of $[n]$, for some n, denoted $\mathfrak{S}_n$. This group is generated by the simple reflections $\{s_1, \ldots, s_{n-1}\}$, subject to the relations in equations (2.5)-(2.6) and (2.8). This is also called the *symmetric group* on n elements.

The symmetric group is studied from a variety of mathematical viewpoints in [29].

While the finite Coxeter group of type A is defined in terms of permutations, types B and D require the additional structure of sign.

Definition 2.2.2. The finite Coxeter group of *type* B is the set of signed permutations of $[\pm n]$, for some n, denoted $\mathfrak{S}_n^B$. This group is generated by the simple reflections $\{s_0, s_1, \ldots, s_{n-1}\}$, subject to the relations in equations (2.5)-(2.6) and (2.8)-(2.9). This is also called the *hyperoctahedral group*.

Definition 2.2.3. The finite Coxeter group of *type* D, denoted $\mathfrak{S}_n^D$, is the subgroup of $\mathfrak{S}_n^B$ consisting of those signed permutations whose one-line notation contains an even number of negative values. This group is generated by the simple reflections $\{s_{1'}, s_1, \ldots, s_{n-1}\}$, subject to the relations in equations (2.5)-(2.8) and (2.10).

Note that the signed permutation in Example 2.1.5 has an even number of negative values in its one-line notation. Therefore $\underline{4}21\underline{3} \in \mathfrak{S}_4^D \subset \mathfrak{S}_4^B$.

The longest element in $\mathfrak{S}_n^D$ is equal to w_0^B if n is even, and $1\underline{23}\cdots\underline{n}$ if n is odd.

It is straightforward to calculate the sizes of these three Coxeter groups, as displayed in Table 2.1.

Group	Cardinality
$\mathfrak{S}_n$	$n!$
$\mathfrak{S}_n^B$	$n! \cdot 2^n$
$\mathfrak{S}_n^D$	$n! \cdot 2^{n-1}$

Table 2.1: Cardinality of the finite Coxeter groups of types A, B, and D.

2.3 Reduced decompositions

The Coxeter groups $\mathfrak{S}_n$, $\mathfrak{S}_n^B$, and $\mathfrak{S}_n^D$ are each generated by a subset of the simple reflections $\mathcal{S}_n$. Therefore, every element in these groups can be written as a product of elements of $\mathcal{S}_n$.

Definition 2.3.1. Let W be a Coxeter group generated by the simple reflections $\mathcal{T} \subset \mathcal{S}_n$. For $w \in W$, if $w = s_{i_1} \cdots s_{i_\ell}$ and ℓ is minimal among all such expressions, then the string $i_1 \cdots i_\ell$ is a *reduced decomposition* of w and ℓ is the *length* of w, denoted $\ell(w)$. The set $R(w)$ consists of all reduced decompositions of w.

This definition of length and the definition in Section 2.1 are in fact equivalent, as discussed in [4] and [21].

Definition 2.3.2. A consecutive substring of a reduced decomposition is a *factor*.

The relations described in equations (2.5)-(2.10) are often called braid relations or Coxeter relations. Similarly, in this thesis, certain factors in reduced decompositions corresponding to the braid relations will be referred to by the following names.

Definition 2.3.3. Consider $s_i, s_j \in \mathcal{S}_n$. If s_i and s_j satisfy equation (2.6) or (2.7), then a factor of the form ij in a reduced decomposition is a *short braid move*. Following the terminology in [26], a factor iji in a reduced decomposition, where s_i and s_j satisfy equation (2.8) or (2.10), is a *Yang-Baxter move*. Finally, a factor of the form 0101 or 1010 in a reduced decomposition, corresponding to equation (2.9), is a 01 *move*.

A factor ij is a short braid move exactly when the reflections s_i and s_j commute.

The order of multiplication for permutations will follow the standard that a function is written to the left of its input. Thus, if $i \in [n-1]$, the permutation $s_i w$ interchanges the positions of the values i and $i+1$ (and $\underline{i}$ and $\underline{i+1}$ if w is a signed permutation) in w, whereas ws_i transposes the values in positions i and $i+1$ in w. Therefore ws_i can be written as $w(1) \cdots w(i+1)w(i) \cdots w(n)$. Similarly, $s_0 w$ transposes $1 \leftrightarrow \underline{1}$ in w, and ws_0 changes the sign of the first position in w, so $ws_0 = \underline{w(1)}w(2) \cdots w(n)$. Products involving $s_{1'}$ can be defined analogously.

Example 2.3.4. The unsigned permutation $4213 \in \mathfrak{S}_4$ decomposes as $s_3 s_2 s_1 s_2$. This permutation has length 4, so there is no shorter such expression. Therefore 3212 is a reduced decomposition of this permutation, and the factor 212 is a Yang-Baxter move.

2.4 Permutation patterns

The classical notion of (unsigned) permutation pattern avoidance is as follows.

Definition 2.4.1. Let $w \in \mathfrak{S}_n$ and $p \in \mathfrak{S}_k$ for $k \leq n$. The permutation w *contains the pattern* p, or *contains a* p*-pattern*, if there exist $i_1 < \cdots < i_k$ such that $w(i_1) \cdots w(i_k)$ is in the same relative order as $p(1) \cdots p(k)$. That is, $w(i_h) < w(i_j)$ if and only if $p(h) < p(j)$. If w does not contain p, then w *avoids* p, or is p*-avoiding*.

The study of permutation patterns has gathered much momentum in recent years, both as mathematical objects in their own right and for their connection to other, not always obviously related, questions. One subject in the former category concerns the enumeration of permutations that contain or avoid a given pattern. Recently the Stanley-Wilf conjecture was resolved affirmatively, proving an exponential upper bound on the number of permutations that avoid a particular pattern (see [24]). An example in the second category is the fact that (unsigned) permutations for which no reduced decomposition contains a Yang-Baxter move are exactly those that avoid the pattern 321 (see [2]).

In Chapter 3 of this thesis, the reduced decompositions of an unsigned permutation are analyzed in conjunction with the notion of pattern containment. This coordinated approach yields significant results for both concepts, including statements about their relationship to each other.

Suppose that w contains the pattern p, with $\{i_1, \ldots, i_k\}$ as defined above. Then $w(i_1) \cdots w(i_k)$ is an *occurrence* of p in w. The notation $\langle p(j) \rangle$ will denote the value $w(i_j)$. If $\overline{p} = p(j)p(j+1) \cdots p(j+m)$, then $\langle \overline{p} \rangle = w(i_j)w(i_{j+1}) \cdots w(i_{j+m})$. If more than one occurrence of the pattern p is being considered, these will be distinguished by subscripts: $\langle \ \rangle_i$.

Example 2.4.2. Let $w = 7413625$, $p = 1243$, and $q = 1234$. Then 1365 is an occurrence of p, with $\langle 1 \rangle = 1$, $\langle 2 \rangle = 3$, $\langle 4 \rangle = 6$, and $\langle 3 \rangle = 5$. Also, $\langle 24 \rangle = 36$. The permutation w is q-avoiding because there is no increasing subsequence of length 4 in the one-line notation of w.

The definition of patterns in signed permutations, as in types B and D, requires an extra clause because of the additional structure that some values can be negative.

Definition 2.4.3. Let $w \in \mathfrak{S}_n^B$ and $p \in \mathfrak{S}_k^B$ for $k \leq n$. The permutation w *contains the pattern* p if there exist $0 < i_1 < \cdots < i_k$ such that

1. $w(i_j)$ and $p(j)$ have the same sign; and
2. $|w(i_1)| \cdots |w(i_k)|$ is in the same relative order as $|p(1)| \cdots |p(k)|$.

If w does not contain p, then w *avoids* p, or is *p-avoiding.*

Example 2.4.4. Let $w = \underline{4}21\underline{3}$, $p = \underline{3}1\underline{2}$, $q = 31\underline{2}$, and $r = 132$. Then $\underline{4}1\underline{3}$ and $\underline{4}2\underline{3}$ are both occurrences of p in w. The signed permutation w is q- and r-avoiding.

2.5 Posets

Standard terminology from the theory of partially ordered sets will be used throughout this thesis. Good sources for information on this topic are [38] and [45].

The notation related to posets that will appear in this thesis is as follows.

- If P is a poset and $x, y \in P$, then $x \lessdot y$ indicates that x is covered by y in P.
- $P \times Q$ denotes the direct product of the posets P and Q.
- P^r indicates the direct product of a poset P with itself r times:

$$P^r \stackrel{\text{def}}{=} \overbrace{P \times P \times \cdots \times P}^{r \text{ copies of } P}.$$

- The unique maximal element in a poset, if it exists, is denoted $\widehat{1}$.

Two aspects of a poset that will be discussed in Chapters 4 and 6 are order ideals and intervals. An *order ideal* in a poset P, sometimes called a down-set, is a subset $I \subset P$ such that if $y \in I$ and $x \leq y$ in P, then $x \in I$. The *principal* order ideal of an element y in P is the order ideal $\{x \in P : x \leq y\}$ in which y is the unique maximal element. If $x \leq y$ in P, then the *interval* $[x, y]$ is $\{z \in P : x \leq z \leq y\}$.

2.6 Bruhat order

The Bruhat order is a partial ordering that can be placed on a Coxeter group. This order will first be defined for an arbitrary Coxeter group.

Definition 2.6.1. Let W be a Coxeter group generated by the simple reflections $\mathcal{T}$. Let $\overline{\mathcal{T}} = \{wtw^{-1} : w \in W, t \in \mathcal{T}\}$. For $w, w' \in W$, write $w \lessdot w'$ if both of the following are true:

1. $\ell(w') = \ell(w) + 1$; and

2. $w' = tw$ for some $t \in \overline{\mathcal{T}}$.

The partial order defined by the covering relations $w \lessdot w'$ gives the Bruhat order for the Coxeter group W.

It should be noted, as discussed in [4], that neither left nor right multiplication is favored by the Bruhat order. That is, restating the second requirement of Definition 2.6.1 as $w' = wt$ gives rise to the same partial order.

There is also a partial order known as the *weak* Bruhat order, or simply the *weak order.* In that order, the clause "$t \in \overline{\mathcal{T}}$" in the second requirement of Definition 2.6.1 is replaced by "$t \in \mathcal{T}$," so that only simple reflections yield covering relations. The weak order is not explicitly studied in this thesis.

In the language of unsigned permutations, the covering relation in Definition 2.6.1 means that the permutation w' covers the permutation w if and only if w' can be obtained from w by transposing two values (equivalently, positions) in the one-line notation of w in such a way so as to increase the length by exactly one.

Example 2.6.2. The permutation $7314625 \in \mathfrak{S}_7$ covers the permutation 7312645. However, it does not cover 1374625, because $\ell(7314625) - \ell(1374625) > 1$. Although not a covering relation, it is true that $7314625 > 1374625$ in the Bruhat order.

This partial order has many properties which are discussed and proved in [4]. The property most relevant to this thesis, the subword property, gives an equivalent definition of the Bruhat order in terms of reduced decompositions.

Theorem 2.6.3 (Subword property). *Let W be a Coxeter group and $w, w' \in W$. Choose a reduced decomposition $i_1 \cdots i_{\ell'} \in R(w')$. Then $w \leq w'$ in the Bruhat order if and only if there exists a reduced decomposition $j_1 \cdots j_\ell \in R(w)$ which is a subword of $i_1 \cdots i_{\ell'}$.*

The two equivalent definitions of the Bruhat order for a Coxeter group indicate that it gives a graded poset where the rank function is equal to the length of an element. The maximal element in $\mathfrak{S}_n$ is w_0, and the maximal element in $\mathfrak{S}_n^B$ is w_0^B. Figures 2-1 and 2-2 give the Hasse diagrams for the Bruhat order on $\mathfrak{S}_4$ and $\mathfrak{S}_2^B$, respectively.

Other properties of the Bruhat order include that it is an Eulerian poset (as shown by Verma in [46]) and that it is CL-shellable (as shown by Björner and Wachs in [5]).

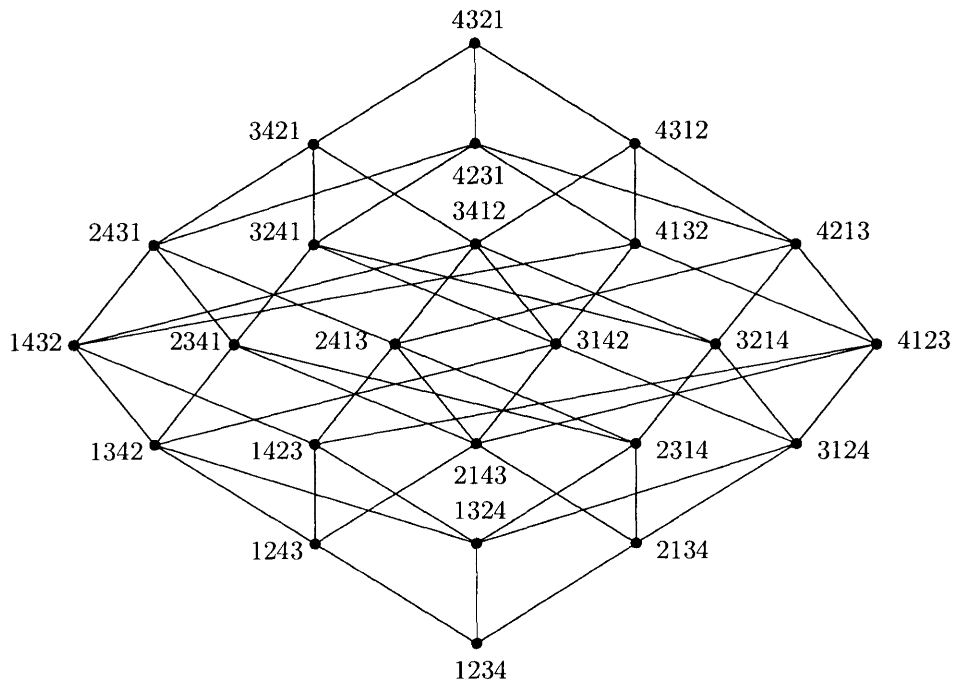

Figure 2-1: The Bruhat order for $\mathfrak{S}_4$.

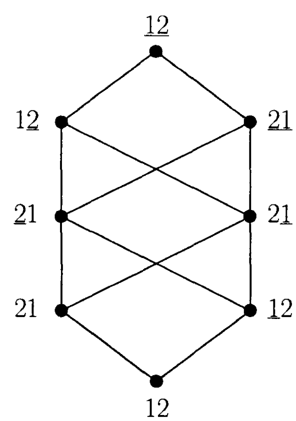

Figure 2-2: The Bruhat order for $\mathfrak{S}_2^B$.

Björner and Wachs also show that every open interval in the poset is topologically a sphere. These properties are not defined here, as they will not be discussed in this thesis. For more information, see [5], [38], and [46].

As mentioned in Chapter 1, there are many open questions concerning the Bruhat order, including the structure of the intervals in this poset. This question is made substantially simpler by Dyer's result (see [8]) that, for any ℓ, there are only finitely many non-isomorphic intervals of length ℓ in the Bruhat order of finite Coxeter groups. The length 4 intervals have been classified, as have the length 5 intervals in the symmetric group, by Hultman in [14] and [15].

Chapter 4 of this thesis considers the Bruhat order on the symmetric group $\mathfrak{S}_n$ and uses reduced decompositions to elucidate many facts about its structure. The principal order ideals in particular are considered, and several facts are shown about their structure which once again emphasize the relationship between permutation patterns and reduced decompositions. Likewise, Chapter 6 discusses the Bruhat order for $\mathfrak{S}_n^B$ and $\mathfrak{S}_n^D$, although in less depth than the type A discussion.

Chapter 3

Reduced decompositions and permutation patterns in type A

3.1 Introduction

Throughout this chapter, all permutations can be assumed to be unsigned. That is, all permutations are in the symmetric group, not the hyperoctahedral group.

In [2], Billey, Jockusch, and Stanley related the two concepts of reduced decompositions and permutation patterns, possibly for the first time. There they showed that 321-avoiding permutations are exactly those permutations where the subsequence $i(i \pm 1)i$ never occurs in a reduced decomposition. That is, a permutation is 321-avoiding if and only if none of its reduced decompositions have any Yang-Baxter moves. Reiner showed in [26] that the total number of $i(i \pm 1)i$ occurrences in reduced decompositions of the longest element in the symmetric group, which has the maximal number of occurrences of 321, is equal to the number of such reduced decompositions. Stanley had previously shown that this is the number of standard Young tableaux of a staircase shape in [35].

Inspired by these results, and more generally by the relationship they suggest between the two aspects of permutations, this chapter studies reduced decompositions of elements of the symmetric group in relation to the concept of permutation patterns. While reduced decompositions and permutation patterns appear extensively in combinatorial literature, they are not often treated together. This chapter strives to remedy that fact, addressing several questions where reduced decompositions and permutation patterns together lead to interesting results.

After introducing basic terminology and notation, Section 3.2 generalizes the result of Billey, Jockusch, and Stanley via a new characterization of vexillary permutations in Theorem 3.2.8. This characterization is based on the reduced decompositions of the permutations *containing* the permutation in question, and is strikingly different from all previous (equivalent) characterizations. In addition to requiring that each of the permutations containing the vexillary permutation has a certain kind of reduced decomposition, the proof of Theorem 3.2.8 explicitly constructs such a reduced decomposition. In the case of a non-vexillary pattern, the proof constructs a permutation, containing the pattern, which has no such reduced decomposition.

There are three algorithms which appear in this chapter, the first of which occurs in the proof of Theorem 3.2.8. It should be noted that these are not deterministic, and include a certain amount of choice. For instance, Example 3.2.9 describes only one possible route that the algorithm VEX may take on a particular input.

There is an equivalence relation, sometimes known as the commutation relation, on the set of reduced decompositions of a particular permutation. This and an associated graph are discussed in Section 3.3. Theorem 3.3.7 and Corollary 3.3.9 characterize permutations with graphs and commutation classes having certain properties, and these results are later strengthened in Theorem 3.5.13.

The results in Sections 3.4 and 3.5 discuss permutation patterns with respect to a polygon defined by Elnitsky in [9], whose rhombic tilings are in bijection with the commutation classes of a permutation. New results include that the number of commutation classes of a permutation is monotonically increasing with respect to pattern containment (Theorem 3.4.10), and several facts pertaining to a poset associated with tilings of the polygon. Finally, Section 3.6 completely describes this poset in the case of a freely braided permutation, as defined by Green and Losonczy in [12] and [13].

In addition to the definitions and notation in Chapter 2, a few concepts particular to this chapter must also be defined.

Definition 3.1.1. Let w contain the pattern p, and let $\langle p \rangle$ be a particular occurrence of p. If $w(j) \in \langle p \rangle$, then $w(j)$ is a *pattern entry* in w. Otherwise $w(j)$ is a *non-pattern entry*. If a non-pattern entry lies between two pattern entries in the one-line notation for w, then it is *inside* the pattern. Otherwise it is *outside* the pattern. "Inside" and "outside" are only defined for non-pattern entries.

Example 3.1.2. Let $w = 7413625$ and $p = 1243$, where $\langle p \rangle = 1365$. In this example, the pattern entries are 1, 3, 5, and 6, while the non-pattern entries are 2, 4, and 7. Of the latter, 4 and 7 are outside the pattern, and 2 is inside the pattern.

Definition 3.1.3. Let $\langle p \rangle$ be an occurrence of $p \in \mathfrak{S}_k$ in w. Suppose that x is inside the pattern, that $\langle m \rangle < x < \langle m+1 \rangle$ for some $m \in [k-1]$, and that the values $\{\langle m \rangle, x, \langle m+1 \rangle\}$ appear in increasing order in the one-line notation for w. Let the non-negative integers $a \leq m-1$ and $b \leq k-m$ be maximal so that the values

$$\{\langle m-a \rangle, \langle m-a+1 \rangle, \ldots, \langle m \rangle, x, \langle m+1 \rangle, \ldots, \langle m+b-1 \rangle, \langle m+b \rangle\}$$

appear in increasing order in the one-line notation for w. The entry x is *obstructed to the left* if a pattern entry smaller than $\langle m-a \rangle$ appears between $\langle m-a \rangle$ and x in w. Likewise, x is *obstructed to the right* if a pattern entry larger than $\langle m+b \rangle$ appears between x and $\langle m+b \rangle$ in w.

Example 3.1.4. Let $w = 32451$ and $p = 3241$. Then 3241 and 3251 are both occurrences of p in w. Obstruction is only defined for the latter, with $x = 4$ and $m = 3$. Then $a = b = 0$, and 4 is obstructed to the left but not to the right.

Example 3.1.5. Let $w = 21354$ and $p = 2143$. Then 2154 is an occurrence of p in w. Using $x = 3$ and $m = 2$ in Definition 3.1.3 shows that $a = b = 0$, and 3 is obstructed both to the left and to the right.

3.2 Vexillary characterization

Vexillary permutations first appeared in [20] and subsequent publications by Lascoux and Schützenberger. They were also independently found by Stanley in [35]. There have since emerged several equivalent definitions of these permutations, and a thorough discussion of these occurs in [21]. The original definition of Lascoux and Schützenberger, and the one of most relevance to this discussion, is the following.

Definition 3.2.1. A permutation is *vexillary* if it is 2143-avoiding.

These permutations were first enumerated by West in [47].

Example 3.2.2. The permutation 3641572 is vexillary, but 3641752 is not vexillary because 3175 is an occurrence of the pattern 2143 in the latter.

The following lemma is key to proving one direction of Theorem 3.2.8.

Lemma 3.2.3. *Let w contain the pattern p. Let x be inside the pattern, with $\langle m\rangle < x < \langle m+1\rangle$ and the values $\{\langle m\rangle, x, \langle m+1\rangle\}$ appearing in increasing order in w. If p is vexillary then x cannot be obstructed both to the left and to the right.*

Proof. Such obstructions would create a 2143-pattern in p. □

Example 3.1.5 illustrates a non-vexillary permutation which has an element x that is obstructed on both sides.

There are several equivalent characterizations of vexillary permutations. These concern the inversion set $I(w)$ of Section 2.1 or the following objects.

Definition 3.2.4. The *diagram* of a permutation w is $D(w) \subset [n]^2$ where

$$(i, j) \in D(w) \text{ if and only if } i < w^{-1}(j) \text{ and } j < w(i).$$

Definition 3.2.5. The *code* of w is the vector $c(w) = (c_1(w), \ldots, c_n(w))$, where $c_i(w) = \#\{j : (i, j) \in I(w)\}$. The *shape* $\lambda(w)$ is the partition formed by writing the entries of the code in non-increasing order.

Proposition 3.2.6. *The following are equivalent definitions of vexillarity for a permutation w:*

(V1) w is 2143-avoiding;

(V2) The set of rows of $I(w)$ is totally ordered by inclusion;

(V3) The set of columns of $I(w)$ is totally ordered by inclusion;

(V4) The set of rows of $D(w)$ is totally ordered by inclusion;

(V5) The set of columns of $D(w)$ is totally ordered by inclusion; and

(V6) $\lambda(w)' = \lambda(w^{-1})$, where $\lambda(w)'$ is the transpose of $\lambda(w)$.

Proof. See [21]. □

This section proves a new characterization of vexillary permutations, quite different from those in Proposition 3.2.6. A partial ordering can be placed on the set of all permutations $\mathfrak{S}_1 \cup \mathfrak{S}_2 \cup \mathfrak{S}_3 \cup \cdots$, where $u < v$ if v contains the pattern u. Definition 3.2.1 determines vexillarity by a condition on the principal order ideal of a permutation. The new characterization, Theorem 3.2.8, depends on a particular condition holding for the principal *dual* order ideal.

Definition 3.2.7. Let $\boldsymbol{i} = i_1 \cdots i_\ell$ be a reduced decomposition of $w = w(1) \cdots w(n)$. For a non-negative integer M, the *shift of $\boldsymbol{i}$ by M* is

$$\boldsymbol{i}^M \stackrel{\text{def}}{=} (i_1 + M) \cdots (i_\ell + M).$$

Note that this is a reduced decomposition of the permutation

$$12 \cdots M\big(w(1) + M\big)\big(w(2) + M\big) \cdots \big(w(n) + M\big) \in \mathfrak{S}_{n+M}.$$

Theorem 3.2.8. *The permutation p is vexillary if and only if, for every permutation w containing a p-pattern, there exists a reduced decomposition $\boldsymbol{j} \in R(w)$ containing some shift of an element $\boldsymbol{i} \in R(p)$ as a factor.*

Proof. First suppose that $p \in \mathfrak{S}_k$ is vexillary. Let $w \in \mathfrak{S}_n$ contain a p-pattern. Assume for the moment that there is a

$$\widetilde{w} = \big(s_{I_1} \cdots s_{I_q}\big)\, w\, (s_{J_1} \cdots s_{J_r}) \in \mathfrak{S}_n \tag{3.1}$$

such that

(R1) $\ell(\widetilde{w}) = \ell(w) - (q + r)$; and

(R2) $\widetilde{w}$ has a p-pattern in positions $\{1 + M, \ldots, k + M\}$ for some $M \in [0, n - k]$.

Choose a reduced decomposition $\boldsymbol{i} \in R(p)$. Let $\widetilde{w}' \in \mathfrak{S}_n$ be the permutation obtained from $\widetilde{w}$ by placing the values $\{\widetilde{w}(1 + M), \ldots, \widetilde{w}(k + M)\}$ in increasing order and leaving all other entries unchanged. Choose any $\boldsymbol{h} \in R(\widetilde{w}')$. Then

$$(I_q \cdots I_1)\, \boldsymbol{h}\boldsymbol{i}^M\, (J_r \cdots J_1) \in R(w). \tag{3.2}$$

It remains only to find a $\widetilde{w} \in \mathfrak{S}_n$ satisfying (R1) and (R2). This will be done by an algorithm VEX that takes as input a permutation $w \in \mathfrak{S}_n$ containing a p-pattern and outputs the desired permutation $\widetilde{w} \in \mathfrak{S}_n$. Because the details of this algorithm can be cumbersome, a brief description precedes each of the major steps.

Algorithm VEX

INPUT: $w \in \mathfrak{S}_n$ with an occurrence $\langle p \rangle$ of the pattern $p \in \mathfrak{S}_k$.

OUTPUT: $\widetilde{w} \in \mathfrak{S}_n$ as in equation (3.1), satisfying (R1) and (R2).

Step 0: Initialize variables.
Set $w_{[0]} \stackrel{\text{def}}{=} w$ and $i \stackrel{\text{def}}{=} 0$.

Step 1: Check if ready to output.
If $w_{[i]}$ has no entries inside the pattern, then OUTPUT $w_{[i]}$. Otherwise, choose $x_{[i]}$ inside the pattern.

Step 2: Move all inside values larger than $\langle k \rangle$ to the right of $\langle p \rangle$.
If $x_{[i]} > \langle k \rangle$, then BEGIN

a. Let $B(x_{[i]}) = \{y \geq x_{[i]} : y \text{ is inside the pattern}\}$.

b. Consider the elements of $B(x_{[i]})$ in decreasing order. Multiply $w_{[i]}$ on the right by simple reflections (changing *positions* in the one-line notation) to move each element immediately to the right of $\langle p \rangle$.

c. Let $w_{[i+1]}$ be the resulting permutation. Set $i \stackrel{\text{def}}{=} i+1$ and GOTO Step 1.

Step 3: Move all inside values smaller than $\langle 1 \rangle$ to the left of $\langle p \rangle$.
If $x_{[i]} < \langle 1 \rangle$, then BEGIN

a. Let $S(x_{[i]}) = \{y \leq x_{[i]} : y \text{ is inside the pattern}\}$.

b. Consider the elements of $S(x_{[i]})$ in increasing order. Multiply $w_{[i]}$ on the right by simple reflections to move each element immediately to the left of $\langle p \rangle$.

c. Let $w_{[i+1]}$ be the resulting permutation. Set $i \stackrel{\text{def}}{=} i+1$ and GOTO Step 1.

Step 4: Determine bounds in the pattern for the inside value.
Let $m \in [1, k-1]$ be the unique value such that $\langle m \rangle < x_{[i]} < \langle m+1 \rangle$.

Step 5: Change the occurrence of $\langle p \rangle$ and the inside value so that it does not lie between its bounds in the pattern.
If the values $\{\langle m \rangle, x_{[i]}, \langle m+1 \rangle\}$ appear in increasing order in $w_{[i]}$, then define a and b as in Definition 3.1.3 and BEGIN

a. If $x_{[i]}$ is unobstructed to the right, then BEGIN

i. Let $R(x_{[i]})$ be the set of non-pattern entries at least as large as $x_{[i]}$ and appearing between $x_{[i]}$ and $\langle m+b \rangle$ in the one-line notation for $w_{[i]}$.

ii. Consider the elements of $R(x_{[i]})$ in decreasing order. For each $y \in R(x_{[i]})$, multiply on the right by simple reflections until y is to the right of $\langle m+b \rangle$, or the right neighbor of y is $z > y$. In the latter case, $z = \langle m+b' \rangle$ for some $b' \in [1, b]$ because all larger non-pattern entries are already to the right of $\langle m+b \rangle$. Interchange the roles of y and $\langle m+b' \rangle$, and move this new y to the right in the same manner, until it is to the right of (the redefined) $\langle m+b \rangle$.

iii. Let $w_{[i+1]}$ be the resulting permutation, with $\langle p \rangle$ redefined as indicated. Let $x_{[i+1]}$ be the non-pattern entry in the final move after any interchange of roles. This is greater than $x_{[i]}$ and the redefined $\langle m+b \rangle$, and occurs to the right of the new $\langle m+b \rangle$. If $x_{[i+1]}$ is outside of the pattern, GOTO Step 1 with $i \stackrel{\text{def}}{=} i+1$. Otherwise GOTO Step 2 with $i \stackrel{\text{def}}{=} i+1$.

b. The entry $x_{[i]}$ is unobstructed to the left (Lemma 3.2.3). BEGIN

i. Let $L(x_{[i]})$ be the set of non-pattern entries at most as large as $x_{[i]}$ and appearing between $\langle m-a \rangle$ and $x_{[i]}$ in the one-line notation for $w_{[i]}$.

ii. Consider the elements of $L(x_{[i]})$ in increasing order. For each $y \in L(x_{[i]})$, multiply on the right by simple reflections until y is to the left of $\langle m-a \rangle$, or the left neighbor of y is $z < y$. In the latter case, $z = \langle m-a' \rangle$ for some $a' \in [0, a]$ because all smaller non-pattern entries are already to the left of $\langle m-a \rangle$. Interchange the roles of y and $\langle m-a' \rangle$, and move this new y to the left in the same manner, until it is to the left of (the redefined) $\langle m-a \rangle$.

iii. Let $w_{[i+1]}$ be the resulting permutation, with $\langle p \rangle$ redefined as indicated. Let $x_{[i+1]}$ be the non-pattern entry in the final move after any interchange of roles. This is less than $x_{[i]}$ and the redefined $\langle m-a \rangle$, and occurs to the left of the new $\langle m-a \rangle$. If $x_{[i+1]}$ is outside of the pattern, GOTO Step 1 with $i \stackrel{\text{def}}{=} i+1$. Otherwise GOTO Step 3 with $i \stackrel{\text{def}}{=} i+1$.

Step 6: Change the occurrence of $\langle p \rangle$, but not its position, so that the value of the inside entry increases but the values of $\langle p \rangle$ either stay the same or decrease.
If $w_{[i]}(s) = \langle m+1 \rangle$ and $w_{[i]}(t) = x_{[i]}$ with $s < t$, multiply $w_{[i]}$ on the left by simple reflections (changing *values* in the one-line notation) to obtain $w_{[i+1]}$ with the values $[x_{[i]}, \langle m+1 \rangle]$ in increasing order. Then $w_{[i+1]}(s)$ is in the half-open interval $[x_{[i]}, \langle m+1 \rangle)$, and $w_{[i+1]}(t)$ is in the half-open interval $(x_{[i]}, \langle m+1 \rangle]$. GOTO Step 2 with $x_{[i+1]} \stackrel{\text{def}}{=} w_{[i+1]}(t)$, the pattern redefined so that $\langle m+1 \rangle \stackrel{\text{def}}{=} w_{[i+1]}(s)$, and $i \stackrel{\text{def}}{=} i+1$.

Step 7: Change the occurrence of $\langle p \rangle$, but not its position, so that the value of the inside entry decreases but the values of $\langle p \rangle$ either stay the same or increase.
If $w_{[i]}(s) = \langle m \rangle$ and $w_{[i]}(t) = x_{[i]}$ with $s > t$, multiply $w_{[i]}$ on the left by simple reflections to obtain $w_{[i+1]}$ with the values $[\langle m \rangle, x_{[i]}]$ in increasing order. Then $w_{[i+1]}(s)$ is in the half-open interval $(\langle m \rangle, x_{[i]}]$, and $w_{[i+1]}(t)$ is in the half-open interval $[\langle m \rangle, x_{[i]})$. GOTO Step 3 with $x_{[i+1]} \stackrel{\text{def}}{=} w_{[i+1]}(t)$, the pattern redefined so that $\langle m \rangle \stackrel{\text{def}}{=} w_{[i+1]}(s)$, and $i \stackrel{\text{def}}{=} i+1$.

Each subsequent visit to Step 1 involves a permutation with strictly fewer entries inside the pattern than on the previous visit. Each multiplication by an adjacent transposition indicated in the algorithm removes an inversion, and so decreases the length of the permutation. This is crucial because of requirement (R1).

Consider the progression of VEX:

- Step 1 $\Longrightarrow$ HALT or begin a pass through VEX;
- Step 2 $\Longrightarrow$ Step 1;
- Step 3 $\Longrightarrow$ Step 1;
- Step 5a $\Longrightarrow$ Steps 2 or 6;
- Step 5b $\Longrightarrow$ Steps 3 or 7;
- Step 6 $\Longrightarrow$ Steps 2, 5, or 6; and
- Step 7 $\Longrightarrow$ Steps 3, 5, or 7.

Step 5a concludes with $x_{[i+1]}$ to the left of its lower pattern bound, and smaller pattern elements lying between $x_{[i+1]}$ and this bound. Therefore, no matter how often Step 6 is next called, the algorithm will never subsequently go to Step 5b before going to Step 1. Likewise, a visit to Step 5b means that Step 5a can never be visited until Step 1 is visited and a new entry inside the pattern is chosen.

Steps 2 and 3 do not change the relative positions of $\langle p \rangle$.

Steps 5a and 6 imply that $x_{[i+1]} > x_{[i]}$, while $x_{[i+1]} < x_{[i]}$ after Steps 5b and 7. Let m be as in Step 4. Until revisiting Step 1, the values $\langle m' \rangle$, for $m' \geq m+1$, do not increase if $x_{[i+1]} > x_{[i]}$. Nor do the values $\langle m' \rangle$, for $m' \leq m$, decrease if $x_{[i+1]} < x_{[i]}$. The other pattern values are unchanged. The definition of m means that the reordering of values in Steps 6 and 7 does not change the positions in which the pattern p occurs. Additionally, these steps change the value of the entry inside the pattern (that is, $x_{[i+1]} \neq x_{[i]}$), but not its position.

These observations indicate not only that VEX terminates, but that it outputs $\widetilde{w} \in \mathfrak{S}_n$ as in equation (3.1) satisfying (R1) and (R2). This completes one direction of the proof.

For the other direction, suppose that $p \in \mathfrak{S}_k$ is not vexillary. There is an occurrence of 2143 such that

$$p = \cdots \langle 2 \rangle \cdots \langle 1 \rangle \big(\langle 2 \rangle + 1\big)\big(\langle 2 \rangle + 2\big) \cdots \big(\langle 3 \rangle - 2\big)\big(\langle 3 \rangle - 1\big)\langle 4 \rangle \cdots \langle 3 \rangle \cdots .$$

Define z to be the index such that $p(z) = \langle 1 \rangle$. Define $w \in \mathfrak{S}_{k+1}$ by

$$w(m) = \begin{cases} p(m) & : \ m \leq z \text{ and } p(m) \leq \langle 2 \rangle; \\ p(m)+1 & : \ m \leq z \text{ and } p(m) > \langle 2 \rangle; \\ \langle 2 \rangle + 1 & : \ m = z+1; \\ p(m-1) & : \ m > z+1 \text{ and } p(m) \leq \langle 2 \rangle; \text{ and} \\ p(m-1)+1 & : \ m > z+1 \text{ and } p(m) > \langle 2 \rangle. \end{cases}$$

For example, if $p = 2143$, then $w = 21354$.

If there is a reduced decomposition $\boldsymbol{j} \in R(w)$ such that $\boldsymbol{j} = \boldsymbol{j_1} \boldsymbol{i}^M \boldsymbol{j_2}$ for $\boldsymbol{i} \in R(p)$ and $M \in \mathbb{N}$, then there is a $\widetilde{w} \in \mathfrak{S}_{k+1}$ as in equation (3.1) satisfying (R1) and (R2).

Keeping the values $\langle 1\rangle$, $\langle 2\rangle$, $\langle 3\rangle$, and $\langle 4\rangle$ as defined above, the permutation w was constructed so that

$$w = \cdots\langle 2\rangle\cdots\langle 1\rangle\big(\langle 2\rangle+1\big)\big(\langle 2\rangle+2\big)\cdots\big(\langle 3\rangle-2\big)\big(\langle 3\rangle-1\big)\langle 3\rangle\big(\langle 4\rangle+1\big)\cdots\big(\langle 3\rangle+1\big)\cdots.$$

One of the values in the consecutive subsequence $(\langle 2\rangle+1)(\langle 2\rangle+2)\cdots\langle 3\rangle$ must move to get a consecutive p-pattern in $\widetilde{w}$. However, the values $\{\langle 2\rangle,\ldots,\langle 3\rangle+1\}$ appear in increasing order in w, and the consecutive subsequence

$$\langle 1\rangle\big(\langle 2\rangle+1\big)\big(\langle 2\rangle+2\big)\cdots\big(\langle 3\rangle-2\big)\big(\langle 3\rangle-1\big)\langle 3\rangle\big(\langle 4\rangle+1\big)$$

in w is increasing. Therefore, there is no way to multiply w by simple reflections, always eliminating an inversion, to obtain a consecutive p-pattern.

Hence, if p is not vexillary then there exists a permutation w containing a p-pattern such that no reduced decomposition of w contains a shift of a reduced decomposition of p as a factor. □

Example 3.2.9. If $w = \mathbf{3}1\mathbf{4}65\mathbf{2}$ and $p = 231$, with the chosen occurrence $\langle p\rangle$ in bold, the algorithm VEX may proceed as follows.

- $w_{[0]} \stackrel{\text{def}}{=} \mathbf{3}1\mathbf{4}65\mathbf{2}$.
- Step 1: $x_{[0]} \stackrel{\text{def}}{=} 1$.
- Step 3: $w_{[0]} \mapsto w_{[0]}s_1 = 1\mathbf{34}65\mathbf{2} \stackrel{\text{def}}{=} w_{[1]}$.
- Step 1: $x_{[1]} \stackrel{\text{def}}{=} 5$.
- Step 6: $w_{[1]} \mapsto s_5 w_{[1]} = 1\mathbf{34}56\mathbf{2} \stackrel{\text{def}}{=} w_{[2]}$; $x_{[2]} \stackrel{\text{def}}{=} 6$.
- Step 2: $w_{[2]} \mapsto w_{[2]}s_5 = 1\mathbf{34}5\mathbf{2}6 \stackrel{\text{def}}{=} w_{[3]}$.
- Step 1: $x_{[3]} \stackrel{\text{def}}{=} 4$.
- Step 5a: $w_{[3]} \mapsto w_{[3]} = 1\mathbf{34}5\mathbf{2}6 \stackrel{\text{def}}{=} w_{[4]}$; $x_{[4]} \stackrel{\text{def}}{=} 5$.
- Step 2: $w_{[4]} \mapsto w_{[4]}s_4 = 1\mathbf{342}56 \stackrel{\text{def}}{=} w_{[5]}$.
- Step 1: output $1\mathbf{342}56$.

Therefore $\widetilde{w} = 134256 = s_5 w s_1 s_5 s_4$, and $\widetilde{w}' = 123456$. Keeping the notation of equation (3.2), $\boldsymbol{h} = \emptyset$ and $M = 1$. The unique reduced decomposition of 231 is 12, and indeed

$$(5)\emptyset(12)^1(451) = 523451 \in R(w).$$

The progression of VEX on this example is also displayed in Table 3.1. The table gives a slightly clearer depiction of which steps affect only the permutation $w_{[i]}$ or only the non-pattern entry $x_{[i]}$, and which affect both.

i	Step	$w_{[i]}$	$x_{[i]}$
0	1	314652	1
1	3	134652	
1	1		5
2	6	134562	6
3	2	134526	
3	1		4
4	5a	134526	5
5	2	134256	
5	1	OUTPUT	

Table 3.1: The progression of VEX on Example 3.2.9.

For an example of how Theorem 3.2.8 acts on a non-vexillary permutation p, consider the smallest non-vexillary permutation $p = 2143$. The permutation w defined in the proof of Theorem 3.2.8 is 21354, where $\langle p \rangle = 2154$.

Example 3.2.10. Let $w = 21354$ and $p = 2143$. No element of $R(w) = \{14, 41\}$ contains a shift of any element of $R(p) = \{13, 31\}$ as a factor.

Suppose that $\boldsymbol{j} \in R(w)$ contains a shift of $\boldsymbol{i} \in R(p)$ as a factor,

$$\boldsymbol{j} = \boldsymbol{j}_1 \boldsymbol{i}^M \boldsymbol{j}_2. \tag{3.3}$$

The factor $\boldsymbol{i} \in R(p)$ can be replaced by any $\boldsymbol{i}' \in R(p)$ in equation (3.3).

Some care must be taken regarding factors in reduced decompositions. This is clarified in the following definition and lemma, the proof of which is straightforward and omitted here.

Definition 3.2.11. Let w be a permutation and $\boldsymbol{i} \in R(w)$. Write $\boldsymbol{i} = \boldsymbol{abc}$, where $\boldsymbol{a} \in R(u)$ and $\boldsymbol{c} \in R(v)$. Suppose that $\boldsymbol{b}$ contains only letters in $S = \{1+M, \ldots, k-1+M\}$. If no element of $R(u)$ has an element of S as its rightmost character and no element of $R(v)$ has an element of S as its leftmost character, then $\boldsymbol{b}$ is *isolated* in $\boldsymbol{i}$. Equivalently, the values $\{1+M, \ldots, k+M\}$ must appear in increasing order in v, and the positions $\{1+M, \ldots, k+M\}$ must comprise an increasing sequence in u.

If $\boldsymbol{b} \in R(w_0^{(k)})$ and a shift $\boldsymbol{b}^M$ appears as a factor in a reduced decomposition of some permutation, then $\boldsymbol{b}^M$ is necessarily isolated. This is because $\boldsymbol{b}$ has maximal reduced length in the letters $\{1, \ldots, k-1\}$, so any factor of length greater than $\binom{k}{2}$ in the letters $\{1+M, \ldots, k-1+M\}$ is not reduced.

Lemma 3.2.12. *If a reduced decomposition of w contains an isolated shift of a reduced decomposition of p, then w contains the pattern p.*

The converse to Lemma 3.2.12 holds if p is vexillary.

The characterization of vexillary in Theorem 3.2.8 differs substantially from those in Proposition 3.2.6. There is not an obvious way to prove equivalence with any of the definitions (V2)-(V6), except via (V1). This raises the question of whether more may be understood about vexillary permutations (or perhaps other classes, such as Grassmannian or dominant permutations) by studying their reduced decompositions or the permutations that contain vexillary permutations as patterns.

Theorem 3.2.8 has a number of consequences, and will be used often in the subsequent sections of this chapter. Most immediately, notice that it generalizes the result of Billey, Jockusch, and Stanley mentioned earlier. The permutation 321 is vexillary and every factor $\boldsymbol{i}^M$ for $\boldsymbol{i} \in R(321)$ must be isolated. Thus, containing a 321-pattern is equivalent to some reduced decomposition containing a factor that is a shift of an element of $R(321) = \{121, 212\}$, which is a Yang-Baxter move.

3.3 The commutation relation

Recall the definitions of short braid moves and Yang-Baxter moves in a reduced decomposition, as well as the braid relations described in equations (2.6) and (2.8). It is well known that any element of $R(w)$ can be transformed into any other element of $R(w)$ by successive applications of these braid relations.

Because the short braid relation represents the commutativity of particular pairs of simple reflections, the following equivalence relation is known as the *commutation relation*.

Definition 3.3.1. For a permutation w and $\boldsymbol{i}, \boldsymbol{j} \in R(w)$, write $\boldsymbol{i} \sim \boldsymbol{j}$ if $\boldsymbol{i}$ can be obtained from $\boldsymbol{j}$ by a sequence of short braid moves. Let $C(w)$ be the set of commutation classes of reduced decompositions of w, as defined by $\sim$.

Example 3.3.2. The commutation classes of $4231 \in \mathfrak{S}_4$ are $\{12321\}$, $\{32123\}$, and $\{13231, 31231, 13213, 31213\}$.

For a given permutation w, there is a representation of its commutation classes $C(w)$ by a particular graph. This graph also takes into account the Yang-Baxter moves that may occur between elements of $R(w)$.

Definition 3.3.3. For a permutation w, the graph $G(w)$ has vertex set equal to $C(w)$, and two vertices share an edge if there exist representatives of the two classes that differ by a Yang-Baxter move.

Example 3.3.4. As shown in [2],

$$\{w : G(w) \text{ is a single vertex}\} = \{w : w \text{ is 321-avoiding}\}.$$

Elnitsky gives a very elegant representation of the graph $G(w)$ in [9], which will be discussed in depth in Section 3.4. A consequence of his description, although not difficult to prove independent of his work, is the following.

Proposition 3.3.5. *The graph $G(w)$ is connected and bipartite.*

Proof. See [9]. □

Despite Proposition 3.3.5, much remains to be understood about the graph $G(w)$. For example, even the size of the graph for w_0 (that is, the number of commutation classes for the longest element) is unknown.

Billey, Jockusch, and Stanley characterize all permutations with a single commutation class, and hence whose graphs are a single vertex, as 321-avoiding permutations. A logical question to ask next is: for what permutations does each reduced decomposition contain at most one Yang-Baxter move? More restrictively: what if this Yang-Baxter move is required to be a specific shift of 121 or 212? Moreover, what are the graphs in these cases?

Definition 3.3.6. Let $U_n = \{w \in \mathfrak{S}_n : \text{no } \boldsymbol{j} \in R(w) \text{ has two Yang-Baxter moves}\}$.

Theorem 3.3.7. *U_n is the set of permutations such that every 321-pattern in w has the same maximal element and the same minimal element.*

Proof. Assume w has a 321-pattern. Suppose that every occurrence of 321 in w has $\langle 3 \rangle = x$ and $\langle 1 \rangle = y$. Suppose that $\boldsymbol{j} \in R(w)$ has at least one long braid move. Choose k so that $j_k j_{k+1} j_{k+2}$ is the first such. Each adjacent transposition in a reduced decomposition increases the length of the product. Then by the supposition,

$$s_{j_{k+2}} s_{j_{k+1}} s_{j_k} \cdots s_{j_1} w$$

is 321-avoiding, so $j_{k+3} \cdots j_\ell$ has no Yang-Baxter moves. It remains only to consider when $j_{k+2} j_{k+3} j_{k+4}$ is also a Yang-Baxter move. The only possible reduced configurations for such a factor $j_k j_{k+1} j_{k+2} j_{k+3} j_{k+4}$ are shifts of 21232 and 23212. If either of these is not isolated, then it is part of a shift of 212321, 321232, 123212, or 232123. Notice that

- $212321, 321232, 123212, 232123 \in R(4321)$;
- $23212 \in R(4312)$; and
- $21232 \in R(3421)$.

If $j_k j_{k+1} j_{k+2} j_{k+3} j_{k+4}$ is isolated in $\boldsymbol{j}$ then w contains a 4312- or 3421-pattern by Lemma 3.2.12. Otherwise, w contains a 4321-pattern. However, every 321-pattern in w has $\langle 3 \rangle = x$ and $\langle 1 \rangle = y$. Therefore $j_k j_{k+1} j_{k+2}$ is the only Yang-Baxter move in $\boldsymbol{j}$, so $w \in U_n$.

Now let w be an element of U_n. If w has two 321-patterns that do not have the same maximal element and the same minimal element, then they intersect at most once or they create a 4321-, 4312-, or 3421-pattern. These three patterns are vexillary. Thus by Theorem 3.2.8 and the examples above, containing one of these patterns would imply that some element of $R(w)$ has more than one long braid move. If the two 321-patterns intersect at most once, their union may be a non-vexillary pattern, so Theorem 3.2.8 does not necessarily apply. However, a case analysis shows that it is possible to shorten w by simple reflections and eliminate one 321-pattern (via a Yang-Baxter move) without destroying the other 321-pattern. Thus an element of $R(w)$ would have more than one Yang-Baxter move, contradicting $w \in U_n$. □

Note that if a permutation has k distinct 321-patterns, and they all have the same maximal element and the same minimal element, then these k patterns together form a $\big((k+2)23\cdots k(k+1)1\big)$-pattern.

Definition 3.3.8. Let $U_n(j)$ consist of permutations with some 321-pattern, where every Yang-Baxter move that occurs must be $j(j+1)j$ or $(j+1)j(j+1)$.

Observe that $U_n(j)$ is a subset of U_n. For if it were not, then an element of $U_n(j)$ would have a reduced decomposition with two Yang-Baxter moves as in Definition 3.3.8. However, to be reduced, these would have to be separated by $j-1$ or $j+2$, and it would be possible to perform Coxeter relations to get a reduced decomposition of the same permutation, containing the Yang-Baxter move $j(j-1)j$ or $(j+1)(j+2)(j+1)$. This contradicts the assumption that the element is in $U_n(j)$.

Corollary 3.3.9. $U_n(j) = \{w \in \mathfrak{S}_n : w$ *has a unique* 321*-pattern and* $\langle 2\rangle = j+1\}$. *If* w *has a unique* 321*-pattern, then* $w(\langle 2\rangle) = \langle 2\rangle$.

Proof. A unique 321-pattern implies that $\{1,\ldots,\langle 2\rangle - 1\} \setminus \langle 1\rangle$ all appear to the left of $\langle 2\rangle$ in $w(1)\cdots w(n)$, and $\{\langle 2\rangle+1,\ldots,n\} \setminus \langle 3\rangle$ all appear to the right of $\langle 2\rangle$, thus $\langle 2\rangle$ must be fixed by w.

Consider the Yang-Baxter moves that may appear for elements of $U_n \supset U_n(j)$. Let $w \in U_n$ have k distinct 321-patterns. These form a pattern $p = (k+2)23\cdots k(k+1)1 \in \mathfrak{S}_{k+2}$ in w. The permutation p is vexillary, so there exists $M \in \mathbb{N}$ and a reduced decomposition $\boldsymbol{j_1 i}^M \boldsymbol{j_2} \in R(w)$ for each $\boldsymbol{i} \in R(p)$. There are elements in $R(p)$ with Yang-Baxter moves $i(i+1)i$ for each $i \in [1,k]$. For example, $12\cdots k(k+1)k\cdots 21 \in R(p)$. Therefore, if $w \in U_n(j)$, then $k=1$. Hence w has a unique 321-pattern.

Suppose that w has a unique 321-pattern. Because $w(\langle 2\rangle) = \langle 2\rangle$, the only possible Yang-Baxter moves in elements of $R(w)$ are $(\langle 2\rangle-1)\langle 2\rangle(\langle 2\rangle-1)$ or $\langle 2\rangle(\langle 2\rangle-1)\langle 2\rangle$. ☐

Corollary 3.3.10. *If* $w \in U_n$ *and* w *has* k *distinct* 321*-patterns, then* $|C(w)| = k+1$ *and the graph* $G(w)$ *is a path of* $k+1$ *vertices connected by* k *edges.*

Proof. Because w contains the pattern $p = (k+2)23\cdots k(k+1)1 \in \mathfrak{S}_{k+2}$, there is a subgraph of $G(w)$ that is a path of $k+1$ vertices connected by k edges. Since p accounts for all of the 321-patterns in w, this is all of $G(w)$. ☐

Corollary 3.3.11. *If* $w \in U_n(j)$, *then* $|C(w)| = 2$ *and the graph* $G(w)$ *is a pair of vertices connected by an edge.*

3.4 Elnitsky's polygon

In his doctoral thesis and in [9], Elnitsky developed a bijection between commutation classes of reduced decompositions of $w \in \mathfrak{S}_n$ and rhombic tilings of a particular $2n$-gon $X(w)$ which he defined. This bijection leads to a number of interesting questions about tilings of $X(w)$ and their relations to the permutation w itself. Several of these ideas are studied in this and the following section.

Definition 3.4.1. For $w \in \mathfrak{S}_n$, let $X(w)$ be the $2n$-gon with all sides of unit length such that

1. Sides of $X(w)$ are labeled $1, \ldots, n, w(n), \ldots, w(1)$ in order;
2. The portion labeled $1, \ldots, n$ is convex; and
3. Sides with the same label are parallel.

Orient the polygon so that the edge labeled 1 lies to the left of the top vertex and the edge labeled $w(1)$ lies to the right. This is *Elnitsky's polygon*.

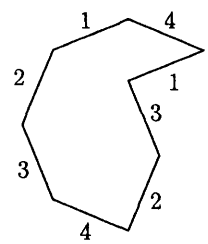

Figure 3-1: The polygon $X(4132)$.

In fact, the sides need not have unit length, provided that sides with the same label have the same length. However, for the sake of simplicity, the assumption of unit length is made here.

Example 3.4.2. The polygon $X(w_0^{(n)})$ is a centrally symmetric $2n$-gon.

Definition 3.4.3. The hexagon $X(321)$ can be tiled by rhombi with sides of unit length in exactly two ways, as in Figure 3-2. Each of these tilings is called the *flip* of the other.

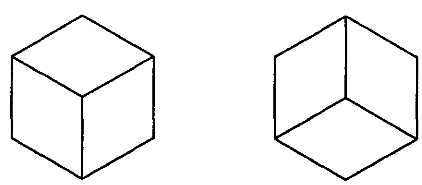

Figure 3-2: The two tilings in $T(321)$. Each is the flip of the other.

Definition 3.4.4. Let $T(w)$ be the set of tilings of $X(w)$ by rhombi with sides of unit length. Define a graph $G'(w)$ with vertex set $T(w)$, and connect two tilings by an edge if they differ by a flip of the tiling of a single sub-hexagon.

Unless otherwise indicated, the term *tiling* refers to an element of $T(w)$.

Theorem 3.4.5 (Elnitsky). *The graphs $G(w)$ and $G'(w)$ are isomorphic.*

Henceforth, both graphs will be denoted $G(w)$.

Before discussing new results related to this polygon, it is important to understand Elnitsky's bijection, outlined in the following algorithm. A more thorough treatment appears in [9].

Algorithm ELN

INPUT: $T \in T(w)$.

OUTPUT: An element of $C_T \in C(w)$.

Step 0. Set the polygon $P_{[0]} \stackrel{\text{def}}{=} X(w)$, the string $\boldsymbol{j}_{[0]} \stackrel{\text{def}}{=} \emptyset$, and $i \stackrel{\text{def}}{=} 0$.

Step 1. If $P_{[i]}$ has no area, then OUTPUT $\boldsymbol{j}_{[i]}$.

Step 2. There is at least one tile t_i that shares two edges with the right side of $P_{[i]}$.

Step 3. If t_i includes the jth and $(j+1)$st edges from the top along the right side of $P_{[i]}$, set $\boldsymbol{j}_{[i+1]} \stackrel{\text{def}}{=} j\boldsymbol{j}_{[i]}$.

Step 4. Let $P_{[i+1]}$ be $P_{[i]}$ with the tile t_i removed. Set $i \stackrel{\text{def}}{=} i+1$ and GOTO Step 1.

ELN yields the entire commutation class because of the choice of tile in Step 2.

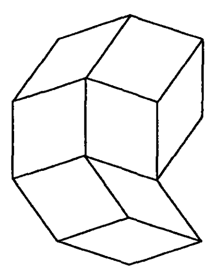

Figure 3-3: A tiling in $T(53241)$.

Example 3.4.6. The tiling in Figure 3-3 corresponds to the equivalence class consisting solely of the reduced decomposition $12343212 \in R(53241)$.

Corollary 3.4.7. *If p is vexillary and w contains a p-pattern, then $G(p)$ is a subgraph of $G(w)$.*

Proof. This follows from Theorems 3.2.8 and 3.4.5. □

Elnitsky's correspondence, described in ELN, combined with Theorem 3.3.7 and Corollary 3.3.9, indicates that any tiling of $X(w)$ for $w \in U_n$ has at most one sub-hexagon (every tiling has exactly one sub-hexagon if w is not 321-avoiding). Moreover, the sub-hexagon has the same vertical position for all elements of $U_n(j)$.

Under certain circumstances, the polygon $X(w)$ for $w \in \mathfrak{S}_n$ can be rotated or reflected to give a polygon $X(w')$ for another $w' \in \mathfrak{S}_n$.

Corollary 3.4.8. *Let $w = w(1)\cdots w(n)$ and $w^R = w(n)\cdots w(1)$. Then $|C(w)| = |C(w^R)|$ and $G(w) \simeq G(w^R)$.*

Corollary 3.4.9. *Let $w = w(1)\cdots w(n)$. If $w(1) = n, w(2) = n-1, \ldots,$ and $w(i) = n+1-i$, then $|C(w)| = |C(w^{(i)})|$ and $G(w) \simeq G(w^{(i)})$ where*

$$w^{(i)} = \big(w(i+1)+i\big)\big(w(i+2)+i\big)\cdots\big(w(n)+i\big)i(i-1)\cdots 21$$

and all values are modulo n.

Likewise, if $w(n) = 1, w(n-1) = 2, \ldots,$ and $w(n-j+1) = j$, then $|C(w)| = |C(w_{(j)})|$ and $G(w) \simeq G(w_{(j)})$ where

$$w_{(j)} = n(n-1)\cdots(n-j+1)\big(w(1)-j\big)\big(w(2)-j\big)\cdots\big(w(n-j)-j\big)$$

and all values are modulo n.

Elnitsky's result interprets the commutation classes of $R(w)$ as rhombic tilings of $X(w)$, with Yang-Baxter moves represented by flipping sub-hexagons. The following theorem utilizes this interpretation, and demonstrates that the number of commutation classes of a permutation is monotonically increasing with respect to pattern containment, thus generalizing one aspect of Corollary 3.4.7. Note that p is not required to be vexillary in Theorem 3.4.10, unlike in Theorem 3.2.8.

Theorem 3.4.10. *If w contains the pattern p, then $|C(w)| \geq |C(p)|$.*

Proof. Consider a tiling $T \in T(p)$. This represents a commutation class of $R(p)$. For an ordering of the tiles in T as defined by ELN, label the tile t_0 by $\ell(p)$, the tile t_1 by $\ell(p)-1$, and so on. If the tile with label r corresponds to the adjacent transposition s_{i_r}, then $i_1\cdots i_{\ell(p)} \in R(p)$.

Algorithm MONO

INPUT: w containing the pattern p and $T \in T(p)$ with tiles labeled as described.

OUTPUT: $T' \in T(w)$.

Step 0. Set $w_{[0]} \stackrel{\text{def}}{=} w$, $p_{[0]} \stackrel{\text{def}}{=} p$, $T_{[0]} \stackrel{\text{def}}{=} T$, $T'_{[0]} \stackrel{\text{def}}{=} \emptyset$, and $i \stackrel{\text{def}}{=} 0$.

Step 1. If $p_{[i]}$ is the identity permutation, then define $T'_{[i+1]}$ to be the tiles of $T'_{[i]}$ together with any tiling of $X(w_{[i]})$. OUTPUT $T'_{[i+1]}$.

Step 2. Let $j_{[i]}$ be such that the tile labeled $\ell(p)-i$ includes edges $p_{[i]}(j_{[i]})$ and $p_{[i]}(j_{[i]}+1)$. Note that $p_{[i]}(j_{[i]}) > p_{[i]}(j_{[i]}+1)$.

Step 3. Define $r < s$ so that $w_{[i]}(r) = \langle p_{[i]}(j_{[i]})\rangle$ and $w_{[i]}(s) = \langle p_{[i]}(j_{[i]}+1)\rangle$. Note that $w_{[i]}(t)$ is a non-pattern entry for $t \in (r,s)$.

Step 4. Let $w_{[i+1]}$ be the permutation defined by

$$w_{[i+1]}(t) = \begin{cases} w_{[i]}(t) & : \quad t < r \text{ or } t > s \\ \widetilde{w}_{[i]}(t) & : \quad r \leq t \leq s \end{cases}$$

where $(\widetilde{w}_{[i]}(r), \ldots, \widetilde{w}_{[i]}(s))$ is $\{w_{[i]}(r), \ldots, w_{[i]}(s)\}$ in increasing order.

Step 5. The right boundaries of $X(w_{[i+1]})$ and $X(w_{[i]})$ differ only in the rth, …, sth edges, and the left side of this difference (part of the boundary of $X(w_{[i+1]})$) is convex. Therefore, this difference has a rhombic tiling $t_{[i]}$. Define $T'_{[i+1]}$ to be the tiles in $T'_{[i]}$ together with the tiles in $t_{[i]}$.

Step 6. Set $i \stackrel{\text{def}}{=} i+1$ and GOTO Step 1.

The algorithm MONO takes a tiling $T \in T(p)$ and outputs one of possibly several tilings $T' \in T(w)$ due to the choice in Steps 1 and 5. A tiling $T' \in T(w)$ so obtained can only come from this T, although possibly with more than one labeling of the tiles. However, this labeling of the tiles merely reflects the choice of a representative from the commutation class, so indeed $|T(w)| \geq |T(p)|$, and $|C(w)| \geq |C(p)|$. □

Example 3.4.11. Let $p = 31542$ and $w = 4617352$. The pattern p occurs in w as $\langle p \rangle = 41752$. Figure 3-4 depicts the output of MONO, given the two tilings of $X(p)$.

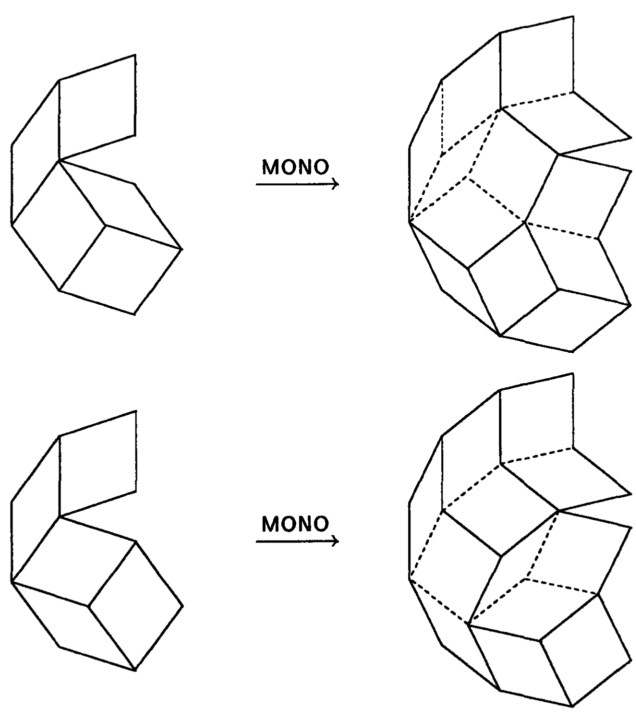

Figure 3-4: The output of MONO, given each of the two elements of $T(31542)$. The dotted lines indicate the choice of tiling in Steps 1 and 5 of the algorithm.

3.5 Zonotopal tilings and the poset of tilings

Elnitsky's bijection considers the rhombic tilings of the polygon $X(w)$. Rhombi are a special case of a more general class of objects known as zonotopes.

Definition 3.5.1. A polytope is a *d-zonotope* if it is the projection of a regular n-cube onto a d-dimensional subspace.

Centrally symmetric convex polygons are exactly the 2-zonotopes. These necessarily have an even number of sides.

Definition 3.5.2. A *zonotopal tiling* of a polygon is a tiling by centrally symmetric convex polygons (2-zonotopes).

Definition 3.5.3. Let $Z(w)$ be the set of zonotopal tilings of Elnitsky's polygon. Rhombi are centrally symmetric, so $T(w) \subset Z(w)$.

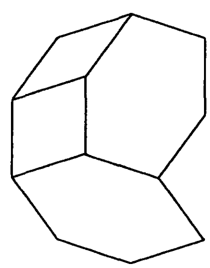

Figure 3-5: A tiling in $Z(53241)$.

Theorem 3.5.4. *There is a tiling in $Z(w)$ containing a $2k$-gon with sides parallel to the sides labeled $i_1 < \cdots < i_k$ if and only if $i_k \cdots i_1$ is an occurrence of $w_0^{(k)}$ in w.*

Proof. Since the zonotopal tiles are convex, a $2k$-gon in the tiling with sides as described has right side labeled $i_k, \ldots, i_1$ from top to bottom and left side labeled $i_1, \ldots, i_k$ from top to bottom. Therefore Elnitsky's bijection shows that this tile (or rather, any decomposition of it into rhombi) transforms the sequence $(i_1, \ldots, i_k)$ into $(i_k, \ldots, i_1)$. Reduced decompositions have minimal length, so no inversions can be "undone" by subsequent simple reflections. Therefore $i_k \cdots i_1$ must be an occurrence of $w_0^{(k)}$ in w.

Conversely, suppose that $i_k \cdots i_1$ is an occurrence of the vexillary pattern $w_0^{(k)}$ in w. For a decreasing pattern, the algorithm VEX can be modified slightly to produce $\widetilde{w}$ as in equation (3.1), where the consecutive occurrence $\langle w_0^{(k)} \rangle$ is $i_k \cdots i_1$. To make this modification, Steps 1-3 do not change, but the rest of the algorithm changes as follows:

Step 4'. Let m be such that $x_{[i]}$ lies between $\langle m+1 \rangle$ and $\langle m \rangle$.

Step 5′. If $x_{[i]} > \langle m \rangle$, then multiply on the right by simple reflections to move $x_{[i]}$ to the right of the pattern $\langle w_0^{(k)} \rangle$. GOTO Step 1 with $i \stackrel{\text{def}}{=} i+1$.

Step 6′. If $x_{[i]} < \langle m \rangle$, then multiply on the right by simple reflections to move $x_{[i]}$ to the left of the pattern $\langle w_0^{(k)} \rangle$. GOTO Step 1 with $i \stackrel{\text{def}}{=} i+1$.

Let $\boldsymbol{i} \in R(w_0^{(k)})$ and $(I_q \cdots I_1)\boldsymbol{h}\boldsymbol{i}^M(J_r \cdots J_1) \in R(w)$ for $\boldsymbol{h} \in R(\widetilde{w}')$. Removing the rhombi that correspond to $s_{J_r} \cdots s_{J_1}$ yields the polygon $X(\widetilde{w})$, and the rhombi that correspond to $\boldsymbol{i}^M$ form a sub-$2k$-gon with sides parallel to the sides labeled $\{i_1, \ldots, i_k\}$ in $X(w)$. □

Less specifically, Theorem 3.5.4 states that a tiling in $Z(w)$ can contain a $2k$-gon if and only if w has a decreasing subsequence of length k.

Using a group theoretic argument, Pasechnik and Shapiro showed in [25] that no element of $Z(w_0^{(n)})$ consists entirely of hexagons for $n > 3$. Their result states that at least one rhombus must be present in a hexagonal/rhombic tiling. Kelly and Rottenberg had previously obtained a better bound in terms of arrangements of pseudolines in [18].

Working with reduced decompositions and Elnitsky's polygons yields a different proof that no element of $Z(w_0^{(n)})$ can consist of entirely hexagonal tiles for $n > 3$, and generalizes the result to other types of tiles. The generalizations in Corollary 3.5.7 and Theorem 3.5.8, are, in a sense, counterparts to Theorem 3.5.4.

Proposition 3.5.5. *If $k \geq 4$, and $\boldsymbol{j}^{M_1} \cdots \boldsymbol{j}^{M_r} \in R(w)$ for $\boldsymbol{j} \in R(w_0^{(k)})$, then the shifts $\{M_i\}$ are all distinct.*

Proof. Fix $k \geq 4$. It is straightforward to show that if $\boldsymbol{j}^{M_1} \cdots \boldsymbol{j}^{M_r} \in R(w)$, then

$$w(x + M_i) = k + 1 - x + M_i \text{ for } x \in [2, k-1], \tag{3.4}$$

for all $i \in [r]$. (In fact, equation (3.4) holds for $k \geq 3$, although the result of the proposition does not.) The result follows from equation (3.4) and the fact that multiplying by subsequent letters in a reduced decomposition must lengthen a permutation.

The result does not hold for $k = 3$ because equation (3.4) says only that $2 + M_i$ is fixed by w for all i. This does not imply any inversions among $\{2+M_i, \ldots, k-1+M_i\}$ as when $k \geq 4$. A reduced decomposition $(121)^{M_1} \cdots (121)^{M_r}$ where the $\{M_i\}$ are not distinct appears after Corollary 3.5.7. □

Proposition 3.5.6. *Fix $k \geq 3$. Every inversion in w is in exactly one $w_0^{(k)}$-pattern if and only if there exists*

$$\boldsymbol{i} = \boldsymbol{j}^{M_1} \cdots \boldsymbol{j}^{M_r} \in R(w) \tag{3.5}$$

for $\boldsymbol{j} \in R(w_0^{(k)})$, where the M_is are distinct. (Consequently there are r occurrences of the pattern $w_0^{(k)}$ in w.) Because $\boldsymbol{i}$ is reduced, $|M_i - M_j| \geq k - 1$ for all $i \neq j$.

Proof. Fix $k \geq 3$. The result is straightforward for permutations with zero or one $w_0^{(k)}$-pattern. Suppose that $w \in \mathfrak{S}_n$ has $r > 1$ occurrences of $w_0^{(k)}$, and that every inversion in w is in exactly one $w_0^{(k)}$-pattern. It is not hard to show that at least one of these patterns occurs in consecutive positions. Therefore, for some M, there exists

$$w' \stackrel{\text{def}}{=} w \cdot \big(1 \cdots M(k+M)(k-1+M)\cdots(1+M)(k+1+M)\cdots n\big)$$

where every inversion in w' is in exactly one $w_0^{(k)}$-pattern, and there are $r-1$ such patterns. Thus, by induction, there exists $\boldsymbol{j}^{M_1}\cdots\boldsymbol{j}^{M_r} \in R(w)$ for $\boldsymbol{j} \in R(w_0^{(k)})$. If $k > 3$, then this direction of the proof is complete by Proposition 3.5.5. If $k = 3$ and the M_is are not distinct, then the permutation w would necessarily have a 4312-, 4231-, or 3421-pattern, which contradicts the original hypothesis.

For the other direction, suppose that w has a reduced decomposition as in equation (3.5), where the M_is are distinct. Consider changing $\boldsymbol{i}$ via short braid and Yang-Baxter moves. It is impossible to get an isolated factor in any $\boldsymbol{i}' \in R(w)$ equal to the shift of any element of $R(4312)$, $R(4231)$, $R(3421)$, or $R(w_0^{(k+1)})$. Therefore, since all of these patterns are vexillary, Theorem 3.2.8 implies that w avoids all four patterns. Consequently, every inversion in w is in exactly one $w_0^{(k)}$-pattern. □

Corollary 3.5.7. *Fix $k \geq 4$. There is a tiling in $Z(w)$ consisting entirely of $2k$-gons if and only if every inversion in w is in exactly one $w_0^{(k)}$-pattern.*

Proof. There being a tiling in $Z(w)$ consisting entirely of $2k$-gons is equivalent to the existence of $\boldsymbol{j}^{M_1}\cdots\boldsymbol{j}^{M_r} \in R(w)$, where $\boldsymbol{j} \in R(w_0^{(k)})$. The result follows from Proposition 3.5.5 and Proposition 3.5.6. □

The implication in Corollary 3.5.7 fails for $k = 3$. For example, $X(5274163)$ can be tiled by four convex hexagons as in Figure 3-6. However, there are six inversions in this permutation that are each in more than one 321-pattern. For example, the inversion $(1,5)$ appears in $\langle 321\rangle_1 = 521$ and $\langle 321\rangle_2 = 541$. This is because the implication in Proposition 3.5.5 does not hold for $k = 3$: $(121)^2(121)^0(121)^4(121)^2 \in R(5274163)$.

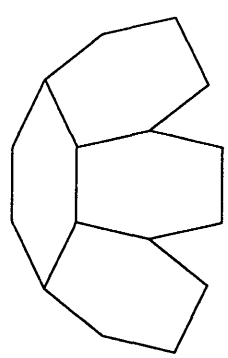

Figure 3-6: A tiling of $X(5274163)$ by four hexagons.

Theorem 3.5.8. *There is a tiling in $Z(w_0^{(n)})$ consisting entirely of $2k$-gons if and only if $k \in \{2, n\}$.*

Proof. If $k = 2$, the result is true for all n: every $X(w_0^{(n)})$ can be tiled by rhombi with unit side length. For the remainder of the proof, assume that $k \geq 3$.

Suppose that there is $Z \in Z(w_0^{(n)})$ consisting entirely of $2k$-gons. Then there exists $\boldsymbol{i} = \boldsymbol{j}^{M_1} \cdots \boldsymbol{j}^{M_r} \in R(w_0^{(n)})$, for some $\boldsymbol{j} \in R(w_0^{(k)})$. If $k \geq 4$, then Corollary 3.5.7 indicates that every inversion in $w_0^{(n)}$ is in exactly one $w_0^{(k)}$-pattern. Thus $n = k$.

Suppose that $k = 3$. Consider $\boldsymbol{i}$ as defined above. To be reduced, $|M_i - M_{i+1}| \geq 2$. Thus it is impossible to obtain a factor $(123121)^M$ by applying short braid and Yang-Baxter moves to $\boldsymbol{i}$. Hence $w_0^{(n)}$ is 4321-avoiding by Theorem 3.2.8, and $n = 3$.

Indeed, there is always a tiling $Z \in Z(w_0^{(n)})$ consisting of a single $2n$-gon. □

Results similar to the above are discussed in Chapter 4, specifically Theorem 4.4.3.

There is a poset $P(w)$ that arises naturally when studying $Z(w)$.

Definition 3.5.9. For a permutation w, let the poset $P(w)$ have elements equal to the zonotopal tilings $Z(w)$, partially ordered by reverse edge inclusion.

Example 3.5.10. In the poset $P(53241)$, the tiling in Figure 3-3 is less than the tiling in Figure 3-5.

Observe that for the longest element $w_0 \in \mathfrak{S}_n$, the poset $P(w_0)$ has a maximal element equal to the tiling in $Z(w_0)$ that consists of a single $2n$-gon.

Consider the meanings of the elements and cover relations in $P(w)$. The minimal elements of $P(w)$ are the rhombic tilings, which are the vertices of the graph $G(w)$. Moreover, edges in the graph $G(w)$ correspond to flipping a single sub-hexagon in the tiling. Therefore these edges correspond to the elements of $P(w)$ that cover the minimal elements.

These relationships are immediately apparent. Another relationship is not as obvious. This follows from a result of Shapiro, Shapiro, and Vainshtein in [30].

Lemma 3.5.11 (Shapiro-Shapiro-Vainshtein). *The set of all* 4- *and* 8-*cycles in* $G(w)$ *form a system of generators for the first homology group* $H_1(G(w), \mathbb{Z}/2\mathbb{Z})$.

Additionally, Björner noted that gluing 2-cells into those 4- and 8-cycles yields a simply connected complex ([3]).

In [30], Lemma 3.5.11 is stated only for $w = w_0$. However, the proof easily generalizes to all $w \in \mathfrak{S}_n$. A straightforward argument demonstrates that a 4-cycle in $G(w)$ corresponds to $Z \in Z(w)$ with rhombi and two hexagons, and an 8-cycle corresponds to $Z \in Z(w)$ with rhombi and an octagon. These are exactly the elements of $P(w)$ which cover those that correspond to edges of $G(w)$.

Corollary 3.5.12. *The elements of* $P(w)$ *that cover the elements (corresponding to edges of* $G(w)$*) covering the minimal elements (corresponding to vertices of* $G(w)$*) correspond to a system of generators for the first homology group* $H_1(G(w), \mathbb{Z}/2\mathbb{Z})$.

Tits had previously proved a statement such as Lemma 3.5.11 for all Coxeter groups (see [44]).

Little is known about the structure of the graph $G(w)$ for arbitrary w. However, in some cases a description can be given via Theorem 3.5.4 and Lemma 3.5.11.

Theorem 3.5.13. *The following statements are equivalent for a permutation w:*

1. *$G(w)$ is a tree;*
2. *$G(w)$ is a path (that is, no vertex has more than two incident edges);*
3. *The maximal elements of $P(w)$ cover the minimal elements; and*
4. *w is 4321-avoiding and any two 321-patterns intersect at least twice.*

Proof. *1* $\Leftrightarrow$ *3* by Corollary 3.5.12. From Theorem 3.5.4 and the discussion preceding Corollary 3.5.12, an 8-cycle in the graph is equivalent to having a 4321-pattern. Similarly, a 4-cycle is equivalent to two sub-hexagons whose intersection has zero area, so some reduced decomposition has two disjoint Yang-Baxter moves. This implies that two 321-patterns intersect in at most one position. Therefore *1* $\Leftrightarrow$ *4*.

Finally, suppose that $G(w)$ is a tree and a vertex has three incident edges. The corresponding tiling has at least three sub-hexagons. However, it is impossible for every pair of these to overlap. This contradicts *1* $\Leftrightarrow$ *3* $\Leftrightarrow$ *4*, so *1* $\Leftrightarrow$ *2*. □

If C_n is the set of all $w \in \mathfrak{S}_n$ for which $G(w)$ is a path, then $U_n(j) \subset U_n \subset C_n$ by Corollaries 3.3.10 and 3.3.11.

Recall from Chapter 2 that the unique maximal element in a poset, if it exists, is denoted $\widehat{1}$. It was noted above that the poset $P(w_0)$ has a $\widehat{1}$. In fact, there are other w for which $P(w)$ has a $\widehat{1}$, as described below.

Theorem 3.5.14. *The poset $P(w)$ has a $\widehat{1}$ if and only if w is* 4231-, 4312-, *and* 3421-*avoiding.*

Proof. The definition of the poset $P(w)$ and Theorem 3.5.4 indicate that $P(w)$ has a $\widehat{1}$ if and only if the union of any two decreasing subsequences that intersect at least twice is itself a decreasing subsequence.

Suppose there are decreasing subsequences in w of lengths $k_1, k_2 \geq 3$ that intersect $i \geq 2$ times, for $i < k_1, k_2$. Let $k = i + 1$, and choose a $k + 1$ element subsequence of $\langle k_1 \cdots 1\rangle \cup \langle k_2 \cdots 1\rangle$ that includes $\langle k_1 \cdots 1\rangle \cap \langle k_2 \cdots 1\rangle$ and one more element from each descending subsequence. Let $p \in \mathfrak{S}_{k+1}$ be the resulting pattern. There being no $\widehat{1}$ in $P(w)$ is equivalent to there being subsequences so that

$$p = (k+1)k \cdots (j+2)j(j+1)(j-1) \cdots 21$$

for some $j \in [1, k]$.

There are two ways to place a $2k$-gon in a zonotopal tiling of $X(p)$, but these overlapping $2k$-gons do not both lie in any larger centrally symmetric polygon. The permutation p is always vexillary, so Theorem 3.2.8 implies that $P(w)$ will not have a $\widehat{1}$ if w contains such a p.

Therefore, considering the permutation p for each possible j, the poset $P(w)$ has a $\widehat{1}$ if and only if w is 4231-, 4312-, and 3421-avoiding. □

The permutations for which $P(w)$ has a $\widehat{1}$ have recently been enumerated by Mansour in [22].

3.6 The freely braided case

Although the graph $G(w)$ and poset $P(w)$ are not known in general, there is a class of permutations for which these objects can be completely described. This chapter concludes with a study of this special case.

In [12] and [13], Green and Losonczy introduce and study "freely braided" elements in simply laced Coxeter groups. In the case of type A, these are as follows.

Definition 3.6.1. A permutation w is *freely braided* if every pair of distinct 321-patterns in w intersects at most once. That is, every inversion in w is in at most one 321-pattern.

Equivalently, w is freely braided if and only if w is 4321-, 4231-, 4312-, and 3421-avoiding. The poset of a freely braided permutation has a unique maximal element by Theorem 3.5.14.

Example 3.6.2. The permutation 35214 is not freely braided because 321 and 521 are both occurrences of the pattern 321, and they intersect twice. The permutation 52143 is freely braided.

Mansour enumerates freely braided permutations in [23].

In [12], Green and Losonczy show that a freely braided w with k distinct 321-patterns has

$$|C(w)| = 2^k. \tag{3.6}$$

Moreover, in [13] they show the following fact for any simply laced Coxeter group, here stated only for type A.

Proposition 3.6.3 (Green-Losonczy). *If a permutation w is freely braided with k distinct 321-patterns, then there exists $\boldsymbol{i} \in R(w)$ with k disjoint Yang-Baxter moves.*

Observe that Proposition 3.6.3 means that there is a tiling of $X(w)$ with k sub-hexagons, none of which overlap. Furthermore, equation (3.6) implies that flipping any sequence of these sub-hexagons does not yield any new sub-hexagons. Hence *every* tiling of $X(w)$ has exactly k sub-hexagons, none of which overlap, and $\widehat{1}$ in $P(w)$ corresponds to the zonotopal tiling with rhombi and k hexagons.

From the above facts, the structures of the graph $G(w)$ and the poset $P(w)$ are clear for a freely braided permutation $w \in \mathfrak{S}_n$.

Theorem 3.6.4. *Let w be freely braided with k distinct 321-patterns. The graph $G(w)$ is the graph of the k-cube, and the poset $P(w)$ is isomorphic to the face lattice of the k-cube without its minimal element.*

Example 3.6.5. The permutation 243196587 is freely braided. Its three 321-patterns are 431, 965, and 987. Figures 3-7 and 3-8 depict its graph and poset.

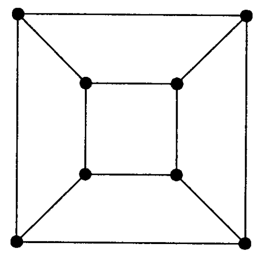

Figure 3-7: $G(243196587)$.

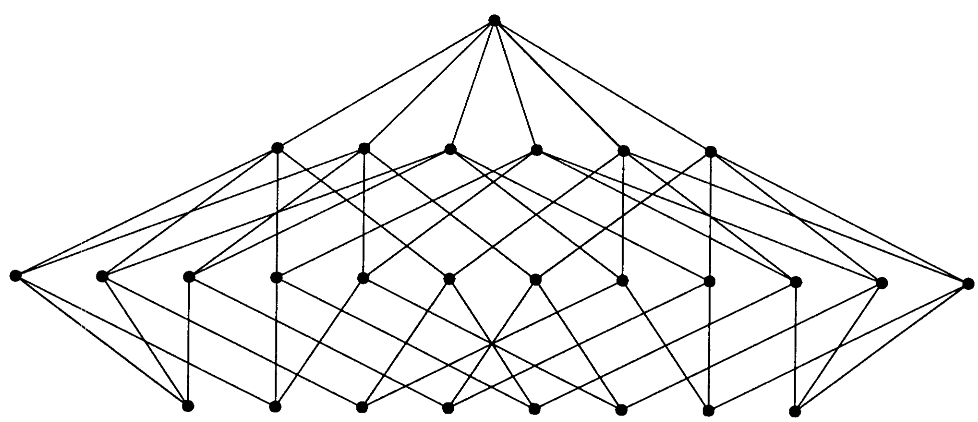

Figure 3-8: $P(243196587)$.

Chapter 4

Pattern avoidance and the Bruhat order for type A

4.1 Introduction

As in the previous chapter, this chapter only considers elements of the symmetric group. All permutations discussed here are unsigned.

Recall the definition of the Bruhat order from Section 2.6. This chapter analyzes the structure of the symmetric group with this partial order, primarily via reduced decompositions as in the subword property definition (Theorem 2.6.3) of the poset. As suggested by the numerous results in Chapter 3 relating reduced decompositions and permutation patterns, several aspects of this structure are strongly linked to patterns.

The structural features of this poset considered here are order ideals and intervals, as defined in Section 2.5. The first subject discussed concerns principal order ideals, with the following notation.

Definition 4.1.1. For $w \in \mathfrak{S}_n$, let

$$B(w) = \{v \in \mathfrak{S}_n : v \leq w\}$$

be the principal order ideal of w in the Bruhat order.

Recent work by Sjöstrand (see [32]) studies $B(w)$ in relation to rook configurations and Ferrers boards. Sjöstrand also provides a polynomial time recurrence for computing $|B(w)|$ in some cases.

One class of posets that will be studied in this chapter is the following.

Definition 4.1.2. The *boolean poset* B_r is the set of subsets of $[r]$ ordered by set inclusion. A poset is *boolean* if it is isomorphic to B_r for some r.

Following the notation of Section 2.5, a poset is boolean if and only if it is isomorphic to $B(21)^r$ for some r. This is because $B(21)$, the connected poset with two elements, is isomorphic to B_1. For example, the poset depicted in Figure 4-1 is isomorphic to $B(21)^3$.

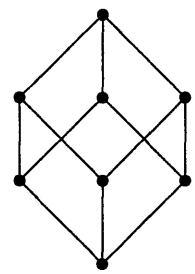

Figure 4-1: The boolean poset B_3.

Section 4.2 considers isomorphism classes of order ideals in the Bruhat order of the symmetric group. The notion of a *decomposable* order ideal is introduced, and this reappears in subsequent discussions of order ideals and intervals. Additionally, the non-isomorphic principals order ideals of length at most 5 are completely described, and tools for doing likewise with longer elements are indicated.

Section 4.3 of this chapter classifies all permutations w for which the principal order ideal $B(w)$ is boolean. Interestingly, as shown in Theorem 4.3.2, these are exactly those permutations that avoid two specific patterns. Using this, the permutations with this property are enumerated by length in Corollary 4.3.5. Additionally, permutations with "nearly boolean" principal order ideals are discussed, along with the size and description of their ideals.

A more general classification is made in Section 4.4. There, the permutations with principal order ideals isomorphic to a power of $B(w_0^{(k)})$, for $k \geq 3$, are entirely classified. This characterization (Theorem 4.4.3) is again stated in terms of patterns, although not exactly pattern avoidance.

Section 4.5 examines sets of permutations avoiding either one or two patterns and determines exactly when these sets are order ideals in the Bruhat order. This property holds in only a few situations, each of which can be enumerated by length.

Finally, Section 4.6 addresses the following question: If w contains a p-pattern, is there necessarily a relationship between $B(w)$ and $B(p)$? Theorems 4.6.10 and 4.6.11 provide some insight into this question, although they do not answer it completely. The chapter concludes with the conjecture that a particular relationship exists if and only if the permutation p avoids sixteen patterns.

4.2 Isomorphism classes of principal order ideals

Prior to classifying permutations with principal order ideals of particular forms, this section examines the issue of isomorphism classes of principal order ideals in the Bruhat order of the symmetric group.

Definition 4.2.1. A permutation $w \in \mathfrak{S}_n$ is *decomposable* if $B(w) \cong B(u) \times B(v)$ for some $u \in \mathfrak{S}_m$ and $v \in \mathfrak{S}_{m'}$ where $m, m' < n$. If w is not decomposable then it is *indecomposable*.

The following proposition is straightforward to show, and its proof is omitted.

Proposition 4.2.2. *A permutation $w \in \mathfrak{S}_n$ is decomposable if and only if there exist $M \in [n-2]$ and $\boldsymbol{gh}$ or $\boldsymbol{hg}$ in $R(w)$, such that $\boldsymbol{g}$ consists only of letters in $[M]$ and $\boldsymbol{h}$ consists only of letters in $[M+1, n-1]$. Equivalently, a permutation $w \in \mathfrak{S}_n$ is indecomposable if and only if there is a substring $M(M+1)M$ or $(M+1)M(M+1)$ in every element of $R(w)$, for all $M \in [n-2]$.*

Suppose that w is decomposable, and keep the notation of Proposition 4.2.2. Let $\boldsymbol{H}$ be such that $\boldsymbol{H}^M = \boldsymbol{h}$, and let $u \in \mathfrak{S}_{M+1}$ and $v \in \mathfrak{S}_{n-M}$ be such that $\boldsymbol{g} \in R(u)$ and $\boldsymbol{H} \in R(v)$. Then $B(w) \cong B(u) \times B(v)$.

The following easy consequence states that the decomposable permutations in $\mathfrak{S}_n$ form an order ideal.

Corollary 4.2.3. *If $w \in \mathfrak{S}_n$ is decomposable, and $v \leq w$, then $v \in \mathfrak{S}_n$ is decomposable as well.*

Proposition 4.2.2 and the subsequent remark greatly simplify the problem of determining the non-isomorphic principal order ideals in the Bruhat order of the symmetric group, as indicated in Table 4.1. Entries for greater lengths can be similarly deduced.

Length	0	1	2	3	4	5
Reduced	∅	1	12	123	1234	12345
decompositions				121	1214	12146
					2132	21325
						12321
						21232

Table 4.1: Representative reduced decompositions giving all non-isomorphic principal order ideals of length at most 5 in $\mathfrak{S}_n$.

Due to Proposition 4.2.2, the entries in Table 4.1 correspond to principal order ideals that can be described by only a few posets. In addition to the boolean posets already mentioned, the principal order ideals in this table are depicted in Figures 4-2, 4-3, 4-4, and 4-5. Table 4.2 indicates the isomorphism class for the principal order ideal of the permutation represented by the corresponding entry in Table 4.1. For example, the top row of Table 4.2 are the boolean posets.

Length	0	1	2	3	4	5
Principal	B_0	B_1	B_2	B_3	B_4	B_5
order				$B(321)$	$B(321) \times B_1$	$B(321) \times B_2$
ideals					$B(3412)$	$B(3412) \times B_1$
						$B(4231)$
						$B(3421)$

Table 4.2: The non-isomorphic principal order ideals of length at most 5 in $\mathfrak{S}_n$. The entries correspond to those in Table 4.1.

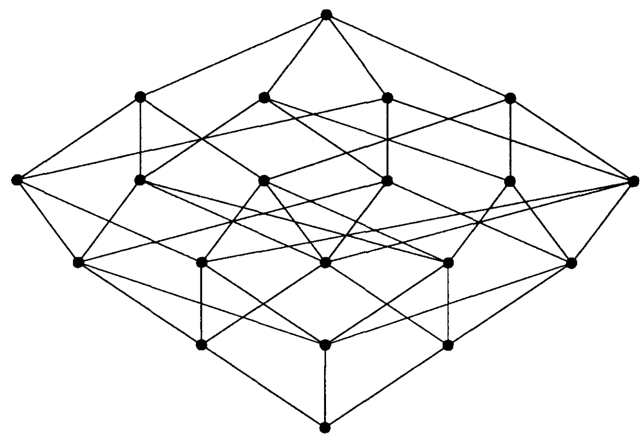

Figure 4-2: The principal order ideal $B(4231)$.

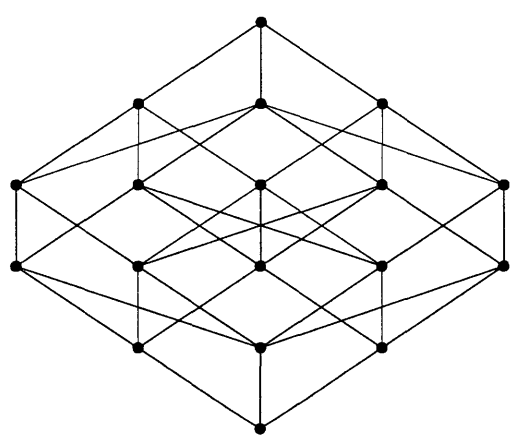

Figure 4-3: The principal order ideal $B(3421)$.

It must be noted that not all intervals that can appear in $\mathfrak{S}_n$ can appear as principal order ideals. To be specific, the discrepancies in the number of non-isomorphic intervals that can appear and the number of non-isomorphic principal order ideals that can appear are displayed in Table 4.3. The theorem that there are only finitely many non-isomorphic intervals of a given length in the Bruhat order is due to Dyer in [8], and the quantitative results for small lengths are due to Jantzen (see [17]) and Hultman (see [14] and [15]).

Length	0	1	2	3	4	5
# Non-isomorphic intervals	1	1	1	3	7	25
# Non-isomorphic $B(w)$	1	1	1	2	3	5

Table 4.3: Comparing the number of non-isomorphic intervals and the number of non-isomorphic principal order ideals of length at most 5 in $\mathfrak{S}_n$.

4.3 Boolean principal order ideals

The goal of this section is to determine exactly when the principal order ideal $B(w)$ is boolean, for $w \in \mathfrak{S}_n$.

Definition 4.3.1. Let w be a permutation in $\mathfrak{S}_n$. If the poset $B(w)$ is boolean, then w is a *boolean* permutation.

Theorem 4.3.2. *The permutation w is boolean if and only if w is* 321- *and* 3412-*avoiding.*

Proof. The subposet $B(w)$ is graded with rank equal to the length of the permutation, $\ell \stackrel{\text{def}}{=} \ell(w)$. Thus, if $B(w)$ is boolean, it must be isomorphic to B_ℓ.

Let $\boldsymbol{i} = i_1 \cdots i_\ell \in R(w)$. By the definition of the Bruhat order in Theorem 2.6.3, it must be possible to delete any subset of $\{i_1, \ldots, i_\ell\}$ from $\boldsymbol{i}$ and obtain a decomposition which is still reduced.

Recall the braid relations for type A, as described in equations (2.5)-(2.6) and (2.8). These indicate that if a decomposition is not reduced, then there must be a letter which occurs more than once. If there exist $j_1 < j_2$ such that $i_{j_1} = i_{j_2}$, then it would be possible to delete all letters except i_{j_1} and i_{j_2} from $\boldsymbol{i}$ and obtain $i_{j_1} i_{j_2}$ which is not reduced. On the other hand, if all the letters in $\boldsymbol{i}$ are distinct, then removing any subset of the letters will always leave a reduced decomposition.

Therefore boolean permutations are exactly those which have a reduced decomposition with no repeated letters. (In fact, if one reduced decomposition has this property, then all reduced decompositions do, because only short braid moves can be performed and these do not change the multiset of letters appearing.)

Suppose that the permutation w is not boolean. Then there is a reduced decomposition of w with a repeated letter. Therefore, there exists a reduced decomposition of w with one of the following factors, for some M.

$$(121)^M \tag{4.1}$$

$$(2132)^M \tag{4.2}$$

A factor as in equation (4.1) is necessary isolated. Thus, by Lemma 3.2.12, such a factor would indicate that w has a 321-pattern.

A factor as in equation (4.2) is not necessarily isolated. If it were, then it would indicate a 3412-pattern. Otherwise, it would indicate a 4312-, 3421-, or 4321-pattern. These latter three all contain the pattern 321, so a repeated letter in a reduced decomposition implies that w has a 321- or a 3412-pattern.

On the other hand, 321 and 3412 are both vexillary. Thus, by Theorem 3.2.8, if either occurs in w then a reduced decomposition of w has a repeated letter.

This completes the proof. □

Boolean permutations were enumerated by the author, although they had previously been enumerated by West in [48] and Fan in [10]. West actually enumerates the number of permutations which avoid the patterns 123 and 2143, but an easy transformation shows that this set of permutations has the same cardinality as the set of boolean permutations.

Definition 4.3.3. The *Fibonacci numbers* are $\{F_0, F_1, F_2, \ldots\}$, where $F_0 = 0$, $F_1 = 1$, and $F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$.

The Fibonacci numbers are sequence A000045 in [34].

Corollary 4.3.4. *The number of boolean permutations in $\mathfrak{S}_n$ is equal to F_{2n-1}.*

The numbers F_{2n-1} are sequence A001519 in [34].

The boolean permutations can also be enumerated by length. That is, the numbers

$$L(n,k) \stackrel{\text{def}}{=} \#\{w \in \mathfrak{S}_n : \ell(w) = k \text{ and } w \text{ is boolean}\} \tag{4.3}$$

have an explicit and concise form.

Corollary 4.3.5. *Let $L(n,k)$ be as defined in equation* (4.3). *Then*

$$L(n,k) = \sum_{i=1}^{k} \binom{n-i}{k+1-i}\binom{k-1}{i-1}, \tag{4.4}$$

where the (empty) sum for $k = 0$ is defined to be 1.

Proof. Equation (4.4) is proved by induction. First, observe that there is exactly one permutation in $\mathfrak{S}_n$ of length 0, and it is boolean. The case $k = 1$ is slightly less trivial. There are $n-1$ permutations in $\mathfrak{S}_n$ of length 1, and these are all boolean as well. Letting $k = 1$ in equation (4.4) yields

$$\sum_{i=1}^{1} \binom{n-i}{1+1-i}\binom{1-1}{i-1} = \binom{n-1}{1}\binom{0}{0} = n-1,$$

so the corollary holds for $k \leq 1$ and any $n > k$.

Suppose that

$$L(n,k) = \sum_{i=1}^{k} \binom{n-i}{k+1-i}\binom{k-1}{i-1}$$

for all $k \in [0, K)$ and $n \in [1, N)$. A boolean permutation avoids the patterns 321 and 3412. Suppose $w \in \mathfrak{S}_N$ is a boolean permutation with $\ell(w) = K$, and consider the location of N in the one-line notation of w.

- If $w(N) = N$, then $w(1)\cdots w(N-1) \in \mathfrak{S}_{N-1}$ can be any boolean permutation of length K.
- If $w(N-1) = N$, then $w(1)\cdots w(N-2)w(N) \in \mathfrak{S}_{N-1}$ can be any boolean permutation of length $K-1$.
- If $w(N-2) = N$, then $w(N) = N-1$. Thus, $w(1)\cdots w(N-3)w(N-1) \in \mathfrak{S}_{N-2}$ can be any boolean permutation of length $K-2$.
- If $w(N-3) = N$, then $w(N) = N-1$ and $w(N-1) = N-2$. Therefore $w(1)\cdots w(N-4)w(N-2) \in \mathfrak{S}_{N-3}$ can be any boolean permutation of length $K-3$.
- ...

This breakdown of cases indicates that $L(N,K) = L(N-1,K) + \sum_{i=1}^{K} L(N-i, K-i)$, and a bit of work turns this into

$$L(N,K) = L(N-1,K-1) + L(N-1,K-1) + L(N-2,K-1) + \cdots + L(K,K-1).$$

By the inductive assumptions and basic facts about binomial coefficients,

$$\begin{aligned}
L(N,K) &= \sum_{i=1}^{K-1}\binom{N-1-i}{K-i}\binom{K-2}{i-1} + \sum_{j=K}^{N-1}\sum_{i=1}^{K-1}\binom{j-i}{K-i}\binom{K-2}{i-1} \\
&= \sum_{i=1}^{K-1}\binom{N-1-i}{K-i}\binom{K-2}{i-1} + \sum_{i=1}^{K-1}\binom{N-i}{K+1-i}\binom{K-2}{i-1} \\
&= \sum_{i=2}^{K}\binom{N-i}{K+1-i}\binom{K-2}{i-2} + \sum_{i=1}^{K-1}\binom{N-i}{K+1-i}\binom{K-2}{i-1} \\
&= \binom{N-K}{1}\binom{K-2}{K-2} + \sum_{i=2}^{K-1}\binom{N-i}{K+1-i}\left(\binom{K-2}{i-2}+\binom{K-2}{i-1}\right) \\
&\qquad + \binom{N-1}{K}\binom{K-2}{0} \\
&= \sum_{i=1}^{K}\binom{N-i}{K+1-i}\binom{K-1}{i-1}.
\end{aligned}$$

□

The numbers $L(n,k)$ are equal to the numbers $T(n,n-k)$ in sequence A105306 of [34]. From this equivalence, it is straightforward to compute the generating function

$$\sum_{n,k} L(n,k)t^k z^n = \frac{z(1-zt)}{1-2zt-z(1-zt)}. \tag{4.5}$$

For small n and k, the number of boolean permutations in $\mathfrak{S}_n$ of length k are displayed in Table 4.4.

$L(n,k)$	$k=0$	1	2	3	4	5	6	7
$n=1$	1							
2	1	1						
3	1	2	2					
4	1	3	5	4				
5	1	4	9	12	8			
6	1	5	14	25	28	16		
7	1	6	20	44	66	64	32	
8	1	7	27	70	129	168	144	64

Table 4.4: The number of boolean permutations of each length in $\mathfrak{S}_1, \ldots, \mathfrak{S}_8$. Missing table entries are equal to 0.

It is interesting to note that a principal order ideal $B(w)$ is boolean if and only if it is a lattice. Certainly being boolean implies that a poset is a lattice, and the other direction follows from Corollary 4.6.1. The ideal $B(w)$ is a lattice if and only if all of the R-polynomials are of the form $(q-1)^{\ell(y)-\ell(x)}$, as discussed by Brenti in [6]. Moreover, Brenti shows that this is equivalent to all of the Kazhdan-Lusztig polynomials equaling the g-polynomials of the duals of the corresponding subintervals. The g-polynomials are defined in [36], and their coefficients are the toric g-vectors.

Subsequent to Theorem 4.3.2, it is natural to ask the following questions. What can be said about the principal order ideal of permutations with exactly one occurrence of exactly one of the patterns 321 or 3412? (Note that these are the permutations with reduced decompositions in which exactly one letter is repeated, and that letter appears exactly twice.) In particular, what are the sizes of these ideals? These questions are answered below. Generalizations allowing more occurrences of 321 and 3412 are not treated here.

- Suppose that w has exactly one 321-pattern and is 3412-avoiding. Let $\ell = \ell(w)$. Then there exists $i_1 \cdots i_\ell \in R(w)$ such that $i_j i_{j+1} i_{j+2}$ is a Yang-Baxter move, and there are no repeated letters besides i_j and i_{j+2}. The subword property dictates the poset $B(w)$ as follows. Consider the poset B_ℓ of subsets of $[\ell]$ ordered by set inclusion. Delete all elements of the poset containing $\{j, j+2\}$ but not $j+1$, and identify all elements of the poset containing j but not $\{j+1, j+2\}$ with those that interchange the roles of $j+2$ and j. The resulting poset is

isomorphic to $B(w)$, and $|B(w)| = 3 \cdot 2^{\ell-2}$.
Figure 4-4 depicts the principal order ideal for the simplest permutation in this category, $w = 321$.

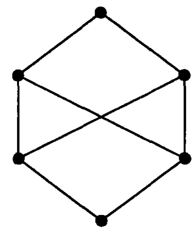

Figure 4-4: The principal order ideal $B(321)$.

- Suppose that w has exactly one 3412-pattern and is 321-avoiding. Let $\ell = \ell(w)$. Then there exists $i_1 \cdots i_\ell \in R(w)$ such that $i_j i_{j+1} i_{j+2} i_{j+3} = (2132)^M$ for some M, and there are no repeated letters besides i_j and i_{j+3}. Again, consider the poset B_ℓ of subsets of $[\ell]$ ordered by set inclusion. Delete all elements of the poset containing $\{j, j+3\}$ but not $\{j+1, j+2\}$, and identify those elements of the poset that contain j but not $\{j+1, j+2, j+3\}$ with those that interchange the roles of $j+3$ and j. The resulting poset is isomorphic to $B(w)$, and $|B(w)| = 7 \cdot 2^{\ell-3}$. Figure 4-5 depicts the principal order ideal for the simplest permutation in this category, $w = 3412$.

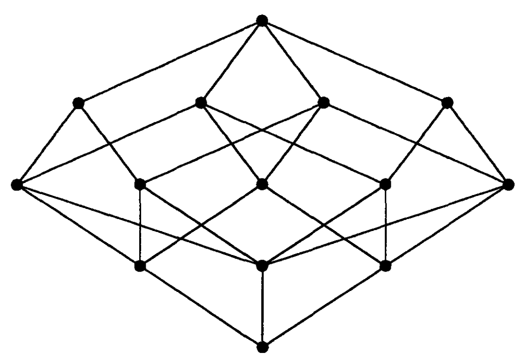

Figure 4-5: The principal order ideal $B(3412)$.

4.4 Principal order ideals isomorphic to a power of $B(w_0^{(k)})$

The previous section characterized all permutations for which $B(w)$ is boolean, where a boolean poset is one which is isomorphic to some power of $B(21)$. This section generalizes the previous work by completely describing all permutations for which $B(w)$ is isomorphic to some power of $B(w_0^{(k)})$ for $k \geq 3$.

Definition 4.4.1. Let $k \geq 3$ be an integer and $w \in \mathfrak{S}_n$ be a permutation. If $B(w) \cong B(w_0^{(k)})^r$ for some r, then w is a *power permutation*. (Note that the specific values of k, n, and r are not recorded.)

If $B(w) \cong B(w_0^{(k)})^r$, then $\ell(w) = r \cdot \binom{k}{2}$.

The reduced decompositions $\emptyset$ and 121 in Table 4.1 represent the only power permutations in the table, as all other power permutations have length at least 6.

As in the previous section, the characterization of power permutations is in terms of patterns, although not in quite the same way as Theorem 4.3.2.

Proposition 4.4.2. *For $x, y \in \mathfrak{S}_n$, suppose that $[x, y] \cong B(w_0^{(k)})$ for some k. Then there exist $\boldsymbol{i} \in R(x)$ and $\boldsymbol{j} \in R(y)$ such that $\boldsymbol{i}$ is obtained by deleting a factor from $\boldsymbol{j}$ which is the shift of an element of $R(w_0^{(k)})$.*

Proof. Let $\boldsymbol{i} \in R(x)$ and $\boldsymbol{j} \in R(y)$ be, by Theorem 2.6.3, such that $\boldsymbol{i}$ is a subword of $\boldsymbol{j}$. Consider the multiset S of the $\binom{k}{2}$ letters deleted from $\boldsymbol{j}$ to form $\boldsymbol{i}$. Because $[u, v] \cong B(w_0^{(k)})$, this S contains exactly $k-1$ distinct letters.

The number of distinct letters in S equals the number of elements covering x in $[x, y]$. Therefore it must be possible to find $\boldsymbol{i}$ and $\boldsymbol{j}$ as above so that the factors in $\boldsymbol{j}$ formed by S have the property that equal elements of S lie in the same factor.

Given T distinct and consecutive letters, the longest reduced decomposition that can be formed by them has length $\binom{T+1}{2}$. Observe that

$$\binom{T_1+1}{2} + \binom{T_2+1}{2} < \binom{T_1+T_2+1}{2}$$

for $T_1, T_2 > 0$. Thus, all of S comprises a single factor in $\boldsymbol{j}$, and the result follows. □

With this groundwork, the main theorem of the section can now be stated.

Theorem 4.4.3. *The permutation w is a power permutation if and only if every inversion in w is in exactly one $w_0^{(k)}$-pattern for some fixed $k \geq 3$.*

Proof. Fix $k \geq 3$ and $w \in \mathfrak{S}_n$. First suppose that every inversion in w is in exactly one $w_0^{(k)}$-pattern, and that w contains R distinct occurrences of the pattern $w_0^{(k)}$. "Undoing" inversions in one of these patterns does not alter the other patterns. Consequently $B(w) \cong B(w_0^{(k)})^R$.

For the other direction of the proof, suppose that $B(w) \cong B(w_0^{(k)})^R$, and proceed by induction on R. The case $R = 0$ is trivial, and the case $R = 1$ was considered in Proposition 4.4.2. Now suppose that the theorem holds for permutations whose principal order ideals are isomorphic to $B(w_0^{(k)})^r$, for all $r \in [0, R)$.

There are R distinct permutations $w_1, \ldots, w_R$, each less than w in the Bruhat order, with

$$B(w_1) \cong \cdots \cong B(w_R) \cong B(w_0^{(k)})^{R-1}.$$

By Proposition 3.5.6 and the inductive hypothesis, each of these R permutations has a reduced decomposition $\boldsymbol{j}^{M_1^h} \cdots \boldsymbol{j}^{M_{R-1}^h} \in R(w_h)$, where the M_i^hs are distinct.

The interval $[w_h, w]$ is isomorphic to $B(w_0^{(k)})$ for all h, so Proposition 4.4.2 indicates that w has a reduced decomposition $\boldsymbol{j}^{M_1} \cdots \boldsymbol{j}^{M_R}$.

The distinct permutations $w_1, \ldots, w_R$ each satisfy the induction hypothesis. Hence the M_is are distinct, and Proposition 3.5.6 completes the proof. □

Corollary 4.4.4. *If the permutation w is a power permutation, then there is a tiling in $Z(w)$ consisting entirely of $2k$-gons for some $k \geq 3$. The converse is true if $k \geq 4$.*

Proof. There is a tiling in $Z(w)$ consisting entirely of $2k$-gons if and only if there is a reduced decomposition $\boldsymbol{j}^{M_1} \cdots \boldsymbol{j}^{M_r} \in R(w)$, where $\boldsymbol{j} \in R(w_0^{(k)})$. By Theorem 4.4.3 and Proposition 3.5.6, power permutations have such reduced decompositions. For $k \geq 4$, the converse follows from Proposition 3.5.5. □

Power permutations avoid the patterns 4312, 3412, and 4231, so their posets $P(w)$, as defined in Chapter 3, have maximal elements by Theorem 3.5.14. Since the tiling of such a w by $2k$-gons, via the above corollary, is maximal in this poset, it is the unique maximal element.

Theorem 4.4.3 gives a concise description of power permutations, again in terms of patterns. Although the flavor of this description differs from that of Theorem 4.3.2, the prominent role of patterns in the power permutation characterization is immediately apparent. It is clear that Theorem 4.4.3 must be restricted to $k \geq 3$, while the $k = 2$ case is treated in Theorem 4.3.2, because to say that "every inversion in w is in exactly one 21-pattern" provides no information.

Observe that in the case $k = 3$, the power permutations are freely braided permutations, as in Section 3.6, that have no "extra" inversions.

It is instructive to consider what it means for every inversion in a permutation to be contained in exactly one $w_0^{(k)}$-pattern. The following facts are straightforward to show for $w \in \mathfrak{S}_n$.

- Distinct occurrences of $w_0^{(k)}$ either do not intersect or share exactly one entry.
- If two occurrences of $w_0^{(k)}$ intersect, then either $\langle k\rangle_1 = \langle k\rangle_2$ or $\langle 1\rangle_1 = \langle 1\rangle_2$.
- Without loss of generality, all values in $\langle w_0^{(k)}\rangle_1$ are at least as large as all values in $\langle w_0^{(k)}\rangle_2$. The non-shared values in $\langle w_0^{(k)}\rangle_1$ all occur to the right of the non-shared values in $\langle w_0^{(k)}\rangle_2$.
- If m is not in any $w_0^{(k)}$-pattern, then it is fixed by w, and the letter m does not occur in any reduced decomposition of w. Moreover, the permutations $w(1)\cdots w(m-1) \in \mathfrak{S}_{m-1}$ and $\big(w(m+1)-m\big)\cdots\big(w(n)-m\big) \in \mathfrak{S}_{n-m}$ are both power permutations with the same parameter k.
- For any occurrence of $w_0^{(k)}$, the values $\langle k-1\rangle, \ldots, \langle 2\rangle$ must occur consecutively in the one-line notation of w.

Example 4.4.5. The permutations $521436 \in \mathfrak{S}_6$ and $432159876 \in \mathfrak{S}_9$ are both power permutations.

4.5 Patterns and order ideals

The previous two sections considered principal order ideals in the Bruhat order of the symmetric group. This section examines order ideals, as defined in Section 2.5, that are not necessarily principal. To be specific, the following questions are completely answered.

1. For what $p \in \mathfrak{S}_k$, where $k \geq 3$, is the set

$$S_n\{p\} = \{w \in \mathfrak{S}_n : w \text{ is } p\text{-avoiding}\}$$

a nonempty order ideal, for some $n > k$?

2. For what $p \in \mathfrak{S}_k$ and $q \in \mathfrak{S}_l$, where $k, l \geq 3$, is the set

$$S_n\{p, q\} = \{w \in \mathfrak{S}_n : w \text{ is } p\text{- and } q\text{-avoiding}\}$$

a nonempty order ideal for some $n \geq k, l$?

Requiring that k and l be at least 3 eliminates trivial cases. The questions are uninteresting if the sets can be empty, although the first of these sets, $S_n\{p\}$, can never be empty. Additionally, if n were permitted to equal k in $S_n\{p\}$, then the set $S_n\{w_0^{(n)}\} = \mathfrak{S}_n \setminus \{w_0^{(n)}\}$ would certainly be an order ideal.

Somewhat surprisingly, there are very few patterns that answer the above questions, and in each case the rank generating function of the resulting order ideal is provided below. Because any principal order ideal in a boolean poset is itself boolean, one answer to the second question is $\{p, q\} = \{321, 3412\}$.

Theorem 4.5.1. *For $k \geq 3$, there is no permutation $p \in \mathfrak{S}_k$ for which there exists $n > k$ such that the set $S_n\{p\}$ is an order ideal.*

Proof. Determining whether $S_n\{p\}$ is an order ideal amounts to determining if there can be $w \in S_n\{p\}$ and $v < w$, where $v \notin S_n\{p\}$. If $w_0^{(n)}$ avoids p, then the set $S_n\{p\}$ cannot be an order ideal because the permutation $p(1)\cdots p(k)(k+1)\cdots n$ is less than $w_0^{(n)}$ but not in $S_n\{p\}$. Thus $p = w_0^{(k)}$.

Let $w = k(k+1)(k-1)(k-2)\cdots 4312(k+2)(k+3)\cdots n \in \mathfrak{S}_n$. There is no p-pattern in w, so $w \in S_n\{p\}$. However, note the following covering relations that exist in the Bruhat order:

$$\begin{aligned}
w &= k(k+1)(k-1)(k-2)\cdots 4312(k+2)(k+3)\cdots n \\
&\gtrdot k(k+1)(k-1)(k-2)\cdots 4132(k+2)(k+3)\cdots n \\
&\gtrdot k(k+1)(k-1)(k-2)\cdots 1432(k+2)(k+3)\cdots n \\
&\gtrdot \cdots \\
&\gtrdot k(k+1)1(k-1)(k-2)\cdots 432(k+2)(k+3)\cdots n \\
&\gtrdot 1(k+1)k(k-1)(k-2)\cdots 432(k+2)(k+3)\cdots n \stackrel{\text{def}}{=} v.
\end{aligned}$$

The permutation v contains the pattern p, where $\langle p\rangle = (k+1)k\cdots 32$, and $v < w$ as demonstrated. Therefore $S_n\{p\}$ is not an order ideal for any $n > k$. $\square$

The set of boolean permutations in $\mathfrak{S}_n$, denoted $S_n\{321, 3412\}$ in this section, is an order ideal. Thus, unlike for $S_n\{p\}$, there do exist permutations p and q for which the set $S_n\{p, q\}$ is an order ideal.

Theorem 4.5.2. *Let $p \in \mathfrak{S}_k$ and $q \in \mathfrak{S}_l$ for $k, l \geq 3$. The only occasions when $S_n\{p, q\}$ is a nonempty order ideal for some $n \geq k, l$ are $S_n\{321, 3412\}$, $S_n\{321, 231\}$, and $S_n\{321, 312\}$. These sets are order ideals for all $n \geq 4$*

Proof. As in the previous proof, it can be assumed that $p = w_0^{(k)} \in \mathfrak{S}_k$.

Suppose that $S_n\{p, q\}$ is a nonempty order ideal for some $n > k, l$. Then the following permutations cannot be in $S_n\{p, q\}$, because they are all larger in the Bruhat order than a permutation containing a p-pattern.

$$k\cdots 3(k+1)12(k+2)\cdots n$$
$$k(k+1)1(k-1)\cdots 32(k+2)\cdots n$$
$$1\cdots(n-k-1)(n-1)\cdots(n-k+2)n(n-k)(n-k+1)$$
$$1\cdots(n-k-1)(n-1)n(n-k)(n-2)\cdots(n-k+2)(n-k+1)$$

These are all p-avoiding, so they must contain the pattern q. The only patterns contained in all of these permutations are the following:

$$q \in \{312, 231, 3412, 12\cdots(l-1)l\}.$$

Suppose $q = 12\cdots(l-1)l$. If $S_n\{p, q\}$ is nonempty, then it cannot be an order ideal because every element in $S_n\{p, q\}$ is greater than $12\cdots n \notin S_n\{p, q\}$. Therefore

$$q \in \{312, 231, 3412\}. \tag{4.6}$$

Suppose that $k > 3$. If $q \in \{231, 312\}$, then $u = 32145\cdots n \in S_n\{p, q\}$. However, $u > q(1)q(2)q(3)45\cdots n \notin S_n\{p, q\}$. Similarly, $v = 342156\cdots n \in S_n\{p, 3412\}$, but $v > 341256\cdots n \notin S_n\{p, 3412\}$. Thus $k = 3$ if $S_n\{p, q\}$ is to be an order ideal.

By Theorems 3.2.8 and 4.3.2, the set $S_n\{321, 231\}$ consists of exactly those permutations that have reduced decompositions $i_1\cdots i_\ell$ for $i_1 > \cdots > i_\ell$. Thus, by Theorem 2.6.3, $S_n\{321, 231\}$ is an order ideal. Similarly, the set $S_n\{321, 312\}$ consists of exactly those permutations that have reduced decompositions $i_1\cdots i_\ell$ for $i_1 < \cdots < i_\ell$. Once again, this is an order ideal. As stated earlier, the set $S_n\{321, 3412\}$ of boolean permutations is also an order ideal. $\square$

The elements of $S_n\{321, 3412\}$ were enumerated by length in Corollary 4.3.5, and their rank generating function is equation (4.5). The enumerations for the sets $S_n\{321, 231\}$ and $S_n\{321, 312\}$ are straightforward.

Corollary 4.5.3. *The number of elements of length k in $S_n\{321, 231\}$ is $\binom{n-1}{k}$. Similarly, the number of elements of length k in $S_n\{321, 312\}$ is $\binom{n-1}{k}$. Consequently these are both rank-symmetric posets, and the rank generating function for each is*

$$\sum_{n,k} \binom{n-1}{k} t^k z^n = \frac{z}{1-(1+t)z}.$$

Proof. The enumeration for $S_n\{321, 231\}$ is due to the fact that each length k element of $S_n\{321, 231\}$ must have a reduced decomposition $i_1 \cdots i_k$ where $i_1 > \cdots > i_k$. Therefore, each element is uniquely determined by choosing k of the $n-1$ possible letters. The enumeration for $S_n\{321, 312\}$ is analogous.

The rank generating function follows. □

Notice that in each instance where $S_n\{p, q\}$ is an order ideal, the rank generating function of this subposet is a rational function. For $S_n\{321, 231\}$ and $S_n\{321, 312\}$, these order ideals are actually *principal*: the maximal element in $S_n\{321, 231\}$ is $n12\cdots(n-1)$ which has a reduced decomposition $(n-1)\cdots 21$, and the maximal element in $S_n\{321, 312\}$ is $23\cdots n1$ which has a reduced decomposition $12\cdots(n-1)$. Results of Lakshmibai and Sandhya (see [19]) and Carrell and Peterson (see [7]) show that the poset $B(w)$ is rank symmetric if and only if w is 3412- and 4231-avoiding, which shows (although it is already clear from Corollary 4.5.3) that $S_n\{321, 231\}$ and $S_n\{321, 312\}$ are both rank symmetric.

The poset of boolean permutations, $S_n\{321, 3412\}$, can be considered in a larger context because of the following definition.

Definition 4.5.4. A finite poset is *simplicial* if it has a unique minimal element and every $B(x)$ is boolean. The *f-vector* of a simplicial poset P is $(f_{-1}, f_0, f_1, \ldots)$, where $f_{-1} \stackrel{\text{def}}{=} 1$ and $f_i \stackrel{\text{def}}{=} \#\{x \in P : B(x) \cong B_{i+1}\}$ for $i \geq 0$. Let $d \stackrel{\text{def}}{=} 1+\max\{i : f_i(P) \neq 0\}$. The *$h$-vector* of P is $(h_0, h_1, \ldots)$, where $h_0 \stackrel{\text{def}}{=} 1$, $h_1 \stackrel{\text{def}}{=} f_0 - d$, and $h_i = f_{i-1} - f_{i-2}$.

The poset $S_n\{321, 3412\}$ is simplicial, and the f-vector was computed by the rank generating function in equation (4.4). In [37], Stanley showed that for a given vector $\boldsymbol{h}$, there exists a Cohen-Macaulay simplicial poset with h-vector equal to $\boldsymbol{h}$ if and only if $h_0 = 1$ and $h_i \geq 0$ for all i. Consider the h-vector of $S_n\{321, 3412\}$. The last coordinate of this is $L(n, n-1) - L(n, n-2)$. This is negative for $n > 3$ (the only n for which $S_n\{321, 3412\}$ is defined), so the simplicial poset $S_n\{321, 3412\}$ is never Cohen-Macaulay.

4.6 Bruhat intervals

The previous sections in this chapter have suggested several relationships between order ideals in the Bruhat order and permutation patterns. This section moves beyond these connections to examine the impact of permutation patterns on *intervals* in the Bruhat order. For example, the following corollary to Theorem 3.2.8 suggests the type of statement that is possible.

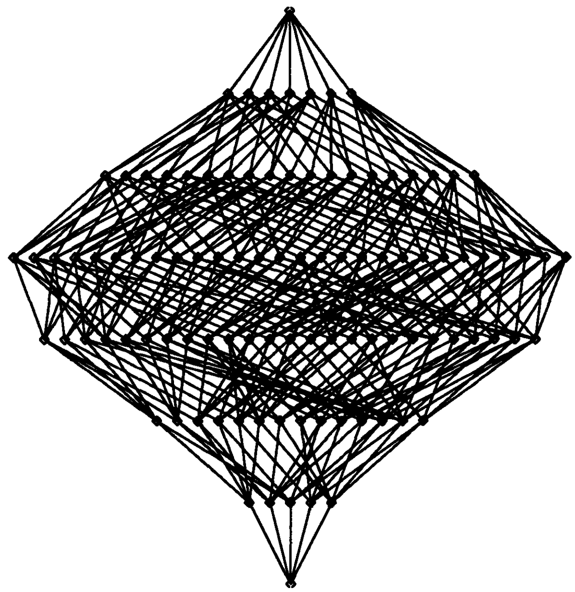

Figure 4-7: The poset $B(351624)$.

Definition 4.6.5. Let $\mathcal{N} \subset \mathfrak{S}_6$ consist of the following sixteen permutations:

$$\begin{aligned}\mathcal{N} = \{&351624, 351642, 352614, 361524, 531624, 352641, 361542, 362514,\\ &531642, 532614, 631524, 362541, 532641, 631542, 632514, 632541\}.\end{aligned}$$

Note that $\mathcal{N}$ is the interval $[351624, 632541]$ in $\mathfrak{S}_6$. This interval is isomorphic to B_4.

Before discussing $\mathcal{P}$, it is important to understand exactly what it means to avoid the patterns in $\mathcal{N}$.

Definition 4.6.6. Let $p \in \mathfrak{S}_k$ and $w \in \mathfrak{S}_n$ be such that w has a p-pattern. Let x be a non-pattern entry in w inside the pattern p, as defined in Chapter 3. The entry x is an *obstacle to* p, or simply an *obstacle*, if the following conditions all hold.

1. There exists $m \in [k-1]$ such that $\langle m \rangle < x < \langle m+1 \rangle$;
2. There exists an inversion (i, j) in p such that $\langle p(i) \rangle > x > \langle p(j) \rangle$;
3. There exist a and b such that $a < m < m+1 < b$;
4. The values $p(i)$, a, b, and $p(j)$ occur from left to right in p;
5. The values m, a, b, and $m+1$ occur from left to right in p; and
6. The values $\langle a \rangle$, x, $\langle b \rangle$ occur from left to right in w.

As with the definitions of *inside* and *outside* from Chapter 3, being an obstacle is only defined for non-pattern entries.

Definition 4.6.7. Let p be a permutation. If there is no permutation that contains the pattern p and has an obstacle to p, then *p has no obstacles.*

Corollary 4.6.1. *Let p be a vexillary permutation. For any w containing a p-pattern, there is a permutation $v \leq w$ such that the interval $[e, v] = B(v)$ in $B(w)$, where e is the identity permutation, is isomorphic to $B(p)$.*

Proof. By Theorem 3.2.8, there exists $\boldsymbol{j} \in R(w)$ with a factor $\boldsymbol{i}^M$ for $\boldsymbol{i} \in R(p)$. Let $\boldsymbol{j} = \boldsymbol{j}_1 \boldsymbol{i}^M \boldsymbol{j}_2$. Then by Theorem 2.6.3, the permutation v, for which $\boldsymbol{i}^M$ is a reduced decomposition, is less than w. Certainly $B(p) \cong B(v)$, which completes the proof. □

The aim of this section is to state conditions for a permutation p so that it is a member of the following set.

Definition 4.6.2. Let p be a permutation. If $B(p)$ is isomorphic to an interval in $B(w)$ for every permutation w containing p, then p is a *pattern-interval permutation*. Let

$$\mathcal{P} = \{p : p \text{ is a pattern-interval permutation}\}. \tag{4.7}$$

Corollary 4.6.1 indicates that $\mathcal{P}$ contains all vexillary permutations. However, it is not the case that $\mathcal{P}$ contains *only* vexillary permutations, as shown below.

Example 4.6.3. Let $p = 2143$, the smallest non-vexillary permutation. The principal order ideal $B(p)$ is shown in Figure 4-6. Indeed, the fact that the Bruhat order is Eulerian implies that *every* interval of length two in the Bruhat order is isomorphic to the poset in Figure 4-6. Therefore, for every w containing p, there is an interval in $B(w)$ isomorphic to $B(p)$, so $2143 \in \mathcal{P}$.

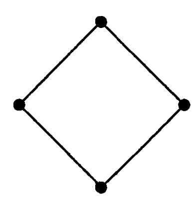

Figure 4-6: The form of every length two interval in the Bruhat order.

Since $\mathcal{P}$ contains all vexillary permutations as well as the permutation 2143, one might suppose that $\mathcal{P}$ in fact contains every permutation. However, this is not the case, as indicated in the next example.

Example 4.6.4. Let $p = 351624$ and $w = 3614725$. The pattern p occurs in w as $\langle p \rangle = 361725$. The posets $B(p)$ and $B(w)$ are depicted in Figures 4-7 and 4-8, respectively. Stembridge's MAPLE packages [40] and [41] indicate that no interval of $B(w)$ is isomorphic to $B(p)$, so $p \notin \mathcal{P}$.

Now that $\mathcal{P}$ is known not to include all permutations, the characterization of its elements becomes a more interesting question. In particular, can $\mathcal{P}$ be described in terms of pattern avoidance? That is, is there a statement "$p \in \mathcal{P}$ if and only if p avoids all of the patterns $q, r, \ldots$"? In fact, there is a statement that if p avoids a particular set of patterns then $p \in \mathcal{P}$. Evidence suggests that the converse of this statement may hold as well.

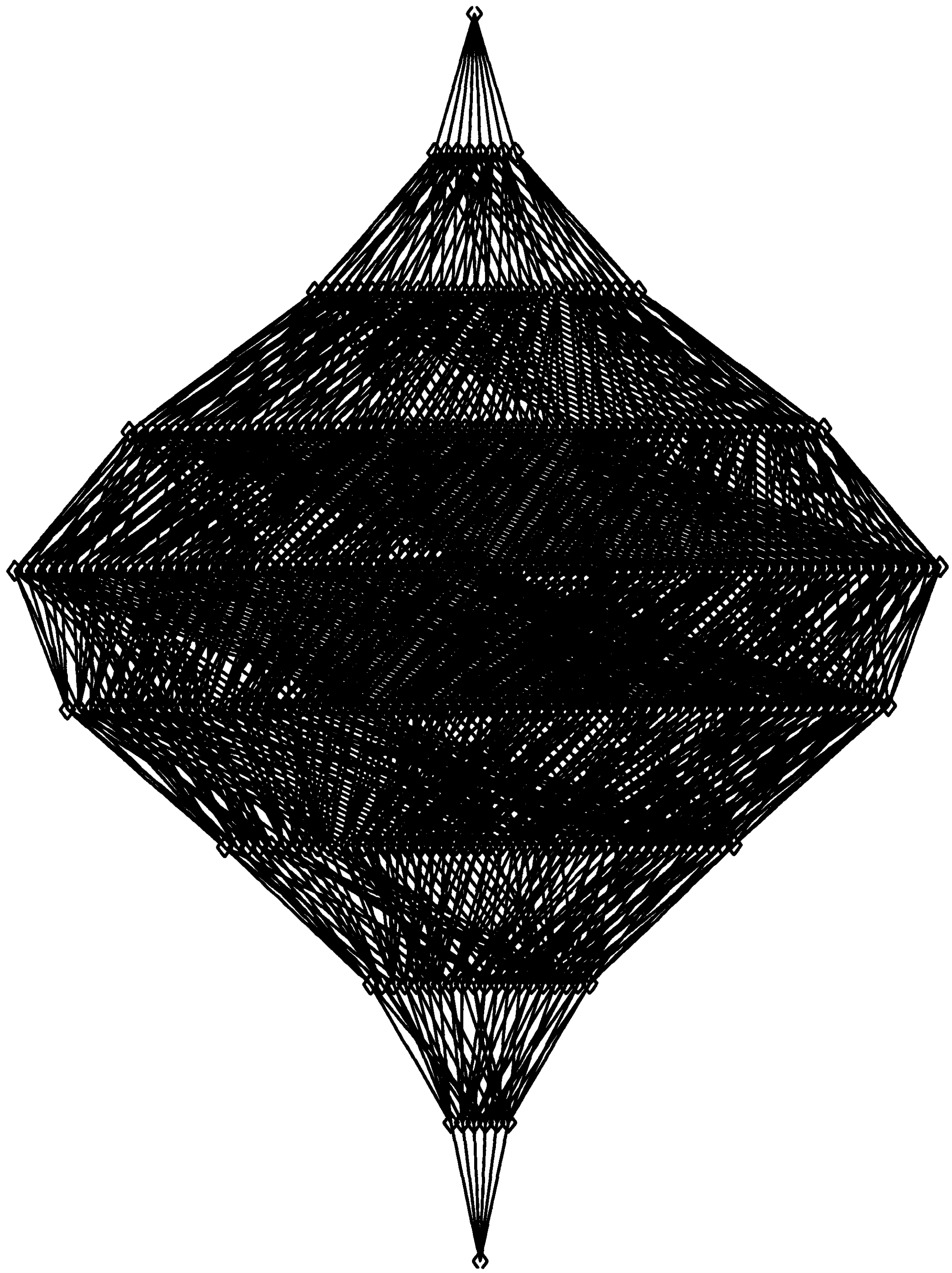

Figure 4-8: The poset $B(3614725)$. The poset in Figure 4-7 does not occur as an interval in this poset.

Proposition 4.6.8. *A permutation p has no obstacles if and only if p avoids all patterns in the set $\mathcal{N}$.*

Proof. Definitions 4.6.6 and 4.6.7 indicate that there can be an obstacle to p if and only if there are substrings $uvwx$ and $yvwz$ occurring from left to right in p, where $v < y < z < w$ and $x < y < z < u$. These amount to sixteen possible patterns in p, which are exactly the elements of $\mathcal{N}$. □

The previous proposition does some of the work towards defining a subset of $\mathcal{P}$ in terms of pattern avoidance. As suggested by this work, the patterns which must be avoided are exactly those in the set $\mathcal{N}$. The proof of this requires some definitions.

Definition 4.6.9. Let q be a permutation in $\mathcal{N} \subset \mathfrak{S}_6$. The *stretch* of q by S is the permutation $q_S \in \mathfrak{S}_{6+S}$ defined by inserting the values $4, 5, \ldots, 3 + S$ between $q(3)$ and $q(4)$, and increasing the values $4, 5, 6$ in q by S. For example,

$$351624_2 = 37145826.$$

Let $\mathcal{N}^* = \{q_S : q \in \mathcal{N} \text{ and } S \geq 0\}$.

In Example 4.6.4, where it was shown that $351624 \notin \mathcal{P}$, the w that prohibits membership in $\mathcal{P}$ is exactly 351624_1. The reason that $B(3614725)$ does not have a subinterval isomorphic to $B(351624)$ is essentially because the 4 in 3614725 is an obstacle to 351624:

- $\langle 3 \rangle < 4 < \langle 4 \rangle$,
- $\langle p(2) \rangle > 4 > \langle p(5) \rangle$,
- $1 < 3 < 4 < 6$,
- $p(2)$, 1, 6, and $p(5)$ occur from left to right in p,
- 3, 1, 6, and 4 occur from left to right in p, and
- $\langle 1 \rangle$, 4, and $\langle 6 \rangle$ occur from left to right in w.

Theorem 4.6.10. *If p has no obstacles, then $p \in \mathcal{P}$.*

Proof. Suppose that p is a permutation which has no obstacles. Let w be any permutation containing p. Unlike in the proof of Theorem 3.2.8, it is not required now to be able to push the pattern p together in some $\widetilde{w} < w$. Rather, it is enough to find $\widetilde{w} < w$ containing p such that for each inversion (i, j) in p, there is no non-pattern entry

$$x \in (\langle p(j) \rangle, \langle p(i) \rangle) \qquad (4.8)$$

lying between $\langle p(i) \rangle$ and $\langle p(j) \rangle$. If there are no such entries x, then there is an interval $[v, \widetilde{w}] \subset B(w)$ isomorphic to $B(p)$.

Recall the algorithm VEX of Chapter 3. The only way that a non-pattern entry x inside the pattern may be impossible to move outside of the pattern is if there is a configuration of the form

$$\cdots \langle m_2 \rangle \cdots \langle m_1 \rangle \cdots x \cdots \langle m_4 \rangle \cdots \langle m_3 \rangle \cdots$$

in the one-line notation of w, where $\langle m_1 \rangle < \langle m_2 \rangle < x < \langle m_3 \rangle < \langle m_4 \rangle$. It is only necessary to move x outside of $\langle p(i) \cdots p(j) \rangle$, thus if p is not a pattern-interval permutation then

- $p(i) > m_3$ and $p(i)$ occurs to the left of m_1 in p, and
- $p(j) < m_2$ and $p(j)$ occurs to the right of m_4 in p.

Equation (4.8) indicates that this only happens if p has an obstacle.

Thus, if p has no obstacles, then there is no such non-pattern entry x, and p is a pattern-interval permutation. □

While Theorem 4.6.10 gives a better idea of what sort of permutations can be in $\mathcal{P}$, note in particular that vexillary permutations are a proper subset of the permutations that have no obstacles, it does not address what permutations are *not* pattern-interval permutations. One step in this direction is the following result.

Theorem 4.6.11. $\mathcal{N}^* \cap \mathcal{P} = \emptyset$.

Proof. Consider $q_S \in \mathcal{N}^*$. The permutation q_{S+1} contains a q_S-pattern. Observe that

$$\boldsymbol{i_S} \stackrel{\text{def}}{=} 21(4+S)(5+S)34\cdots(3+S)\cdots 43(4+S)2 \in R(351624_S).$$

Therefore $\boldsymbol{a i_S b} \in R(q_S)$, where $\boldsymbol{a}, \boldsymbol{b} \in \{\emptyset, 1, 5+S, 1(5+S)\}$. Likewise $\boldsymbol{A i_S B} \in R(q_{S+1})$, where $\boldsymbol{A}$ is $\boldsymbol{a}$ after changing any $5+S$ in $\boldsymbol{a}$ to $6+S$, and $\boldsymbol{B}$ is defined similarly.

Observe that $\ell(q_{S+1}) = \ell(q_S) + 2$. Therefore, if an interval $[x,y] \subset B(q_{S+1})$ is isomorphic to $B(q_S)$, then $\ell(x) \leq 2$. Also note that the all elements of $\mathcal{N}^*$, in particular the permutation $q_S \in \mathfrak{S}_{6+S}$, are indecomposable.

Keeping in mind this indecomposability, consider the reduced decompositions $\boldsymbol{a i_S b} \in R(q_S)$ and $\boldsymbol{A i_{S+1} B} \in R(q_{S+1})$, and examine the three cases $\ell(x) \in \{0,1,2\}$. The details are omitted here, but it is relatively straightforward to show that there is never an appropriate y yielding an interval $[x,y] \cong B(q_S)$.

Therefore the ideal $B(q_{S+1})$ never has an interval isomorphic to $B(q_S)$, and q_S is not a pattern-interval permutation.

□

Theorem 4.6.10 states that avoiding the patterns in $\mathcal{N}$ implies membership in $\mathcal{P}$. On the other hand, Theorem 4.6.11 demonstrates that containing an element of $\mathcal{N}$ as a very specific type of pattern prohibits membership in $\mathcal{P}$. Taken together these results suggest the following conjecture.

Conjecture 4.6.12. *The set $\mathcal{P}$ is the set of permutations that have no obstacles.*

Chapter 5

Computing expected values in type B

5.1 Introduction

Recall that $\mathfrak{S}_n^B$ denotes the signed permutations of $[\pm n]$, the finite Coxeter group of type B. This group is generated by the simple reflections $\{s_0, s_1, \ldots, s_{n-1}\}$, which satisfy the braid relations in equations (2.5)-(2.6) and (2.8)-(2.9). As a reminder, these are repeated below:

$$s^2 = e \text{ for all } s; \tag{2.5}$$
$$s_i s_j = s_j s_i \text{ for } |i-j| > 1; \tag{2.6}$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \text{ for } i \in [n-2]; \text{ and} \tag{2.8}$$
$$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0. \tag{2.9}$$

Recall further that every element in $\mathfrak{S}_n^B$ can be written as a product of the simple reflections $\{s_i : i \in [0, n-1]\}$, and the minimum number of simple reflections required for a product to equal w is the length of w, denoted $\ell(w)$. The longest element in $\mathfrak{S}_n^B$ is $w_0^B = \underline{1}\underline{2} \cdots \underline{n}$, and $\ell(w_0^B) = n^2$.

The symmetric group $\mathfrak{S}_n$ of unsigned permutations is generated by the simple reflections $\{s_i : i \in [n-1]\}$, which are subject to the relations in equations (2.5)-(2.6) and (2.8). The longest element in $\mathfrak{S}_n$ is $w_0 = n(n-1)\cdots 1$, which has length $\binom{n}{2}$. In [26], Reiner computes the following somewhat surprising result.

Theorem 5.1.1 (Reiner). *The expected number of Yang-Baxter moves in a reduced decomposition of $w_0 \in \mathfrak{S}_n$ is 1 for all $n \geq 3$.*

This chapter presents results for finite Coxeter groups of type B that are analogous to Theorem 5.1.1. In type A, the expectation of factors corresponding to the braid relation in equation (2.8) was computed. For the hyperoctahedral group, factors corresponding to the braid relations in each of equations (2.8) and (2.9) will be treated. Theorem 5.3.1 calculates that the expected number of Yang-Baxter moves in a reduced decomposition of $w_0^B \in \mathfrak{S}_n^B$ is $2 - 4/n$, and Theorem 5.4.1 shows that the expected number of 01 moves is $2/(n^2-2)$. Unlike Reiner's result, both of these expectations are dependent upon n. Moreover, in the context of Theorem 5.1.1, the value $2 - 4/n$ seems quite plausible since the length of $w_0^B \in \mathfrak{S}_n^B$ is approximately twice that of $w_0 \in \mathfrak{S}_n$.

A variety of tools are used to prove Theorems 5.3.1 and 5.4.1, several of which are discussed in Section 5.2. Section 5.3 computes the expected number of Yang-Baxter moves in elements of $R(w_0^B)$, and Section 5.4 does the same for 01 moves.

Prior to these discussion, a few terms must be defined.

Definition 5.1.2. A *shape*, or *partition*, λ is a sequence of integers $(\lambda_1, \ldots, \lambda_r)$ where

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0.$$

The shape λ may be represented by a diagram with λ_1 boxes in the top row, λ_2 boxes in the second row, and so on, with the left sides justified. Figure 5-1 depicts the shape $(5, 5, 1)$.

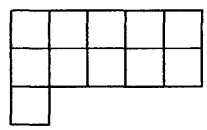

Figure 5-1: The shape $\lambda = (5, 5, 1)$.

Definition 5.1.3. A *shifted shape* λ^B is a sequence of integers $(\lambda_1^B, \ldots, \lambda_r^B)$ where

$$\lambda_1^B > \lambda_2^B > \cdots > \lambda_r^B > 0.$$

The shifted shape λ^B may be represented by a diagram with λ_1^B boxes in the top row, λ_2^B boxes in the second row, and so on, with the left edge of each row shifted to the right one unit from the left edge of the row above it. Figure 5-2 depicts the shifted shape $(5, 4, 1)$.

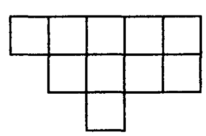

Figure 5-2: The shifted shape $\lambda^B = (5, 4, 1)$.

Definition 5.1.4. A *standard Young tableau*, or *standard tableau*, is a filling of the n boxes in the diagram of a (shifted) shape by the integers $1, 2, \ldots, n$, such that the values in each row are strictly increasing from left to right, and the values in each column are strictly increasing from top to bottom.

A good reference for Young tableaux, with an emphasis on their connections to representation theory, is [11].

5.2 Vexillary elements, shapes, and hook lengths in type B

In [35], Stanley shows that for a vexillary element $v \in \mathfrak{S}_n$,

$$\#R(v) = f^{\lambda(v)}, \tag{5.1}$$

where $f^{\lambda(v)}$ is the number of standard Young tableaux of a particular shape $\lambda(v)$. This result is central to the proof of Theorem 5.1.1.

There are numerous definitions of vexillarity in type A, as discussed in Section 3.2. Billey and Lam define a notion of vexillary for type B in [1]. Their definition is in terms of Stanley symmetric functions and Schur Q-functions, and they prove its equivalence with a statement about pattern avoidance, now of signed permutations. This latter statement will be given as the definition here, and it follows from the work of Billey and Lam that it generalizes equation (5.1) to type B in the appropriate way.

Definition 5.2.1. An element $w \in \mathfrak{S}_n^B$ is *vexillary for type B* if $w = w(1) \cdots w(n)$ avoids the following patterns:

$$\begin{array}{lll}
21 & \underline{3}2\underline{1} & 2\underline{3}4\underline{1} \\
\underline{23}4\underline{1} & 3\underline{412} & \underline{34}1\underline{2} \\
\underline{3412} & \underline{4}1\underline{2}3 & \underline{412}3
\end{array}$$

Recall from Section 2.4 that patterns in signed permutations must maintain their signs. For example, $\underline{12}$ is not an instance of the pattern 21, even though $\underline{1} > \underline{2}$.

Example 5.2.2. $2\underline{14}3 \in \mathfrak{S}_4^B$ is vexillary for type B, but $2\underline{1}43 \in \mathfrak{S}_4^B$ is not.

To each element $w \in \mathfrak{S}_n^B$, Billey and Lam define a shifted shape $\lambda^B(w)$ as follows.

Definition 5.2.3. Let $w = w(1) \cdots w(n)$ be in $\mathfrak{S}_n^B$.

1. Write $\{w(1), \ldots, w(n)\}$ in increasing order and call this $u \in \mathfrak{S}_n^B$.

2. Let $v \in \mathfrak{S}_n$ be the (vexillary) permutation $u^{-1}w$.

3. Let μ be the partition with (distinct) parts $\{|u_i| : u_i < 0\}$.

4. Let U be any standard shifted Young tableau of shape μ, and let V be any standard Young tableau whose shape is the transpose of the shape $\lambda(v)$ as defined in Chapter 3.

5. Embed U in the shifted shape $\delta = (n, n-1, \ldots, 1)$.

6. Fill in the rest of δ with $1', \ldots, k'$ starting from the rightmost column and labeling each column from bottom to top. This gives the tableau R.

7. Obtain S by adding $|\mu|$ to each entry of V, and glue R to the left side of S to obtain T.

8. Delete the box containing $1'$ from T. If the remaining tableau is not shifted, apply jeu de taquin to fill in the box. (The procedure known as *jeu de taquin* is described at length in [39].) Do likewise for the box containing $2'$, then $3'$, and so on, stopping after completing the procedure for the box containing k'.

The (shifted) shape of the resulting tableau is $\lambda^B(w)$.

Example 5.2.4. Suppose $w = 2\underline{14}3 \in \mathfrak{S}_4$. Then $u = \underline{41}23$, $v = 3214$, $\mu = (4,1)$, and the tableau V has shape $(2,1)$. Five boxes of δ will be filled by primed numbers, and the algorithm proceeds as depicted in Figure 5-3.

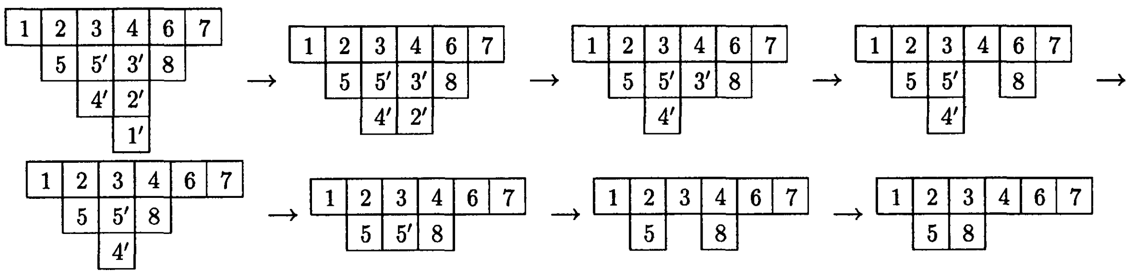

Figure 5-3: Determining the shifted shape of $2\underline{14}3$.

The final tableau has shifted shape $\lambda^B(2\underline{14}3) = (6,2)$.

Proposition 5.2.5 (Billey-Lam). *If $w \in \mathfrak{S}_n^B$ is vexillary for type B, then*

$$\#R(w) = f^{\lambda^B(w)} \tag{5.2}$$

where $f^{\lambda^B(w)}$ is the number of standard tableaux of shifted shape $\lambda^B(w)$.

Equation (5.2) will play an analogous role in the proofs of this chapter to that played by equation (5.1) in [26]. Hooks and hook-lengths for shifted shapes will also be important tools, as they facilitate the calculation of $f^{\lambda^B(w)}$. Recall the hook-length formula for straight shapes (see [39] for a more extensive treatment).

Proposition 5.2.6. *For a shape $\lambda \vdash N$,*

$$f^{\lambda} = \frac{N!}{\prod_{u\in\lambda} h(u)},$$

where $h(u)$ is the number of squares in λ that are

1. In the same column as u but no higher; or

2. In the same row as u but no farther to the left.

There is an analogous formula for shifted shapes (for more information, see [28]).

Proposition 5.2.7. *For a shifted shape $\lambda^B \vdash N$,*

$$f^{\lambda^B} = \frac{N!}{\prod_{u\in\lambda^B} h^B(u)},$$

where $h^B(u)$ is the total number of the squares in λ^B that are

1. *In the same column as u but no higher;*

2. *In the same row as u but no farther to the left; or*

3. *In the $(k+1)$st row of λ^B if u is in the kth column of λ^B.*

9	6	5	3	2
	5	4	2	1
		1		

Figure 5-4: Hook lengths for $\lambda^B = (5,4,1)$. $f^{\lambda^B} = 56$.

The final preliminary to proving the main results of this chapter is the following lemma.

Lemma 5.2.8. *For $w_0^B \in \mathfrak{S}_n^B$ and $i \in [0, n-1]$,*

$$s_i w_0^B s_i = w_0^B.$$

Proof. For $i \in [n-1]$,

$$s_i w_0^B = \underline{1} \cdots \underline{(i-1)\ (i+1)\ i\ (i+2)} \cdots \underline{n} = w_0^B s_i.$$

Also,

$$s_0 w_0^B = 1\underline{23} \cdots \underline{n} = w_0^B s_0.$$

□

This indicates a $\mathbb{Z}/n^2\mathbb{Z}$-action on the set $R(w_0^B)$ defined by

$$s_{i_1} s_{i_2} \cdots s_{i_{n^2}} \mapsto s_{i_2} \cdots s_{i_{n^2}} s_{i_1}.$$

As with the other machinery discussed in this section, Lemma 5.2.8 has an analogous, although not identical, statement in type A which is used in [26].

5.3 Expectation of Yang-Baxter moves

Consider the set $R(w_0^B)$ with uniform probability distribution. Let X_n^B be the random variable on reduced decompositions of $w_0^B \in \mathfrak{S}_n^B$ which counts the number of Yang-Baxter moves.

Theorem 5.3.1. *For all $n \geq 3$, $E(X_n^B) = 2 - 4/n$.*

Proof. Fix $n \geq 3$ and let w_0^B be the longest element in $\mathfrak{S}_n^B$. For $k > 0$, let $X_n^B[j,k]$ be the indicator random variable which determines whether the factor $i_j i_{j+1} i_{j+2}$ in a reduced decomposition $i_1 \cdots i_{n^2} \in R(w_0^B)$ is of either form $k(k+1)k$ or $(k+1)k(k+1)$. Therefore

$$E(X_n^B) = \sum_{j=1}^{n^2-2} \sum_{k=1}^{n-2} E(X_n^B[j,k]).$$

The variables $X_n^B[j,k]$ and $X_n^B[j',k]$ have the same distribution by Lemma 5.2.8, so in fact

$$E(X_n^B) = (n^2-2) \sum_{k=1}^{n-2} E(X_n^B[1,k]).$$

If $X_n^B[1,k](\boldsymbol{i}) = 1$ for $\boldsymbol{i} = i_1 \cdots i_{n^2} \in R(w_0^B)$, then

$$\boldsymbol{i} \in \{k(k+1)ki_4 \cdots i_{n^2}, (k+1)k(k+1)i_4 \cdots i_{n^2}\}.$$

In both cases, $i_4 \cdots i_{n^2}$ is a reduced decomposition of

$$w_k \stackrel{\text{def}}{=} s_k s_{k+1} s_k w_0^B = \underline{1} \cdots \underline{(k-1)\ (k+2)\ (k+1)\ k\ (k+3)} \cdots \underline{n}.$$

Notice that w_k is vexillary for type B for all k, as is w_0^B. Therefore, by Proposition 5.2.5,

$$E(X_n^B) = 2(n^2-2) \sum_{k=1}^{n-2} \frac{\#R(w_k)}{\#R(w_0^B)} = 2(n^2-2) \sum_{k=1}^{n-2} \frac{f^{\lambda^B(w_k)}}{f^{\lambda^B(w_0^B)}} \tag{5.3}$$

The shifted shapes $\lambda^B(w_0^B)$ and $\lambda^B(w_k)$ are easy to determine, as the signed permutation u in Definition 5.2.3 is $\underline{n} \cdots \underline{1}$ in both cases, so no boxes contain primed entries in the shifted tableau T. Thus, the shifted shapes are

$$\begin{aligned} \lambda^B(w_0^B) &= (2n-1, 2n-3, \ldots, 3, 1) \text{ and} \\ \lambda^B(w_k) &= (2n-1, 2n-3, \ldots, 2k+5, 2k+1, 2k, 2k-1, \ldots, 3, 1). \end{aligned} \tag{5.4}$$

Recall the hook-length formula of Proposition 5.2.7, particularly the definition of the hooks h^B in shifted shapes. The only hook-lengths that do not cancel in the ratio $f^{\lambda^B(w_k)}/f^{\lambda^B(w_0^B)}$ correspond to the shaded boxes in Figure 5-5.

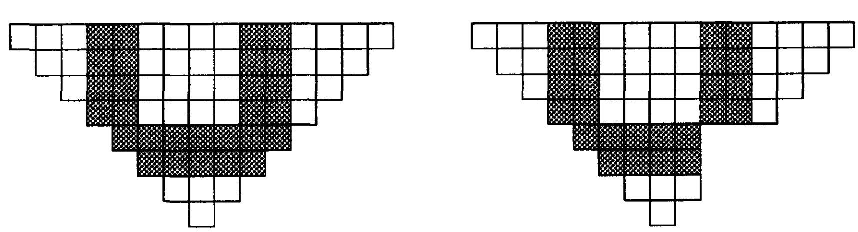

Figure 5-5: The shifted shapes $\lambda^B(w_0^B)$ and $\lambda^B(w_k)$ for $n = 8$ and $k = 2$. The shaded boxes are where the hook-lengths are unequal.

Consequently, equation (5.3) can be written as

$$E(X_n^B) = \frac{1}{3}\binom{n^2}{2}^{-1} \sum_{k=1}^{n-2} C_k, \tag{5.5}$$

where

$$C_k = \frac{3\cdot 5\cdots(2k+3)}{2\cdot 4\cdots(2k)} \cdot \frac{3\cdot 5\cdots(2n-2k-1)}{2\cdot 4\cdots(2n-2k-4)} \cdot \frac{(2k+4)(2k+6)\cdots(4k+4)}{(2k+1)(2k+3)\cdots(4k+1)} \cdot \frac{(4k+8)(4k+10)\cdots(2n+2k+2)}{(4k+5)(4k+7)\cdots(2n+2k-1)},$$

and empty products are defined to be 1.

Notice that

$$\frac{C_{k+1}}{C_k} = \frac{(2k+3)(4k+7)(2k+1)(n-k-2)(n+k+2)}{(4k+3)(k+2)(2n+2k+1)(2n-2k-1)(k+1)}$$

is a rational function in k. Therefore, $\sum_{k=1}^{n-2} C_k$ is a hypergeometric series. Following the notation in [27], equation (5.5) can be rewritten as

$$E(X_n^B) = \frac{1}{3}\binom{n^2}{2}^{-1} C_0 \left({}_5F_4\left(\begin{matrix}3/2, 7/4, 1/2, 2-n, 2+n \\ 3/4, 2, 1/2+n, 1/2-n\end{matrix}\right) - 1\right).$$

The hypergeometric series in question can be computed via Dougall's theorem, as discussed in [33]. The theorem states that

$${}_5F_4\left(\begin{matrix}a, 1+a/2, b, c, d \\ a/2, 1+a-b, 1+a-c, 1+a-d\end{matrix}\right) = \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}.$$

It is not immediately obvious that Dougall's theorem applies to this particular series because of a potential pole. However, the theorem does show that

$$\begin{aligned} {}_5F_4\left(\begin{matrix}3/2, 7/4, 1/2, 2-n, 2+x \\ 3/4, 2, 1/2+n, 1/2-x\end{matrix}\right) &= \frac{\Gamma(2)\Gamma(1/2+n)\Gamma(1/2-x)\Gamma(n-x-2)}{\Gamma(5/2)\Gamma(n)\Gamma(-x)\Gamma(n-x-3/2)} \\ &= \frac{(-x)_{n-2}(5/2)_{n-2}}{(n-1)!(1/2-x)_{n-2}}, \end{aligned}$$

where $(y)_m = y(y-1)\cdots(y-m+1)$ is a *falling factorial*. Therefore there is no pole in this situation. Letting x approach n shows that the desired hypergeometric series has sum $n/2$.

Finally,

$$C_0 = 3 \cdot \frac{3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots (2n-4)} \cdot \frac{4}{1} \cdot \frac{8 \cdot 10 \cdots (2n+2)}{5 \cdot 7 \cdots (2n-1)} = 6n(n^2-1),$$

which completes the proof:

$$E(X_n^B) = \frac{1}{3}\binom{n^2}{2}^{-1} 6n(n^2-1)(n/2-1) = 2 - 4/n.$$

□

As suggested earlier, it is appropriate that Yang-Baxter moves are approximately twice as common in elements of $R(w_0^B)$ as in elements of $R(w_0)$, as

$$\ell(w_0^B) = n^2 \approx 2\binom{n}{2} = 2\ell(w_0)$$

for $w_0^B \in \mathfrak{S}_n^B$ and $w_0 \in \mathfrak{S}_n$.

5.4 Expectation of 01 moves

As in the previous section, consider the set $R(w_0^B)$ with uniform probability distribution. Let Y_n^B be the random variable on reduced decompositions of $w_0^B \in \mathfrak{S}_n^B$ which counts the number of 01 moves.

Theorem 5.4.1. *For all* $n \geq 2$, $E(Y_n^B) = 2/(n^2-2)$.

Proof. Fix $n \geq 2$ and let w_0^B be the longest element in $\mathfrak{S}_n^B$. Let $Y_n^B[j]$ be the indicator random variable which determines whether the factor $i_j i_{j+1} i_{j+2} i_{j+3}$ in a reduced decomposition $i_1 \cdots i_{n^2} \in R(w_0^B)$ is of either form 0101 or 1010. As in the proof of Theorem 5.3.1, Lemma 5.2.8 implies that

$$E(Y_n^B) = \sum_{j=1}^{n^2-3} E(Y_n^B[j]) = (n^2-3)E(Y_n^B[1]).$$

If $Y_n^B[1](\boldsymbol{i}) = 1$ for $\boldsymbol{i} = i_1 \cdots i_{n^2} \in R(w_0^B)$, then $\boldsymbol{i}$ is either $0101 i_5 \cdots i_{n^2}$ or $1010 i_5 \cdots i_{n^2}$. The string $i_5 \cdots i_{n^2}$ is a reduced decomposition of

$$w' = 12\underline{3}\underline{4} \cdots \underline{n}$$

in both situations. As with w_k, the signed permutation w' is vexillary for type B. Therefore

$$E(Y_n^B) = 2(n^2-3)\frac{\#R(w')}{\#R(w_0^B)} = 2(n^2-3)\frac{f^{\lambda^B(w')}}{f^{\lambda^B(w_0^B)}}.$$

The shifted shape $\lambda^B(w_0^B)$ is as in equation (5.4). Applying Definition 5.2.3 to w' proceeds as follows:

1. The signed permutation u is $\underline{n}\cdots\underline{3}12$.
2. The vexillary permutation v is $(n-1)n(n-2)(n-3)\cdots 21$.
3. The partition μ is $(n, n-1, \ldots, 4, 3)$.
4. The shifted tableau U has shape $(n, n-1, \ldots, 4, 3)$ and the straight tableau V has shape $(n-1, n-2, \ldots, 3, 2)$.

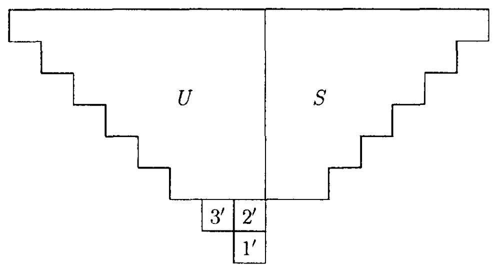

Figure 5-6: Step (7) of Definition 5.2.3 applied to w'.

Unlike the cases of w_0^B or w^k in the proof of Theorem 5.3.1, there will be boxes of T containing primed numbers, specifically $1'$, $2'$, and $3'$, as in Figure 5-6. However, removing $1'$ leaves a shifted tableau so jeu de taquin is not applied. Similarly, $2'$ and then $3'$ can each be removed without performing jeu de taquin. Thus

$$\lambda^B(w') = (2n-1, 2n-3, \ldots, 7, 5).$$

Having determined $\lambda^B(w')$, it remains to compute the ratio $f^{\lambda^B(w')}/f^{\lambda^B(w_0^B)}$ via Proposition 5.2.7. As in the proof of Theorem 5.3.1, many of the hook-lengths cancel. Figure 5-7 depicts the only boxes in the two shapes where the hook-lengths differ.

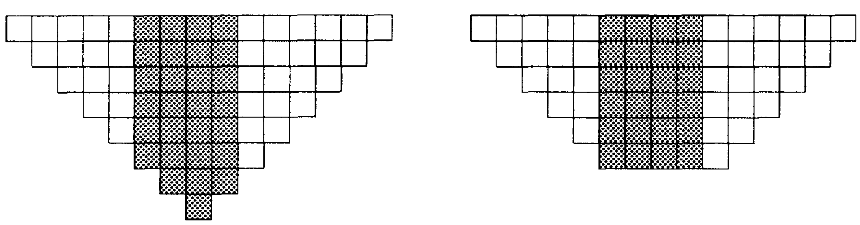

Figure 5-7: The shifted shapes $\lambda^B(w_0^B)$ and $\lambda^B(w')$ for $n = 8$. The shaded boxes indicate unequal hook-lengths.

From here it is not hard to compute that

$$\begin{aligned} E(Y_n^B) &= 2(n^2-3)\frac{(n^2-4)!}{(n^2)!}\cdot 3\cdot(2n)\cdot\frac{(2n-2)(2n)(2n+2)}{2\cdot 4\cdot 6} \\ &= \frac{2}{n^2-2}. \end{aligned}$$

□

n	2	3	4	5	6	7	8
$E(X_n^B)$	-	.6667	1	1.2	1.3333	1.4286	1.5
$E(Y_n^B)$	1	.2857	.1429	.0870	.0588	.0426	.0323

Table 5.1: The expected values of X_n^B and Y_n^B for $n \leq 8$, rounded to four decimal places.

Given Reiner's result for the longest element in $\mathfrak{S}_n$ and the results stated above for $\mathfrak{S}_n^B$, it is natural to try to make analogous calculations for the group $\mathfrak{S}_n^D$. Some of the framework from types A and B carries over to type D, but it is not as straightforward. For example, the correspondence between reduced decompositions of a vexillary element and Young tableaux of a particular shape is no longer bijective, as objects are weighted by different powers of two.

Chapter 6

Boolean order ideals in the Bruhat order for types B and D

6.1 Introduction

As Section 4.3 looks at the permutations in $\mathfrak{S}_n$ whose principal order ideals in the Bruhat order are boolean, this chapter answers similar questions for signed permutations.

Recall that the finite Coxeter groups of types B and D consist of signed permutations, where $\mathfrak{S}_n^D \subset \mathfrak{S}_n^B$ is the subset of elements that have an even number of negative signs when written in one-line notation.

Example 6.1.1. $\mathfrak{S}_2^B = \{12, 21, \underline{1}2, \underline{2}1, 1\underline{2}, 2\underline{1}, \underline{12}, \underline{21}\}$ and $\mathfrak{S}_2^D = \{12, 21, \underline{12}, \underline{21}\}$.

Like in Section 4.3, the central object in this discussion is the principal order ideal of a signed permutation in the Bruhat order of the respective groups $\mathfrak{S}_n^B$ and $\mathfrak{S}_n^D$. While the definition of this ideal is the same as earlier, it is repeated here in slightly more generality.

Definition 6.1.2. Let W be a finite Coxeter group of type A, B, or D. For $w \in W$, let

$$B(w) = \{v \in W : v \leq w\}$$

be the principal order ideal of w in the Bruhat order for W.

The specific group W will be apparent from the context of the discussions below.

The aim of this chapter is to describe exactly when $B(w)$ is boolean for $w \in \mathfrak{S}_n^B$ and $w \in \mathfrak{S}_n^D$. The following definition was included in Section 4.3 for $W = \mathfrak{S}_n$.

Definition 6.1.3. Let W be a finite Coxeter group of type A, B, or D. The (unsigned or signed) permutation $w \in W$ is *boolean* if the poset $B(w)$ is a boolean poset.

Recall the proof of Theorem 4.3.2, which began by showing that w is boolean if and only if some (every) reduced decomposition of w contains no repeated letter. In fact, the argument for $w \in \mathfrak{S}_n$ holds for elements of $\mathfrak{S}_n^B$ and $\mathfrak{S}_n^D$ as well.

Proposition 6.1.4. *Let W be a finite Coxeter group of type A, B, or D. An element $w \in W$ is boolean if and only if some (every) reduced decomposition of w contains no repeated letter.*

There is a certain resemblance between Proposition 6.1.4 and a result of Fan's in [10] for an arbitrary Weyl group W. Fan showed that if the reduced decompositions of $w \in W$ avoid factors of the form sts, then the corresponding Schubert variety X_w is smooth if and only if some (every) reduced decomposition of w contains no repeated letter.

The classifications of the boolean elements in $\mathfrak{S}_n^B$ and $\mathfrak{S}_n^D$ in the subsequent sections rely on Proposition 6.1.4. The next section of this chapter examines the boolean elements in the group $\mathfrak{S}_n^B$, enumerating them for each n as well as by length. The following section contains analogous results for the boolean elements in the group $\mathfrak{S}_n^D$. As with boolean elements in $\mathfrak{S}_n$, these characterizations are in terms of patterns, although the type B case is more complicated than type A, and type D is more complicated still.

6.2 Type B

In this section, the Coxeter group W in the definitions of Section 6.1 is equal to $\mathfrak{S}_n^B$. This section answers the question: when is the principal order ideal $B(w)$ boolean, for $w \in \mathfrak{S}_n^B$?

Theorem 6.2.1. *The signed permutation $w \in \mathfrak{S}_n^B$ is boolean if and only if w avoids all of the following patterns.*

$$\begin{array}{ll} \underline{1}\underline{2} & \underline{2}\underline{1} \\ 321 & 3412 \\ 32\underline{1} & 34\underline{1}2 \\ \underline{3}21 & \underline{3}412 \\ 1\underline{2} & 3\underline{2}1 \end{array}$$

Proof. By Proposition 6.1.4, a reduced decomposition of a boolean element can contain at most one 0. Therefore boolean elements in $\mathfrak{S}_n^B$ have at most one negative value. Thus the patterns $\underline{1}\underline{2}$ and $\underline{2}\underline{1}$ must be avoided. Also by Proposition 6.1.4, the element $w \in \mathfrak{S}_n^B$ is boolean if and only if there is a reduced decomposition $\boldsymbol{i} \in R(w)$ having one of the following forms:

1. An ordered subset of $[n-1]$;

2. 0 {an ordered subset of $[n-1]$}; or

3. {an ordered subset of $[n-1]$} 0.

Theorem 4.3.2 indicates that $\boldsymbol{i} \in R(v)$, for $v \in \mathfrak{S}_n$, consists of an ordered subset of $[n-1]$ if and only if v is 321- and 3412-avoiding. Multiplying such a permutation on the left by s_0 changes the sign of the value 1, while multiplying it on the right by

s_0 changes the sign of the value in the first position. Therefore, in addition to the patterns $\underline{1}\underline{2}$, $\underline{2}\underline{1}$, 321, and 3412, a boolean permutation in $\mathfrak{S}_n^B$ also avoids $32\underline{1}$, $34\underline{1}2$, $\underline{3}21$, and $\underline{3}412$.

Finally, since a negative value can appear in a boolean permutation in $\mathfrak{S}_n^B$ only if it is $\underline{1}$ or occurs in the first position, the permutation must also avoid $1\underline{2}$ and $3\underline{2}1$. □

Proposition 6.1.4 states that $w \in \mathfrak{S}_n^B$ is boolean if and only if it has a reduced decomposition whose letters are all distinct. Given previous results, the enumeration of these elements is straightforward. The simple reflections generating $\mathfrak{S}_n^B$ are $\{s_0, s_1, \ldots, s_{n-1}\}$. Since each of $\{0, 1, \ldots, n-1\}$ can appear at most once in a reduced decomposition of a boolean element, it is necessary only to understand when two ordered subsets of $\{0, 1, \ldots, n-1\}$ (necessarily having the same length) correspond to the same permutation. That is, to understand when two such subsets differ only by a sequence of short braid moves.

Recall the Coxeter relations as defined in Section 2.2. There is a natural bijection between pairs of commuting elements in $\{s_0, s_1, \ldots, s_{n-1}\}$ and pairs of commuting elements in $\{s_1, \ldots, s_{n-1}, s_n\}$. Therefore, the work of enumerating boolean elements in $\mathfrak{S}_n^B$ by length was already done in Section 4.3.

Corollary 6.2.2. *The number of boolean signed permutations in $\mathfrak{S}_n^B$ is equal to F_{2n+1}.*

Proof. The number of boolean signed permutations in $\mathfrak{S}_n^B$ is equal to the number of boolean unsigned permutations in $\mathfrak{S}_{n+1}$, which is F_{2n+1} by Corollary 4.3.4. □

The previous result was also obtained by Fan in [10].

Corollary 6.2.3. *The number of boolean signed permutations in $\mathfrak{S}_n^B$ of length k is equal to*

$$\sum_{i=1}^{k} \binom{n+1-i}{k+1-i}\binom{k-1}{i-1},$$

where the (empty) sum for $k = 0$ is defined to be 1.

Proof. The number of boolean signed permutations in $\mathfrak{S}_n^B$ of length k is equal to the number of boolean unsigned permutations in $\mathfrak{S}_{n+1}$ of length k. By Corollary 4.3.5, this is exactly $L(n+1, k)$, as defined in equation (4.3). □

6.3 Type D

Throughout this section, the Coxeter group W of Section 6.1 is the finite group $\mathfrak{S}_n^D$ of type D. Once again, the boolean elements of this group are defined and enumerated. As for types A and B, this characterization is in terms of patterns avoidance.

Theorem 6.3.1. *The signed permutation $w \in \mathfrak{S}_n^D$ is boolean if and only if w avoids all of the following patterns*

$\underline{123}$	$\underline{132}$	$\underline{213}$	$\underline{231}$	$\underline{312}$	$\underline{321}$
321	3412				
$32\underline{1}$	$3\underline{12}$	$34\underline{1}2$	$34\underline{21}$		
$\underline{3}21$	$\underline{23}1$	$\underline{3}412$	$\underline{43}12$		
$1\underline{2}$	$3\underline{2}1$				
$\underline{3}2\underline{1}$	$\underline{3}4\underline{1}2$				

Note that not all of these patterns are themselves in $\mathfrak{S}_n^D$, as some of them have an odd number of negative values in one-line notation.

Proof. By Proposition 6.1.4, a reduced decomposition of a boolean element can contain at most one $1'$. Therefore boolean elements in $\mathfrak{S}_n^D$ have at most two negative values. Thus the patterns $\underline{123}$, $\underline{132}$, $\underline{213}$, $\underline{231}$, $\underline{312}$, and $\underline{321}$ must be avoided. Also by Proposition 6.1.4, the element $w \in \mathfrak{S}_n^D$ is boolean if and only if there is a reduced decomposition $\boldsymbol{i} \in R(w)$ having one of the following forms:

1. An ordered subset of $[n-1]$;

2. $1'$ {an ordered subset of $[n-1]$}; or

3. {an ordered subset of $[n-1]$} $1'$.

Theorem 4.3.2 indicates that $\boldsymbol{i} \in R(v)$, for $v \in \mathfrak{S}_n$, consists of an ordered subset of $[n-1]$ if and only if v is 321- and 3412-avoiding. Multiplying such a permutation on the left by $s_{1'}$ maps the value 1 to $\underline{2}$ and the value 2 to $\underline{1}$, while multiplying it on the right by $s_{1'}$ sends the permutation $v \in \mathfrak{S}_n$ to $\underline{v(2)v(1)}v(3)\cdots v(n)$. Therefore in addition to the patterns mentioned previously, a boolean permutation in $\mathfrak{S}_n^D$ also avoids the patterns $32\underline{1}$, $3\underline{12}$, $34\underline{1}2$, $34\underline{21}$, $\underline{3}21$, $\underline{23}1$, $\underline{3}412$, and $\underline{43}12$.

Finally, since negative values in a boolean permutation in $\mathfrak{S}_n^D$ can only appear either as $\underline{1}$ and $\underline{2}$ or in the first two positions, it is not hard to see that the permutation must also avoid the patterns $1\underline{2}$, $3\underline{2}1$, $\underline{3}2\underline{1}$, and $\underline{3}4\underline{1}2$. □

It is instructive to compare those patterns that must be avoided by boolean elements in the three Coxeter groups considered in this thesis. In the case of types B and D, these fall into two categories, depending on whether or not they concern the number of negative signs that can appear in a boolean element. Table 6.1 displays this comparison.

As in types A and B, the boolean elements in type D can be enumerated. However, this enumeration is not as simple to state as in the other types. Fan computed these values in [10], with the following results.

Corollary 6.3.2 (Fan). *For $n \geq 4$, the number of boolean elements in $\mathfrak{S}_n^D$ is*

$$\frac{13-4b}{a^2(a-b)}a^n + \frac{13-4a}{b^2(b-a)}b^n,$$

where $a = (3+\sqrt{5})/2$ and $b = (3-\sqrt{5})/2$.

	$\mathfrak{S}_n$	$\mathfrak{S}_n^B$	$\mathfrak{S}_n^D$		
Not describing signs	321	321	321		
	3412	3412	3412		
		$32\underline{1}$	$32\underline{1}$	$3\underline{12}$	
		$34\underline{1}2$	$34\underline{1}2$	$34\underline{21}$	
		$\underline{3}21$	$\underline{3}21$	$\underline{23}1$	
		$\underline{3}412$	$\underline{3}412$	$\underline{43}12$	
		$1\underline{2}$	$1\underline{2}$		
		$3\underline{2}1$	$3\underline{2}1$		
			$\underline{3}2\underline{1}$		
			$\underline{3}4\underline{1}2$		
Describing signs		$\underline{12}$	$\underline{123}$	$\underline{132}$	$\underline{213}$
		$\underline{21}$	$\underline{231}$	$\underline{312}$	$\underline{321}$

Table 6.1: Patterns that must be avoided by boolean elements in $\mathfrak{S}_n$, $\mathfrak{S}_n^B$, and $\mathfrak{S}_n^D$.

Corollary 6.3.3. *For $n > 1$, the number of boolean elements in $\mathfrak{S}_n^D$ having length $k \leq n$ is*

$$L^D(n,k) \stackrel{\text{def}}{=} L(n,k) + 2L(n,k-1) - L(n-2,k-1) - L(n-2,k-2), \qquad (6.1)$$

where $L(n,k)$ is as defined in equation (4.3), and $L(n,k)$ is 0 for any (n,k) on which it is undefined. $L^D(1,0) = 1$ and $L^D(1,1) = 0$.

Proof. These enumerative results follow from Theorem 6.3.1 and Corollary 4.3.5. The subtracted terms in equation (6.1) resolve the overcounting that occurs when the reduced decompositions of a boolean element contain $1'$ but not 2. In such a situation, either the leftmost letter or the rightmost letter can be $1'$. The case $n = 1$ must be treated separately because the only element in $\mathfrak{S}_1^D$ is the identity. □

For small n and k, the number of boolean elements in $\mathfrak{S}_n^D$ of length k are displayed in Table 6.2.

$L^D(n,k)$	$k=0$	1	2	3	4	5	6	7	8
$n=1$	1	0							
2	1	2	1						
3	1	3	5	4					
4	1	4	9	13	8				
5	1	5	14	26	30	16			
6	1	6	20	45	69	68	32		
7	1	7	27	71	133	176	152	64	
8	1	8	35	105	230	373	436	336	128

Table 6.2: The number of boolean elements of each length in $\mathfrak{S}_1^D, \ldots, \mathfrak{S}_8^D$. Missing table entries are equal to 0.

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