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Multiscale Estimation of Regular Image Contours via a Graph of Local Edgel Hypotheses

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Abstract

The problem of estimating the regular portions of the contours in an image is formulated in a probabilistic and multiscale framework. The objective is to compute a small set of polygonal lines which, with high probability, contains an approximation to every contour in the scene. These polygonal lines are represented by paths in a graph whose arcs represent local contour hypotheses. The main difficulty of this problem is that, in order to achieve high probability of accurate reconstruction according to a global metric, it is necessary to deal with a combinatorially large number of contour hypotheses. To control the complexity of the search, the notion of a compressible graph is introduced and an efficient contour estimation algorithm based on graph compression is proposed.

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1 Introduction

Edge detection, or image segmentation, is an important ingredient of many image analysis applications and consists in estimating the projection on the image plane of the contours of the objects in a scene from the brightness intensities in the image. For certain classes of images, the basic source of information to perform this task is given by the sharp variations of brightness perpendicular to the edge which often occur in the vicinity of the boundaries between objects. However, this source of information is quite unreliable since in a typical image a substantial portion of the contours do not have a sufficiently large brightness contrast compared to the noise ("invisible" boundaries). Also, even though most of the contours in the image can be represented by smooth curves with low curvature, singularities such as corners and junctions are present and contribute to make the edge estimation problem quite difficult.

One possible approach to deal with these difficulties is to proceed in stages. First, the regular and visible portions of the contours are recovered. Then, the missing parts are estimated with the aid of the regular visible contours detected initially. Thus, even though an edge representation containing only regular visible contours is certainly incomplete for most practical purposes, it is nevertheless an important intermediate stage of image segmentation.

To deal with the problem of contour estimation, we adopt a probabilistic model-based approach. That is, we introduce a probabilistic model of the relationship between the contours in the image and the brightness data and then use this model to design an estimation algorithm and analyze its performance. The model is constructed in two steps. First, a single-contour model of the brightness data is defined. The probability measure induced by a single contour on the brightness image carries information about the data only in a neighborhood of the contour. Thus, the support of the measure induced by a contour consists of the σ-algebra associated with a subregion of the image domain. Then, a composite model is constructed which specifies a probability measure on the entire image given a set of contours. The construction of the composite model is straightforward if one assumes an *exclusion principle* according to which only sets of contours with disjoint domains are allowed.

The formulation of the estimation problem proposed here is different from the standard Bayesian method, in which the conditional density of the desired parameters is obtained by multiplying the prior density of these parameters with the likelihood of the data given the parameters. First, since we restrict our attention to only a subset of the contours in the image, namely the visible regular contours, we are not in a position to specify a complete probabilistic model of the data given the contours. In fact, the available information is restricted to a neighborhood of the given contours. Thus, the support of this measure is too small to define a density. Second, we do not assume a prior measure on the contours; instead, we restrict the family of allowed sets of contours and evaluate the performance of the proposed algorithm for all the contour sets in this class.

The overall structure of the algorithm and most of the mathematical results are rather independent from the specific brightness model of the contours. One simple such model is obtained by assuming that the one dimensional discontinuity profile across the edge is described by a step function smoothed by a gaussian filter with variance s, where s is the *blur scale* of the contour. Since s is not known a priori, and since contours of different scales might co-exist at the same location, a multiscale estimation approach is needed.

For simplicity, some of the formal analysis will be restricted to *flat* contours, namely contours with zero curvature (straight lines) of infinite length and such that the brightness model is constant along to the edge. However, the algorithms (and many of the mathematical results) are developed without these restrictions and require only that the curvature is sufficiently small and that the variations of scale and of brightness along the contour are sufficiently slow.

The main objective of the proposed algorithm is to obtain a *global* representation of the contours in the scene. That is, contours in the scene will be approximated by planar curves and not merely by a set of independently estimated points. This requirement complicates the estimation problem since the estimated curves have quite arbitrary shapes and therefore are characterized by a large number of parameters. A global representation of this type is important for the subsequent stages of edge detection dealing with invisible contours and curve singularities.

The general structure of the algorithm is driven by three requirements: low approximation error; high confidence of success; and computational efficiency. The first stage of the algorithm, which is briefly described in Section 4, computes a graph whose vertices are candidate contour point and whose arcs are candidate contour fragments. The major property of this graph is that, with high probability, it *covers* the set of contours in the scene, that is, it contains at least one approximating path for each contour in the scene. Due to noise, in order for the graph to cover all the contours with high probability, it must contain many redundant vertices and arcs. As a consequence, each contour might be represented by a combinatorially large number of paths. This paper addresses the issue of how to compute efficiently a small subset of these paths so that, with high probability, all contours are approximated by at least one path in the subset. An important notion in this regard is that of *compressibility* of a graph. A compressible sub-graph which preserves its covering properties can be computed from a covering graph by estimating arc likelihoods and removing "divergent" arcs with locally minimum likelihood.

1.1 Outline and contributions of the paper

This paper proposes a multiscale probabilistic framework to formulate and solve the contour estimation problem according to a global performance criterion defined on curvilinear sets. Section 3 introduces a sparse probabilistic model which allows to isolate the performance of the algorithm near each scene contour from the data outside a neighborhood of the contour itself. Section 4 describes the local contour hypotheses and how they are computed from the brightness image. Section 5 introduces the notion of compressible graphs which ensures that a globally accurate representation of all the paths in a graph can be computed efficiently. Sufficient local conditions for compressibility are described. Section 6 proposes a method to make the graph of local contour hypotheses compressible by comparing the brightness model of these hypotheses with the actual brightness data and pruning certain hypotheses with locally minimum likelihood. The error probability of this method is analyzed. Section 7 describes an algorithm which computes a covering of the set of paths in a compressible graph. Section 8 generalizes many of the results to the case where multiple contours at the same location are allowed. This generalization allows to detect contours simultaneously at different scale passing through the same point. Section 9 presents some experimental results and Section 10 draws some conclusions.

2 Previous work

The contour estimation algorithm proposed in this paper is based on estimation performance requirements — low localization error and high probability of accurate detection — defined on a *curve* representation of contours. Previous work on performance analysis and algorithm design based on these statistical estimation criteria had been carried out by modeling a contour as a set of small independent fragments which, essentially, reduces edge detection to a one dimensional problem. Optimal linear operators for the estimation of the discontinuity point along the direction perpendicular to the gradient have been developed for step edges [4], and more complicated brightness models [24]. Surface fitting methods have also been proposed [12] which are essentially equivalent to linear convolution schemes. Substantial work has been done to assess analytically the one dimensional estimation performance [27, 26, 14] of these local edge detectors. However, since most of this performance analysis is carried out for point-based models of contours only, the stage of constructing curve representation from these edge-point fragments is most of the time rather heuristic, with very little theoretical analysis of the overall performance of the algorithm. In the end, performance of the algorithm is usually assessed by means of human judgment [13].

Several statistical frameworks have been proposed for contour estimation and image analysis in general [10, 21, 11, 28]. Most of these methods are based on Bayes' formula. That is, the problem specification must provide a prior density defined on the desired representation and a conditional density of the data given this representation. Estimation consists then in maximizing the a-posteriori probability of the representation given the data. These methods can incorporate global information quite effectively but often result in hard optimization problems. Moreover, these approaches do not usually allow to obtain information about the probability distribution of the errors. Most variational and regularization approaches [3, 22, 25] to contour estimation can also be viewed within this statistical framework.

Recently, a statistical approach based on multiscale recursive estimation on trees has been proposed which yields efficient algorithms as well information about the covariance of the errors [1]. This method has been successfully applied to texture modeling and segmentation.

Wavelets provide an important tool to analyze a multiscale signal [19] and wavelet-based representations can also be used to model non-stationary processes [17].

The importance of multiscale representations for contour estimation has been acknowledged for a long time. Some multiscale algorithms for edge detection proceed in a coarse to fine fashion [20, 2, 25] whereas others are more similar to the approach proposed here in that they emphasize the importance of detecting *all* the relevant scales [18], with priority given to the lowest one [8].

The proposed algorithms exploits the curvilinear nature of contours to augment the information provided by brightness variation. Relaxation labeling has also been used successfully for this purpose [23] as well as "snake" and curve evolution methods [15, 16].

Some of the results in this paper have already been reported and proved within a non probabilistic framework, and under the assumption that the scale of the contours is fixed and known [7]. The compressibility condition introduced here is very similar to the stability property discussed in [7]. Some ideas on how to enhance the regular curve representation with corners and junctions are discussed in [5].

3 Contour models

A regular visible contour γ in the scene, shortly a *scene contour*, induces a probability measure on the *observed image* I. The image I consists of an array of real values, which for simplicity can be assumed to be infinite. Its entries are denoted $I(i,j)$, $(i,j) \in \mathbb{Z} \times \mathbb{Z}$. A set of scene contours is denoted Γ. To obtain a probabilistic model of the observed image given Γ, one needs to combine the probability measures associated with the contours $\gamma \in \Gamma$ into a composite measure on the observed image. The goal of the estimation algorithm is to compute an estimate $\hat{\Gamma}$ of the scene contours Γ from the observed image I. The estimate $\hat{\Gamma}$ consists of a set of contour *descriptors*, which in the proposed algorithm are given by polygonal curves. The performance of the algorithm is evaluated by three parameters: complexity, which can be measured by either the size of the estimate $\hat{\Gamma}$ or the time complexity of the algorithm, or both; accuracy, namely the distance from the scene contours to the contour descriptors; and confidence, namely the probability that the scene contours are within the tolerated accuracy of the computed descriptors. In short, the objective is to compute efficiently a compact estimate which contains with high probability at least one accurate approximation for every scene contour.

3.1 Probabilistic model of a scene contour

The projection on the image plane of a scene contour γ can be represented by a planar curve. The image of this curve is a subset of the real plane and will be called the *trace* of γ and denoted $T(\gamma)$. The simplest model for a scene contour is given by a contour with zero curvature, infinite length and brightness model constant along the contour.

Such a contour, called *flat* contour, is characterized by the following parameters:

- The orientation of the contour with respect to the x-axis, denoted $\gamma.\theta$;
- The distance of the contour from the origin, $\gamma.\rho$;
- the brightness intensities on the left and right sides of the contour, denoted $\gamma.b_1$ and $\gamma.b_2$;
- the blur scale of the one dimensional discontinuity profile across the contour, $\gamma.s$.
- The intensity (variance) of the noise, denoted $\gamma.\nu$.

The set of flat contours is denoted $\mathbf{\Gamma}_0$. A more general class of contours, denoted $\mathbf{\Gamma}_1$ and called regular contours, is given by contours with bounded curvature and with b_1, b_2, s, ν slowly varying along the contour. Notice that the definition of $\mathbf{\Gamma}_1$ depends on the upper bound on curvature and the upper bounds on the rate of variation of b_1, b_2, s, ν.

Most of the following discussion and results are rather independent of the discontinuity profile model across the contours. However, for the sake of concreteness, let us fix a specific profile, namely a step function smoothed by a gaussian filter. Thus the image data model associated with a flat contour γ is

$$I(i,j) = \beta(i,j|\gamma) + \eta(i,j), \qquad (i,j) \in D(\gamma) \tag{1}$$

where

$$\begin{aligned}\beta(i,j|\gamma) &= \gamma.b_1 + (\gamma.b_2 - \gamma.b_1) \cdot (G_{\gamma.s} * \delta^{(-1)})(\xi(i,j)) \\ &= \gamma.b_1 + (\gamma.b_2 - \gamma.b_1) \cdot \operatorname{erf}\left(\frac{\xi(i,j)}{\gamma.s}\right);\end{aligned} \tag{2}$$

$\delta^{(-1)}$ is the unit step function: $\delta^{(-1)}(x) = 0$ for $x \leq 0$, $\delta^{(-1)}(x) = 1$ for $x > 0$; $G_{\gamma.s}$ is a gaussian smoothing filter with variance $\gamma.s$; $\xi(i,j)$ is the signed distance from $T(\gamma)$ to the the pixel (i,j); η is a noise field assumed to be i.i.d. and gaussian with variance $\gamma.\nu$; $\operatorname{erf}(\cdot)$ is the error function given by

$$\operatorname{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2} du;$$

and $D(\gamma) \subset \mathbb{Z} \times \mathbb{Z}$ is the *domain* of γ, namely the set of image measurement affected by the presence of the contour γ in the scene. This model can be easily generalized to the class of regular contours $\mathbf{\Gamma}_1$.

The signal to noise ratio of γ is defined to be the brightness contrast divided by the variance of the noise,

$$SNR(\gamma) = \frac{|\gamma.b_1 - \gamma.b_2|}{\gamma.\nu}.$$

In order to be "visible" a contour must have sufficiently large signal to noise ratio.

Let $\mu(I|\gamma)$ be the probability measure associated with the brightness model (1). Its support is the σ-algebra generated by the family of sets

$$\{\{I \ : \ I(i,j) \leq u_{i,j}, (i,j) \in D(i,j)\} \ : \ u_{i,j} \in \mathbb{R}\}$$

and its value on one of these sets is

$$\prod_{(i,j) \in D(i,j)} \operatorname{erf}\left(\frac{u_{i,j} - \beta(i,j|\gamma)}{\gamma.\nu}\right).$$

3.2 Composite models and the exclusion principle

The construction of composite probabilistic models from primitive ones has been studied mostly for the problem of recognition [9]. We believe that a compositional approach can be useful also to deal with tasks usually considered to be more "low level", such as contour estimation. In this paper, only a trivial compositional model of contours will be considered.

In order to define a composite probability model associated to a set of scene contours, we are going to assume that *each scalar measurement $I(i,j)$ can be affected by at most one scene contour* (exclusion principle). This assumption is satisfied if we only allow sets of scene contours Γ whose domains are pairwise disjoint:

$$D(\gamma_1) \cap D(\gamma_2) = \emptyset, \quad \text{for all } \ \gamma_1, \gamma_2 \in \Gamma \ \text{ such that } \ \gamma_1 \neq \gamma_2.$$

A contour set satisfying this property will be said to be *free*, since its element do not interact. The collection of all free sets of regular contours is denoted $\mathcal{G}_1$. The measure associated to a free contour set $\Gamma \in \mathcal{G}_1$ is simply given by the product measure:

$$\mu(I|\Gamma) = \prod_{\gamma \in \Gamma} \mu(I|\gamma).$$

Notice that this is a rather "incomplete" probabilistic model of the data. In fact, for any given contour set Γ the image is completely arbitrary outside its domain $D(\Gamma)$ given by:

$$D(\Gamma) = \bigcup_{\gamma \in \Gamma} D(\gamma).$$

Furthermore, we are not going to assume that a prior measure on Γ is available. Rather, we simply assume that $\Gamma \in \mathcal{G}_1$ and that the signal to noise ratio of the elements of Γ is sufficiently high. For these two reasons, it is not possible to define a conditional probability of Γ given the data I. Nevertheless, we still want to guarantee that good performance is achieved with high probability. More precisely, the goal is to prove an upper bound on the estimation error of Γ which is independent of the data outside its domain $D(\Gamma)$. To do this, we are going to formulate conditions (of a non-probabilistic nature) on the data inside $D(\Gamma)$ which are sufficient to guarantee that the proposed algorithm computes an accurate estimate of Γ. Then, the probability of attaining this performance, which depends on Γ, is simply the conditional probability that this conditions are satisfied given Γ.

4 Graph of local contour hypotheses

4.1 Edgels

The contour descriptors computed by the estimation algorithm are polygonal lines represented by paths in a graph. The vertices of this graph represent local contour hypotheses and are obtained by least-square fitting polynomial brightness models to the data in small regions in the image. Regions of different sizes are used because the scale s and the noise intensity ν of the contours in the scene are unknown. If the set of regions used for fitting is dense enough, both in the size and space dimension, then any contour γ will be sampled by a set of regions, denoted $\mathcal{R}(\gamma)$, sufficiently close to its trace and of size approximately tuned to the blur scale of the contour so that the parameters computed in these regions are good estimates of the local parameters of the contour. Polynomial brightness models are used for fitting because they yield fast least-square estimators and because the original brightness model, e.g. (2), can be approximated by its Taylor expansion in a sufficiently small neighborhood of the contour.

Initially, for each region, a linear brightness model is used to estimate the local orientation of the contour. Then, a cubic brightness model is fitted to a larger rectangular region (aligned to the estimated orientation of the contour) to refine the estimate of the location of the contour and to estimate the scale s and the contrast $|b_1 - b_2|$. The parameters estimated from each region are gathered to form an object, called *edgel*, and the set of these edgels constitute the set of vertices of the graph. Each edgel e is characterized by the following parameters:

- The estimated orientation of the contour, $e.\theta$.
- The estimated location of the point on the contour nearest to the center of the region used to estimate e, $e.p$.
- The estimated blur scale of the contour, $e.s$.
- The estimated brightness values on the two sides of the contour, $e.b_1$, $e.b_2$.
- The fitting residual, $e.\nu$.

The total number of edgels is of the order of the number of pixels in the image, times the number of scales. The edgels computed from the regions $\mathcal{R}(\gamma)$ are denoted $E(\gamma)$. If γ is scene contour, $\gamma \in \Gamma$, then $E(\gamma)$ provides a good sampled approximation of γ.

Some of computed edgels are thresholded out by estimating locally the probability that there exists a scene contour $\gamma \in \Gamma$ such that the edgel belongs to $E(\gamma)$. If this estimated probability is less than some threshold $P_{\text{edgel,min}}$, then the edgel is purged.

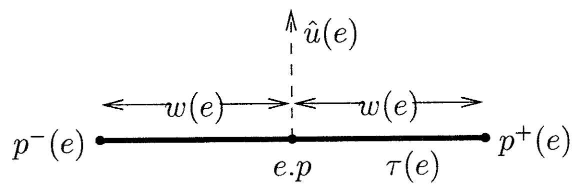


Figure 1: An edgel e. The segment of length $2w(e)$, perpendicular to the orientation of e, and centered at $e.p$, is denoted $\tau(e)$. The end-points of $\tau(e)$ are denoted $p^-(e)$ and $p^+(e)$.

For each edgel e we introduce two quantities, $w(e)$ and $w_{\mathsf{sc}}(e)$, which represent estimates of the localization and scale uncertainty respectively. More precisely, if $e \in E(\gamma)$, then we assume that $w(e)$ is large enough so that, with high probability, the distance from $e.p$ to $T(\gamma)$ is less than $w(e)$. An essentially equivalent assumption is that, with high probability, the trace $T(\gamma)$ intersects $\tau(e)$, where $\tau(e)$ is the straight line segment with end-points $p^-(e)$, $p^+(e)$ given by (see Fig. 1)

$$p^-(e) = e.p - w(e) \cdot \hat{u}_\perp(e), \tag{3}$$

$$p^+(e) = e.p + w(e) \cdot \hat{u}_\perp(e). \tag{4}$$

Here, $\hat{u}_\perp(e)$ denotes the versor perpendicular to the orientation of the edgel, $e.\theta$. Similarly, it is assumed that, with high probability, the scale of the contour $\gamma.s$ lies in the interval

$$\tau_{\mathsf{sc}}(e) = [e.s - w_{\mathsf{sc}}(e), e.s + w_{\mathsf{sc}}(e)]. \tag{5}$$

4.2 Edgel-arcs and edgel-paths

An *edgel-arc* a is a pair of edgels $a = (e_1, e_2)$. A set of edgel-arcs A is called an *edgel-graph.* To construct an edgel-graph A, the algorithm creates an edgel-arc between every pair of edgels computed from adjacent regions, and then removes edgel-arcs which are non valid according to Definition 1. In order for an arc to be valid, the parameters of its edgels must be sufficiently close (see Definition 1). The set of vertices of the edgel-graph A, is denoted $V(A)$.

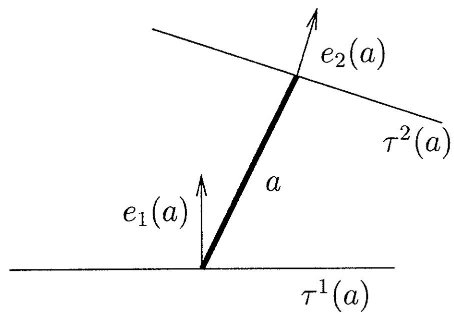


Figure 2: An edgel-arc a. Its two vertices are denoted $e_1(a), e_2(a)$. We have also $\tau^1(a) = \tau(e_1(a))$, $\tau^2(a) = \tau(e_2(a))$.

Let's introduce the following notation (see Fig. 2). Let $a = (e_1, e_2)$ be an edgel-arc. Then for $i = 1, 2$, let $p_i(a) = e_i.p$; $\theta_i(a) = e_i.\theta$; $s_i(a) = e_i.s$, $\tau^i(a) = \tau(e_i)$. Let $\hat{u}(e)$ be the unit versor in the direction of $e.\theta$ and let $\hat{u}(a)$ be the unit versor from $p_1(a)$ to $p_2(a)$. Let $\Theta \in [0, \pi]$, $S_\Delta > 0$, $L_\Delta > 0$ be constants. Let us assume that

$$w_{\mathsf{sc}}(e) > S_\Delta, \qquad \forall e \tag{6}$$

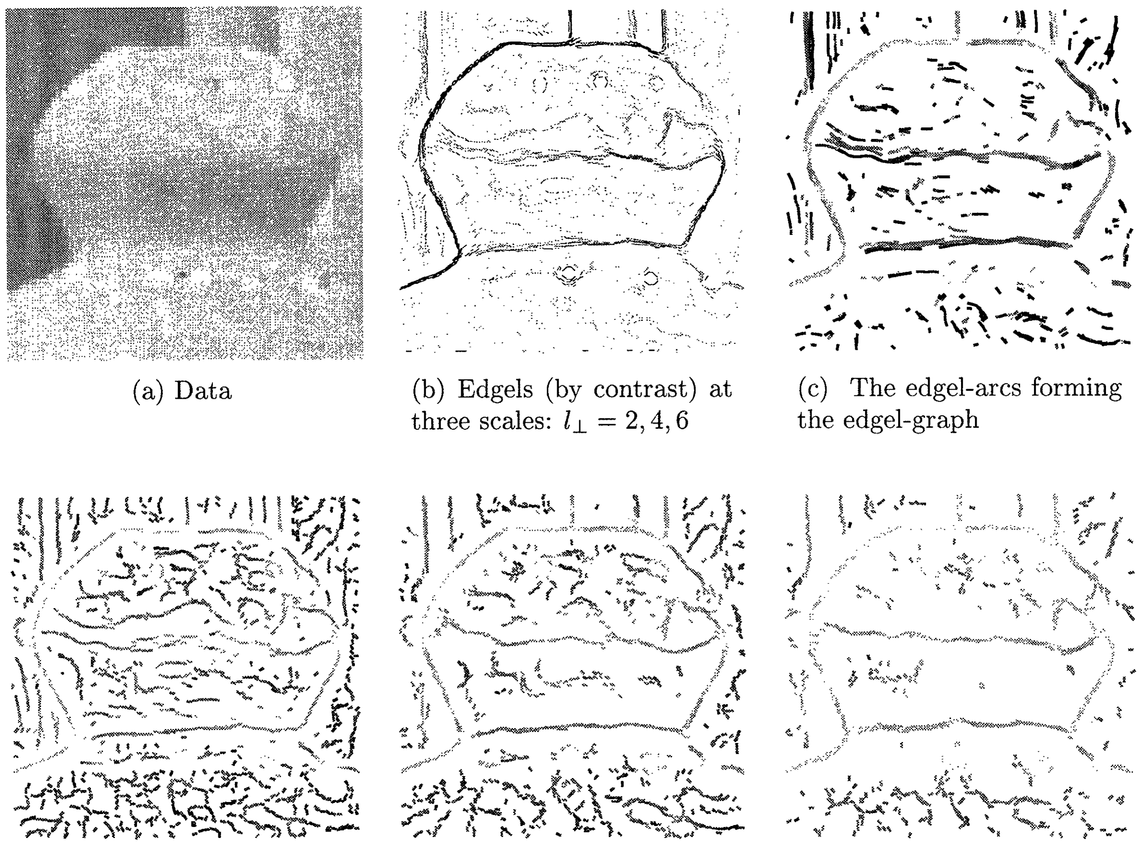
Definition 1 *An arc a is said to be* valid *if*

$$\begin{aligned}
\hat{u}(e_1(a)) \cdot \hat{u}(e_2(a)) &> \cos\Theta; && (7)\\
\hat{u}(a) \cdot \hat{u}(e_i(a)) &> 0, \qquad i = 1, 2; && (8)\\
|s_1(a) - s_2(a)| &< S_\Delta; && (9)\\
||p_1(a) - p_2(a)|| &< L_\Delta; && (10)\\
\tau^1(a) \cap \tau^2(a) &= \emptyset. && (11)
\end{aligned}$$

A set of arcs A is said to be valid if all its arcs are valid.

To achieve scale invariance, the upper bound L_Δ on the length of an arc in (10) should depend on the scales of the arc, $s_1(a)$ and $s_1(a)$: $L_\Delta = L_\Delta(s_1(a), s_1(a))$. However, for simplicity, this dependence is not denoted explicitly.

An edgel-path π is *simple* if the polygonal line $T(\pi)$ does not self-intersect (more precisely, if $T(\pi)$ is homeomorphic to a straight line segment).



(a) Data

(b) Edgels (by contrast) at three scales: $l_\perp = 2, 4, 6$

(c) The edgel-arcs forming the edgel-graph

(d) Edgels, $l_\perp = 2$

(e) Edgels, $l_\perp = 4$

(f) Edgels, $l_\perp = 6$

Figure 3: The edgel-graph (shown in (c)) computed from the brightness data in (a) by using three scales, $l_\perp = 2, 4, 6$. The parameter $2l_\perp$ is the width of the rectangular region used for the cubic fit, as described in Section 4.1. The vertices of the edgel-graph are shown in (b) with gray intensity proportional to contrast and in (d),(e),(f) with gray intensity proportional to the likelihood indicator, given by equation (34). In (c),(d),(e),(f), an edgel e is represented by a needle centered at $e.p$ and with orientation $e.\theta$. The gray intensity of the edgel-arcs in (c) are proportional to the arc likelihood indicator used in the current implementation of the algorithm. See Section 9 for more details.

Definition 2 *Let A be an edgel-graph. A path in A with vertices $e_0, \ldots, e_l$ is said to be* regular *if it is simple and if $T(\pi) \cap \tau(e_i) = \{e_i.p\}$, $i = 0, \ldots, l$.*

For any $p \in \mathbb{R}^2$ and $U \subset \mathbb{R}^2$ let $d(p \to U)$ denote the distance from the point p to the set U. For any two sets $U_1, U_2 \subset \mathbb{R}^2$, let $d(U_1 \to U_2)$ be the directed Hausdorff distance from U_1 to U_2:

$$d(U_1 \to U_2) = \max_{p_1 \in U_1} d(p_1 \to U_2) = \max_{p_1 \in U_1} \min_{p_2 \in U_2} ||p_1 - p_2||. \tag{12}$$

For any path π, the polygonal line passing through $p(e_1), \ldots, p(e_n)$, where $e_1, \ldots, e_n$ are the vertices of π, is denoted $T(\pi)$.

Definition 3 *The graph A is said to cover γ if there exists a regular path π in A with vertices $e_1, \ldots, e_n$ such that*

$$d(T(\gamma) \to T(\pi)) < X_0, \tag{13}$$

and, for $1 \leq i \leq n$,

$$\begin{aligned} d(e_i.p \to T(\gamma)) &< X_0; & (14)\\ d(e_i.\theta, \gamma.\theta) &< \Theta_0; & (15)\\ |e_i.s - \gamma.s| &< S_0. & (16) \end{aligned}$$

If A covers γ, the covering sub-graph *of γ, denoted $A|\gamma$, is given by the set of arcs whose two vertices e_1 and e_2 satisfy (14)-(16). If $a \in A|\gamma$ for some $\gamma \in \Gamma$ then a is said to be a* covering arc.

It should be noted that the covering property of a graph is a monotone property, that is, if $A \supset A'$ and A' covers γ, then A also covers γ. Furthermore, once the constants X_0, Θ_0 and S_0 have been fixed, if the graph A were computed by sampling the space of all vertices and the space of all valid arcs with a sufficiently small resolution (rather than the estimation procedure just described), then one would obtain a valid graph A "dense" enough to cover *any* possible contour (not just the scene contours).

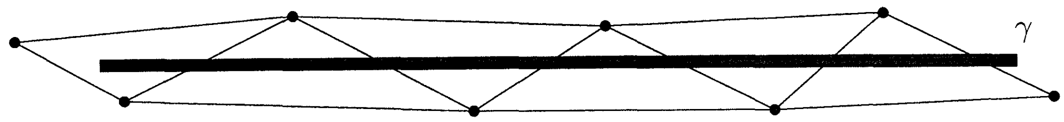


Figure 4: An edgel-graph might contain several paths approximating the same scene contour.

If the graph A is computed from the brightness image I by using the estimation procedure just described (as it is in practice), then A is a random variable whose probability measure depends on Γ. Then, we define $P_{\text{cov}}(\Gamma)$ to be the probability that A covers Γ. It is quite plausible that if the signal to noise ratios of the contours in Γ is sufficiently large, then $P_{\text{cov}}(\Gamma)$ can be made arbitrarily close to one by letting $P_{\text{edgel,min}}$ be sufficiently small, provided that the parameters in Definitions 1 and 3 are suitably constrained and the domain of the image is finite.

A consequence of $P_{\text{edgel,min}}$ being small is that there might be multiple redundant edgels in correspondence of the same point on a contour. Therefore, there might be multiple arcs in A approximating any fragment of a contour $\gamma \in \Gamma$, and the average degree of a vertex in the graph might be large. Thus, each contour in Γ might be covered by a combinatorially large number of paths (see Fig. 4). The presence of so many redundant paths makes it difficult to extract a small number of paths from the graph which approximate every contour in Γ. All the paths which approximate the same scene contour are equivalent to each other and therefore should be compressed into a unique representative in order to obtain a complete representation of the contours with minimal complexity. In the next sections the notion of compressibility of a graph will be introduced together with sufficient properties for compressibility which yield an efficient algorithm to compress a covering graph. This algorithm is based on a feedback function which estimates the likelihood that an arc is a covering arc, namely that it belongs to the sub-graph of some scene contour. Sufficient properties of this feedback function that guarantee good performance are discussed and related to the brightness model of the contours.

5 Compressible graphs

For any two subset of the real plane, U_1 and U_2, $d(U_1, U_2)$ denotes the Hausdorff distance between U_1 and U_2:

$$d(U_1, U_2) = \max\{d(U_1 \to U_2), d(U_2 \to U_1)\}. \tag{17}$$

Definition 4 *An edgel-graph A is* ϵ-compressible *if for any two regular paths π_1, π_2 having the same initial vertex and the same last vertex we have*

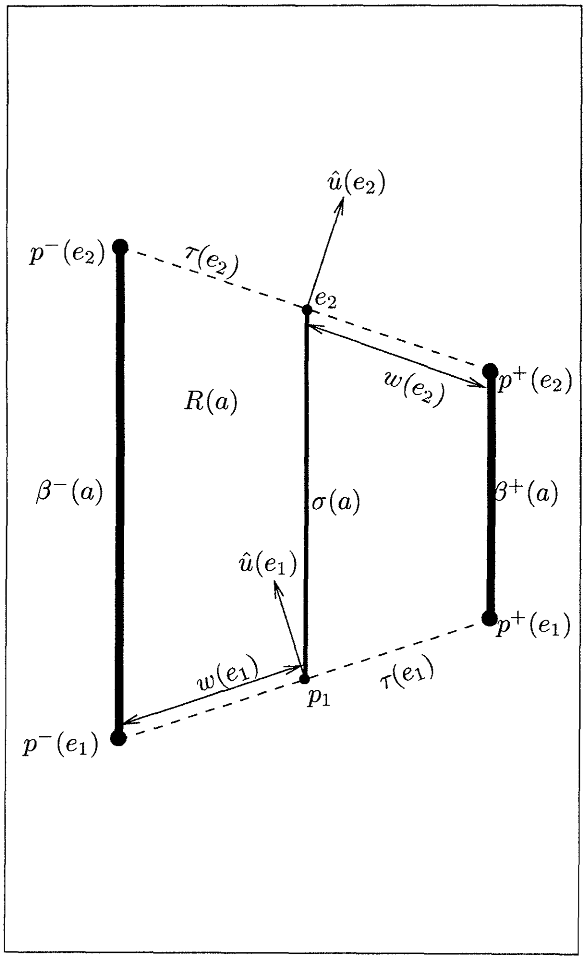
$$d(T(\pi_1), T(\pi_2)) < \epsilon.$$

According to this definition of compressibility an edgel-graph is compressible if all the regular paths connecting two nodes are close to each other. Therefore, for our purposes, these paths are all equivalent approximations of a scene contour so that only one of them needs to be considered during path exploration. An efficient algorithm which computes an approximation to every contour from a compressible graph is described in Section 7.

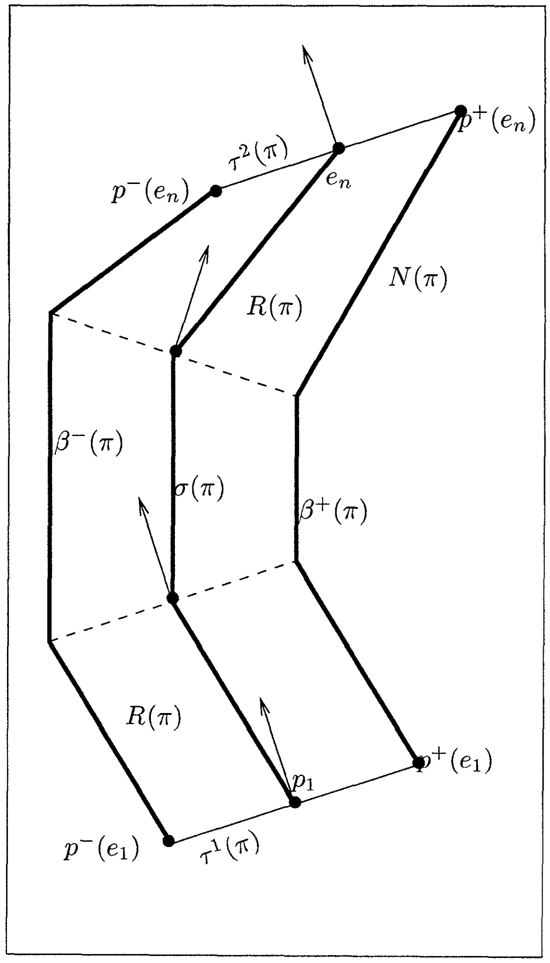
5.1 Sufficient conditions for compressibility

It turns out that compressibility is a local property of a graph and can be enforced by a simple computation involving only pairs of nearby arcs. The sufficient condition for compressibility described in this section are useful if the exclusion principle as defined earlier holds, that is, if one assumes that the contours in the scene have disjoint domains. The definitions and results of this section should be compared with the similar ones in [7]. More general conditions which allow for multiple contours in the same spatial neighborhood, as long as these contours are well-separated in some other dimension (such as scale), will be described in Section 8.

Before describing the sufficient condition, we introduce some notation. For any $a = (e_1, e_2) \in A$ let $\beta^-(a)$, $\beta^+(a)$, be the straight line segments with end-points $p^-(e_1)$,



(a) Lateral segments of arc $a = (e_1, e_2)$



(b) Lateral boundaries of path π

Figure 5: Notation.

$p^-(e_2)$ and $p^+(e_1)$, $p^+(e_2)$ respectively (see Fig. 5(a)). Let $\beta(a) = \beta^-(a) \cup \beta^+(a)$. Let $\sigma(a)$ be the straight line segment with end-points $e_1.p$, $e_2.p$. Let $R(a)$ be the closed region inside the quadrilateral with vertices $p^-(e_1)$, $p^-(e_2)$,$p^+(e_2)$, $p^+(e_1)$. For any path π in A with vertices E_π and arcs A_π let (see Fig. 5(b))

$$\begin{aligned} \beta(\pi) &= \bigcup_{a \in A_\pi} \beta(a); \\ \sigma(\pi) &= \bigcup_{a \in A_\pi} \sigma(a); \\ R(\pi) &= \bigcup_{a \in A_\pi} R(a); \\ w^{\max}(\pi) &= \max_{e \in E_\pi} w(e). \end{aligned}$$

Similar definitions hold for $\beta^-(\pi)$ and $\beta^+(\pi)$. Notice that $\sigma(\pi) = T(\pi)$.

Theorem 1 (Sufficient condition for compressibility) *Let A be a valid edgel-graph. If*

$$\sigma(a) \cap \beta(a') = \emptyset, \qquad \forall a, a' \in A, \tag{18}$$

then, for any two regular paths π_1, π_2 in A with the same initial vertex and final vertex we have

$$d(T(\pi_1), T(\pi_2)) < \min\left\{w^{\max}(\pi_1), w^{\max}(\pi_2)\right\}.$$

For the proof of Theorem 1 we need the following proposition.

Proposition 1 *Let π be a regular path in A and let $p \in R(\pi)$. Then,*

$$d(p \to \sigma(\pi)) < w^{\max}(\pi).$$

Proof of Theorem 1. From (18) we have $\sigma(\pi_1) \cap \beta(\pi_2) = \emptyset$ and $\sigma(\pi_2) \cap \beta(\pi_1) = \emptyset$. Let e_{fi} and e_{la} be the first and last vertex of π_1 and π_2 and let

$$\sigma^\circ(\pi_i) = \sigma(\pi_i) \setminus \{e_{\text{fi}}.p, e_{\text{la}}.p\}, \qquad i = 1, 2.$$

Since the paths π_i, $i = 1, 2$ are regular, we have

$$\sigma^\circ(\pi_i) \cap \tau(e_{\text{fi}}) = \sigma^\circ(\pi_i) \cap \tau(e_{\text{la}}) = \emptyset, \qquad i = 1, 2.$$

Thus,

$$\sigma^\circ(\pi_1) \cap \partial R(\pi_2) = \sigma^\circ(\pi_2) \cap \partial R(\pi_1) = \emptyset.$$

Then, for $i = 1, 2$, $\sigma^\circ(\pi_i)$ is contained in either $R(\pi_{\bar{i}})$ (where $\bar{i} = i + 1(mod\ 2)$) or in the complement of $R(\pi_{\bar{i}})$. From (8) it follows that $\sigma^\circ(\pi_i) \cap R(\pi_{\bar{i}}) \neq \emptyset$, $i = 1, 2$, so that $\sigma^\circ(\pi_1) \subset R(\pi_2)$ and $\sigma^\circ(\pi_2) \subset R(\pi_1)$. The result then follows from Proposition 1. $\square$

From Theorem 1 we have the following corollary. Let

$$W = \max_{e \in V(A)} w(e). \tag{19}$$

Corollary 1 *If (18) holds then A is W-compressible.*

If the arcs a and a' satisfy condition (18) in Theorem 1, the we say that a' is *non-divergent in space* from a and we denote this $a' \parallel a$. Otherwise, we say that a' is *divergent in space* from a, denoted $a' \nparallel a$. Notice that $a' \nparallel a$ does not imply $a \nparallel a'$ in general.

5.2 Computing a compressible edgel-graph

A simple algorithm to obtain a compressible sub-graph of an arbitrary edgel-graph is to detect all divergent pairs of arcs and remove one of the two arcs from the graph (compare with the similar algorithm proposed in [6, 7]). If care is used to decide which of these two arcs is removed, one can ensure that covering arcs are seldom removed so that, with high probability, the resulting sub-graph still covers Γ if the original graph covers Γ. To make this decision, one can use an "edginess" function $\phi_A(a)$, e.g. some estimate of the likelihood that an arc is a covering arc, and then remove the arc with the smallest edginess. Let A^* be the edgel-graph produced by this algorithm. That is,

$$A^* = A \setminus A^\dagger, \qquad A^\dagger = \{a \in A \ : \ \exists a' \in \text{div}(a), \phi_A(a) \leq \phi_A(a')\} \tag{20}$$

where $\mathrm{div}(a)$ denotes the set of arcs *divergent with* a:

$$\mathrm{div}(a) = \{a' \in A \ : \ a' \nparallel a \ \vee \ a \nparallel a'\} . \tag{21}$$

Proposition 2 *A^* is W-compressible.*

Proof. Let a, a' be two edgel-arcs in A such that $a' \nparallel a \vee a \nparallel a'$. Let us assume, without loss of generality, that $\phi_A(a) \leq \phi_A(a')$. Since $a' \in \mathrm{div}(a)$, from (20) we have $a \in A^\dagger$ and therefore $a \notin A^*$. Hence, for every pair of edgel-arcs $a, a' \in A^*$, $a' \parallel a$, that is, $\sigma(a) \cap \beta(a') = \emptyset$. Then from Corollary 1, it follows that A^* is W-compressible. □

Definition 5 *Let $\phi_A : A \to \mathbb{R}$ and let A^* be given by* (20). *Let γ be a contour covered by A. If $A|\gamma \subset A^*$ then ϕ_A is said to* preserve γ.

Proposition 3 *Let γ be a contour covered by A. Then ϕ_A preserves γ if and only if*

$$\phi_A(a) > \phi_A(a'), \qquad \forall a' \in \mathrm{div}(a).$$

Proof. Let $a \in A|\gamma$. If $\phi_A(a) > \phi_A(a')$ for every $a' \in \mathrm{div}(a)$ then, from (20), $a \notin A^\dagger$ and $a \in A^*$. On the other hand, if $\phi_A(a) \leq \phi_A(a')$ for some $a' \in \mathrm{div}(a)$ then $a \in A^\dagger$ and $a \notin A^*$. □

Notice that Proposition 2 holds true for any function ϕ_A. Clearly, ϕ_A has to be chosen so that the probability of removing covering arcs is small. Let's consider edginess functions of the form

$$\phi_A(a) = \min\{\phi_E(e_1(a)), \phi_E(e_2(a))\} \tag{22}$$

where, for any edgel e,

$$\phi_E(e) = \phi(e.p, e.\theta, e.s), \tag{23}$$

and $\phi(p, \theta, s)$ is an edginess function of three continuous variables, $p \in \mathbb{R}^2$, $\theta \in [0, 2\pi]$, $s > 0$, computed from the observed image I. Lemma 1 at the end of this section, which is based purely on geometrical arguments, gives sufficient conditions for the edginess function to be capable of discriminating covering arcs from spurious divergent arcs. Roughly speaking, the lemma asserts that if the edginess function near a contour is larger than the edginess function at points whose distance from the contour is in a certain range of values, then the arc suppression procedure just described will preserve the covering sub-graph of the contour. The lemma contains the parameters Θ_0, X_0 and S_0 introduced in Definition 3 (covering edgel-graph) and the new parameters Θ_1, X_1 and X_2, which are assumed to satisfy the following conditions

$$
\begin{aligned}
\Theta_0 &< \Theta_1, && (24)\\
X_0 &< X_1 < \frac{\cos\Theta_1}{2} w(e), \qquad \forall e \in V(A) && (25)\\
X_2 &> X_1 + \max\{L_\Delta, W\}, && (26)
\end{aligned}
$$

where L_Δ is the maximum arc length in a valid graph and $W = \max_{e \in V(A)} w(e)$. Let L_{div} be the maximum distance between any two points $p \in \sigma(a)$, $p' \in \sigma(a')$, over all $a \in A$ and $a' \in \mathrm{div}(a)$.

Lemma 1 (Preservation of covering arcs) *Let A be a valid edgel-graph. Let Θ_1, X_1 and X_2 be constants such that (24), (25) and (26) hold true. Let γ be a flat contour covered by A and let $\phi_A : A \to \mathbb{R}$ be given by (22),(23). Then ϕ_A preserves γ if for any p, θ, s and p', θ', s' we have $\phi(p,\theta,s) > \phi(p',\theta',s')$ whenever*

$$
\begin{aligned}
d(p \to T(\gamma)) &\leq X_0, && (27)\\
d(\theta, \gamma.\theta) &\leq \Theta_0, && (28)\\
|s, \gamma.s| &\leq S_0, && (29)\\
||p' - p|| &\leq L_{\mathrm{div}}, && (30)
\end{aligned}
$$

and at least one of the two conditions hold true

$$d(p' \to T(\gamma)) \in [X_1, X_2]; \tag{31}$$

$$d(p', T(\gamma)) \leq X_1 \,\wedge\, d(\theta', \gamma.\theta) \geq \Theta_1. \tag{32}$$

Proof. Let γ be a flat contour for which the sufficient condition of the lemma is satisfied. From Proposition 3, we have to prove that $\phi_A(a) > \phi_A(a')$ for every $a \in A|\gamma$ and $a' \in \text{div}(a)$. Let then $a \in A|\gamma$ and $a' \in \text{div}(a)$. One needs to prove that for every $i \in \{1, 2\}$ there exists $j \in \{1, 2\}$ such that

$$\phi(p_i(a), \theta_i(a), s_i(a)) > \phi(p_j(a'), \theta_j(a'), s_j(a')).$$

A stronger statement will be proved, namely that there exists $j \in \{1, 2\}$ for which this inequality holds for both $i \in \{1, 2\}$. Let us make the following substitutions in (27)-(32): $p = p_i(a), \theta = \theta_i(a), s = s_i(a)$; $p' = p_j(a'), \theta' = \theta_j(a'), s' = s_j(a)$. Notice that since $a \in A|\gamma$ we have that (27)-(29) hold with the above substitutions for $i = 1, 2$. Furthermore, for $i = 1, 2$ and $j = 1, 2$, we have $||p_i(a) - p_j(a')|| \leq L_{\text{div}}$ from the definition of L_{div}. Thus both vertices of a satisfy (27)-(30). It remains to prove that one of the two vertices of a' satisfies either (31) or (32). First, let us assume that

$$d(p_j(a') \to T(\gamma)) \leq X_1 \implies d(\theta_j(a'), \gamma.\theta) < \Theta_1, \quad j = 1, 2, \tag{33}$$

so that condition (69) of Proposition 6 holds true. Then, from Proposition 6 it follows that one of the two vertices of a' satisfies (31). Let then assume that (33) is false, namely that there exists $j \in \{1, 2\}$ such that $d(p_j(a') \to T(\gamma)) \leq X_1$ and $d(\theta_j(a'), \gamma.\theta) \geq \Theta_1$. Then (32) is satisfied by the vertex j. $\square$

6 Likelihood indicator for graph compression

A natural way to construct an edginess function is to estimate the likelihood that an arc belongs to the covering sub-graph of some scene contour. In order to use Lemma 1, let's define ϕ_A to be given by (22) and (23), where $\phi(p, \theta, s)$ is the likelihood that a scene contour with scale s and orientation θ passes through the point p. Notice that, since the edginess function is used only to *compare* competing arcs, it is not

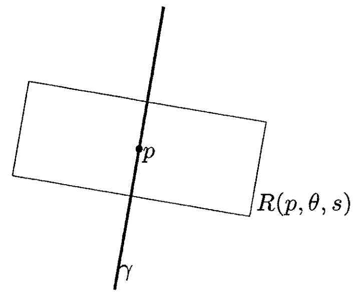


Figure 6: Region $R(p, \theta, s)$ used to compute $\phi(p, \theta, s)$.

necessary to normalized this likelihood into a probability measure. In fact, the proposed arc suppression procedure to obtain a compressible graph is invariant to monotone transformations of the edginess function. An increasing function of likelihood will said to be a *likelihood indicator*.

The likelihood indicator $\phi(p, \theta, s)$ is constructed by least-square fitting a brightness contour model with scale s and orientation θ passing through the point p to the observed image in a rectangular region $R(p, \theta, s)$ centered at p, with orientation θ and dimensions proportional to s. Then, $\phi(p, \theta, s)$ is defined as minus the residual of the optimal fit:

$$\phi(p, \theta, s) = -\min_{b_1, b_2} \frac{1}{|R(p, \theta, s)|} \left(\int_{R(p,\theta,s)} (I(p') - \beta(p'|p, \theta, s, b_1, b_2))^2 dp' \right)^{\frac{1}{2}} . \tag{34}$$

The expression (34) needs to be discretized appropriately if the observed image is discrete. Since $\beta(p'|p, \theta, s, b_1, b_2)$ is a linear function of b_1 and b_2 (compare with (2)), the minimization can be performed by linear convolution with two appropriate filters, which depend on θ and s. Notice that, if a gaussian noise model is assumed, and the proper normalization is chosen, then the quantity in (34) is indeed an increasing function of the likelihood that a contour with orientation θ and scale s passes through the point p, maximized over the nuisance parameters b_1 and b_2. Fig. 3 (d),(e),(f) illustrate the value of $\phi(p, \theta, s)$, which is coded into the intensity of the edgels by means of (23).

6.1 1D analysis of the likelihood indicator

To illustrate the behavior of ϕ, let's consider its one dimensional projection, which is given by:

$$\phi(x,s) = -\min_{b_1,b_2} \frac{1}{2\alpha s} \left(\int_{x-\alpha s}^{x+\alpha s} (I(x') - \beta(x'|x,s,b_1,b_2))^2 dx' \right)^{\frac{1}{2}}, \tag{35}$$

where $\alpha > 0$ is a constant. To analyze the behavior of the likelihood indicator in the neighborhood of a scene contour, let's substitute for $I(x')$ in (35) the noise-free brightness model (2) centered at the origin and with unit scale:

$$I(x') = \beta(x'|0,1,-1,1) = \text{erf}(x') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x'} e^{-\frac{1}{2}u^2} du \tag{36}$$

and let $\phi_0(x,s)$ be the likelihood indicator function obtained in this way. Notice that whereas $\phi(x,s)$ is a random variable, $\phi_0(x,s)$ is a fixed function. More generally, we can define $\phi_\nu(x,s)$ to be $\phi(x,s)$ computed with $I(x')$ equal to the same brightness model (36) plus i.i.d. gaussian noise with variance ν. The indicator function $\phi_0(x,s)$ is shown in Fig. 7.

To clarify the relationship between the likelihood indicator and the conditions of Lemma 1, let's fix the scale $s = 1$ and let $\phi(x) \equiv \phi(x,1)$, $\phi_\nu(x) \equiv \phi_\nu(x,1)$, and let's assume that the noise is the image has variance ν. Let's write $\phi_\nu(x)$ as

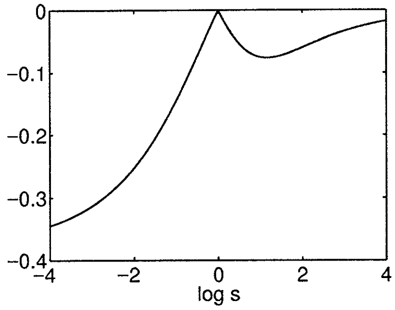
$$\phi_\nu(x) = \phi_0(x) + f_\nu(x),$$

where, for each $x \in \mathbb{R}$, $f_\nu(x)$ is a random variable whose variance vanishes for $\nu \to 0$. In order for the sufficient condition of Lemma 1 to be violated when the distance to γ from the two points p, p' are given by $x < X_0$ and $x' \in [X_1, X_2]$ respectively, it must be that $\phi_\nu(x') > \phi_\nu(x)$, that is

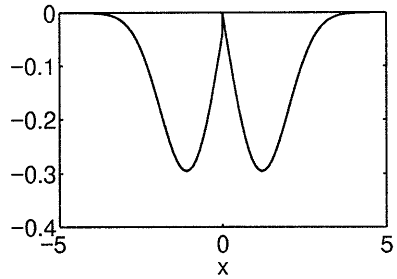
$$f_\nu(x') - f_\nu(x) > \phi_0(x) - \phi_0(x') > \Phi_0(X_0, X_1, X_2), \tag{37}$$

where

$$\Phi_0(X_0, X_1, X_2) = \min_{u \in [0, X_0]} \phi_0(u) - \max_{u \in [X_1, X_2]} \phi_0(u).$$



(a) $\phi_0(0, s)$ vs. $\log s$.



(b) $\phi_0(x, 1)$ vs. x.

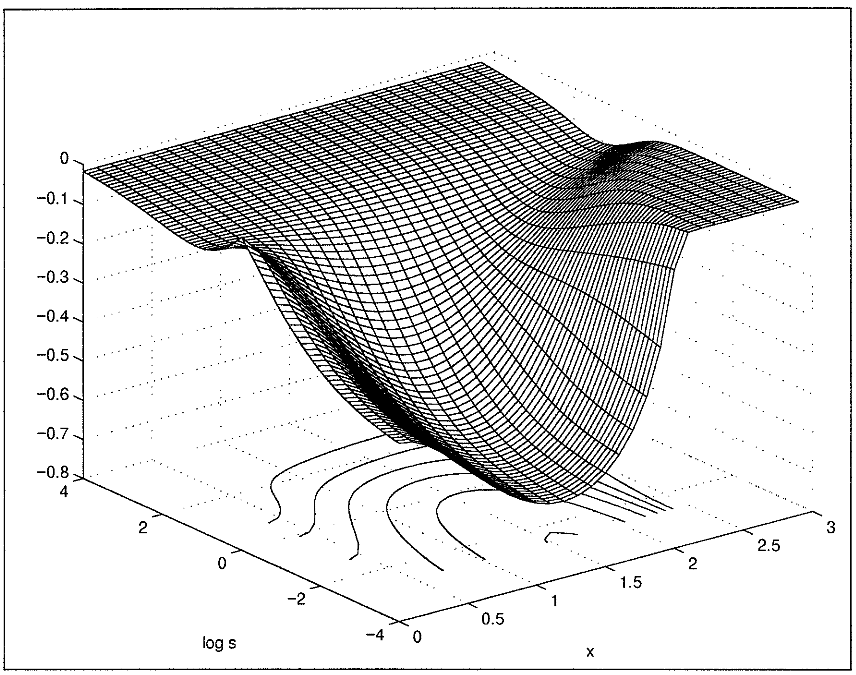


Figure 7: The indicator function $\phi_0(x, s)$.

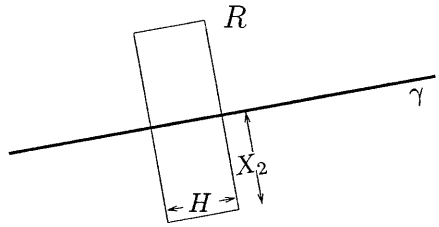


Figure 8: Region across scene contour where data obeys one dimensional model.

Let $P^{\mathsf{x}}_{\mathrm{stb},0}(\gamma)$ be the probability that the event (37) does not occur. Since the variance of $f_\nu(x') - f_\nu(x)$ vanishes for $\nu \to 0$, if the constants $X_0 < X_1 < X_2$ are chosen so that $\Phi_0(X_0, X_1, X_2)$ is a positive number and if ν is sufficiently small, then the probability that (37) occurs is small so that $P^{\mathsf{x}}_{\mathrm{stb},0}(\gamma)$ is close to one. Thus, the likelihood indicator ϕ can be viewed as a stabilizing feedback signal which separates correctly edgel-arcs covering a scene contour from spurious divergent arcs, with probability $P^{\mathsf{x}}_{\mathrm{stb},0}(\gamma)$.

Similar stabilizing properties of the likelihood indicator exist also in the scale dimension, as can be seen in Fig. 7, and in the orientation dimension. Let $P^{\theta}_{\mathrm{stb},0}(\gamma)$ and $P^{\mathsf{s}}_{\mathrm{stb},0}(\gamma)$ be the corresponding probabilities. Presumably, one has

$$\lim_{\nu\to 0} P^{\mathsf{x}}_{\mathrm{stb},0}(\gamma) = \lim_{\nu\to 0} P^{\theta}_{\mathrm{stb},0}(\gamma) = \lim_{\nu\to 0} P^{\mathsf{s}}_{\mathrm{stb},0}(\gamma) = 1.$$

In general, let us define $P_{\mathrm{stb},0}(\gamma)$ be the probability that the sufficient condition of Lemma 1 is satisfied at one fixed location along the contour $\gamma \in \Gamma$.

6.2 Arc suppression error probability in 2D

To evaluate the probability that $\phi(p, \theta, s)$ violates the sufficient condition of Lemma 1, let $\Gamma \in \mathcal{G}_1$ be a free contour set and let $\gamma \in \Gamma$. Let us assume that the domain of γ, $D(\gamma)$, is defined so that it contains all the pixels with distance to $T(\gamma)$ less than X_2. Then, consider a rectangular region R with width $2X_2$ across the contour and with height H in the direction along the contour (see Fig. 8). The probability distribution of the data in R is governed by (1) and (2). Thus, if H is sufficiently small, with probability $P_{\mathrm{stb},0}$, the condition of Lemma 1 will not be violated in R. Let $L_{\mathrm{stb}}(\gamma)$

be the maximum height H for which this is true. $L_{\text{stb}}(\gamma)$ is some sort of correlation length of $\phi(p, \theta, s)$. To evaluate the probability that no error occurs anywhere along the contour γ, let's cover the whole contour with disjoint rectangular regions of height $L_{\text{stb}}(\gamma)$. Then, the probability than no errors occur in any of these regions is

$$[P_{\text{stb},0}(\gamma)]^{\frac{|T(\gamma)|}{L_{\text{stb}}(\gamma)}},$$

where $|T(\gamma)|$ is the length of the contour. Then, the probability that the likelihood indicator preserves all the covering edgel-arcs of the contours in Γ, denoted $P_{\text{stb}}(\Gamma)$, is:

$$P_{\text{stb}}(\Gamma) = \prod_{\gamma \in \Gamma} [P_{\text{stb},0}(\gamma)]^{\frac{|T(\gamma)|}{L_{\text{stb}}(\gamma)}}.$$

7 Computation of a covering estimate

To provide an accurate estimate of Γ, a set of edgel-paths should contain at least one approximating path for every contour $\gamma \in \Gamma$. Since the proposed probabilistic model for a contour γ does not provide any information outside the domain $D(\gamma)$, nor specific constraints to model contour end-points, it is possible that the path approximating some scene contour γ extends beyond the end-points of γ. Thus, an appropriate distance function to measure how well a path approximates a contour is the directed Hausdorff distance $d(T(\gamma) \to T(\pi))$ defined by (12). Then, then distance from a contour set Γ to a set of paths $\hat{\Gamma}$ is defined as:

$$d(\Gamma \to \hat{\Gamma}) = \max_{\gamma \in \Gamma} \min_{\pi \in \hat{\Gamma}} d(T(\gamma) \to T(\pi)).$$

The goal of an estimation algorithm is to compute a $\hat{\Gamma}$ with small complexity such that $d(\Gamma \to \hat{\Gamma})$ is small with high probability. The notion of compressible graphs is an important tool for controlling the complexity of the computed representation. In the previous sections we have described how to compute a compressible graph A^* from the observed image which covers the scene contours with probability $P_{\text{cov}}(\Gamma) \cdot P_{\text{stb}}(\Gamma)$. This probability ultimately depends on the signal to noise ratio in the brightness data and the total length of the scene contours in Γ.

Definition of $paths(e)$

1 If $A_{\text{out}}(e) = \emptyset$

2 $\quad Q(e) := \{nil\}$

3 $\quad$ return;

4 else

5 $\quad$ For every $a \in A_{\text{out}}(e)$

6 $\quad\quad paths(e_2(a))$;

7 $\quad Q(e) := compress\left(\bigcup_{a \in A_{\text{out}}(e)} \bigcup_{\pi \in Q(e_2(a))} a \circ \pi \right)$;

8 $\quad$ return;

Table 1: The procedure $paths(e)$ computes a covering subset, denoted $Q(e)$, of the set of paths originating from the vertex e. Let $A_{\text{out}}(e)$ denote the set of arcs leaving from the vertex e and for any edgel-arc a let $e_2(a)$ denote the second vertex of a (i.e. e_2 if $a = (e_1, e_2)$). If $A_{\text{out}}(e)$ is empty (lines 1,2), then $Q(e)$ contains only the zero-length path nil. If $A_{\text{out}}(e)$ is not empty (lines 5,6), then the procedure $paths(\cdot)$ is called recursively on all the vertices connected to e by an arc, namely on all the elements of the set $\{e_2(a) \ : \ a \in A_{\text{out}}(e)\}$. It is assumed here that the graph does not contain cycles. Then (line 7), for every $a \in A_{\text{out}}(e)$, all the paths in the compressed representations $Q(e_2(a))$ are composed with the arc a. The resulting set of paths is then compressed by selecting a unique representative in every set of paths which terminate at the same vertex.

If A^* indeed covers Γ, then the set of all regular paths in A^* is certainly an accurate representation of Γ. From Definition 3, it turns out that the approximation error of this exhaustive representation is X_0. Clearly, the representation containing only maximal regular paths, namely regular paths which are not sub-paths of any other path, has the same approximation error. However, the set of all maximal regular paths can still be a combinatorially large set, as discussed in Section 4 and illustrated by Fig. 4.

Since A^* is a W-compressible edgel-graph, where W is given by (19), it is possible to compress significantly the maximal-path representation without increasing the approximation error by more than W. To simplify the discussion let us assume that A^* does not contain cycles. Now, for any source e_1, and any sink e_2 (a source is a vertex with no in-arcs and a sink is a vertex with no out-arcs) let $\Pi(e_1, e_2)$ be the set of all regular paths from e_1 to e_2. Clearly, all elements of $\Pi(e_1, e_2)$ are maximal paths. Then, since A^* is W-compressible, the symmetric distance between any two paths $\Pi(e_1, e_2)$ is less than W. Thus, $\Pi(e_1, e_2)$ can be safely compressed down to one element without increasing the approximation distance by more than W. Let then $\hat{\Gamma}$ be the set of paths in A^* obtained by selecting arbitrarily one path in $\Pi(e_1, e_2)$ for each source e_1 and sink e_2. Under all the assumptions made we have the following theorem.

Theorem 2 *Let $\Gamma \in \mathcal{G}_1$ be a free set of regular contours. Then, with probability $P_{\text{cov}}(\Gamma) \cdot P_{\text{stb}}(\Gamma)$,*

$$d(\Gamma \to \hat{\Gamma}) \leq X_0 + W.$$

A simple recursive procedure to compute the paths $Q(e)$ originating from a vertex e is described in Table 1. The estimate $\hat{\Gamma}$ is then given by the union of all $Q(e)$ over all sources e in the graph A^*.

8 Generalization of the exclusion principle

8.1 Composite models with overlapping primitives

The arguments and results in Sections 6 and 7 are based on the assumption that the exclusion principle holds, namely that the scene is composed of regular contours having disjoint domains in the image plane so that the components of the scene do not interfere with each other. However, in order to construct more general contour models, it can be useful to allow the primitive contours to overlap in space. For instance, some non-regular contours, such as contours with corners and junctions, can be modeled as sets of regular contours which intersect at a point. Another example of overlapping contours is given by contours of different scales intersecting at some point in space (e.g. a shadow cast on a surface with a reflectance discontinuity edge).

One important issue, which is not dealt with here, is how to compose the probability measures of the primitives when their domain overlap and therefore the support of these measures are not independent. One possibility is to consider all the non-empty intersections of the atoms of the primitive measures, multiply together the primitive measures of these, and then renormalize. These composite probability models can be quite different from the original ones if the amount of overlap is significant.

Another way to construct models when the domain of the primitives overlap is to use an "occlusion" approach. That is, the primitive with highest "priority" determines the probability measure of the data in the common regions. This method could be used to model intersecting contours of different scales.

8.2 Compressibility in scale-space

Rather than pursuing these modeling issues further, we show how the arc suppression algorithm described in Sections 5 and the two main results therein (Theorem 1 and Lemma 1) can be generalized so that intersecting contours can be estimated independently from each other, provided that they can be separated reliably along some other dimension, such as scale.

Recall that according to the arc suppression algorithm of Sections 5, two arcs $a, a' \in A$ are considered to be incompatible with each other whenever $\sigma(a) \cap \beta(a') = \emptyset$. Thus, a compressible graph can not contain two paths which intersect and diverge from each other. One can relax this restriction by allowing spatially divergent pairs of arcs, provided that they can be labeled in such a way that the two arcs can be processed independently from each other. For instance, if all vertices are labeled either "red" or "green" and if only arcs with vertices of the same color are allowed, then a spatially divergent pair of arcs of different color is acceptable, since one can compute red and green paths independently from each other, e.g. in two independent stages.

Scale (or other contour features such as orientation, brightness intensity, etc.) can be used as a label to separate paths which overlap spatially. If the scale of scene contours is constant along each contour and if scale can be estimated perfectly, then one can construct a graph containing only arcs whose two vertices have identical scales and then restrict the arc suppression step to only pairs of divergent arcs with the same scale. Pair of divergent arcs with different scales can co-exist since paths of different scales can be processed independently of each other.

Implementing this idea in a more realistic situation, when scale estimates are noisy and scale can vary along the contour requires some extra work. Notice that in the definition of a valid graph, the scale difference between the two end-points of an arc is required to be bounded but it can be non-zero.

As indicated by the results below, the roles of the space and scale dimensions are quite symmetric in determining what pairs of arcs are incompatible as well as in using the likelihood indicator function to select which arc to remove. Roughly speaking, a pair of arcs is incompatible if either the two arcs are divergent in space and overlapping in scale or divergent in scale and overlapping in space.

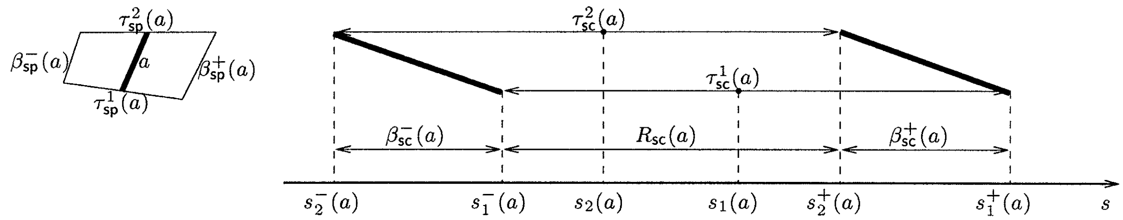


Figure 9: Left: Space entities of and edgel-arc a. Right: scale entities of a.

8.3 Notation

The geometric entities introduced in Section 5.1 for the space dimension, $\beta(a)$, $\tau(a)$, $R(a)$, etc., can also be defined in a similar way for the scale dimension with only minor modifications. To distinguish between the two sets of symbols, we'll use a sp subscript for the entities in the space dimension and a sc subscript for those in the scale dimension. Thus $\beta_{sp}(a)$, $\tau_{sp}(a)$, $R_{sp}(a)$, $||_{sp}$, etc. will replace the notation $\beta(a)$, $\tau(a)$, $R(a)$, $||$, etc., See Fig. 9 for a graphical explanation.

For any $e \in V(A)$, let $\tau_{sc}(e)$ be the interval of the real line with end-points $s^-(e)$, $s^+(e)$, where

$$\begin{aligned} s^-(e) &= e.s - w_{sc}(e), \\ s^+(e) &= e.s + w_{sc}(e). \end{aligned}$$

For any $a = (e_1, e_2) \in A$ let $\tau^1_{sc}(a) = \tau_{sc}(e_1)$, $\tau^2_{sc}(a) = \tau_{sc}(e_2)$,

$$\begin{aligned} \sigma_{sc}(a) &= [e_1.s, e_2.s], \\ R_{sc}(a) &= \tau^1_{sc}(a) \cap \tau^2_{sc}(a), \\ \beta_{sc}(a) &= \begin{cases} (\tau^1_{sc}(a) \cup \tau^2_{sc}(a)) \setminus R_{sc}(a) & \text{if } R_{sc}(a) \neq \emptyset \\ \mathbb{R} & \text{if } R_{sc}(a) = \emptyset \end{cases} \end{aligned}$$

If $R_{sc}(a)$ is not empty, then $\beta_{sc}(a)$ is composed of two disjoint connected components, denoted $\beta^-_{sc}(a)$, $\beta^+_{sc}(a)$ and given by:

$$\begin{aligned} \beta^-_{sc}(a) &= [\min\{s_1^-(a), s_2^-(a)\}, \max\{s_1^-(a), s_2^-(a)\}], \\ \beta^+_{sc}(a) &= [\min\{s_1^+(a), s_2^+(a)\}, \max\{s_1^+(a), s_2^+(a)\}], \end{aligned}$$

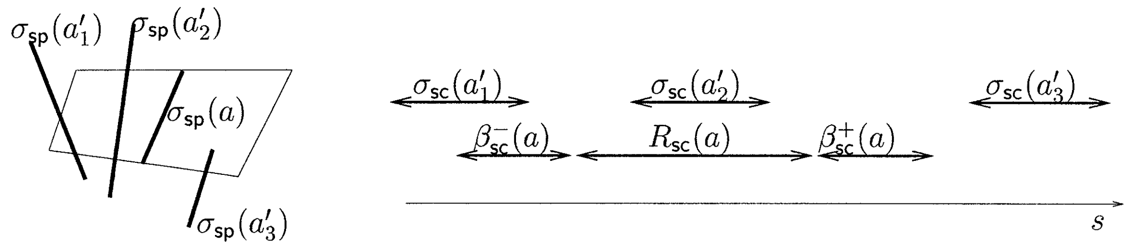


Figure 10: Left: a and a'_1 are space divergent and space non-overlapping, $a' \nparallel_{sp} a \wedge a' \not\diamond_{sp} a$. a and a'_2 are space non-divergent and space overlapping, $a' \parallel_{sp} a \wedge a' \diamond_{sp} a$. a and a'_2 are space non-divergent and space overlapping, $a' \parallel_{sp} a \wedge a' \diamond_{sp} a$. Right: a and a'_1 are scale divergent and scale non-overlapping, $a' \nparallel_{sc} a \wedge a' \not\diamond_{sc} a$. a and a'_2 are scale non-divergent and scale overlapping, $a' \parallel_{sc} a \wedge a' \diamond_{sc} a$. a and a'_2 are scale non-divergent and scale non-overlapping, $a' \parallel_{sc} a \wedge a' \not\diamond_{sc} a$.

where $s_i^-(a) = s^-(e_i)$, $s_i^+(a) = s^+(e_i)$, $i = 1, 2$. If $R_{sc}(a) = \emptyset$, then let $\beta^-_{sc}(a) = \beta^+_{sc}(a) = \mathbb{R}$.

8.4 Sufficient condition for compressibility in scale-space

For the following definition refer to Fig. 10 for examples.

Definition 6 *Let a, a' be two arcs in A. Then*

- a' *is* non-divergent in space from a, *denoted* $a' \parallel_{sp} a$, *if*

$$\sigma_{sp}(a') \cap \beta_{sp}(a) = \emptyset; \tag{38}$$

- a' *is* non-divergent in scale from a, *denoted* $a' \parallel_{sc} a$, *if*

$$\sigma_{sc}(a') \cap \beta_{sc}(a) = \emptyset; \tag{39}$$

- a' overlaps a in space, *denoted* $a' \diamond_{sp} a$, *if*

$$\begin{aligned} &[p_1(a') \in R_{sp}(a)] \vee \\ &[p_2(a') \in R_{sp}(a)] \vee \\ &[\sigma_{sp}(a') \cap \tau^1_{sp}(a) \neq \emptyset \wedge \sigma_{sp}(a') \cap \tau^2_{sp}(a) \neq \emptyset] \end{aligned} \tag{40}$$

- a' overlaps a in scale, *denoted* $a' \diamond_{\mathsf{sc}} a$, *if*

$$\sigma_{\mathsf{sc}}(a') \subset R_{\mathsf{sc}}(a) \cup \beta_{\mathsf{sc}}(a). \tag{41}$$

Notice that these four relations are not symmetric in general. Most of the proof of the following result is in appendix B.

Theorem 3 (Sufficient condition for compressibility) *Let A be a valid edgel-graph. If*

$$a' \diamond_{\mathsf{sc}} a \implies a' \parallel_{\mathsf{sp}} a, \qquad \forall a, a' \in A, \tag{42}$$

$$a' \diamond_{\mathsf{sp}} a \implies a' \parallel_{\mathsf{sc}} a, \qquad \forall a, a' \in A, \tag{43}$$

then, for any two regular paths π_1, π_2 in A with the same initial vertex and final vertex,

$$d(T(\pi_1), T(\pi_2)) < \min\left\{w_{\mathsf{sp}}^{\max}(\pi_1), w_{\mathsf{sp}}^{\max}(\pi_2)\right\}.$$

Proof. Let e_{fi} and e_{la} be the first and last vertex of π_1 and π_2. Clearly, $e_{\mathrm{fi}}.p \in R_{\mathsf{sp}}(\pi_1)$ and $e_{\mathrm{fi}}.s \in R_{\mathsf{sp}}^{\pi_1}(e_{\mathrm{fi}}.p)$. Since π_1 and π_2 are regular paths, it follows that $\sigma_{\mathsf{sp}}^{\circ}(\pi_2) \cap \tau_{\mathsf{sp}}(\pi_1) = \emptyset$, where $\tau_{\mathsf{sp}}(\pi_1)$ denotes $\tau_{\mathsf{sp}}(e_{\mathrm{fi}}) \cup \tau_{\mathsf{sp}}(e_{\mathrm{la}})$. From (8) we have that $\sigma_{\mathsf{sp}}(a_1') \cap R_{\mathsf{sp}}^{\circ}(\pi_1) \neq \emptyset$ where a_1' is the first arc of π_2. Then, from Proposition 11,

$$\sigma_{\mathsf{sp}}(\pi_2) \subset R_{\mathsf{sp}}(\pi_1).$$

Similarly, by interchanging π_1 with π_2 in the above argument,

$$\sigma_{\mathsf{sp}}(\pi_1) \subset R_{\mathsf{sp}}(\pi_2).$$

The result then follows from Proposition 1. □

Let

$$W_{\mathsf{sp}} = \max_{e \in V(A)} w_{\mathsf{sp}}(e).$$

Corollary 2 *If* (42), (43) *hold then A is W_{sp}-compressible.*

8.5 Computing a compressible edgel-graph in scale-space

The algorithm to compute a compressible edgel-graph A^* described in Section 5.2 can be easily adapted to deal with the more general condition of Theorem 3. We still have $A \setminus A^\dagger$ but now $A^\dagger$ is (compare with (20)):

$$A^\dagger = \{a \in A \; : \; \exists a' \in \mathrm{div}_{\mathsf{sp}}(a) \cup \mathrm{div}_{\mathsf{sc}}(a), \phi_A(a) \leq \phi_A(a')\} \tag{44}$$

where $\mathrm{div}_{\mathsf{sp}}(a)$ and $\mathrm{div}_{\mathsf{sc}}(a)$ are given by:

$$\mathrm{div}_{\mathsf{sp}}(a) = \left\{a' \in A \; : \; a' \not\parallel_{\mathsf{sp}} a \wedge a' \diamond_{\mathsf{sc}} a\right\} \cup \left\{a' \in A \; : \; a \not\parallel_{\mathsf{sp}} a' \wedge a \diamond_{\mathsf{sc}} a'\right\}; \tag{45}$$

$$\mathrm{div}_{\mathsf{sc}}(a) = \{a' \in A \; : \; a' \not\parallel_{\mathsf{sc}} a \wedge a' \diamond_{\mathsf{sp}} a\} \cup \{a' \in A \; : \; a \not\parallel_{\mathsf{sc}} a' \wedge a \diamond_{\mathsf{sp}} a'\}. \tag{46}$$

Recall that, for any contour γ covered by A, ϕ_A is said to preserve γ if $A|\gamma \subset A^*$ (see Definition 5). The proof of the following proposition is analogous to the corresponding one in Section 5.2.

Proposition 4 *Let γ be a contour covered by A. Then ϕ_A preserves γ if and only if*

$$\phi_A(a) > \phi_A(a'), \qquad \forall a' \in \mathrm{div}_{\mathsf{sp}}(a) \cup \mathrm{div}_{\mathsf{sc}}(a).$$

The following result is a generalization of Lemma 1. Recall that Θ_0, X_0 and S_0 are the parameters introduced in Definition 3 (covering edgel-graph) and that Θ_1, X_1 and X_2 are parameters satisfying (24)-(26). Furthermore, let

$$L_{\mathrm{div}} = \max_{a \in A} \max_{a' \in \mathrm{div}_{\mathsf{sp}}(a) \cup \mathrm{div}_{\mathsf{sc}}(a)} \max_{i,j \in \{1,2\}} ||p_i(a) - p_j(a')||; \tag{47}$$

$$S_3 = \max_{\gamma} \max_{a \in A|\gamma} \max_{a' \in \mathrm{div}_{\mathsf{sp}}(a)} \max_{i \in \{1,2\}} |s_i(a') - \gamma.s|; \tag{48}$$

$$X_3 = \max_{\gamma} \max_{a \in A|\gamma} \max_{a' \in \mathrm{div}_{\mathsf{sc}}(a)} \max_{i \in \{1,2\}} |\xi_\gamma(p_i(a'))|. \tag{49}$$

where $\xi_\gamma(p)$ is the signed distance from $T(\gamma)$ to the the point p. Let us introduce two constants $0 < S_1, S_2$ which satisfy

$$S_1 \leq W_{\mathsf{sc}}^{\min} - S_\Delta - S_0; \tag{50}$$

$$S_2 \geq W_{\mathsf{sc}}^{\max} + S_\Delta + S_0, \tag{51}$$

where

$$W_{\mathsf{sc}}^{\min} = \min_{e\in V(A)} w_{\mathsf{sc}}(e); \tag{52}$$

$$W_{\mathsf{sc}}^{\max} = \max_{e\in V(A)} w_{\mathsf{sc}}(e), \tag{53}$$

and S_Δ is the maximum scale change between the vertices of a valid edgel-arc (see Definition 1).

Lemma 2 (Preservation of covering arcs) *Let A be a valid edgel-graph. Let γ be a flat contour covered by A. Then ϕ_A preserves γ if for any p, θ, s and p', θ', s' we have $\phi(p, \theta, s) > \phi(p', \theta', s')$ whenever*

$$d(p, T(\gamma)) \leq X_0, \tag{54}$$

$$d(\theta, \gamma.\theta) \leq \Theta_0, \tag{55}$$

$$d(s, T(\gamma)) \leq S_0, \tag{56}$$

$$||p' - p|| \leq L_{\text{div}}, \tag{57}$$

and at least one of the following three sets of condition hold

$$d(p' \to T(\gamma)) \in [X_1, X_2], \tag{58}$$

$$|s' - \gamma.s| \leq S_3; \tag{59}$$

$$|s' - \gamma.s| \in [S_1, S_2] \tag{60}$$

$$d(p' \to T(\gamma)) \leq X_3; \tag{61}$$

$$d(p' \to T(\gamma)) \leq X_1, \tag{62}$$

$$d(\theta', \gamma.\theta) \geq \Theta_1, \tag{63}$$

$$|s' - \gamma.s| \leq S_3. \tag{64}$$

Proof. The proof is similar to the proof of Lemma 1. Let γ be a flat contour for which the sufficient condition of the lemma is satisfied. From Proposition 4, we have to prove that $\phi_A(a) > \phi_A(a')$ for every $a \in A|\gamma$ and $a' \in \text{div}_{\text{sp}}(a) \cup \text{div}_{\text{sc}}(a)$. Let then $a \in A|\gamma$ and $a' \in \text{div}_{\text{sp}}(a) \cup \text{div}_{\text{sc}}(a)$. One needs to prove that for every $i \in \{1, 2\}$ there exists $j \in \{1, 2\}$ such that

$$\phi(p_i(a), \theta_i(a), s_i(a)) > \phi(p_j(a'), \theta_j(a'), s_j(a')).$$

Notice that (54)-(57) are identical to (27)-(30) of Lemma 1. If $a' \in \text{div}_{\text{sp}}(a)$ then the results follows immediately from Lemma 1 because (59) and (64) follow from (48) and therefore the trigger condition (58)$\wedge$(59) implies (31) and (62)$\wedge$(63)$\wedge$(64) implies (32). Let then $a' \in \text{div}_{\text{sc}}(a)$. It is sufficient to prove that for each vertex i of a there exists a vertex j of a' which satisfies the trigger condition (60)$\wedge$(61). (61) follows immediately from (49) for $j = 1, 2$. From $a' \in \text{div}_{\text{sc}}(a)$ it follows that $a' \not\parallel_{\text{sc}} a \vee a \not\parallel_{\text{sc}} a'$. Hence, from the corollary of Lemma 3 in Appendix C, we have that for each i there exists j such that

$$|s_i(a) - s_j(a')| \in [w_{\text{sc}}^{\min}(a, a') - S_\Delta, w_{\text{sc}}^{\max}(a, a') + S_\Delta] \subset [W_{\text{sc}}^{\min} - S_\Delta, W_{\text{sc}}^{\max} + S_\Delta].$$

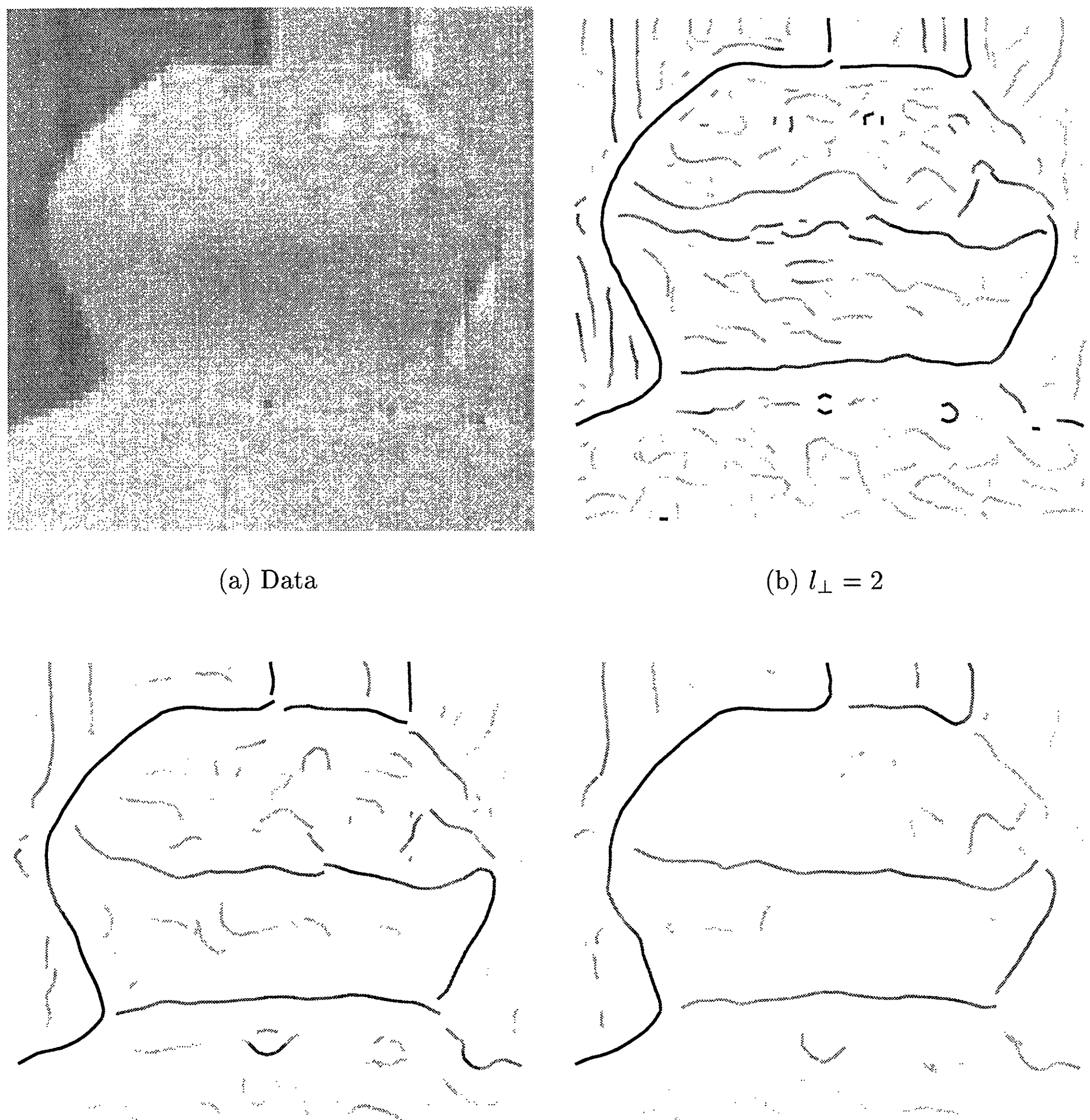
Since $a \in A|\gamma$, we have that $|s_i(a) - s.\gamma| < S_0$, $i = 1, 2$, and therefore

$$|s_j(a') - \gamma.s| \in [W_{\text{sc}}^{\min} - S_\Delta - S_0, W_{\text{sc}}^{\max} + S_\Delta + S_0].$$

From (50) and (51) it follows then that $|s_j(a') - \gamma.s| \in [S_1, S_2]$. $\square$

9 Experiments

A set of edgels has been computed from the image 11(a) by using the method described in Section 4.1. The rectangular regions used to estimate the edgels are three pixel long in the direction of the contour and $2l_\perp$ wide across the contour. Three values of the parameter $l_\perp$ have been used, 2, 4 and 6. The likelihood indicator has been computed by assuming the blurred step model (2) and by using regions of width equal to twice



(a) Data

(b) $l_\perp = 2$

(c) $l_\perp = 4$

(d) $l_\perp = 6$

Figure 11: Polygonal approximations of regular scene contours.

the estimated blur scale. These edgels are shown in Fig. 3(d),(e),(f). The three sets of edgels are shown together in Fig. 3(b), where the gray value intensity is proportional to brightness contrast rather than the likelihood indicator. Fig. 3(c) shows the arcs obtained by connecting edgels computed from adjacent regions after removing invalid arcs. Finally, Fig. 11 shows the polygonal approximations of the regular scene contours obtained by processing each of the three sets of arcs independently from each other. The parameter $w_{\mathsf{sp}}(e)$ has been set to 0.75 for all edgels. The gray value intensity is proportional to the local brightness contrast.

10 Conclusions and future work

A multiscale probabilistic model of regular scene contours has been proposed, together with a computationally efficient algorithm to approximate these contours by means of polygonal curves. An important notion in developing an efficient algorithm which guarantees high probability of accurate reconstruction is that of a compressible graph. It has been argued that if the signal to noise ratio in the image is sufficiently high then, with high probability, the proposed algorithm computes a representation which is guaranteed to contain at least one good approximation for each scene contour, as measured by the directed Hausdorff distance.

Future work will focus on using the basic regular-and-visible contour model studied here to construct composite probabilistic models of more complex contours, such as contours with corners and junctions and partially invisible (i.e. with low signal to noise ratio) contours. Our general approach to develop efficient and reliable algorithm to estimate these complex models is based on three components: composition of descriptors (e.g. arcs composed into paths); pruning of unlikely descriptors (e.g. edgel thresholding); and compression of equivalent hypotheses.

A Lemmas for Section 5

For any flat contour γ and $X > 0$, let $N_X(\gamma)$ be the neighborhood of γ of radius X:

$$N_X(\gamma) = \left\{p \in \mathbb{R}^2 \ : \ d(p \to T(\gamma)) \leq X\right\}.$$

The following proposition has been proved in [7] (see Lemma 1 in [7]). Recall the following notation. For any $a = (e_1, e_2)$, let $p_i(a) = e_i.p$, $\theta_i(a) = e_i.\theta$, $w_i(a) = w(e_i)$, etcetera.

Proposition 5 *Let γ be flat contour; let $a \in A$ be such that*

$$\begin{aligned} p_i(a) &\in N_{X_1}(\gamma), \quad i = 1, 2, && (65)\\ d(\theta_i(a), \gamma.\theta) &< \Theta_1, \quad i = 1, 2 && (66)\\ \frac{2X_1}{\cos\Theta_1} &< w_i(a), \quad i = 1, 2, && (67)\\ X_2 - X_1 &> w_i(a), \quad i = 1, 2. && (68) \end{aligned}$$

Then, $\beta^+(a) \cup \beta^-(a) \ \subset \ N_{X_2}(\gamma) \setminus N_{X_1}(\gamma)$.

By using this proposition, one can prove the following result. The proof is similar to the proof of Proposition 4 in [7].

Proposition 6 *Let γ be a flat contour and let $a \in A|\gamma$. Let $a' \in \mathrm{div}(a)$ be such that*

$$d(p_i(a') \to T(\gamma)) \leq X_1 \implies d(\theta_i(a'), \gamma.\theta) < \Theta_1, \quad i = 1, 2. \tag{69}$$

If

$$\begin{aligned} \Theta_0 &< \Theta_1, && (70)\\ X_0 &< X_1 < \frac{\cos\Theta_1}{2} \cdot \min\left\{w_1(a), w_2(a), w_1(a'), w_2(a')\right\}, && (71)\\ X_2 &> X_1 + \max\left\{||p_2(a') - p_1(a')||, w_1(a), w_2(a), w_1(a'), w_2(a')\right\}, && (72) \end{aligned}$$

then there exists $p' \in \{p_1(a'), p_2(a')\}$ such that

$$d(p' \to T(\gamma)) \in [X_1, X_2]. \tag{73}$$

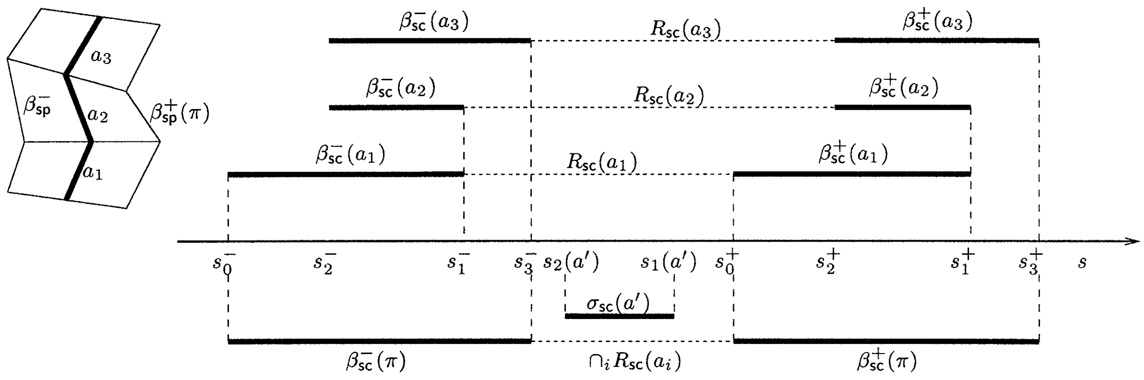


Figure 12: An edgel-path composed of three arcs a_1, a_2, a_3. Left: the edgel-path's entities in $\mathbb{R}^2$. Right: the edgel-path's entities in the scale domain.

B Lemmas for Section 8.4

Definition 7 *A graph for which* (42) *and* (43) *hold is said to be* separated.

More explicitly, a graph is separated whenever the following conditions are satisfied:

(S1) If $p_1(a') \in R_{\mathsf{sp}}(a)$ or $p_2(a') \in R_{\mathsf{sp}}(a)$ then $\sigma_{\mathsf{sc}}(a') \cap \beta_{\mathsf{sc}}(a) = \emptyset$.

(S2) If $\sigma_{\mathsf{sp}}(a') \cap \tau^1_{\mathsf{sp}}(a) \neq \emptyset$ and $\sigma_{\mathsf{sp}}(a') \cap \tau^2_{\mathsf{sp}}(a) \neq \emptyset$ then $\sigma_{\mathsf{sc}}(a') \cap \beta_{\mathsf{sc}}(a) = \emptyset$.

(S3) If $\sigma_{\mathsf{sc}}(a') \subset R_{\mathsf{sc}}(a) \cup \beta_{\mathsf{sc}}(a)$ then $\sigma_{\mathsf{sp}}(a') \cap \beta_{\mathsf{sp}}(a) = \emptyset$.

For any regular edgel-path π with arcs $(a_1, \ldots, a_n)$ let

$$\begin{aligned} \beta^-_{\mathsf{sc}}(\pi) &= \bigcup_{i=1}^{n} \beta^-_{\mathsf{sc}}(a), \\ \beta^+_{\mathsf{sc}}(\pi) &= \bigcup_{i=1}^{n} \beta^+_{\mathsf{sc}}(a), \end{aligned}$$

and let $\beta_{\mathsf{sc}}(\pi) = \beta^-_{\mathsf{sc}}(\pi) \cup \beta^+_{\mathsf{sc}}(\pi)$. Notice that $\beta^-_{\mathsf{sc}}(\pi)$ and $\beta^+_{\mathsf{sc}}(\pi)$ are connected sets.

Proposition 7 *Let π be a regular edgel-path with arcs $(a_1, \ldots, a_n)$, $n \geq 1$, and let $a' = (e'_1, e'_2)$ be an edgel-arc such that $\sigma_{\mathsf{sc}}(a') \cap \beta_{\mathsf{sc}}(\pi) = \emptyset$ and*

$$\{e'_1.s, e'_2.s\} \cap \bigcup_{i=1}^{n} R_{\mathsf{sc}}(a_i) \neq \emptyset.$$

Then,

$$\sigma_{\mathsf{sc}}(a') \subset \bigcap_{i=1}^{n} R_{\mathsf{sc}}(a_i). \tag{74}$$

Proof. Without loss of generality, let $e'_1.s \in \cup_{i=1}^{n} R_{\mathsf{sc}}(a_i)$ and let k be such that $e'_1.s \in R_{\mathsf{sc}}(a_k)$, $1 \leq k \leq n$. Since $\beta^-_{\mathsf{sc}}(a_k) \leq R_{\mathsf{sc}}(a_k)$, namely all the real numbers in $\beta^-_{\mathsf{sc}}(a_k)$ are less or equal to any real number in $R_{\mathsf{sc}}(a_k)$, from $e'_1.s \in R_{\mathsf{sc}}(a_k)$ it follows that $\beta^-_{\mathsf{sc}}(a_k) \leq e'_1.s$. Thus, since $e'_1.s \in \sigma_{\mathsf{sc}}(a')$ and $\sigma_{\mathsf{sc}}(a') \cap \beta^-_{\mathsf{sc}}(a_k) = \emptyset$, we have $\beta^-_{\mathsf{sc}}(a_k) < \sigma_{\mathsf{sc}}(a')$. Therefore, since $\beta^-_{\mathsf{sc}}(a_k) \subset \beta^-_{\mathsf{sc}}(\pi)$ and $\beta^-_{\mathsf{sc}}(\pi)$ is a connected set, from $\sigma_{\mathsf{sc}}(a') \cap \beta^-_{\mathsf{sc}}(\pi) = \emptyset$ it follows that $\beta^-_{\mathsf{sc}}(\pi) < \sigma_{\mathsf{sc}}(a')$. Similarly, $\sigma_{\mathsf{sc}}(a') < \beta^+_{\mathsf{sc}}(\pi)$. Thus $\beta^-_{\mathsf{sc}}(a_i) < \sigma_{\mathsf{sc}}(a') < \beta^+_{\mathsf{sc}}(a_i)$, $i = 1, \ldots, n$, that is, $\sigma_{\mathsf{sc}}(a') \in R_{\mathsf{sc}}(a_i)$, $i = 1, \ldots, n$. $\square$

The following notation will be used when dealing with properties and assumptions holding for sets of integers. The set of integers i such that $i \geq k$ and $i \leq k$ is denoted $\{k, \ldots, l\}$. If $l < k$ then this set is empty and therefore a property which holds true $\forall i \in \{k, \ldots, l\}$ is always true if $l < k$. The notation $i = k, \ldots, l$ is equivalent to $k \leq i \leq l$ and therefore requires that $k \leq l$.

Proposition 8 *Let π be a regular path with arcs $(a_1, \ldots, a_n)$, $n \geq 1$, in a separated edgel-graph A and let $a' = (e'_1, e'_2) \in A$ be an arc such that $e'_1.p \in R_{\mathsf{sp}}(a_1)$, $e'_1.s \in R_{\mathsf{sc}}(a_1)$ and (see Fig. 13(a))*

$$\sigma_{\mathsf{sp}}(a') \cap \tau^h_{\mathsf{sp}}(a_i) \neq \emptyset, \quad h = 1, 2; \quad \forall i \in \{2, \ldots, n-1\}. \tag{75}$$

Then,

$$\sigma_{\mathsf{sc}}(a') \subset \bigcap_{i=1}^{n-1} R_{\mathsf{sc}}(a_i). \tag{76}$$

Furthermore, if $e'_2.p \in R_{\mathsf{sp}}(a_n)$ (see Fig. 13(b)),

$$\sigma_{\mathsf{sc}}(a') \subset \bigcap_{i=1}^{n} R_{\mathsf{sc}}(a_i). \tag{77}$$

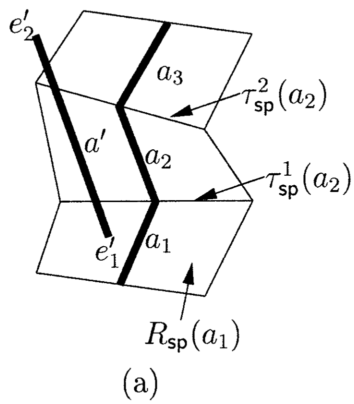


Figure 13: Proposition 8. Condition (75) holds for $i = 2$: $\sigma_{\mathsf{sp}}(a') \cap \tau^1_{\mathsf{sp}}(a_2) \neq \emptyset$, $\sigma_{\mathsf{sp}}(a') \cap \tau^2_{\mathsf{sp}}(a_2) \neq \emptyset$. By Proposition 8, if $e'_1.s \in R_{\mathsf{sc}}(a_1)$ this implies that $\sigma_{\mathsf{sc}}(a') \subset \cap^2_{i=1} R_{\mathsf{sc}}(a_i)$. In (b) we also have $e'_2.p \in R_{\mathsf{sp}}(a_3)$ so that $\sigma_{\mathsf{sc}}(a') \subset \cap^3_{i=1} R_{\mathsf{sc}}(a_i)$.

Proof. From the separation condition (S1) and $e'_1.p \in R_{\mathsf{sp}}(a_1)$ we have

$$\sigma_{\mathsf{sc}}(a') \cap \beta_{\mathsf{sc}}(a_1) = \emptyset. \tag{78}$$

If $n \geq 3$, from (75) and the separation condition (S2) we have

$$\sigma_{\mathsf{sc}}(a') \cap \beta_{\mathsf{sc}}(a_i) = \emptyset, \qquad i = 2, \dots, n-1 \tag{79}$$

which, together with (78), yields

$$\sigma_{\mathsf{sc}}(a') \cap \beta_{\mathsf{sc}}(\langle a_1, \dots, a_{n-1} \rangle) = \emptyset, \qquad n \geq 2, \tag{80}$$

where $\langle a_1, \dots, a_{n-1} \rangle$ denotes the edgel-path with arcs $a_1, \dots, a_{n-1}$. Then, (76) follows from (80) and Proposition 7.

To prove the second part, let $e'_2.p \in R_{\mathsf{sp}}(a_n)$. Then, from the separation condition (S1) we get $\sigma_{\mathsf{sc}}(a') \cap \beta_{\mathsf{sc}}(a_n) = \emptyset$ and therefore, by combining this with (80), it follows that $\sigma_{\mathsf{sc}}(a') \cap \beta_{\mathsf{sc}}(\pi) = \emptyset$ which, by Proposition 7 implies (77). □

For any regular edgel-path π and any $p \in R_{\mathsf{sp}}(\pi)$ let $R^{\pi}_{\mathsf{sc}}(p) = R_{\mathsf{sc}}(a)$, where a is the unique arc in π such that $p \in R_{\mathsf{sp}}(a)$.

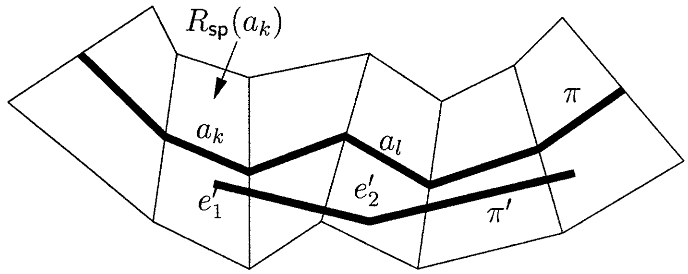


Figure 14: Proposition 9.

Proposition 9 *Let π, π' be regular paths in a separated edgel-graph A such that $\sigma_{\mathsf{sp}}(\pi') \subset R_{\mathsf{sp}}(\pi)$ and $e'_1.s \in R^{\pi}_{\mathsf{sc}}(e'_1.p)$ where e'_1 is the first vertex of π'. then*

$$e'.s \in R^{\pi}_{\mathsf{sc}}(e'.p)$$

for every vertex e' of π'.

Proof. Refer to Fig. 14. First, let's assume that π' consists of one arc $a' = (e'_1, e'_2)$. Let $\{a_i\}$ be the arcs of π. Let k, l be such that $e'_1.p \in R_{\mathsf{sp}}(a_k)$ and $e'_2.p \in R_{\mathsf{sp}}(a_l)$. Without loss of generality, let's assume that $l \geq k$ (otherwise, interchange e'_1 with e'_2 in the argument). Notice that, since $\sigma_{\mathsf{sp}}(\pi') \subset R_{\mathsf{sp}}(\pi)$,

$$\sigma_{\mathsf{sp}}(a') \cap \tau^h_{\mathsf{sp}}(a_i) \neq \emptyset, \qquad h = 1, 2; \quad \forall i \in \{k+1, \dots, l-1\}.$$

Thus, from the second part of Proposition 8 applied to the edgel-path $\langle a_k, \dots, a_l \rangle$,

$$\sigma_{\mathsf{sc}}(a') \subset R_{\mathsf{sc}}(a_l)$$

from which

$$e'_2.s \subset R_{\mathsf{sc}}(a_l) = R^{\pi}_{\mathsf{sc}}(e'_2.p).$$

For paths with more than arc the result follows by recursion. □

Proposition 10 *Let π be a regular path with arcs $(a_1, \dots, a_n)$, $a_i = (e_i, e_{i+1})$ in a separated edgel-graph A. Let $a' = (e'_1, e'_2) \in A$. If $e'_1.p \in R_{\mathsf{sp}}(\pi)$ and $e'_1.s \in R^{\pi}_{\mathsf{sc}}(e'_1.p)$ then $\sigma_{\mathsf{sp}}(a') \cap \beta_{\mathsf{sp}}(\pi) = \emptyset$.*

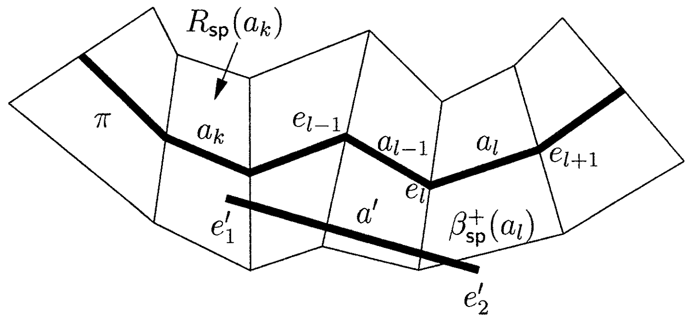


Figure 15: Proposition 10. The situation shown, namely $\sigma_{\mathsf{sp}}(a') \cap \beta_{\mathsf{sp}}(\pi) \neq \emptyset$, is prohibited by the proposition.

Proof. Refer to Fig. 15. Let k be such that $e_1'.p \in R_{\mathsf{sp}}(a_k)$. For the purpose of contradiction, let $\sigma_{\mathsf{sp}}(a') \cap \beta_{\mathsf{sp}}(\pi) \neq \emptyset$ and let a_l be the first exit arc for $\sigma_{\mathsf{sp}}(a')$. That is, $\sigma_{\mathsf{sp}}(a') \cap \beta_{\mathsf{sp}}(a_l) \neq \emptyset$ and

$$\sigma(a') \cap \beta_{\mathsf{sp}}(a_i) = \emptyset, \qquad \forall i \in \{k, \dots, l-1\}, \tag{81}$$

where we have assumed for simplicity that $k \leq l$. From the separation condition (S3), it is sufficient to prove that

$$\sigma_{\mathsf{sc}}(a') \subset R_{\mathsf{sc}}(a_l) \cup \beta_{\mathsf{sc}}(a_l). \tag{82}$$

Note that from $e_1'.s \in R^{\pi}_{\mathsf{sc}}(e_1'.p) = R_{\mathsf{sc}}(a_k)$ and the separation condition (S1) it follows that

$$\sigma_{\mathsf{sc}}(a') \subset R_{\mathsf{sc}}(a_k). \tag{83}$$

Thus (82) holds if $k = l$. If $l \geq k+1$, we will first prove that

$$\sigma_{\mathsf{sc}}(a') \subset R_{\mathsf{sc}}(a_{l-1}). \tag{84}$$

If $l = k+1$ then (84) follows immediately from (83). If $l \geq k+2$ then from (81) we have

$$\sigma_{\mathsf{sp}}(a') \cap \tau^h_{\mathsf{sp}}(a_i) \neq \emptyset, \qquad h = 1, 2; \quad i = k+1, \dots, l-1$$

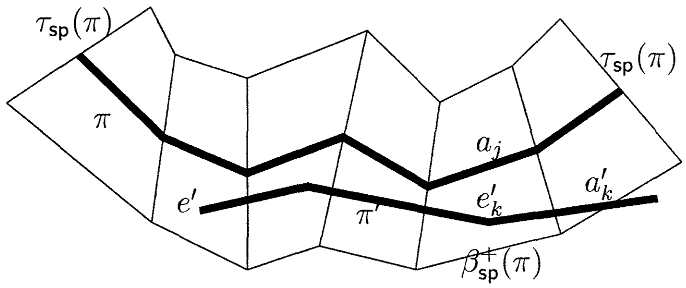


Figure 16: Proposition 11. The situation shown, namely $\sigma_{\mathsf{sp}}(\pi') \not\subset R_{\mathsf{sp}}(\pi)$, is prohibited by the proposition.

which, together with the first part of Proposition 8, proves (84) for $l \geq k+2$. Then, since $R_{\mathsf{sc}}(a_{l-1}) = \tau_{\mathsf{sc}}(e_{l-1}) \cap \tau_{\mathsf{sc}}(e_l)$, from (84) we have

$$\sigma_{\mathsf{sc}}(a') \subset R_{\mathsf{sc}}(a_{l-1}) \subset \tau_{\mathsf{sc}}(e_l) \subset (\tau_{\mathsf{sc}}(e_l) \cup \tau_{\mathsf{sc}}(e_{l+1})) = R_{\mathsf{sc}}(a_l) \cup \beta_{\mathsf{sc}}(a_l).$$

$\square$

For any edgel-path π, let $\tau_{\mathsf{sp}}(\pi) = \tau_{\mathsf{sp}}(e_{\mathrm{fi}}) \cup \tau_{\mathsf{sp}}(e_{\mathrm{la}})$ where e_{fi} and e_{la} and the first and last vertex of π. Let $R^\circ_{\mathsf{sp}}(\pi)$ denote the interior of $R_{\mathsf{sp}}(\pi)$ and let $\sigma^\circ_{\mathsf{sp}}(\pi)$ denote the interior of the polygonal line $\sigma_{\mathsf{sp}}(\pi)$, namely $\sigma_{\mathsf{sp}}(\pi)$ less its two end-points.

Proposition 11 *Let π, π' be paths in a separated edgel-graph A such that $e'.p \in R_{\mathsf{sp}}(\pi)$ and $e'.s \in R^\pi_{\mathsf{sc}}(e'.p)$ where e' is a vertex of π'. If $\sigma^\circ_{\mathsf{sp}}(\pi') \cap \tau_{\mathsf{sp}}(\pi) = \emptyset$ and $\sigma_{\mathsf{sp}}(\pi') \cap R^\circ_{\mathsf{sp}}(\pi) \neq \emptyset$, then*

$$\sigma_{\mathsf{sp}}(\pi') \subset R_{\mathsf{sp}}(\pi)$$

and $e'_i.s \in R^\pi_{\mathsf{sc}}(e'_i.p)$ for all the vertices e'_i of π'.

Proof. Refer to Fig. 16. For the purpose of contradiction let us assume that $\sigma_{\mathsf{sp}}(\pi') \not\subset R_{\mathsf{sp}}(\pi)$. Since $R_{\mathsf{sp}}(\pi)$ is a closed set, then $\sigma^\circ_{\mathsf{sp}}(\pi')$ must contain a point outside $R_{\mathsf{sp}}(\pi)$. Then, since $\sigma^\circ_{\mathsf{sp}}(\pi') \cap R^\circ_{\mathsf{sp}}(\pi) \neq \emptyset$, it follows that $\sigma^\circ_{\mathsf{sp}}(\pi')$ must intersect the boundary of $R_{\mathsf{sp}}(\pi)$, which is given by $\tau_{\mathsf{sp}}(\pi) \cup \beta_{\mathsf{sp}}(\pi)$. Since $\sigma^\circ_{\mathsf{sp}}(\pi') \cap \tau_{\mathsf{sp}}(\pi) = \emptyset$ we must have $\sigma^\circ_{\mathsf{sp}}(\pi') \cap \beta_{\mathsf{sp}}(\pi) \neq \emptyset$ and therefore $\sigma_{\mathsf{sp}}(\pi') \cap \beta_{\mathsf{sp}}(\pi) \neq \emptyset$. For simplicity let us assume that e' is the first vertex of π', e'_1, and that $\sigma_{\mathsf{sp}}(a'_1) \cap R^\circ_{\mathsf{sp}}(\pi) \neq \emptyset$. Let a'_k be the first arc in the path π' such that $\sigma_{\mathsf{sp}}(a'_k) \cap \beta_{\mathsf{sp}}(\pi) \neq \emptyset$. First, let us prove that

$$e'_k.s \in R^{\pi}_{\mathsf{sc}}(e'_k.p). \tag{85}$$

If $k = 1$ then this follows immediately from $e'_1.s \in R^{\pi}_{\mathsf{sc}}(e'_1.p)$. If $k > 1$ we have $\sigma_{\mathsf{sp}}(\pi'') \subset R_{\mathsf{sp}}(\pi)$ where π'' is the subpath with arcs $a'_1, \ldots, a'_{k-1}$ and vertices $e'_1, \ldots, e'_k$. Then (85) follows from Proposition 9. From (85) we have $e'_k.s \in R_{\mathsf{sc}}(a_j)$ where j is such that $e'_k.p \in R_{\mathsf{sp}}(a_j)$. Then, by substituting $a'_k = (e'_k, e'_{k+1})$ for (e'_1, e'_2) in Proposition 10 one gets $\sigma(a'_k) \cap \beta(\pi) = \emptyset$ which is a contradiction.

C Lemma for Section 8.5

For any $a = (e_1, e_2) \in A$ let

$$w^{\min}_{\mathsf{sc}}(a) = \min\{w_{\mathsf{sc}}(e_1), w_{\mathsf{sc}}(e_2)\}, \tag{86}$$

$$w^{\max}_{\mathsf{sc}}(a) = \max\{w_{\mathsf{sc}}(e_1), w_{\mathsf{sc}}(e_2)\}. \tag{87}$$

Lemma 3 *Let $a, a' \in A$ be such that $a' \not\parallel_{\mathsf{sc}} a$ and*

$$|s_1(a) - s_2(a)| < S_\Delta, \tag{88}$$

$$|s_1(a') - s_2(a')| < S_\Delta. \tag{89}$$

Then,

$$\forall j \in \{1,2\}, \exists i \in \{1,2\}, \quad |s_j(a') - s_i(a)| \in [w^{\min}_{\mathsf{sc}}(a) - S_\Delta, w^{\max}_{\mathsf{sc}}(a) + S_\Delta] \tag{90}$$

$$\forall i \in \{1,2\}, \exists j \in \{1,2\}, \quad |s_j(a') - s_i(a)| \in [w^{\min}_{\mathsf{sc}}(a) - S_\Delta, w^{\max}_{\mathsf{sc}}(a) + S_\Delta] \tag{91}$$

Proof. From $a' \not\parallel_{\mathsf{sc}} a$ it follows that

$$\sigma_{\mathsf{sc}}(a') \cap \beta_{\mathsf{sc}}(a) \neq \emptyset,$$

that is,

$$\sigma_{\mathsf{sc}}(a') \cap \left((\tau^1_{\mathsf{sc}}(a) \cup \tau^2_{\mathsf{sc}}(a)) \setminus (\tau^1_{\mathsf{sc}}(a) \cap \tau^2_{\mathsf{sc}}(a))\right) \neq \emptyset.$$

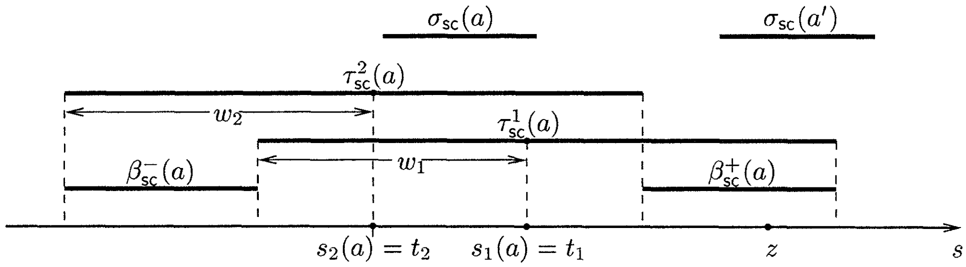


Figure 17: Some notation for the proof of Lemma 3. z denotes a point in $\sigma_{\mathsf{sc}}(a') \cap \beta_{\mathsf{sc}}(a)$.

Therefore, there exists $z \in \sigma_{\mathsf{sc}}(a')$ such that z belongs to one of the two intervals $\tau^i_{\mathsf{sc}}(a)$ but not to the other. Without loss of generality, let then $z \in \tau^1_{\mathsf{sc}}(a)$, $z \notin \tau^2_{\mathsf{sc}}(a)$ (see Fig. 17), that is, by letting $t_1 = s_1(a)$, $t_2 = s_2(a)$,

$$|z - t_1| \leq w^1_{\mathsf{sc}}(a), \qquad |z - t_2| \geq w^2_{\mathsf{sc}}(a), \tag{92}$$

from which,

$$|z - t_1| \leq w^{\max}_{\mathsf{sc}}(a), \qquad |z - t_2| \geq w^{\min}_{\mathsf{sc}}(a). \tag{93}$$

Since $|t_1 - t_2| < S_\Delta$ by assumption, (93) yields, by letting $w_1 = w^{\min}_{\mathsf{sc}}(a)$, $w_2 = w^{\max}_{\mathsf{sc}}(a)$,

$$\begin{array}{rcccl} w_1 - S_\Delta & < & |z - t_1| & \leq & w_2, \\ w_1 & \leq & |z - t_2| & < & w_2 + S_\Delta. \end{array} \tag{94}$$

Let's introduce the two functions $\delta_i(s') = |s' - t_i|$, $i = 1, 2$. We have

$$w_1 - S_\Delta < \delta_1(z) \leq w_2, \tag{95}$$

$$w_1 \leq \delta_2(z) < w_2 + S_\Delta. \tag{96}$$

Notice that

$$|\delta_i(x) - \delta_i(x')| < |x - x'|, \qquad \forall x, x' \in \mathbb{R}, \quad i = 1, 2, \tag{97}$$

Also, since $|t_1 - t_2| < S_\Delta$,

$$|\delta_1(x) - \delta_2(x)| < S_\Delta, \qquad \forall x \in \mathbb{R}. \tag{98}$$

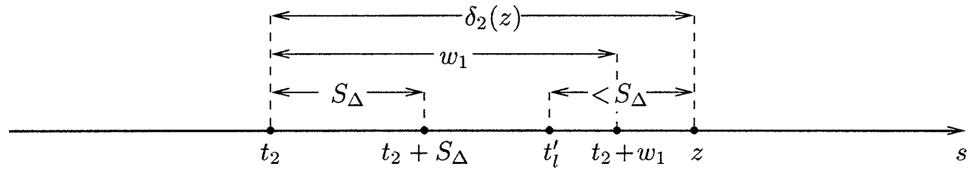


Figure 18: Lemma 3: existence of t'_l such that $\delta_2(t'_l) \le \delta_2(z)$.

To prove (90) let's fix, $t'_k \in \{t'_1, t'_2\} = \{s_1(a'), s_2(a')\}$. Since $|t'_1 - t'_2| < S_\Delta$ and $z \in \sigma_{\mathsf{sc}}(a') = [t'_1, t'_2]$, we have $|t'_k - z| < S_\Delta$ and therefore, from (97),

$$|\delta_1(t'_k) - \delta_1(z)| \;<\; S_\Delta, \tag{99}$$

$$|\delta_2(t'_k) - \delta_2(z)| \;<\; S_\Delta. \tag{100}$$

These, together with (95) and (96) yield

$$\delta_1(t'_k) \;<\; w_2 + S_\Delta, \tag{101}$$

$$w_1 - S_\Delta \;<\; \delta_2(t'_k). \tag{102}$$

These two inequalities and $|\delta_1(t'_k) - \delta_2(t'_k)| < S_\Delta$ (which comes from (98)), imply that $\delta_1(t'_k)$ and $\delta_2(t'_k)$ can not be both outside the interval $[w_1 - S_\Delta, w_2 + S_\Delta]$, which proves (90).

To prove the second part, let's fix $t_k \in \{t_1, t_2\} = \{s_1(a), s_2(a)\}$. If $k = 1$, then let t'_l be the point in $\{t'_1, t'_2\}$ where the function δ_1 is *greater* or equal to $\delta_1(z)$ (one of the two points has this property because $z \in [t'_1, t'_2]$; notice that t'_l is the point in $\{t'_1, t'_2\}$ furthest away from t_1). Thus, $\delta_1(z) \le \delta_1(t'_l) < \delta_1(z) + S_\Delta$ (the second inequality follows from (97) and $|z - t'_l| \le |t_1 - t_2| < S_\Delta$). Then, from (95) we have that $\delta_1(t'_l)$ is in the interval $[w_1 - S_\Delta, w_2 + S_\Delta]$.

Let now $k = 2$. From (96) and (6) we have $\delta_2(z) = |z - t_2| \ge w_1 > S_\Delta$. Thus, since $z \in [t'_1, t'_2]$ and $|t'_1 - t'_2| < S_\Delta$, by moving on the real axis from z towards t_2 so that δ_2 decreases, either t'_1 or t'_2 is reached before t_2. Let t'_l be this point. Clearly, $\delta_2(t'_l) \le \delta_2(z)$. Hence, from $z \in [t'_1, t'_2]$ and $|t'_1 - t'_2| < S_\Delta$, it follows that $\delta_2(z) - S_\Delta < \delta_2(t'_l) \le \delta_2(z)$. Thus, from (96), $\delta_2(t'_l) \in [w_1 - S_\Delta, w_2 + S_\Delta]$. $\square$

Corollary 3 *If* (88) *and* (89) *hold and* $a' \nparallel_{\mathsf{sc}} a \vee a \nparallel_{\mathsf{sc}} a'$ *then*

$$\forall i \in \{1,2\}, \exists j \in \{1,2\}, \quad |s_j(a') - s_i(a)| \in [w_{\mathsf{sc}}^{\min}(a,a') - S_\Delta, w_{\mathsf{sc}}^{\max}(a,a') + S_\Delta].$$

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