November 1979 LIDS-R-958

THE STOCHASTIC ANALYSIS OF DYNAMIC SYSTEMS MOVING THROUGH RANDOM FIELDS*

by

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ABSTRACT

In this paper we consider dynamic systems that move along specified trajectories across random fields, where the field acts as a driving force to the dynamic system. For a specific class of random fields we develop equations for the evolution of the covariance of the state of the dynamic system, and in the special case in which the trajectory is a straight line path followed by a 180° turn (i.e. an "over and back" trajectory) we develop a Markovian model that involves a change in the dimension of the state after the turn. For this case we also discuss the estimation problem using recently developed results on "real-time smoothing."

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* This work was performed in part at the M.I.T. Laboratory for Information and Decision Systems, and at The Analytic Sciences Corporation, Reading, Mass.

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I. Introduction

The problem we consider in this paper is depicted in Figure 1.1. We have an object that traverses a specified trajectory $(\eta_1(t), \eta_2(t))$ over a planar surface. Aboard this object is a dynamic system which is affected by a random field $f(\eta_1, \eta_2)$. We would like to consider the statistical description of the state $x(t)$ of the system in terms of the specified trajectory and the statistical description of the random field. Problems of this general type arise in applications such as inertial navigation [1,2] where f represents the errors in our knowledge of the variations in gravity and $x(t)$ consists of the errors in an inertial navigation system. Since the inertial system's accelerometers measure actual acceleration plus gravity, an estimate of gravity, from a gravity map of some sort, must be subtracted from the accelerometer outputs. Thus map errors directly drive the dynamics of the navigation system.

In the next section we develop equations for the evolution of the covariance of $x(t)$ for a particular class of random fields. For the special case of a straight line trajectory that reverses on itself, we develop in Section III a novel Markovian representation for the process $x(t)$, and in Section IV we use this representation, together with recent results on the real-time updating of smoothed estimates, to solve an estimation problem.

II. Covariance Analysis for Motion Through a Two-Dimensional Random Field

Let $f(\eta_1, \eta_2)$ be a two-dimensional stationary Gaussian random field which for simplicity we assume to be zero mean. The correlation matrix for this field is

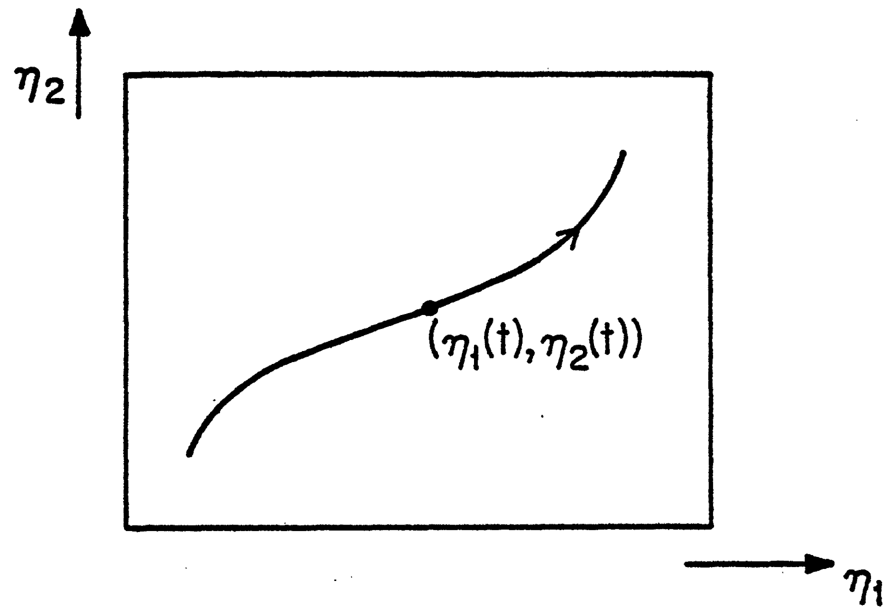

FIGURE 1.1:

$$E[f(t,s)f'(0,0)] = R(t,s) \tag{2.1}$$

It is easily seen from (2.1) that

$$R(t,s) = R'(-t,-s) \tag{2.2}$$

Let $(\eta_1(t),\eta_2(t))$ be a specified trajectory through the plane and consider a dynamic system driven by the field along the trajectory

$$\dot{x}(t) = Ax(t) + f(\eta_1(t),\eta_2(t)) + w(t) \tag{2.3}$$

where $w(t)$ is a zero mean white Gaussian process with

$$E[w(t)w'(\tau)] = Q\delta(t-\tau) \tag{2.4}$$

We assume that the initial condition $x(0)$ is zero mean and Gaussian and that $x(0)$, w, and f are mutually independent. We would like to determine the evolution of

$$P(t) = E[x(t)x'(t)] \tag{2.5}$$

We will put further restrictions on the field f that, as we will see, lead to $P(t)$ being specified by a finite set of matrix differential equations. Specifically we will assume that the covariance R is separable

$$R(t,\tau) = R_1(t)R_2(s) \tag{2.6}$$

where we assume that R_1 and R_2 are square and that

$$R_1(t) = R_1'(-t) \ , \qquad R_2(s) = R_2'(-s) \tag{2.7}$$

and

$$R_i(t) = H_i e^{F_i t} G_i \;, \quad t \geq 0, \quad i=1,2 \tag{2.8}$$

From (2.1), (2.6) and (2.7) we can also deduce that R_1 and R_2 commute for any values of their arguments. This model is the continuous-time analog of the model examined by Attasi [3]. Specifically, the 2-D spectrum of f is separable and rational.

As a first step in obtaining the desired equations for P(t), define

$$Q(t,s) = R(\eta_1(t)-\eta_1(s),\; \eta_2(t)-\eta_2(s)) \tag{2.9}$$

Then, writing

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}[f(\eta_1(\tau),\eta_2(\tau))+w(\tau)]d\tau \tag{2.10}$$

we can obtain and expression for P(t) from (2.5). Differentiating we obtain the basic equations

$$\dot{P}(t) = AP(t) + P(t)A' + L(t) + L'(t) \tag{2.11}$$

$$L(t) = \int_0^t Q(t,\tau)e^{A'(t-\tau)}d\tau \tag{2.12}$$

The problem then becomes one of determining a set of differential equations for L(t). This calculation depends upon the nature of the trajectory. There are several cases to be examined. For simplicity, we will assume throughout that $\eta_1(0) = \eta_2(0)=0$.

Case 1: This is the simplest case in which we don't change quadrants in which we're heading. That is, if we choose the northeast as the direction of motion, we have the situation depicted in Figure 2.1a where

$$\begin{aligned} \eta_1(t) - \eta_1(s) &\geq 0 \\ \eta_2(t) - \eta_1(s) &\geq 0 \end{aligned} \qquad \forall t \geq s \tag{2.13}$$

In this case, using (2.6)-(2.9) we find that (2.12) can be written as

$$\begin{aligned} L(t) &= \int_0^t H_1 e^{F_1(\eta_1(t)-\eta_1(\sigma))} G_1 H_2 e^{F_2(\eta_2(t)-\eta_2(\sigma))} G_2 e^{A'(t-\sigma)} d\sigma \\ &\triangleq H_1 B_1(t) \end{aligned} \tag{2.14}$$

Differentiating $B_1(t)$, we obtain

$$\begin{aligned} \dot{B}_1(t) &= \dot{\eta}_1(t) F_1 B_1(t) + B_1 A' + G_1 H_2 G_2 \\ &\quad + \dot{\eta}_2(t) \int_0^t e^{F_1(\eta_1(t)-\eta_1(\sigma))} G_1 H_2 F_2 e^{F_2(\eta_2(t)-\eta_2(\sigma))} A(t-\sigma) d\sigma \end{aligned} \tag{2.15}$$

Note the F_2 factor in the middle of the last term of (2.15). This leads to the following. Define

$$B_j(t) = \int_0^t e^{F_1(\eta_1(t)-\eta_1(\sigma))} G_1 H_2 F_2^{j-1} e^{F_2(\eta_2(t)-\eta_2(\sigma))} G_2 e^{A'(t-\sigma)} d\sigma \tag{2.16}$$

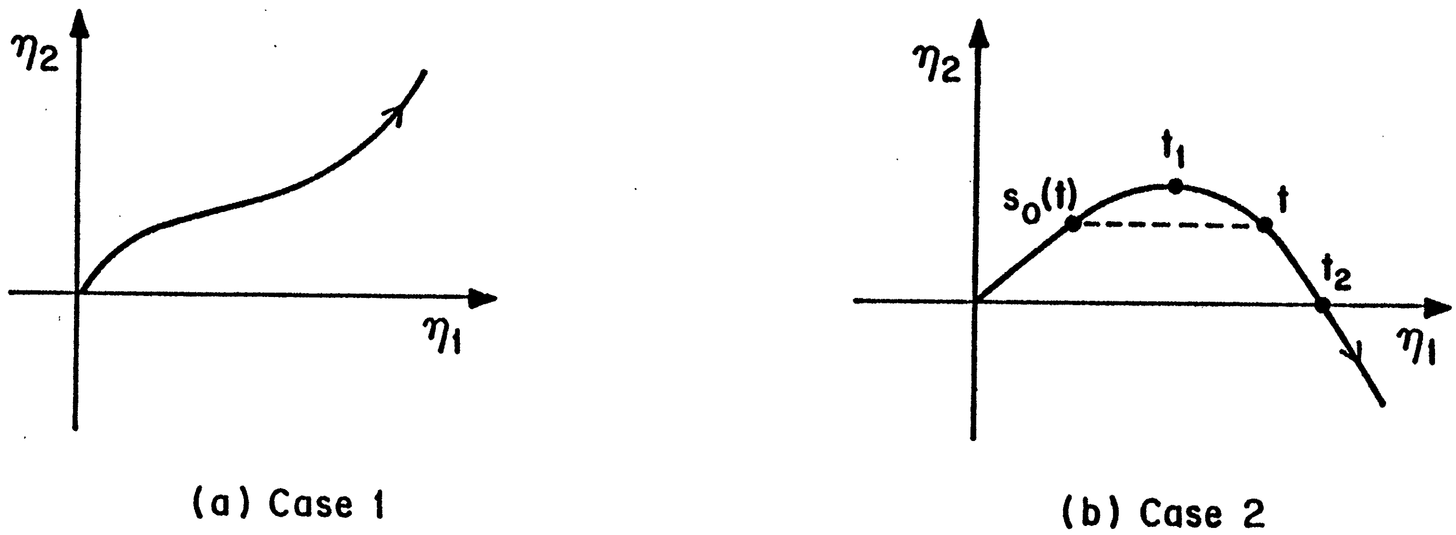

FIGURE 2.1: Two Simple Trajectory Types.

We know that there is an integer r and coefficients $\rho_0, \ldots, \rho_{r-1}$ such that

$$F_2^r = \sum_{j=0}^{r-1} \rho_j F_2^j \tag{2.17}$$

Then

$$L(t) = H_1 B_1(t)$$

$$\dot{B}_j(t) = \dot{\eta}_1(t) F_1 B_j(t) + B_j(t) A' + \dot{\eta}_2(t) B_{j+1}(t) + G_1 H_2 F_2^{j-1} G_2$$

$$1 \le j \le r-1 \tag{2.19}$$

$$\dot{B}_r(t) = \dot{\eta}_1(t) F_1 B_r(t) + B_r(t) A' + \dot{\eta}_2(t) \sum_{j=1}^{r} \rho_{j-1} B_j(t) + G_1 H_2 F_2^{r-1} G_2 \tag{2.20}$$

$$B_j(0) = 0 \;, \qquad j = 1, \ldots, r \tag{2.21}$$

Note that we can obtain analogous equations with the roles of F_1 and F_2 reversed if we use the commutativity of $R_1(t)$ and $R_2(t)$. Thus in this case we obtain a finite set of linear matrix differential equations for L and therefore for P. Note that if the trajectory is a straight line -- i.e. $\dot{\eta}_1(t)=\alpha$, $\dot{\eta}_2(t)=\beta$ -- then these equations are time-invariant and are equivalent to one higher-dimensional Lyapunov equation for x and for the state of a shaping filter for f along this line.

Case 2: In this case, illustrated in Figure 2.1b, we have a change of quadrants from northeast to southeast. Clearly the following analysis also holds for any turn from one quadrant into an adjacent one. Mathematically,

$$\begin{aligned} &\eta_1(t) - \eta_1(s) \geq 0 && t \geq s \\ &\eta_2(t) - \eta_2(s) \geq 0 && s \leq s_0(t) \\ &\eta_2(t) - \eta_2(s) \leq 0 && s \geq s_0(t) \end{aligned} \tag{2.22}$$

where $s_0(t)$ is defined in the figure. Here t_1 is the time at which our turn takes us into another quadrant in direction, and t_2 is the time at which $s_0(t)=0$.

For $t \leq t_1$, the analysis of this case is identical to that for Case 1. Thus, consider $t_1 \leq t \leq t_2$ and let us break up the integral expression for $L(t)$:

$$L(t) = \int_0^{s_0(t)} H_1 e^{F_1(\eta_1(t)-\eta_1(s))} G_1 H_2 e^{F_2(\eta_2(t)-\eta_2(s))} G_2 e^{A'(t-s)} ds$$

$$+ \int_{s_0(t)}^{t} H_1 e^{F_1(\eta_1(t)-\eta_1(s))} G_1 G_2' e^{F_2'(\eta_2(s)-\eta_2(t))} H_2' e^{A'(t-s)} ds \tag{2.23}$$

where we have used the fact that $R_2(\eta_2(t)-\eta_2(s)) = R_2'(\eta_2(s)-\eta_2(t))$. In differentiating (2.23) we will need to calculate $\dot{s}_0(t)$. This can be done as follows. By definition

$$\eta_2(s_0(t)) = \eta_2(t) \tag{2.24}$$

Therefore

$$\dot{\eta}_2(s_0(t))\dot{s}_0(t) = \dot{\eta}_2(t) \tag{2.25}$$

or

$$\dot{s}_0(t) = \frac{\eta_2(t)}{\dot{\eta}_2(s_0(t))} \tag{2.26}$$

Note that if $\dot{\eta}(s_0(t))=0$, as it is in Figure 2.2, we will have to evaluate higher derivatives. This causes no conceptual difficulty but it simply complicates the development. Therefore we will assume for simplicity that there are no inflection points in the trajectory over the interval $[0,t_1)$.

Let

$$B_j(t) = \int_0^{s_0(t)} e^{F_1(\eta_1(t)-\eta_1(s))} G_1H_2F_2^{j-1} e^{F_2(\eta_2(t)-\eta_2(s))} G_2 e^{A'(t-s)} ds$$

$$j=1,\ldots,r \tag{2.27}$$

Note that if we define $s_0(t)=t$ for $t<t_1$, then B_j is precisely the quantity in equation (2.16) and thus the initial condition at time t_1 for B_j in (2.27) is $B_j(t_1)$ calculated from (2.19)-(2.21). If we now differentiate (2.27) and use (2.17) we find

$$\dot{B}_j(t) = \dot{\eta}_1(t)F_1B_j(t) + B_j(t)A' + \dot{\eta}_2(t)B_{j+1}(t)$$
$$+ \frac{\dot{\eta}_2(t)}{\dot{\eta}_2(s_0(t))} e^{F_1[\eta_1(t)-\eta_1(s_0(t))]} G_1H_2F_2^{j-1}Ge^{A'(t-s_0(t))}$$

$$j=1,\ldots,r-1 \tag{2.28}$$

$$\dot{B}_r(t) = \dot{\eta}_1(t)F_1B_r(t) + B_r(t)A' + \dot{\eta}_2(t) \sum_{j=1}^{r} \rho_{j-1}B_j(t)$$
$$+ \frac{\dot{\eta}_2(t)}{\dot{\eta}_2(s_0(t))} e^{F_1[\eta_1(t)-\eta_1(s_0(t))]} G_1H_2F_2^{r-1}G_2e^{A'(t-s_0(t))} \tag{2.29}$$

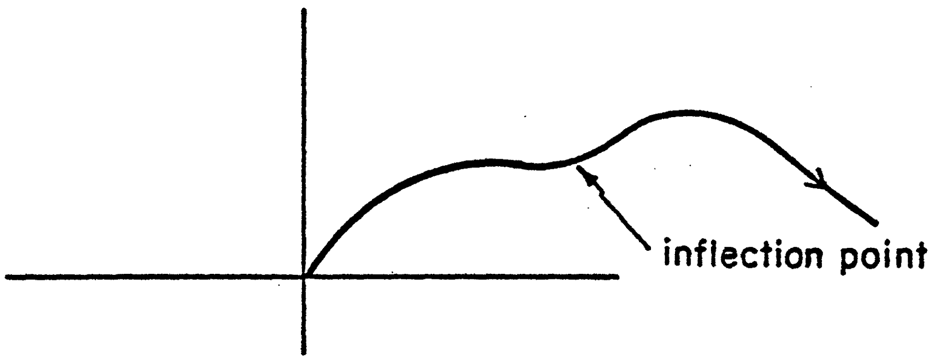

FIGURE 2.2:

Note that

$$B_j(t_2)=0 \tag{2.30}$$

since $s_0(t_2)=0$.

Now let

$$C_j(t) = \int_{s_0(t)}^{t} e^{F_1(\eta_1(t)-\eta_1(s))} G_1G_2'(F_2')^{j-1} e^{F_2'(\eta_2(s)-\eta_2(t))} H_2' e^{A'(t-s)} ds \tag{2.31}$$

Then

$$\dot{C}_j(t) = \dot{\eta}_1(t)F_1C_j(t) + C_j(t)A' - \dot{\eta}_2(t)C_{j+1}(t) + G_1G_2(F_2')^{j-1}H_2' - \frac{\dot{\eta}_2(t)}{\dot{\eta}_2(s_0(t))} e^{F_1(\eta_1(t)-\eta_1(s_0(t)))} G_1G_2'(F_2')^{j-1}H_2e^{A'(t-s_0(t))} \tag{2.32}$$

$$\dot{C}_r(t) = \dot{\eta}_1(t)F_1C_r(t) + C_r(t)A' - \dot{\eta}_2(t) \sum_{j=1}^{r} \rho_{j-1}C_j(t) + G_1G_2'(F_2')^{r-1}H_2' - \frac{\dot{\eta}_2(t)}{\dot{\eta}_2(s_0(t))} e^{F_1(\eta_1(t)-\eta_1(s_0(t)))} G_1G_2'(F_2')^{r-1}H_2e^{A'(t-s_0(t))} \tag{2.33}$$

$$C_j(t_1)=0 \quad , \qquad j=1,\ldots r \tag{2.34}$$

Then

$$L(t) = H_1[B_1(t) + C_1(t)] \qquad t_1\leq t\leq t_2 \tag{2.35}$$

Note that in the case of a piecewise linear trajectory, such as

$$\dot{\eta}_1(t) = \begin{cases} \alpha_1 & t<t_1 \\ \alpha_2 & t>t_1 \end{cases} \tag{2.36}$$

$$\dot{\eta}_2(t) = \begin{cases} \beta_1 & t<t_1 \\ \beta_2 & t>t_1 \end{cases} \tag{2.37}$$

$$\dot{s}_0(t) = \frac{\beta_2}{\beta_1} \tag{2.38}$$

which is negative here since $\beta_1>0$, $\beta_2<0$.

We now need only piece together the situation for $t>t_2$. In this case

$$L(t) = \int_0^t H_1 e^{F_1(\eta_1(t)-\eta_1(s))} G_1 G_2' e^{F_2'(\eta_2(s)-\eta_2(t))} H_2' e^{A'(t-s)} ds \tag{2.39}$$

Thus in this region

$$L(t) = H_1 C_1(t) \qquad t \geq t_2 \tag{2.40}$$

where $C_1(t_2)$ is obtained from (2.32)-(2.34), and, for $t>t_2$

$$\dot{C}_j(t) = \dot{\eta}_1(t) F_1 C_j(t) + C_j(t) A' - \dot{\eta}_2(t) C_{j+1}(t) + G_1 G_2' (F_2')^{j-1} H_2'$$

$$j=1,\ldots r-1 \tag{2.41}$$

$$\dot{C}_r(t) = \dot{\eta}_1(t)F_1C_r(t) + C_r(t)A' - \dot{\eta}_2(t) \sum_{j=1}^{r} \rho_{j-1}C_j(t)$$
$$+ G_1G_2'(F_2')^{r-1}H_2' \qquad (2.42)$$

Thus in Case 2, over the time interval $[0,t_1]$ we have one set of equations to calculate L; over the interval $[t_1,t_2]$, while the η_2 coordinate of the trajectory retraces its path over $[0,t_1]$, we have two sets of equations; and for $t>t_2$ we are back to a single equation which is essentially the equation obtained in Case 1, except here we have a southeasterly trajectory as opposed to the northeasterly trajectory of Figure 2.1a. It should be clear that we can do this for arbitrary trajectories. Only during the "transient" of a turn do we pick up additional equations. In the Appendix we describe one somewhat more complex case in which both η_1 and η_2 coordinates simultaneously retrace previous values (this does not mean a trajectory that retraces itself -- see Figure A.1). In that case, there are two additional sets of equations. From the cases considered in this section and in the Appendix it is not difficult to see that at any time we must include m additional sets of equations, where m is the total number of previous times in the trajectory that either the η_1 or η_2 coordinate of the trajectory equals the corresponding coordinate at the present time. If A is a stable matrix, then the effect on L(t) (and hence P(t)) of a trajectory turn far in the past becomes insignificant. This can be seen in (2.28) where the driving term goes to zero exponentially as $t-s_0(t) \to \infty$ (the matrices F_1 and F_2 are stable since f is a stationary process with finite covariance). Thus in practice we need only keep track of turns within a certain number of time constants of A and correlation distances of the field (inverses of the magnitudes of the eigenvalues of F_1 and F_2).

III. Markov-Type Models for Over-and-Back Trajectories

In the preceding section and in the appendix we performed some relatively straightforward calculations to obtain sets of differential equations for the propagation of the covariance of the state of a dynamic system moving through a random field. The primary contribution of that analysis is to provide some understanding of how the geometry of the trajectory affects the state covariance. While this is of some use, there is still a great deal left to understand about the fundamental way in which the uncertainty in the field affects the statistics of the process x(t). In this section we will develop a Markov-like description for the special case of over-and-back trajectories. This not only provides us with further insight into the evolution of x(t) but it also forms a basis for solving estimation problems of this type, a topic which is considered in the next section.

The case that we will examine in this and in the next section involves a trajectory consisting of a straight line path followed by a reversal of direction and a return trajectory over the same path. We also will assume a constant velocity (normalized to 1) over both segments of the path, but this assumptions is made only for clarity in our exposition as is our assumption that the dynamic system is time-invariant. It should be clear from our analysis as to how our results can be modified to account for nonuniform velocity and time-varying systems.

Consider the model

$$\dot{x}(t) = Ax(t) + u(t) + w_1(t), \qquad 0 \le t \le 2T \tag{3.1}$$

where w_1 is a white noise process with

$$E[w_1(t)w_1'(\tau)] = S_1\delta(t-\tau) \tag{3.2}$$

and where

$$u(t) = \begin{cases} f(t) & 0 \le t \le T \\ f(2T-t) & T \le t \le 2T \end{cases} \tag{3.3}$$

Here f is a one-dimensional process (representing the field along the track, and we assume the f can be modeled as the output of a finite-dimensional shaping filter

$$\dot{\xi}(t) = F\xi(t) + w_2(t) \qquad 0 \le t \le T \tag{3.4}$$

$$f(t) = H\xi(t) \tag{3.5}$$

where w_2 is white noise, with

$$E[w_2(t)w_2'(\tau)] = S_2\delta(t-\tau) \tag{3.6}$$

We assume that all of the processes above are zero mean and Gaussian and that $x(0)$, w_1, $\xi(0)$, and w_2 are mutually independent.

For $0 \le t \le T$ we have the same situation as in Case 1 considered in the preceding section. Over this time interval, while we are going forward, the joint process

$$x_2(t) = \begin{bmatrix} \xi(t) \\ x(t) \end{bmatrix} \tag{3.7}$$

is Markovian, with the following state equation

$$\dot{x}_2(t) = \begin{bmatrix} F & 0 \\ H & A \end{bmatrix} x_2(t) + \begin{bmatrix} w_2(t) \\ w_1(t) \end{bmatrix} \tag{3.8}$$

$$0 \le t \le T \quad \text{(forward)}$$

The meaning of the notation (forward) in (3.8) will become clear shortly. Thus the covariance $\Sigma_2(t)$ of $x_2(t)$ can be obtained from the differential equation

$$\dot{\Sigma}_2(t) = \begin{bmatrix} F & 0 \\ H & A \end{bmatrix} \Sigma_2(t) + \Sigma_2(t) \begin{bmatrix} F' & H' \\ 0 & A' \end{bmatrix} + \begin{bmatrix} S_2 & 0 \\ 0 & S_1 \end{bmatrix} \tag{3.9}$$

As we saw in Case 1, we needed one additional set of equations in order to calculate the covariance of $x(t)$. Here we see that that set of equations essentially comes about by augmenting the state $x(t)$ with a shaping filter model for the field in order to obtain a process that is Markovian. Once that is done, as in (3.8), we can use standard results to write down the covariance equation (3.9).

The interesting part of this analysis occurs over the time interval $T \le t \le 2T$, since here we are reversing over the same sample path of ξ. Again our goal is to augment $x(t)$ with something in order to obtain a Markov model over this time interval. In order to do this we clearly must consider a model for ξ that runs in reverse. Using the results in [4] we can write a reverse time model for the augmented process $x_2(t)$:

$$-\dot{x}_2(t) = -\left\{\begin{bmatrix} F & 0 \\ H & A \end{bmatrix} + \begin{bmatrix} S_2 & 0 \\ 0 & S_1 \end{bmatrix} \Sigma_2^{-1}(t)\right\} x_2(t) - \begin{bmatrix} \tilde{w}_2(t) \\ \tilde{w}_1(t) \end{bmatrix}$$

$0 \le t \le T$ (backward) (3.10)

Here $(\tilde{w}_2'(t), \tilde{w}_1'(t))$ is a white noise process backward in time independent of $x_2'(T) = (\xi'(T), x'(T))$.* The processes $\tilde{w}_i$ have the same statistics as the w_i. Note one interesting aspect of this model. If we examine (3.8) or (3.4) we see that ξ is a Markov process by itself forward in time -- i.e. it is decoupled from $x(t)$. However this is not true in the reverse-time model (3.10) since Σ_2^{-1} is not block diagonal. The reason is that going forward in time the process ξ drives the process x. Then, since the reverse process is a Markovian representation of x_2 given its future, we should expect to see coupling, since the present value of ξ is certainly not independent of the future of x.

* The use of white noise here makes our derivation somewhat informal. However it is conceptually correct. To be precise we should replace w_i by $d\beta_i$ where the β_i are Brownian motion processe forward in time. Then $\tilde{w}_i$ is replaced by $d\tilde{\beta}_i$ which is a Brownian motion backward in time, independent of $x_2'(T) = x_2'(T) = (\xi'(T), x_1'(T))$. This process is obtained by subtracting from $d\beta_i$ that part which is predictable given the future of x_2. See [4] for details.

If we now let

$$x_3(t) = x_2(2T-t),\ \eta_i(t) = \tilde{w}_i(2T-t) \tag{3.11}$$

we obtain a model <u>forward</u> in time over the time interval $T \leq t \leq 2T$:

$$\dot{x}_3(t) = -\left\{\begin{bmatrix} F & 0 \\ H & A \end{bmatrix} + \begin{bmatrix} S_2 & 0 \\ 0 & S_1 \end{bmatrix} \Sigma_2^{-1}(t)\right\} x_3(t) + \begin{bmatrix} \eta_2(t) \\ \eta_1(t) \end{bmatrix}$$

$$T \leq t \leq 2T \text{ (forward)} \tag{3.12}$$

The initial condition for this process is $x_3(T) = x_2(T)$, with covariance $\Sigma_2(t)$.

Consider now the following augmented process over the time interval $T \leq t \leq 2T$:

$$x_4(t) = \begin{bmatrix} x_3(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} \xi(2T-t) \\ x(2T-t) \\ x(t) \end{bmatrix} \tag{3.13}$$

Then, using (3.1), (3.3), (3.5), and (3.11) we obtain a <u>Markovian</u> representation for the behavior of this augmented state

$$\dot{x}_4 = \left[\begin{array}{c:c} -\left\{\begin{pmatrix} F & 0 \\ H & A \end{pmatrix} + \begin{pmatrix} S_2 & 0 \\ 0 & S_1 \end{pmatrix} \Sigma_2^{-1}(t)\right\} & 0 \\ & \\ (0,\ H) & A \end{array}\right] x_4(t) + \begin{pmatrix} \eta_2(t) \\ \eta_1(t) \\ w_1(t) \end{pmatrix} \tag{3.14}$$

where

$$E\begin{bmatrix}\eta_2(t)\\ \eta_1(t)\\ w_1(t)\end{bmatrix}\left[\eta_2'(\tau),\eta_1'(\tau),w_1'(\tau)\right] = \begin{bmatrix}S_2 & 0 & 0\\ 0 & S_1 & 0\\ 0 & 0 & S_1\end{bmatrix}\delta(t-\tau)$$

Basically (3.14) describes a method, starting at $t=T$, for simultaneously generating the future ($t>T$) of $x(t)$ and its past ($t<T$). In this fashion we can take into account the fact that the trajectory has reversed its direction.

We can use (3.14) as the basis for determining the covariance for $x(t)$. Specifically, define $N(t)$ as

$$N(t) = \begin{bmatrix} -\left\{\begin{bmatrix}F & 0\\ H & A\end{bmatrix}+\begin{bmatrix}S_2 & 0\\ 0 & S_1\end{bmatrix}\Sigma_2^{-1}(t)\right\} & 0\\ (0,\ H) & A\end{bmatrix} \tag{3.15}$$

Then, letting $\Sigma_4(t)$ denote the covariance of $x_4(t)$, we obtain

$$\dot{\Sigma}_4(t) = N(t)\Sigma_4(t) + \Sigma_4(t)N'(t) + \begin{bmatrix}S_2 & 0 & 0\\ 0 & S_1 & 0\\ 0 & 0 & S_1\end{bmatrix} \tag{3.16}$$

To obtain the initial condition for this equation, note that

$$x_4(T) = \begin{bmatrix}\xi(T)\\ x(T)\\ x(T)\end{bmatrix} \tag{3.17}$$

Thus, if we write

$$\sum_2(T) = \begin{pmatrix} \left(\sum_2(T)\right)_{11} & \left(\sum_2(T)\right)_{12} \\ \left(\sum_2(T)\right)_{12}' & \left(\sum_2(T)\right)_{22} \end{pmatrix} \tag{3.18}$$

then

$$\sum_4(T) = \begin{pmatrix} \left(\sum_2(T)\right)_{11} & \left(\sum_2(T)\right)_{12} & \left(\sum_2(T)\right)_{12} \\ \left(\sum_2(T)\right)_{12}' & \left(\sum_2(T)\right)_{22} & \left(\sum_2(T)\right)_{22} \\ \left(\sum_2(T)\right)_{12}' & \left(\sum_2(T)\right)_{22} & \left(\sum_2(T)\right)_{22} \end{pmatrix} \tag{3.19}$$

Thus we see that, as in Case II, a reversal of motion leads to an additional equation. Also, one can regard the over-and-back example as a degenerate form of Cases b and c which are examined in the Appendix (referring to the notation in the Appendix, in the over-and-back case $s_1(t) = s_2(t)$ for all t and $t_x = t_y$). Thus the straightforward analysis of the Appendix will lead to equivalent equations in this case.

Note that based on the understanding gained in this and in the preceding section, we can see what will happen for more general over-and-back trajectories. For example, as illustrated in Figure 4.1a, consider the case in which we continue the process for $t \geq 2T$ without any further change in course. It is not difficulty to show that for $t \geq 2T$ we can once again obtain a Markovian representation for the joint process

$$x_2(t) = \begin{bmatrix} \xi(t) \\ x(t) \end{bmatrix}$$

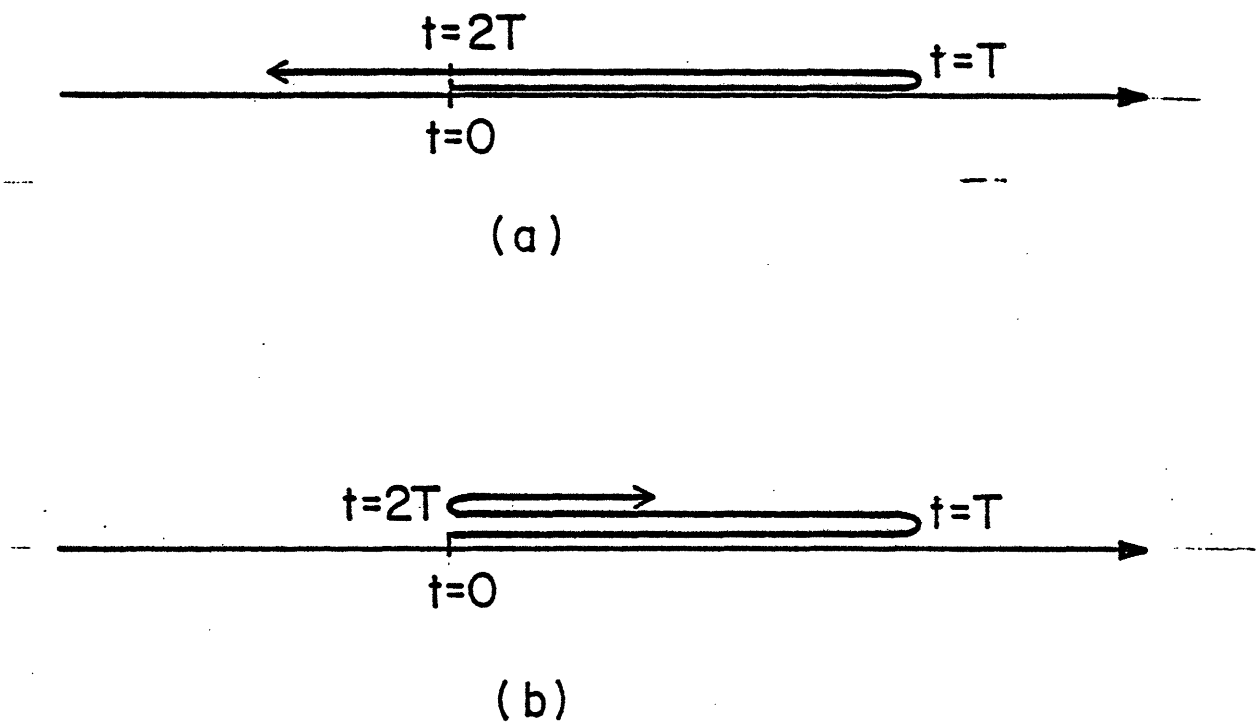

FIGURE 4.1: Two Over-and-Bask Trajectories.

where the initial covariance $\sum_2(2T)$ for this process at time 2T is obtained from the solution to (3.16)

$$\sum_2(2T) = \begin{bmatrix} \left(\sum_4(2T)\right)_{11} & \left(\sum_4(2T)\right)_{13} \\ \left(\sum_4(2T)\right)_{13}' & \left(\sum_4(2T)\right)_{33} \end{bmatrix} \tag{3.20}$$

In this case the time period [T,2T] represents a transient due to the turn, whose effective will become negligible if A is stable. In fact in this case as $t\to\infty$, x will achieve the same steady-state covariance in this situation as one would from a trajectory that moves to the left for $t>0$ without any turns. Similarly, if we consider a second course reversal as in Figure 4.1b, we must obtain a reverse time model for x_4, reverse time once again to obtain an equation for $x_5(t) = x_4(4T-t)$ and augment this with $x(t)$ to obtain a Markovian model over the time period $2T\leq t\leq 3T$. Thus in this case we obtain another additional equation for the covariance evolution.

IV. Over-and-Back Estimation

In this section we consider the problem of estimating the process described by (3.1)-(3.6) given measurements. Specifically, suppose we assume that the random field has been mapped by a previous survey

$$y_1(t) = C_1\xi(t) + v_1(t), \qquad 0\leq t\leq T \tag{4.1}$$

where $E[v_1(t)v_1(\tau)] = R_1\delta(t-\tau)$, so that we have the smoothed estimates

$$\hat{\xi}_s(t) = E[\xi(t)|\ y_1(\tau),\ 0\leq\tau\leq T] \tag{4.2}$$

Consider now a set of real-time measurements

$$z(t) = C_2x(t) + v_2(t), \qquad 0\leq t\leq 2T \tag{4.3}$$

where $E[v_2(t)v_2(t)] = R_2\delta(t-\tau)$ and v_1 and v_2 are independent. We wish to consider the problem of using the previously mapped information (4.2), together with the new data (4.3) to estimate $x(t)$.

As in the preceding section, this problem is best analyzed by considering the two intervals [0.T] and [T,2T] separately. Thus, let

$$y_2(t) = z(t),\ t\in[0,T],\ y_3(t) = z(t),\ t\in[T,2T] \tag{4.4}$$

The problem over the first time interval is a real-time smoothing problem, that is, we have smoothed estimates for part of the state (here the ξ part of x_2 as given in (3.7) and (3.8)) from previously taken data, and wish to incorporate these estimates into an overall state estimate given new real-time data. A real-time smoothing problem of this type was solved in [5]. In order to apply that solution here, define

$$\hat{x}_2(t|T,\bar{t}) = E[x_2(t)|\ y_1(\tau),\ 0\leq\tau\leq T;\ y_2(\tau),\ 0\leq\tau\leq t] \tag{4.5}$$

Then

$$\hat{x}_2(t|T,t) = P_2(t)[P_f^{-1}(t)q_f(t) + P_b^{-1}(t)q_b(t)] + \begin{bmatrix}\hat{\xi}_s(t)\\ 0\end{bmatrix} \tag{4.6}$$

where

$$\dot{q}_f(t) = F_f(t)q_f(t) + \begin{bmatrix} 0 \\ H \end{bmatrix} \hat{\xi}_s(t) + P_f(t) \begin{bmatrix} 0 \\ C_2'R_2^{-1} \end{bmatrix} y_2(t) \qquad (4.7)$$

$$q_f(0)=0 \qquad (4.8)$$

$$-\dot{q}_b(t) = F_b(t)q_b(t) + (P_b(t) \Sigma_2^{-1}(t)-I) \begin{bmatrix} 0 \\ H \end{bmatrix} \hat{\xi}_s(t) \qquad (4.9)$$

$$q_b(T)=0 \qquad (4.10)$$

where $\Sigma_2(t)$ is the unconditional covariance of $x_2(t)$, as calculated from (3.9). The remaining quantities in (4.5)-(4.10) are deterministic and are determined from the equations

$$\dot{P}_f(t) = \begin{bmatrix} F & 0 \\ H & A \end{bmatrix} P_f(t) + P_f(t) \begin{bmatrix} F' & H' \\ 0 & A' \end{bmatrix} + \begin{bmatrix} S_2 & 0 \\ 0 & S_1 \end{bmatrix} - P_f(t) \begin{bmatrix} C_1R_1^{-1}C_1 & 0 \\ 0 & C_2R_2^{-1}C_2 \end{bmatrix} P_f(t) \qquad (4.11)$$

$$P_f(0) = \Sigma_2(0) \qquad (4.12)$$

$$-\dot{P}_b(t) = -\left\{ \begin{bmatrix} F & 0 \\ H & A \end{bmatrix} + \begin{bmatrix} S_2 & 0 \\ 0 & S_1 \end{bmatrix} \Sigma_2^{-1}(t) \right\} P_b(t) - P_b(t) \left\{ \begin{bmatrix} F' & H' \\ 0 & A' \end{bmatrix} + \Sigma_2^{-1}(t) \begin{bmatrix} S_2 & 0 \\ 0 & S_1 \end{bmatrix} \right\} + \begin{bmatrix} S_2 & 0 \\ 0 & S_1 \end{bmatrix} - P_b(t) \begin{bmatrix} C_1'R_1^{-1}C_1 & 0 \\ 0 & 0 \end{bmatrix} P_b(t) \qquad (4.13)$$

$$P_b(T) = \Sigma_2(T) \tag{4.14}$$

$$P_2^{-1}(t) = P_f^{-1}(t) + P_b^{-1}(t) - \Sigma_2^{-1}(t) \tag{4.15}$$

$$F_f(t) = \begin{bmatrix} F & 0 \\ H & A \end{bmatrix} - P_f(t) \begin{bmatrix} C_1'R_1^{-1}C_1 & 0 \\ 0 & C_2'R_2^{-1}C_2 \end{bmatrix} \tag{4.16}$$

$$F_b(t) = -\begin{bmatrix} F & 0 \\ H & A \end{bmatrix} - \begin{bmatrix} S_2 & 0 \\ 0 & S_1 \end{bmatrix} \Sigma_2^{-1}(t) - P_b(t) \begin{bmatrix} C_1'R_1^{-1}C_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{4.17}$$

Let us briefly interpret these equations. For a detailed discussion, we refer the reader to [5]. Here $P_f(t)$ is the estimation error covariance for $x_2(t)$ given $y_1(\tau)$ and $y_2(t)$, $\tau \leq t$ -- i.e. the causal estimate of $x_2(t)$ using only causal information from the previously collected data y_1 and the new data y_2. Similarly, $P_b(t)$ is the estimation error for a reverse-time filter which estimates $x_2(t)$ given only $y_1(\tau)$, $t \leq \tau \leq T$. The set of information used in these two estimates comprise all the information used in computing $\hat{x}_2(t|T,t)$. The forward filter (resulting in an estimate with covariance $P_f(t)$) is the usual Kalman filter and has the form

$$\dot{\hat{x}}_f(t) = F_f(t)\hat{x}_f(t) + P_f(t) \begin{bmatrix} C'R^{-1} & 0 & y_1(t) \\ 0 & C_2'R_2^{-1} & y_2(t) \end{bmatrix} \tag{4.18}$$

Similarly the backward Kalman filter, with estimation error covariance $P_b(t)$, has the form

$$-\dot{\hat{x}}_b(t) = F_b(t)\hat{x}_b(t) + P_b(t)\begin{bmatrix} C_1'R_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_1(t) \\ 0 \end{bmatrix} \quad (4.19)$$

The estimation error covariance for $\hat{x}_2(t|T,t)$ is $P_2(t)$, and the expression for it in (4.15) is taken from Wall [6]. In fact, using the smoothing equations in [6] and the result of [5], $\hat{x}_2(t|t,T)$ can be written as

$$\hat{x}_2(t|T,t) = P_2(t)[P_f^{-1}(t)\hat{x}_f(t) + P_b^{-1}(t)\hat{x}_b(t)] \quad (4.20)$$

The fact that $\sum_2^{-1}(t)$ is subtracted on the right-hand side of (4.15) reflects the fact that the estimates $\hat{x}_f(t)$ and $\hat{x}_b(t)$ of the state are correlated, as they both use the a priori information which has an uncertainty specified by the unconditional statistics of $x_2(t)$.

Finally, (4.6)-(4.10) are obtained from (4.18)-(4.19) by some manipulations aimed at replacing $y_1(t)$ in (4.18), (4.19) by the previously determined map $\hat{\xi}_s(t)$. The details of these calculations in a somewhat more general context can be found in [5]. Note that only $q_f(t)$ is driven by the new, real-time data $y_2(t)$, while $q_b(t)$ is a functional only of the smoothed map $\hat{\xi}_s(t)$ and in principal can be precomputed. In practice, what this means is that once the one-dimensional trajectory has been charted, we can integrate (4.9) backward along this trajectory, store the result, and combine it with the stored map $\hat{\xi}_s(t)$ and $q_f(t)$ through (4.6) in order to determine $\hat{x}_2(t|T,t)$.

Consider now the estimation problem over the time interval [T,2T]. This is again a real-time smoothing problem, thanks to our augmented

Markovian model (3.13), (3.14). Specifically using the data $\{y_1(\tau), y_2(\tau)\}$ we have computed

$$\hat{x}_{3s}(t) = E[x_3(t)|y_1(\tau), y_2(\tau), \quad 0\leq\tau\leq T]$$
$$= E[x_2(2T-t)\Big| y_1(\tau), y_2(\tau), \quad 0\leq\tau\leq T] = \hat{x}_{2s}(2T-t) \qquad (4.21)$$

The quantity that we wish to compute is

$$\hat{x}_4(t|T,T,t) = E[x_4(t)|y_1(\tau), \quad 0\leq\tau\leq T, \; y_2(\tau), \quad 0\leq\tau\leq T, \; y_3(\tau), \; T\leq\tau\leq t]$$
$$= E[x_4(t)|y_1(\tau), \quad 0\leq\tau\leq T, \; z(\tau), \quad 0\leq\tau\leq t] \qquad (4.22)$$

Using the real-time smoothing formulas of [5], we can express $\hat{x}_4(t|T,T,t)$ in terms of $\hat{x}_{3s}$ and y_3 for $t\in[T,2T]$. However, there is one complication caused by the fact that $\Sigma_4(T)$ is not invertible (see (3.19)). This is due to the fact that $x(2T-t)=x(t)$ for $t=T$. In order to make the necessary modifications, it is convenient to change basis. Let

$$\rho(t) = \begin{bmatrix} x_3(t) \\ \\ x(t)-x(2T-t) \end{bmatrix} = \begin{bmatrix} \xi(2T-t) \\ x(2T-t) \\ x(t)-x(2T-t) \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -I & I \end{bmatrix} x_4(t) \qquad (4.23)$$

(see (3.18)). Also, define

$$T(t) = -\left\{\begin{bmatrix} F & 0 \\ H & A \end{bmatrix} + \begin{bmatrix} S_2 & 0 \\ 0 & S_1 \end{bmatrix} \Sigma_2^{-1}(t)\right\} = \begin{bmatrix} T_{11}(t) & T_{12}(t) \\ T_{21}(t) & T_{22}(t) \end{bmatrix} \qquad (4.24)$$

Then, from (3.14), (4.23), and (4.24)

$$\dot{\rho}(t) = \left[\begin{array}{c:c} T(t) & 0 \\ (-T_{21}(t),\ H+A-T_{22}(t)) & A \end{array}\right]\rho(t) + \begin{bmatrix} \eta_2(t) \\ \eta_1(t) \\ w_1(t)-\eta_1(t) \end{bmatrix}$$

$$\triangleq D(t)\rho(t) + \beta(t) \tag{4.25}$$

and the unconditional covariance, $V(t)$, of $\rho(t)$ is given by

$$V(t) = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -I & I \end{bmatrix} \quad \Sigma_4(t) \begin{bmatrix} I & 0 & 0 \\ 0 & I & -I \\ 0 & 0 & I \end{bmatrix} \tag{4.26}$$

Thus, from (3.19)

$$V(T) = \begin{bmatrix} \left(\Sigma_2(T)\right)_{11} & \left(\Sigma_2(T)\right)_{12} & 0 \\ \left(\Sigma_2(T)\right)_{12}' & \left(\Sigma_2(T)\right)_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.27}$$

Further, let us assume that $x(t)$ as defined in (3.1) is controllable from the noise $w_1(t)$, i.e. that (A,S_1) is a completely controllable pair. In this case it is not difficult to see that for <u>any</u> $t>T$, $V(t)>0$. Then, defining

$$\hat{\rho}(t|T,T,t) = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -I & I \end{bmatrix} \hat{x}_4(t|T,T,t) \tag{4.28}$$

we can adapt the results in [5] to obtain

$$\hat{\rho}(t|T,T,t) = E(t)\left[M_f^{\dagger}(t)r_f(t) + M_b^{\dagger}(t)r_b(t)\right] + \begin{bmatrix} \hat{x}_{3s}(t) \\ 0 \end{bmatrix} \tag{4.29}$$

$$\dot{r}_f(t) = G_f(t)r_f(t) + \begin{bmatrix} 0 \\ (-T_{21}(t), H+A-T_{22}(t)) \end{bmatrix} \hat{x}_{3s}(t) + M_f(t)\begin{bmatrix} 0 \\ C_2'R_2^{-1} \end{bmatrix} y_3(t) \tag{4.30}$$

$$r_f(T)=0 \tag{4.31}$$

$$-r_b(t) = G_b(t)r_b(t) + (M_b(t)V^{\dagger}(t)-I)\begin{bmatrix} 0 \\ (-T_{21}(t), H+A-T_{22}(t)) \end{bmatrix} \hat{x}_{3s}(t) \tag{4.32}$$

$$r_b(2T)=0 \tag{4.33}$$

where M_f, M_b, E, G_f, and G_b are given by

$$\dot{M}_f(t) = D(t)M_f(t) + M_f(t)D'(t) + \begin{bmatrix} S_2 & 0 & 0 \\ 0 & S_1 & -S_1 \\ 0 & -S_1 & 2S_1 \end{bmatrix} - M_f(t)\begin{bmatrix} C_1'R_1'C_1 & 0 & 0 \\ 0 & 2C_2'R_2^{-1}C_2 & C_2'R_2^{-1}C_2 \\ 0 & C_2'R_2^{-1}C_2 & C_2'R_2^{-1}C_2 \end{bmatrix} M_f(t) \tag{4.34}$$

$$M_f(T) = V(T) \tag{4.35}$$

$$-\dot{M}_b(t) = -\left\{ D(t) + \begin{bmatrix} S_2 & 0 & 0 \\ 0 & S_1 & -S_1 \\ 0 & -S_1 & 2S_1 \end{bmatrix} V^{\dagger}(t) \right\} M_b(t)$$

$$-M_b(t) \left\{ D'(t) + V^{\dagger}(t) \begin{bmatrix} S_2 & 0 & 0 \\ 0 & S_1 & -S_1 \\ 0 & -S_1 & 2S_1 \end{bmatrix} \right\} + \begin{bmatrix} S_2 & 0 & 0 \\ 0 & S_1 & -S_1 \\ 0 & -S_1 & 2S_1 \end{bmatrix}$$

$$-M_b(t) \begin{bmatrix} C_1'R_1^{-1}C_1 & 0 & 0 \\ 0 & C_2'R_2^{-1}C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.36}$$

$$M_b(2T) = V_4(2T) \tag{4.37}$$

$$E^{\dagger}(t) = M_f^{\dagger}(t) + M_b^{\dagger}(t) - V^{\dagger}(t) \tag{4.38}$$

$$G_f(t) = D(t) - M_f(t) \begin{bmatrix} C_1'R_1^{-1}C_1 & 0 & 0 \\ 0 & 2C_2'R_2^{-1}C_2 & C_2'R_2^{-1}C_2 \\ 0 & C_2'R_2^{-1}C_2 & C_2'R_2^{-1}C_2 \end{bmatrix} \tag{4.39}$$

$$G_b(t) = -D(t) - \begin{bmatrix} S_2 & 0 & 0 \\ 0 & S_1 & -S_1 \\ 0 & -S_1 & 2S_1 \end{bmatrix} V^{\dagger}(t) - M_b(t) \begin{bmatrix} C_1'R_1^{-1}C_1 & 0 & 0 \\ 0 & C_2'R_2^{-1}C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.40}$$

The various quantities here play analogous roles to those played by the corresponding quantities in (4.6)-(4.17). For example, E(t) is the error covariance associated with the estimate $\hat{\rho}(t|T,T,t)$. The only difference in this case is the use of pseudo-inverses for M_f, M_b, V, and E. These all represent covariances of ρ, and thus from (4.23) at time T they are all singular. However because of our noise-controllability assumption, all of these quantities are positive definite for t>T. Furthermore, it is not difficult to check that the estimate ρ at time T, as defined by (4.29) does have its last block-component equal to zero.

Finally, let us comment on the issue of computing $\hat{x}_{3s}(t)$ or, equivalently $\hat{x}_{2s}(t)$, as defined in (4.21). Recall that what we calculated over the interval [0,T] was the real-time smoothing estimate $\hat{x}_2(t|T,t)$, as defined by (4.5), and using results from [5] we displayed an algorithm for performing this calculation in terms of the new data y_2 and the previous smoothed estimate trajectory $\hat{\xi}_s$. The problem we wish to solve now is the calculation of the smoothed estimate $\hat{x}_{2s}(t)$ using y_2 and ξ_s. This is a problem in the updating of smoothed estimates, which is also examined in [5]. Using those results, we find that

$$\hat{x}_{2s}(t) = P_{2s}(t)\left[P_f^{-1}(t)q_f(t) + P_r^{-1}(t)q_r(t)\right] + \begin{bmatrix} \hat{\xi}_s(t) \\ 0 \end{bmatrix} \tag{4.41}$$

where P_f, and q_f are as before, and

$$-\dot{q}_r(t) = F_r(t)q_r(t) + [P_r(t)\textstyle\sum_2^{-1}(t)-I]\begin{bmatrix} 0 \\ H \end{bmatrix}\hat{\xi}_s(t) + P_r(t)\begin{bmatrix} 0 \\ C_2'R_2^{-1} \end{bmatrix}y_2(t) \tag{4.42}$$

$$q_r(T)=0 \tag{4.43}$$

and P_r, P_{2s}, and F_r are defined by

$$-\dot{P}_r(t) = -\left\{\begin{pmatrix} F & 0 \\ H & A \end{pmatrix} + \begin{bmatrix} S_2 & 0 \\ 0 & S_1 \end{bmatrix} \textstyle\sum_2^{-1}(t)\right\} P_r(t)$$

$$-P_r(t)\left\{\begin{bmatrix} F' & H' \\ 0 & A' \end{bmatrix} + \textstyle\sum_2^{-1}(t) \begin{bmatrix} S_2 & 0 \\ 0 & S_1 \end{bmatrix}\right\} + \begin{bmatrix} S_2 & 0 \\ 0 & S_1 \end{bmatrix}$$

$$- P_r(t) \begin{bmatrix} C_1'R_1^{-1}C_1 & 0 \\ 0 & C_2'R_2^{-1}C_2 \end{bmatrix} P_r(t) \tag{4.44}$$

$$P_r(T) = \textstyle\sum_2(t) \tag{4.45}$$

$$P_{2s}^{-1}(t) = P_f^{-1}(t) + P_r^{-1}(t) - \textstyle\sum_2^{-1}(t) \tag{4.46}$$

$$F_r(t) = -\begin{bmatrix} F & 0 \\ H & A \end{bmatrix} - \begin{bmatrix} S_2 & 0 \\ 0 & S_1 \end{bmatrix} \textstyle\sum_2^{-1}(t) - P_r(t) \begin{bmatrix} C_1'R_1^{-1}C_1 & 0 \\ 0 & C_2'R_2^{-1}C_2 \end{bmatrix} \tag{4.47}$$

Comparing (4.41)-(4.47) to (4.6)-(4.17) we see that the only change in computing the full smoothed estimate $\hat{x}_{2s}(t)$ versus the real-time smoothed estimate $\hat{x}_2(t|T,t)$ is the incorporation of the new data (y_2) into the reverse-time processor (4.42), (4.47) and into the corresponding reverse-time error covariance P_r and the overall smoothed error covariance P_{2s}.

V. Conclusions

In this paper we have examined the effect of a random field on a linear dynamic system moving through the field. We have developed a methodology for calculating the covariance of the state of the dynamic system along any trajectory. The evolution of this covariance is clearly dependent upon the nature of the trajectory, and our results indicate explicitly how this dependence is reflected in the differential equations that must be solved to determine the covariance.

In the case of one-dimensional motion we have gone several steps farther in our understanding and analysis of over-and-back trajectories. Specifically, with the use of the technique for constructing backwards Markovian models we have developed Markov models over each separate undirectional segment of the trajectory. The dimension of these models decreases when the trajectory goes beyond the region covered in previous segments and increases when there is a turn. Using this model and results on real-time smoothing we then were able to solve an over-and-back estimation problem.

Several directions for further work suggest themselves. The first is the detailed investigation of the estimation problem discussed in Section IV. While we have described the solution to this problems we have not exploited its structure as fully as is possible, either in terms of obtaining efficient on-line solutions or of gaining insight. For example, it is clear that the measurements of the state x of the dynamic system provide information about the field ξ. How is this information incorporated in the solution of the real-time smoothing problem? This is potentially important in problems in which we wish to use the dynamic system to estimate the random field. Gravity mapping using inertial instruments is a potential application.

A more significant extension of our work is the development of Markov-type models and the corresponding estimation algorithms for more general 2-D trajectories. This will involve a significant extension of the notion of a backwards Markov process.

Finally, an important generalization of the problems considered in this paper are to systems moving along trajectories which are random themselves. Specifically, referring to our general model, suppose that $(\eta_1(t),\eta_2(t))$ are in fact components of x. This is in fact a more realistic model in some applications. While our results do not address this problem, they may be of value in the case in which the trajectory is only slightly disturbed from some nominal. In that situation our analysis might form the basis for a perturbation analysis of the random trajectory problem.

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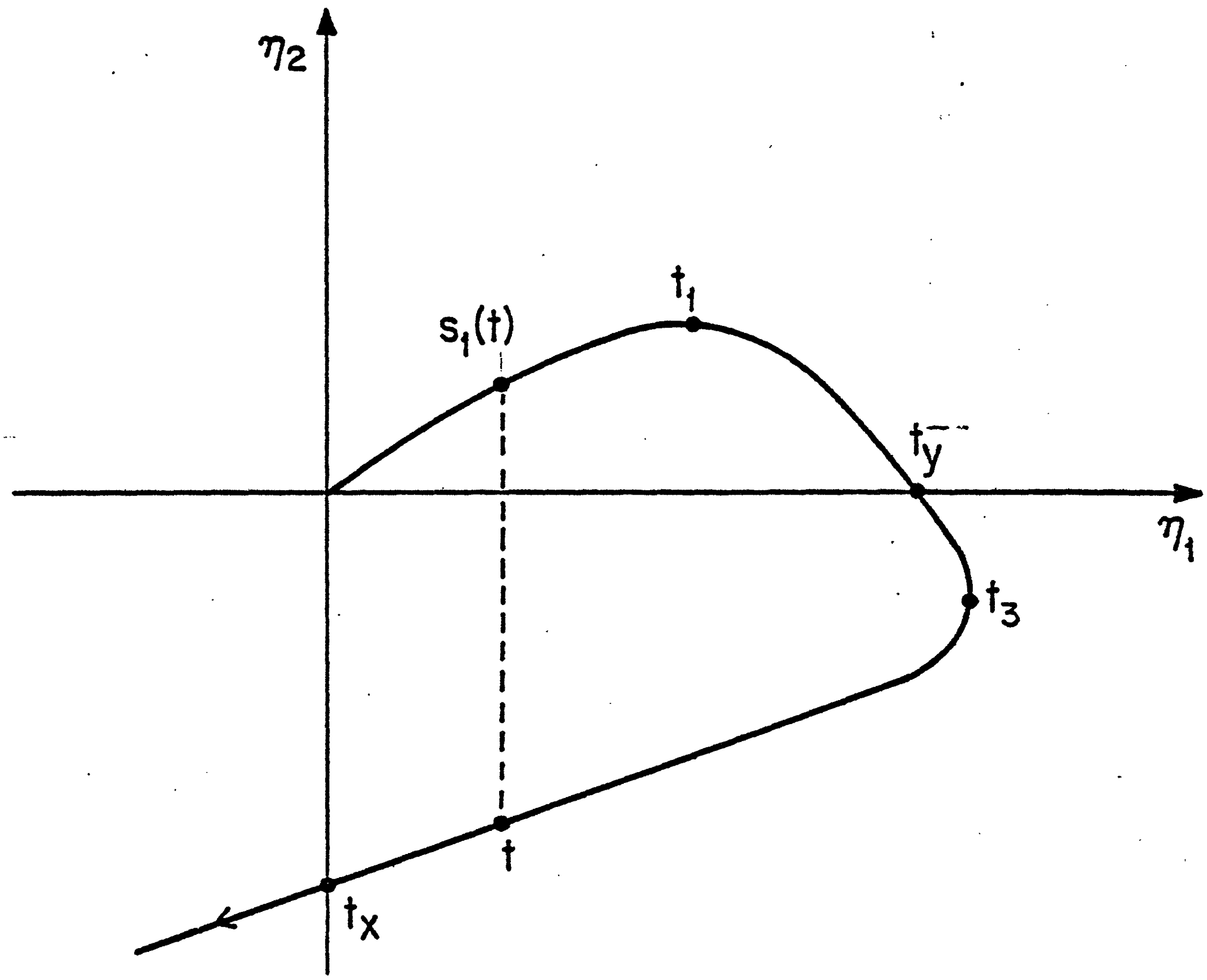
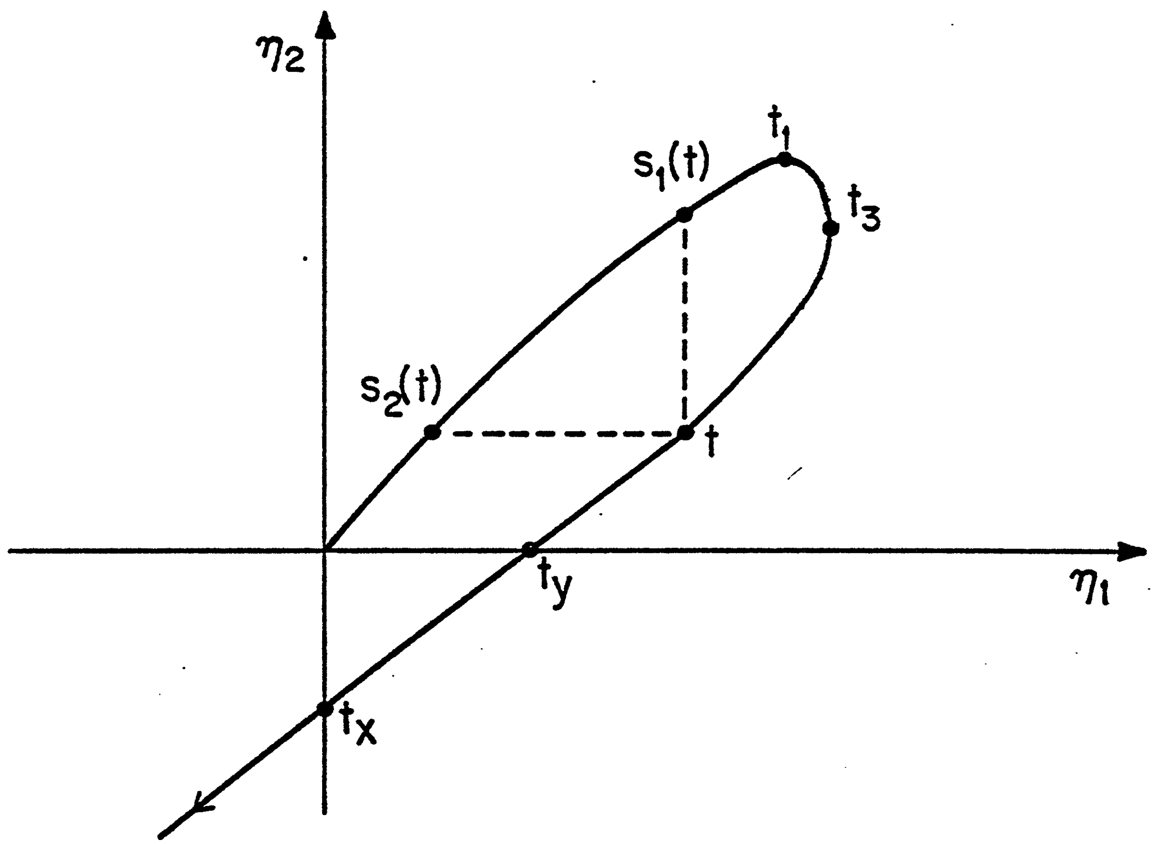
APPENDIX

We consider one final case of the trajectory-covariance problem of Section II. In contrast to Case 2 in which the turn takes us into a neighboring quadrant, we now consider a trajectory that has a sharp angle and takes us into the opposite quadrant -- i.e. northeast (NE) to southwest (SW). Three cases of this will be considered, and these are illustrated in Figure A.1. As can be seen, these cases represent successively sharper turns. In each, t_1 is the time when the direction of the direction of the trajectory changes from NE to SE, and t_3 is the time we change from SE to SW. Also t_x denotes the time at which the trajectory crosses the η_2 axis in the southwesterly direction. This corresponds to the time at which the η_1 coordinate of the trajectory has evolved from 0 to its maximum value $\eta_1(t_3)$ and has decreased back to zero. The time t_y is defined in an analogous fashion.

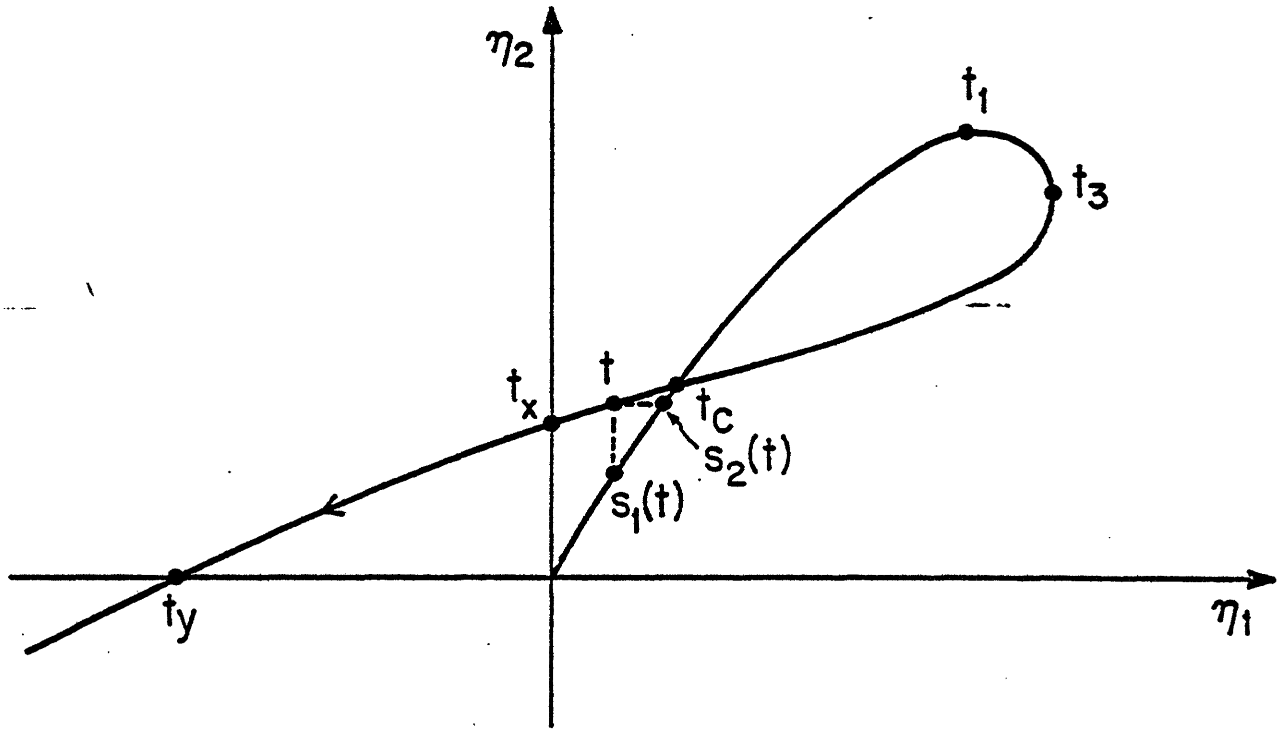
The distinguishing characteristics of these three trajectories are as follows: in trajectory (a) only one of the coordinates at a time retraces past values -- first $\eta_2(t)$ and then $\eta_1(t)$; in trajectories (b) and (c) both coordinates retrace past values over a common portion of the trajectory; in case (b) $\eta_2(t)$ completes its retracing before $\eta_1(t)$, while exactly the opposite is true in case (c). As we mentioned in Section II, we will see that at any time we get one set of additional equations for each component that is retracing past values. Thus in case (a), we will have one additional set of equations beginning at t_1 and ending at t_y.



(a)



(b)



(c)

FIGURE A.1: The Three Cases Considered in the Appendix.

A different single set of equations begins at t_3 and end at t_x. In case (b) we have an additional set of equations beginning at t_1, a second set is added at time t_3, the first set ends at t_y, leaving us with the second set which ends at t_x. In case (c) the only differences are that there is a change in the equations at time t_c, and also the second set of equations ends first ($t_x < t_y$). This verbal description, together with the following analysis, should make clear the approach that can be taken in analyzing a general trajectory.

Case a: Referring to Figure 2.1b and Figure A.1a, we see that the present case is identical to Case 2 considered in Section II up to time t_3. That is, for $0 \le t \le t_1$ we use the equations of Case 1, with $L(t) = H_1 B_1(t)$ (see (2.18)-(2.21)). For $t_1 \le t \le t_2$, $L(t) = H_1[B_1(t) + C_1(t)]$ (equations (2.28)-(2.35)), while for $t_2 \le t \le t_3$, $L(t) = H_1 C_1(t)$ ((2.40)-(2.42)). For $t_3 \le t \le t_x$

$$
\begin{aligned}
\eta_1(t) - \eta_1(s) &\ge 0 && \text{for} \quad 0 \le s \le s_1(t) \\
\eta_1(t) - \eta_1(s) &\le 0 && \text{for} \quad s_1(t) \le s \le t \qquad \text{(A.1)} \\
\eta_2(t) - \eta_2(s) &\le 0 && \text{for} \quad 0 \le s \le t
\end{aligned}
$$

Comparing (A.1) to (2.22) we see that this is quite similar to Case 2, except here we are turning from SE to SW. Thus, in analogy with Case 2 we can write

$$L(t) = \int_0^{s_1(t)} H_1 e^{F_1(\eta_1(t)-\eta_1(s))} G_1 G_2' e^{F_2'(\eta_2(s)-\eta_2(t))} H_2' e^{A'(t-s)} ds$$

$$+ \int_{s_1(t)}^{t} G_1' e^{F_1'(\eta_1(s)-\eta_1(t))} H_1' G_2' e^{F_2'(\eta_2(s)-\eta_2(t))} A_2' e^{A'(t-s)} ds \qquad (A.2)$$

or

$$L(t) = H_1 C_1(t) + G_1' D_1(t) \qquad t \geq t_3 \qquad (A.3)$$

where

$$C_j(t_3) = \text{values calculated using (2.40)-(2.42)}$$
$$D_j(t_3) = 0 \qquad (A.4)$$

and

$$\dot{C}_j(t) = \dot{\eta}_1(t) F_1 C_j(t) + C_j(t) A' - \dot{\eta}_2(t) C_{j+1}(t)$$
$$+ \frac{\dot{\eta}_1(t)}{\dot{\eta}_1(s_1(t))} G_1 G_2' (F_2')^{j-1} e^{F_2'(\eta_2(s_1(t))-\eta_2(t))} H_2' e^{A'(t-s_1(t))}$$

$$j=1,\ldots r-1 \qquad (A.5)$$

$$\dot{C}_r(t) = \dot{\eta}_1(t) F_1 C_r(t) + C_r(t) A' - \dot{\eta}_2(t) \sum_{j=1}^{r} \rho_{j-1} C_j(t)$$
$$+ \frac{\dot{\eta}_1(t)}{\dot{\eta}_1(s_1(t))} G_1 G_2' (F_2')^{r-1} e^{F_2'(\eta_2(s_1(t))-\eta_2(t))} H_2' e^{A'(t-s_1(t))} \qquad (A.6)$$

Note that $C_j(t_x)=0$. Also

$$\dot{D}_j(t) = -\dot{\eta}_1(t)F_1'D_j(t) + D_j(t)A' - \dot{\eta}_2(t)D_{j+1}(t) + H_1'G_2'(F_2')^{j-1}H_2'$$

$$- \frac{\dot{\eta}_1(t)}{\dot{\eta}_1(s_1(t))} H_1'G_2'(F_2')^{j-1} e^{F_2(\eta_2(s_1(t))-\eta_2(t))} H_2'e^{A'(t-s_1(t))}$$

$$j=1,\dots,r-1 \qquad (A.7)$$

$$\dot{D}_r(t) = -\dot{\eta}_1(t)F_1D_r(t) + D_r(t)A' - \dot{\eta}_2(t)\sum_{j=1}^{r} \rho_{j-1}D_r(t) + H_1'G_2'(F_2')^{r-1}H_2'$$

$$- \frac{\dot{\eta}_1(t)}{\dot{\eta}_1(s_1(t))} H_1'G_2'(F_2')^{r-1} e^{F_2'(\eta_2(s_1(t))-\eta_2(t))} H_2'e^{A'(t-s_1(t))} \qquad (A.8)$$

Then, for $t \geq t_x$

$$L(t) = \int_0^t G_1'e^{F_1'(\eta_1(s)-\eta_1(t))} H_1'G_2'e^{F_2'(\eta_2(s)-\eta_2(t))} H_2'e^{A'(t-s)}ds \qquad (A.9)$$

Thus

$$L(t) = G_1'D_1'(t) \qquad (A.10)$$

where $D_1(t_x)$ is obtained from (A.7), (A.8), and for $t>t_x$

$$\dot{D}_j(t) = -\dot{\eta}_1(t)F_1'D_j(t) + D_j(t)A' - \dot{\eta}_2(t)D_{j+1}(t) + H_1'G_2'(F_2')^{j-1}H_2' \qquad (A.11)$$

$$\dot{D}_r(t) = -\dot{\eta}_1(t)F_1'D_r(t) + D_r(t)A' - \dot{\eta}_2(t) \sum_{j=1}^{r}\rho_{j-1}D_j(t) + H_1'G_2'(F_2')^{r-1}H_2' \qquad (A.12)$$

Case b: As in Case a, this case is the same as Case 2 up until time t_3, using equations (2.28)-(2.35) to provide us with initial conditions, at time t_3 on B_j and C_j. For $t_3 \le t \le t_y$

$$L(t) = \int_0^{s_2(t)} H e^{F_1(\eta_1(t)-\eta_1(s))} G_1 H_2 e^{F_2(\eta_2(t)-\eta_2(s))} G_2 e^{A'(t-s)} ds$$

$$+ \int_{s_2(t)}^{s_1(t)} H_1 e^{F_1(\eta_1(t)-\eta_1(s))} G_1 G_2' e^{F_2'(\eta_2(s)-\eta_2(t))} H_2' e^{A'(t-s)} ds$$

$$+ \int_{s_1(t)}^{t} G_1' e^{F_1'(\eta_1(s)-\eta_1(t))} H_1' G_2' e^{F_2'(\eta_2(s)-\eta_2(t))} H_2' e^{A'(t-s)} ds \tag{A.13}$$

Thus

$$L(t) = H_1[B_1(t) + C_1(t)] + G_1' D_1(t) \tag{A.14}$$

where the B_j satisfy (2.28), (2.29) with $s_0(t)$ replaced by $s_2(t)$. Note that $B_j(t_y)=0$. Also the D_j satisfy (A.7),(A.8) with $D_j(t_3)=0$. The only new equation is for the C_j over the time interval $t_3 \le t \le t_y$.

$$\dot{C}_j(t) = \dot{\eta}_1(t) F_1 C_j(t) + C_j(t) A' - \dot{\eta}_2(t) C_{j+1}(t)$$
$$+ \frac{\dot{\eta}_1(t)}{\dot{\eta}_1(s_1(t))} G_1 G_2' (F_2')^{j-1} e^{F_2'(\eta_2(s_1(t))-\eta_2(t))} H_2' e^{A'(t-s_1(t))}$$
$$- \frac{\dot{\eta}_2(t)}{\dot{\eta}_2(s_2(t))} e^{F_1(\eta_1(t)-\eta_1(s_2(t)))} G_1 G_2' (F_2')^{j-1} H_2' e^{A'(t-s_2(t))} \tag{A.15}$$

$$\dot{C}_r(t) = \dot{\eta}_1(t)F_1C_r(t) + C_r(t)A' - \dot{\eta}_2(t)\sum_{j=1}^{r}\rho_{j-1}C_j(t)$$

$$+ \frac{\dot{\eta}_1(t)}{\dot{\eta}_1(s_1(t))} G_1G_2(F_2')^{r-1} e^{F_2'(\eta_2(s_1(t))-\eta_2(t))} H_2'e^{A'(t-s_1(t))}$$

$$- \frac{\dot{\eta}_2(t)}{\dot{\eta}_2(s_2(t))} e^{F_1(\eta_1(t)-\eta_1(s_2(t))} G_1G_2'(F_2')^{r-1}H_2'e^{A'(t-s_2(t))} \qquad (A.16)$$

For $t_y \le t \le t_x$, $L(t) = H_1C_1(t) + G_1'D_1(t)$ and the equations for C_j and D_j are as in Case 3c - i.e. (A.5)-(A.8), and $C_j(t_x)=0$. Also, for $t>t_x$, $L(t) = G_1'D_1(t)$, and the D_j are calculated from (A.11) and (A.12).

Case c: Up until t_3 we are in Case 2, and thus use (2.28)-(2.35). Over the interval $t_3 \le t \le t_c$ we have the same situation as in Case b, and equations (2.28), (2.29) (with $s_0(t)$ replaced by $s_2(t)$), (A.7), (A.8), and (A.14)-(A.16) apply. At time t_c, since $s_1(t_c) = s_2(t_c)$, $C_j(t_c)=0$ (see (A.13)), while $B_j(t_c)$ and $D_j(t_c)$ take on values calculated from the previously mentioned equations.

Now consider the time interval $t_c \le t \le t_x$. Note that for $t<t_c$ $s_1(t)>s_2(t)$, while for $t>t_c$, $s_1(t)<s_2(t)$. Thus

$$L(t) = \int_0^{s_1(t)} H_1e^{F_1(\eta_1(t)-\eta_1(s))} G_1H_2e^{F_2(\eta_2(t)-\eta_2(s))} G_2e^{A'(t-s)}ds$$

$$+ \int_{s_1(t)}^{s_2(t)} G_1'e^{F_1'(\eta_1(s)-\eta_1(t))} H_1'H_2e^{F_2(\eta_2(t)-\eta_2(s))} G_2e^{A'(t-s)}ds$$

$$+ \int_{s_2(t)}^{t} G_1'e^{F_1'(\eta_1(s)-\eta_1(t))} H_1'G_2'e^{F_2'(\eta_2(s)-\eta_2(t))} H_2'e^{A'(t-s)}ds \qquad (A.17)$$

or

$$L(t) = H_1B_1(t) + G_1'[D_1(t)+E_1(t)] \tag{A.18}$$

where

$$\dot{B}_j(t) = \dot{\eta}_1(t)F_1B_j(t) + B_j(t)A' + \dot{\eta}_2(t)B_{j+1}(t) + \frac{\dot{\eta}_1(t)}{\dot{\eta}_1(s_1(t))} G_1H_2F_2^{j-1} e^{F_2(\eta_2(t)-\eta_2(s_1(t))} G_2e^{A'(t-s_1(t))} \tag{A.19}$$

$$\dot{B}_r(t) = \dot{\eta}_1(t)F_1B_r(t) + B_r(t)A' + \dot{\eta}_2(t) \sum_{j=1}^{r} \rho_{j-1}B_j(t) + \frac{\dot{\eta}_1(t)}{\dot{\eta}_1(s_1(t))} G_1H_2F_2^{r-1}e^{F_2(\eta_2(t)-\eta_2(s_1(t))}G_2e^{A'(t-s_1(t))} \tag{a.20}$$

$$\dot{E}_j(t) = -\dot{\eta}_1(t)F_1'E_j(t) + E_j(t)A' + \dot{\eta}_2(t)E_{j+1}(t) + \frac{\dot{\eta}_2(t)}{\dot{\eta}_2(s_2(t))} e^{F_1'(\eta_1(s_2(t))-\eta_1(t))} H_1'H_2F_2^{j-1}G_2e^{A'(t-s_2(t))} - \frac{\dot{\eta}_1(t)}{\dot{\eta}_1(s_1(t))} H_1'H_2F_2^{j-1}e^{F_2(\eta_2(t)-\eta_2(s_1(t))} G_2e^{A'(t-s_1(t))} \tag{A.21}$$

$$\dot{E}_r(t) = -\dot{\eta}_1(t)F_1'E_r(t) + E_r(t)A' + \dot{\eta}_2(t) \sum_{j=1}^{r} \rho_{j-1}E_j(t) + \frac{\dot{\eta}_2(t)}{\dot{\eta}_2(s_2(t))} e^{F_1'(\eta_1(s_2(t))-\eta_1(t))} H_1'H_2F_2^{r-1}G_2e^{A'(t-s_2(t))} - \frac{\dot{\eta}_1(t)}{\dot{\eta}_1(s_1(t))} H_1'H_2F_2^{r-1}e^{F_2(\eta_2(t)-\eta_2(s_1(t))} G_2e^{A'(t-s_1(t))} \tag{A.22}$$

$$E_j(t_c)=0 \qquad j=1,\ldots,r \tag{A.23}$$

$$\dot{D}_j(t) = -\dot{\eta}_1(t)F_1'D_j(t) + D_j(t)A' - \dot{\eta}_2(t)D_{j+1}(t) + H_1'G_2'(F_2')^{j-1}H_2'$$
$$- \frac{\dot{\eta}_2(t)}{\dot{\eta}_2(s_2(t))} e^{F_1'(\eta_1(s_2(t))-\eta_1(t))} H_1'G_2'(F_2')^{j-1}H_2'e^{A'(t-s_2(t))} \tag{A.24}$$

$$\dot{D}_r(t) = -\dot{\eta}_1(t)F_1'D_r(t) + D_r(t)A' - \dot{\eta}_2 \sum_{j=1}^{r} \rho_{j-1}D_j(t) + H_1'G_2'(F_2')^{r-1}H_2'$$
$$- \frac{\dot{\eta}_2(t)}{\dot{\eta}_2(s_2(t))} e^{F_1'(\eta_1(s_2(t))-\eta_1(t))} H_1'G_2'(F_2')^{r-1}H_2'e^{A'(t-s_2(t))} \tag{A.25}$$

Note that $B_j(t_x)=0$.

Now consider the interval $t_x \leq t \leq t_y$. In this case

$$L(t) = \int_0^{s_2(t)} G_1'e^{F_1'(\eta_1(s)-\eta_1(t))} H_1'H_2e^{F_2(\eta_2(t)-\eta_2(s))} G_2e^{A'(t-s)}ds$$
$$+ \int_{s_2(t)}^{t} G_1'e^{F_1'(\eta_1(s)-\eta_1(t))} H_1'G_2'e^{F_2'(\eta_2(s)-\eta_2(t))} H_2'e^{A'(t-s)}ds \tag{A.26}$$

Here

$$L(t) = G_1'[D_1(t) + E_1(t)] \tag{A.27}$$

where $D_j(t)$ satisfies (A.24), (A.25), with $D_j(t_x)$ given by the previous step. Similarly, $E_j(t_x)$ is given by the preceding step, but then over the interval $t_x \leq t \leq t_y$

$$\dot{E}_j(t) = -\dot{\eta}_1(t)F_1'E_j(t) + E_j(t)A' + \dot{\eta}_2(t)E_{j+1}(t)$$

$$+ \frac{\dot{\eta}_2(t)}{\dot{\eta}_2(s_2(t))} e^{F_1'(\eta_1(s_2(t))-\eta_1(t))} H_1'H_2F_2^{j-1}G_2e^{A'(t-s_2(t))} \qquad (A.28)$$

$$\dot{E}_r(t) = -\dot{\eta}_1(t)F_1'E_r(t) + E_r(t)A' + \dot{\eta}_2(t) \sum_{j=1}^{r} \rho_{j-1}E_j(t)$$

$$+ \frac{\dot{\eta}_2(t)}{\dot{\eta}_2(s_2(t))} e^{F_1'(\eta_1(s_2(t))-\eta_1(t))} H_1'H_2F_2^{r-1}G_2e^{A'(t-s_2(t))} \qquad (A.29)$$

Note that $E_j(t_y)=0$. Then, for $t\geq t_y$, $L(t) = G_1'D_1(t)$, where $D_j(t_y)$ is obtained from the preceding equations, and for $t\geq t_y$ the D_j satisfy (A.11),(A.12).