REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO. N/A	3. RECIPIENT'S CATALOG NUMBER N/A
4. TITLE (and Subtitle) BROWNIAN MOTION		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER LIDS-R-1485
7. AUTHOR(s) IOANNIS KARATZAS STEVEN E. SHREVE		8. CONTRACT OR GRANT NUMBER(s) ARO Grant DAAG-29-84-K-0005
9. PERFORMING ORGANIZATION NAME AND ADDRESS M.I.T. LABORATORY FOR INFORMATION AND DECISION SYSTEMS CAMBRIDGE, MA 02139		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office Post Office Box 12211 Research Triangle Park, NC 27709		12. REPORT DATE JULY 1985
		13. NUMBER OF PAGES 469
14. MONITORING AGENCY NAME & ADDRESS(If different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

NA

18. SUPPLEMENTARY NOTES

The view, opinions, and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

BROWNIAN MOTION

A Graduate Course in Stochastic Processes

by

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PREFACE

The most fundamental concepts in the theory of stochastic processes are the <u>Markov property</u> and the <u>martingale property</u>. This book is written for those who are familiar with both of these ideas in the discrete-time setting, and who now wish to explore stochastic processes in the continuous-time context. It has been our goal to write a systematic and thorough exposition of this subject, leading in many instances to the frontiers of knowledge. At the same time, we have endeavored to keep the mathematical prerequisites as low as possible, namely, knowledge of measure-theoretic probability and some acquaintance with discrete-time processes. The vehicle we have chosen for this task is Brownian motion, which we present as the canonical example of both a Markov process and a martingale in continuous time. We support this point of view by showing how, by means of stochastic integration and random time change, all continuous martingales and a multitude of continuous Markov processes can be represented in terms of Brownian motion. This approach forces us to leave aside those processes which do not have continuous paths. Thus, the Poisson process is not a primary object of study, although it is developed in Chapter 1 to be used as a tool when we later study passage times of Brownian motion.

At this time, only the first three chapters of this book are complete.[*] We provide, however, a table of contents for the entire work. The material in Chapters 6 and 7 on Brownian

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[*]The complete book will be published by Springer-Verlag.

local time and its applications to stochastic control will be appearing in a form suitable as a text for the first time. It is our desire to give an account of these topics which motivates the entire book.

We are greatly indebted to Sanjoy Mitter and Dimitri Bertsekas for generously extending to us the invitation to work this past year at M.I.T., for their support and encouragement during the writing of this book, and for providing the intellectual environment which made this task more agreeable than it might otherwise have been. We also wish to acknowledge the allowances made by our respective home departments and institutions, which made this year of close collaboration possible. Parts of the book grew out of notes on lectures given by one of us at Columbia University over several years, and we owe much to the audiences in those courses.

Typing of this manuscript was done with remarkable care and efficiency by Doodmatie Kalicharan, Stella DeVito, Katherine Tougher, and Muriel Knowles. We wish to thank them all.

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1.1: STOCHASTIC PROCESSES AND σ-FIELDS

A <u>stochastic process</u> is a mathematical model for the occurrence, at each moment after the initial time, of a random phenomenon. The randomness is captured by the introduction of a measurable space $(\Omega, \mathcal{F})$, called the <u>sample space</u>, on which probability measures can be placed. Thus, a stochastic process is a collection of random variables $X = \{X_t; \; 0 \leq t < \infty\}$ on $(\Omega, \mathcal{F})$, which take values in a second measurable space $(S, \mathcal{A})$, called the <u>state space</u>. For our purposes, the state space $(S, \mathcal{A})$, will be the d-dimensional Euclidean space equipped with the σ-field of Borel sets, i.e., $S = R^d$, $\mathcal{A} = \mathcal{B}(R^d)$, where $\mathcal{B}(U)$ will always be used to denote the smallest σ-field containing all open sets of a topological space U. The index $t \in [0, \infty)$ of the random variables X_t admits a convenient interpretation as <u>time</u>.

For a fixed sample point $\omega \in \Omega$, the function $t \mapsto X_t(\omega)$, $t \geq 0$, is the <u>sample path</u> (realization, trajectory) of the process X associated with ω. It provides the mathematical model for a random experiment, whose outcome can be observed continuously in time (e.g., the number of customers in a queue observed and recorded over a period of time, the trajectory of a molecule subjected to the random disturbances of its neighbours, the output of a communications channel operating in noise, etc).

Let us consider two stochastic processes X and Y defined on the same probability space $(\Omega, \mathcal{F}, P)$. When regarded as functions of t and ω, we would say X and Y were the same if and only if $X_t(\omega) = Y_t(\omega)$ for all $t \geq 0$ and all $\omega \in \Omega$. However, in the

presence of the probability measure P, we could weaken this requirement in at least three different ways to obtain three related concepts of "sameness" between two processes. We list them here.

1.1. Definition: Y is a _modification_ of X if, for every $t \geq 0$, we have $P[X_t = Y_t] = 1$.

1.2. Definition: X and Y have _the same finite-dimensional distributions_ if, for any integer $n \geq 1$, real numbers $0 \leq t_1 < t_2 < \ldots < t_n < \infty$, and $A \in \mathscr{B}(\mathbb{R}^{nd})$, we have:

$$P[(X_{t_1}, \ldots, X_{t_n}) \in A] = P[(Y_{t_1}, \ldots, Y_{t_n}) \in A].$$

1.3. Definition: X and Y are called _indistinguishable_ if almost all their sample paths agree:

$$P[X_t(\omega) = Y_t(\omega), \quad \forall \ 0 \leq t < \infty] = 1.$$

The third property is the strongest; it implies trivially the first one, which in turn yields the second. On the other hand, two processes can be modifications of one another and yet have completely different sample paths. Here is a standard example:

1.4. Example: Consider a positive random variable T with a continuous distribution, put $X_t \equiv 0$, and let $Y_t = \begin{cases} 0; & t \neq T \\ 1; & t = T. \end{cases}$

Y is a modification of X, since for every $t \geq 0$ we have

$P[Y_t = X_t] = P[T \neq t] = 1$, but on the other hand: $P[Y_t = X_t; \forall t \geq 0] = 0$.

A positive result in this direction is the following.

1.5. Problem: Let Y be a modification of X, and suppose that both processes have a.s. right-continuous sample paths. Then X and Y are indistinguishable. □

It does not make sense to ask if Y is a modification of X, or if Y and X are indistinguishable, unless X and Y are defined on the same probability space and have the same state space. However, if X and Y have the same state space but are defined on different probability spaces, we can ask if they have the same finite dimensional distributions.

1.2. Definition: Let X and Y be stochastic processes defined on probability spaces $(\Omega, \mathfrak{F}, P)$ and $(\widetilde{\Omega}, \widetilde{\mathfrak{F}}, \widetilde{P})$ respectively, and having the same state space $(R^d, \mathfrak{B}(R^d))$. X and Y have <u>the same finite-dimensional distributions</u> if, for any integer $n \geq 1$, real numbers $0 \leq t_1 < t_2 < \ldots < t_n < \infty$, and $A \in \mathfrak{B}(R^{nd})$, we have

$$P[(X_{t_1}, \ldots, X_{t_n}) \in A] = \widetilde{P}[(Y_{t_1}, \ldots, Y_{t_n}) \in A].$$
□

Many processes, including d-dimensional Brownian motion, are defined in terms of their finite-dimensional distributions irrespective of their probability space. Indeed, in Chapter 2 we will construct a standard d-dimensional Brownian motion B on a canocical probability space and then state that any process, on any prob-

ability space, which has state space $(R^d, \mathcal{B}(R^d))$ and which has the same finite-dimensional distributions as B, is a standard d-dimensional Brownian motion. □

For technical reasons in the theory of Lebesgue integration, probability measures are defined on σ-fields and random variables are assumed to be measurable with respect to these σ-fields. Thus, implicit in the statement that a random process $X = \{X_t; 0 \leq t < \infty\}$ is a collection of $(R^d, \mathcal{B}(R^d))$ - valued random variables on $(\Omega, \mathcal{F})$, is the assumption that each X_t is $\mathcal{F}/\mathcal{B}(R^d)$ - measurable. However, X is really a function of the pair of variables (t, ω), and so for technical reasons, it is often convenient to have some joint measurability properties.

1.6. Definition: The stochastic process X is called <u>measurable</u>
   if, for every $A \in \mathcal{B}(R^d)$, the set $\{(t, \omega); X_t(\omega) \in A\}$ belongs to
   the product σ-field $\mathcal{B}([0, \infty)) \otimes \mathcal{F}$; in other words, if the
   mapping

$$(t, \omega) \to X_t(\omega): ([0, \infty) \times \Omega, \mathcal{B}([0, \infty)) \otimes \mathcal{F}) \to (R^d, \mathcal{B}(R^d))$$

   is measurable. □

It is an immediate consequence of Fubini's Theorem that the trajectories of such a process are Borel-measurable functions of $t \in [0, \infty)$, and provided that the components of X have defined expectations, then the same is true for the function $m(t) = EX_t$, where E denotes expectation with respect to a probability measure P on $(\Omega, \mathcal{F})$ that integrates X_t for all $t \geq 0$. Moreover, if X

takes values in R and I is an inteval of $[0, \infty)$ such that $\int_I E|X_t| dt < \infty$, then

$$\int_I |X_t| dt < \infty \quad \text{a.s.P,} \quad \text{and:} \quad \int_I EX_t dt = E\int_I X_t \, dt.$$

$\square$

There is a very important, nontechnical reason to include σ-fields in the study of stochastic processes, and that is to keep track of information. The temporal feature of a stochastic process suggests a flow of time, in which, at every moment $t \geq 0$, we can talk about a _past_, _present_ and _future_ and can ask how much an observer of the process knows about it at present as compared to how much he knew at some point in the past or will know at some point in the future. We equip our sample space $(\Omega, \mathfrak{F})$ with a _filtration_, i.e., a nondecreasing family $\{\mathfrak{F}_t; t \geq 0\}$ of sub-σ-fields of $\mathfrak{F}: \mathfrak{F}_s \subseteq \mathfrak{F}_t \subseteq \mathfrak{F}$ for $0 \leq s < t < \infty$. We set $\mathfrak{F}_\infty = \sigma(\bigcup_{t \geq 0} \mathfrak{F}_t)$.

Given a stochastic process, the simplest choice of a filtration is that generated by the process itself, i.e.,

$$\mathfrak{F}_t^X \triangleq \sigma(X_s; 0 \leq s \leq t).$$

We interpret $A \in \mathfrak{F}_t^X$ to mean that by time t, an observer of X knows whether or not A has occurred. The next two problems illustrate this point.

1.7. Problem: Let X be a process with every sample path right-continuous. Let A be the event that X is continuous on $[0, t_o)$. Show $A \in \mathfrak{F}_{t_o}^X$.

1.8. Problem: Let X be a process whose sample paths are right-continuous a.s., and let A be the event that X is continuous on $[0, t_o)$. Show that A can fail to be in $\mathcal{F}_{t_o}^X$, but if $\{\mathcal{F}_t; \ t \geq 0\}$ is a filtration satisfying $\mathcal{F}_t^X \subseteq \mathcal{F}_t$, $t \geq 0$, and $\mathcal{F}_{t_o}$ is complete under P, then $A \in \mathcal{F}_{t_o}$.

Let $\{\mathcal{F}_t; \ t \geq 0\}$ be a filtration. We define $\mathcal{F}_{t-} \triangleq \sigma(\underset{s < t}{\cup} \mathcal{F}_s)$ to be the σ-field of events strictly prior to $t > 0$ and

$\mathcal{F}_{t+} \triangleq \underset{\epsilon > 0}{\cap} \mathcal{F}_{t+\epsilon}$ to be the σ-field of events immediately after $t \geq 0$. We decree $\mathcal{F}_{o-} \triangleq \mathcal{F}_o$ and say that the filtration $\{\mathcal{F}_t\}$ is right (left) continuous if $\mathcal{F}_t = \mathcal{F}_{t+}$ (resp., $\mathcal{F}_t = \mathcal{F}_{t-}$) holds for every $t \geq 0$.

The concept of measurability for a stochastic process, introduced in Definition 1.6, is a rather weak one. The introduction of a filtration $\{\mathcal{F}_t\}$ opens up the possibility for more interesting and useful concepts.

1.9. Definition: The stochastic process X is <u>adapted</u> to the filtration $\{\mathcal{F}_t\}$ if, for each $t \geq 0$, X_t is an $\mathcal{F}_t$-measurable random variable.

$\square$

Obviously, every process X is adapted to $\{\mathcal{F}_t^X\}$. Moreover, if X is adapted to $\{\mathcal{F}_t\}$ and Y is a modification of X, then Y is also adapted to $\{\mathcal{F}_t\}$ provided that $\mathcal{F}_o$ contains all the P-negligible sets in $\mathcal{F}$. Note that this requirement is not the same as saying that $\mathcal{F}_o$ is complete, since some of the P-negligible sets in $\mathcal{F}$ may not be in the completion of $\mathcal{F}_o$.

1.10 Problem: Let X be a process with every sample path left-continuous, and let A be the event that X is continuous on $[0, t_o]$. Let X be adapted to a right-continuous filtration $\{\mathfrak{F}_t\}$. Show that $A \in \mathfrak{F}_{t_o}$.

1.11 Definition: The stochastic process X is called progressively measurable with respect to the filtration $\{\mathfrak{F}_t\}$ if, for each $t \geq 0$ and $A \in \mathcal{B}(R^d)$, the set $\{(s, \omega); 0 \leq s \leq t, \omega \in \Omega, X_s(\omega) \in A\}$ belongs to the product σ-field $\mathcal{B}([0, t]) \otimes \mathfrak{F}_t$; in other words, if the mapping $(s, \omega) \mapsto X_s(\omega): ([0, t] \times \Omega, \mathcal{B}([0, t]) \otimes \mathfrak{F}_t) \to (R^d, \mathcal{B}(R^d))$ is measurable, for each $t \geq 0$. $\square$

The terminology here comes from Chung & Doob [3], which is a basic reference for this section and the next. Evidently, any progressively measurable process is measurable and adapted; the following theorem of Chung & Doob [3] provides the extent to which the converse is true.

1.12 Proposition: If the stochastic process X is measurable and adapted to the filtration $\{\mathfrak{F}_t\}$, then it has a progressively measurable modification.

The reader is referred to the book of Meyer [16; p. 68] for the (lengthy, and rather demanding) proof of this result. It will be used only once in the sequel, and then again in a tangential fashion. Nearly all processes of interest are either right or left continuous, and for them the proof of a stronger result is easier and will now be given.

<u>1.13 Proposition:</u>   If the stochastic process X is right (left)

continuous and adapted to the filtration $\{\mathfrak{F}_t\}$, then it is

also progressively measurable with respect to $\{\mathfrak{F}_t\}$.

<u>Proof</u>:   With $t>0$, $n\geq 1$, $k = 0,1,\ldots,2^n-1$ and $0\leq s\leq t$, we define:

$X_s^{(n)}(\omega) = X_{\frac{k+1}{2^n}t}(\omega)$ for $\frac{k}{2^n}\, t < s \leq \frac{k+1}{2^n}\, t$, as well as $X_0^{(n)}(\omega) = X_0(\omega)$.

The so-constructed map $(s,\omega) \mapsto X_s^{(n)}(\omega)$ from $[0,t] \times \Omega$ into R^d

is demonstrably $\mathfrak{B}([0,t]) \otimes \mathfrak{F}_t$ - measurable. Besides, by right-

continuity we have: $\lim\limits_{n\to\infty} X_s^{(n)}(\omega) = X_s(\omega)$, $\forall$ $(s,\omega) \in [0,t] \times \Omega$.

Therefore, the (limit) map $(s,\omega) \mapsto X_s(\omega)$ is also $\mathfrak{B}([0,t]) \otimes \mathfrak{F}_t$ -

measurable.

$\square$

<u>1.14 Remark:</u> If the stochastic process X is right (or left)

continuous, but not necessarily adapted to $\{\mathfrak{F}_t\}$, then the

same argument shows that X is measurable.

A <u>random time</u> T is an $\mathfrak{F}$ - measurable random variable, with

values in $[0,\infty]$.

<u>1.15 Definition:</u>   If X is a stochastic process and T is a

random time, we define the function X_T on the event $\{T<\infty\}$

by

$$X_T(\omega) \triangleq X_{T(\omega)}(\omega).$$

If $X_\infty(\omega)$ is defined for all $\omega\in\Omega$, then X_T can also be

defined on Ω, by setting $X_T(\omega) \triangleq X_\infty(\omega)$, on $\{T=\infty\}$.

1.16 Problem: If the process X is measurable and the random time
   T is finite, then the function X_T defined above is a random
   variable.

   We shall devote our next section to a very special and
extremely useful class of random times, called stopping times.
These are of fundamental importance in the study of stochastic
processes, since they constitute our most effective tool in the
effort to "tame the continuum of time", as Chung [2] puts it.

1.2: STOPPING TIMES

Let us keep in mind the interpretation of the parameter t as time, and of the σ-field $\mathfrak{F}_t$ as the accumulated information up to t. Let us also imagine that we are interested in the occurrence of a certain phenomenon: an earthquake with intensity above a certain level, a number of customers exceeding the safety requirements of our facility, and so on. We are thus forced to pay particular attention to the instant $T(\omega)$, at which the phenomenon manifests itself _for the first time_. It is quite intuitive then that the event $\{\omega; T(\omega) \leq t\}$, which occurs if and only if the phenomenon has appeared prior to (or at) time t, should be part of the information accumulated by that time.

We can now formulate these heuristic considerations as follows:

2.1 Definition: Let us consider a measurable space $(\Omega, \mathfrak{F})$ equipped with a filtration $\{\mathfrak{F}_t\}$. A random time T is a _stopping time_ of the filtration, if the event $\{T \leq t\}$ belongs to the σ-field $\mathfrak{F}_t$, for every $t \geq 0$. A random time T is an _optional time_ of the filtration, if $\{T < t\} \in \mathfrak{F}_t$, for every $t \geq 0$. □

2.2 Problem: Let X be a stochastic process and T be an $\{\mathfrak{F}_t^X\}$ stopping time. Choose $\omega, \omega' \in \Omega$ and suppose $X_t(\omega) = X_t(\omega')$ for all $t \in [0, T(\omega)] \cap [0, \infty)$. Show that $T(\omega) = T(\omega')$.

2.3 Proposition: Every random time equal to a positive constant is a stopping time. Every stopping time is optional, and

the two concepts coincide if the filtration is right-continuous.

Proof: The first statement is trivial; the second is based on the observation: $\{T<t\}= \bigcup\limits_{n=1}^{\infty} \{T\le t - \frac{1}{n}\}\in\mathcal{F}_t$, because if T is a stopping time, then $\{T\le t - \frac{1}{n}\}\in\mathcal{F}_{t-\frac{1}{n}} \subseteq \mathcal{F}_t$ for $n\ge 1$. For the third claim, suppose that T is an optional time of the right-continuous filtration $\{\mathcal{F}_t\}$. Since $\{T\le t\} = \bigcap\limits_{\epsilon>0} \{T<t+\epsilon\}$, we have $\{T\le t\}\in\mathcal{F}_{t+\epsilon}$ for every $t\ge 0$ and every $\epsilon>0$; whence $\{T\le t\}\in\mathcal{F}_{t+} = \mathcal{F}_t$. $\square$

Corollary: T is an optional time of the filtration $\{\mathcal{F}_t\}$, if and only if it is a stopping time of the (right-continuous!) filtration $\{\mathcal{F}_{t+}\}$.

2.4 Example: Consider a stochastic process X with right-continuous paths, which is adapted to a filtration $\{\mathcal{F}_t\}$. Consider a subset $\Gamma\in\mathcal{B}(\mathbb{R}^d)$ of the state space of the process, and define the <u>hitting time</u>

$$H_{\Gamma}(\omega) = \begin{cases} \inf\{t\ge 0; X_t(\omega)\in\Gamma\}; & \text{if this set is nonempty} \\ + \infty & ; \text{otherwise.} \end{cases}$$

2.5 Problem: If the set Γ in Example 2.4 is open, show that H_{Γ} is an optional time.

2.6 Problem: If the set Γ in Example 2.4 is closed and the sample paths of the process X are continuous, then H_{Γ} is a stopping time.

Let us establish some simple properties of stopping times.

2.7 Lemma: If T is optional and θ is a positive constant, then T+θ is a stopping time.

Proof: If $0 \leq t < \theta$, then $\{T+\theta \leq t\} = \emptyset \in \mathfrak{F}_t$.

If $t \geq \theta$, then

$$\{T+\theta \leq t\} = \{T \leq t-\theta\} \in \mathfrak{F}_{(t-\theta)+} \subset \mathfrak{F}_t.$$ □

2.8 Lemma: If T,S are stopping times, then so are $T \wedge S$, $T \vee S$, T+S.

Proof: The first two assertions are trivial. For the third, start with the decomposition, valid for $t>0$:

$$\{T+S > t\} = \{T=0; \, S>t\} \cup \{0<T<t, \, T+S>t\} \cup$$

$$\cup \{T>t, \, S=0\} \cup \{T \geq t, \, S>0\}.$$

The first, third and fourth events in this decomposition are in $\mathfrak{F}_t$, either trivially or by virtue of Proposition 2.3. As for the second event, we rewrite it as:

$$\bigcup_{\substack{r \in Q \\ 0<r<t}} \{t>T>r, \, S>t-r\},$$

where Q is the set of rational numbers in $[0, \infty)$. Membership in $\mathfrak{F}_t$ is now trivial. □

2.9 Problem: Let T,S be optional times; then T+S is optional.
It is a stopping time, if one of the following conditions holds:

(i)   T>0, S>0.

(ii)   T>0, T is a stopping time.

2.10 Lemma: Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of optional times; then
the random times

$$\sup_{n \geq 1} T_n, \quad \inf_{n \geq 1} T_n, \quad \overline{\lim_{n \to \infty}} T_n, \quad \underline{\lim_{n \to \infty}} T_n$$

are all optional. Furthermore, if the T_n's are stopping
times, then so is $\sup_{n \geq 1} T_n$.

Proof: Obvious, from Corollary to Proposition 2.3 and from the
identities

$$\{\sup_{n \geq 1} T_n \leq t\} = \bigcap_{n=1}^{\infty} \{T_n \leq t\} \quad \text{and} \quad \{\inf_{n \geq 1} T_n < t\} = \bigcup_{n=1}^{\infty} \{T_n < t\}.$$

$\square$

How can we measure the information accumulated up to a stopping
time T? In order to broach this question, let us suppose that an
event A is part of this information, i.e., that the occurrence
or nonoccurrence of A has been decided by time T. Now if by
time t one observes the value of T, which can happen only if
$T \leq t$, then one must also be able to tell whether A has occurred.
In other words, $A \cap \{T \leq t\}$ and $A^c \cap \{T \leq t\}$ must both be
$\mathfrak{F}_t$-measurable, and this must be the case for any $t \geq 0$. Since

$$A^c \cap \{T \leq t\} = \{T \leq t\} \cap (A \cap \{T \leq t\})^c,$$

it is enough to check only that $A \cap \{T \le t\} \in \mathfrak{F}_t$, $t \ge 0$.

2.11 Definition: Let T be a stopping time of the filtration $\{\mathfrak{F}_t\}$. The $\underline{\sigma\text{-field}}$ $\mathfrak{F}_T$ $\underline{\text{of events determined prior to the}}$ $\underline{\text{the stopping time}}$ T consists of those events $A \in \mathfrak{F}$ for which $A \cap \{T \le t\} \in \mathfrak{F}_t$ for every $t \ge 0$.

2.12 Problem: Verify that $\mathfrak{F}_T$ is actually a σ-field and T is $\mathfrak{F}_T$-measurable.

2.13 Problem: Let T be a stopping time and S a random time such that $S \ge T$ on Ω. If S is $\mathfrak{F}_T$-measurable, then it is also a stopping time.

2.14 Lemma: For any two stopping times T and S, and for any $A \in \mathfrak{F}_S$, we have: $A \cap \{S \le T\} \in \mathfrak{F}_T$.

In particular, if $S \le T$ on Ω, we have $\mathfrak{F}_S \subseteq \mathfrak{F}_T$.

Proof: It is not hard to verify that, for every stopping time T and positive constant t, $T \wedge t$ is an $\mathfrak{F}_t$-measurable random variable. With this in mind, the claim follows from the decomposition:

$$A \cap \{S \le T\} \cap \{T \le t\} = [A \cap \{S \le t\}] \cap \{T \le t\} \cap \{S \wedge t \le T \wedge t\},$$

which shows readily that the left-hand side is an event in $\mathfrak{F}_t$. $\square$

2.15 Lemma: Let T and S be stopping times. Then each of the events

$$\{T<S\}, \ \{S<T\}, \ \{T \leq S\}, \ \{S \leq T\}, \ \{T = S\}$$

belongs to $\mathfrak{F}_T \cap \mathfrak{F}_S$. Besides, $\mathfrak{F}_{T \wedge S} = \mathfrak{F}_T \cap \mathfrak{F}_S$.

Proof: For the last claim, we notice first that $T \wedge S \leq T$, so, by Lemma 2.14, $\mathfrak{F}_{T \wedge S} \subseteq \mathfrak{F}_T \cap \mathfrak{F}_S$. In order to establish the opposite inclusion, let us take $A \in \mathfrak{F}_S \cap \mathfrak{F}_T$ and observe that

$$A \cap \{S \wedge T \leq t\} = A \cap [\{S \leq t\} \cup \{T \leq t\}]$$

$$= [A \cap \{S \leq t\}] \cup [A \cap \{T \leq t\}] \in \mathfrak{F}_t, \quad \text{and}$$

therefore $A \in \mathfrak{F}_{S \wedge T}$.

From Lemma 2.14 we have $\{S \leq T\} \in \mathfrak{F}_T$, and thus $\{S>T\} \in \mathfrak{F}_T$. On the other hand, consider the stopping time $R = S \wedge T$ which, again by virtue of Lemma 2.14, is measurable with respect to $\mathfrak{F}_T$. Therefore, $\{S<T\} = \{R<T\} \in \mathfrak{F}_T$. Interchanging the roles of S, T we see that $\{T>S\}, \{T<S\}$ belong to $\mathfrak{F}_S$, and thus we have shown that both these events belong to $\mathfrak{F}_T \cap \mathfrak{F}_S$. But then the same is true for their complements, and consequently also for $\{S=T\}$. □

2.16 Problem: Let T, S be stopping times and Z an integrable random variable. We have

(i) $E[Z|\mathfrak{F}_T] = E[Z|\mathfrak{F}_{S \wedge T}]$, P-a.s. on $\{T \leq S\}$

(ii) $E[E(Z|\mathfrak{F}_T)|\mathfrak{F}_S] = E[Z|\mathfrak{F}_{S \wedge T}]$, P-a.s. □

Now we can start to appreciate the usefulness of the concept of stopping time in the study of stochastic processes.

2.17 Proposition: Let $X = \{X_t;\ 0 \le t < \infty\}$ be a progressively

measurable process with respect to $\{\mathfrak{F}_t\}$, and let T be a

stopping time of the filtration $\{\mathfrak{F}_t\}$. Then the random

variable X_T of Definition 1.14 is $\mathfrak{F}_T$-measurable and the

"stopped process" $\{X_{T \wedge t};\ 0 \le t < \infty\}$ is progressively measurable.

Proof: For the first claim, one has to show that, for any

$B \in \mathfrak{B}(\mathbb{R}^d)$ and any $t \ge 0$, the event $\{X_T \in B\} \cap \{T \le t\}$ is in $\mathfrak{F}_t$;

but this event can also be written in the form $\{X_{T \wedge t} \in B\} \cap \{T \le t\}$,

and so it is sufficient to prove the progressive measurability

of the stopped process.

To this end, one observes that the mapping $(s, \omega) \mapsto (T(\omega) \wedge s, \omega)$

of $[0,t] \times \Omega$ into itself is $\mathfrak{B}([0,t]) \otimes \mathfrak{F}_t$-measurable. Besides,

by the assumption of progressive measurability, the mapping

$$(s, \omega) \mapsto X_s(\omega):\ ([0,t] \times \Omega,\ \mathfrak{B}([0,t]) \otimes \mathfrak{F}_t) \to (\mathbb{R}^d,\ \mathfrak{B}(\mathbb{R}^d)).$$

is measurable, and therefore the same is true for the composite

mapping

$$(s, \omega) \mapsto X_{T(\omega) \wedge s}(\omega):\ ([0,t] \times \Omega,\ \mathfrak{B}([0,t]) \otimes \mathfrak{F}_t) \to (\mathbb{R}^d,\ \mathfrak{B}(\mathbb{R}^d)).$$

□

2.18 Problem: Under the same assumptions as in Proposition 2.17,

and with $f(t,x):\ [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ a bounded, $\mathfrak{B}([0, \infty)) \otimes \mathfrak{B}(\mathbb{R}^d)$ -

measurable function, show that the process $Y_t = \int_0^t f(s, X_s)ds;$

$t \ge 0$ is progressively measurable with respect to $\{\mathfrak{F}_t\}$, and

that Y_T is an $\mathfrak{F}_T$-measurable random variable.

□

2.19 Definition: Let T be an optional time of the filtration $\{\mathfrak{F}_t\}$. The $\underline{\sigma\text{-field}}$ $\mathfrak{F}_{T+}$ $\underline{\text{of events determined}}$ $\underline{\text{immediately after the optional time}}$ T consists of those events $A \in \mathfrak{F}$ for which $A \cap \{T \leq t\} \in \mathfrak{F}_{t+}$ for every $t \geq 0$.

2.20 Problem: Verify that the class $\mathfrak{F}_{T+}$ is indeed a σ-field with respect to which T is measurable, that it coincides with $\{A \in \mathfrak{F}; A \cap \{T < t\} \in \mathfrak{F}_t, \ \forall \ t \geq 0\}$, and that if T is a stopping time (so that both $\mathfrak{F}_T$, $\mathfrak{F}_{T+}$ are defined), then $\mathfrak{F}_T \subseteq \mathfrak{F}_{T+}$.

2.21 Problem: Verify that analogues of Lemmas 2.14 and 2.15 hold if T and S are assumed to be optional and $\mathfrak{F}_T$, $\mathfrak{F}_S$ and $\mathfrak{F}_{T \wedge S}$ are replaced by $\mathfrak{F}_{T+}$, $\mathfrak{F}_{S+}$ and $\mathfrak{F}_{(T \wedge S)+}$, respectively.

Prove that if S is an optional time and T is a stopping time with $S \leq T$, and $S < T$ on $\{S < \infty\} \cap \{T > 0\}$, then $\mathfrak{F}_{S+} \subseteq \mathfrak{F}_T$.

2.22 Problem: Show that if $\{T_n\}_{n=1}^{\infty}$ is a sequence of optional times and $T = \inf_{n \geq 1} T_n$, then $\mathfrak{F}_{T+} = \bigcap_{n=1}^{\infty} \mathfrak{F}_{T_n+}$. Besides, if each T_n is a stopping time and $T < T_n$ on $\{T < \infty\} \cap \{T_n > 0\}$, then we have $\mathfrak{F}_{T+} = \bigcap_{n=1}^{\infty} \mathfrak{F}_{T_n}$.

2.23 **Problem**: Given an optional time T for the family of σ-fields $\{\mathfrak{F}_t\}$, consider the sequence $\{T_n\}_{n=1}^{\infty}$ of random times given by

$$T_n(\omega) = T(\omega); \quad \text{on} \quad \{\omega; \ T(\omega) = +\infty\}$$

$$= \frac{k}{2^n} \quad ; \quad \text{on} \quad \{\omega; \ \frac{k-1}{2^n} \leq T(\omega) < \frac{k}{2^n}\}$$

for $n \geq 1$, $k \geq 1$. Obviously $T_n \geq T_{n+1} \geq T$, for every $n \geq 1$. Show that each T_n is a stopping time, that $\lim\limits_{n \to \infty} T_n = T$, and that for every $\Lambda \in \mathfrak{F}_{T+}$, $n \geq 1$, $k \geq 1$ we have: $\Lambda \cap \{T_n = \frac{k}{2^n}\} \in \mathfrak{F}_{\frac{k}{2^n}}$.

THE FOLLOWING SECTION OF THIS REPORT (1.3) IS MISSING THE
EVEN NUMBERED PAGES. THIS IS THE MOST COMPLETE VERSION
AVAILIBLE.

1.3: CONTINUOUS - TIME MARTINGALES

We assume in this section that the reader is familiar with the concept and basic properties of martingales in discrete time. An excellent presentation of this material can be found in Chung [1, §9.3 and 9.4, pp. 319-341] and we shall cite from this source frequently. The purpose of this section is to extend the discrete-time results to continuous-time martingales.

The standard example of a continuous-time martingale is one-dimensional Brownian motion. This process can be regarded as the continuous-time version of the one-dimensional symmetric random walk, as we shall see in Chapter 2. Since we have not yet introduced Brownian motion, we shall take instead the compensated Poisson process as a continuing example developed in the problems throughout this section. The compensated Poisson process is a martingale which will serve us later in the construction of Poisson random measures, a tool necessary for the treatment of excursions of Brownian motion.

In this section we shall consider exclusively real-valued processes $X = \{X_t; 0 \le t < \infty\}$ on a probability space $(\Omega, \mathcal{F}, P)$, adapted to a given filtration $\{\mathcal{F}_t\}$ and such that $E|X_t| < \infty$ holds for every $t \ge 0$.

3.1. Definition: $\{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ as above is said to be a submartingale (respectively, a supermartingale) if, for every $0 \le s < t < \infty$ we have, a.s. P: $E(X_t | \mathcal{F}_s) \ge X_s$ (respectively, $E(X_t | \mathcal{F}_s) \le X_s$).

We shall say that $\{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ is a martingale if it

(ii) Show that for $0 \leq s < t$, $N_t - N_s$ is independent of $\mathfrak{F}_s^N$.

(Hint: It suffices to show that for arbitrary positive integer m,

$$P[S_{N_s+1} - s > t_1, \ T_{N_s+2} > t_2, \ \ldots, \ T_{N_s+m} > t_m \,|\, \mathfrak{F}_s^N]$$

is constant. Indeed, it equals $P[T_1 > t_1, \ T_2 > t_2, \ \ldots, \ T_m > t_m])$.

(iii) Prove that for $0 \leq s < t$, $N_t - N_s$ is a Poisson random variable with mean $\lambda(t-s)$.

3.3. Definition: A _Poisson process with intensity_ $\lambda > 0$ $\{N_t, \ t \geq 0\}$ is an integer-valued, right-continuous process such that $N_0 = 0$ a.s., and for $0 \leq s < t$, $N_t - N_s$ is independent of $\mathfrak{F}_s$ and is Poisson distributed with mean $\lambda(t-s)$.

We have demonstrated in Problem 3.2 that Poisson processes exist. Given a Poisson process N_t with intensity λ, we define the _compensated Poisson process_

$$M_t = N_t - \lambda t, \quad t \geq 0.$$

Note that the filtrations $\{\mathfrak{F}_t^M\}$ and $\{\mathfrak{F}_t^N\}$ agree.

3.4. Problem: Prove that a compensated Poisson process $\{M_t, \ \mathfrak{F}_t^M; \ t \geq 0\}$ is a martingale.

The following theorem extends to the continuous-time case certain well-known results of discrete martingales.

3.6. <u>Theorem:</u> Let $\{X_t, \mathfrak{F}_t; 0 \leq t < \infty\}$ be a right-continuous submartingale, $[\sigma, \tau]$ an interval of $[0, \infty)$ and $\alpha < \beta$, $\lambda > 0$ given real numbers. We have the following results:

(i) <u>First submartingale inequality:</u>

$$\lambda \cdot P[\sup_{\sigma \leq t \leq \tau} X_t \geq \lambda] \leq E(X_\tau^+).$$

(ii) <u>Second submartingale inequality:</u>

$$\lambda \cdot P[\inf_{\sigma \leq t \leq \tau} X_t \leq -\lambda] \leq E(X_\tau^+) - E(X_\sigma).$$

(iii) <u>Upcrossings inequality:</u>

$$E\nu_{[\sigma, \tau]}(\omega; [\alpha, \beta]) \leq \frac{E(X_\tau^+) + |\alpha|}{\beta - \alpha}.$$

(iv) <u>Doob's maximal inequality:</u>

$$E(\sup_{\sigma \leq t \leq \tau} X_t)^p \leq (\frac{p}{p-1})^p E(X_\tau^p), \quad p > 1,$$

provided $X_t \geq 0$ a.s. P for every $t \geq 0$, and $E(X_\tau^p) < \infty$.

(v) <u>Regularity of the paths:</u> Almost every sample path $\{X_t(\omega); 0 \leq t < \infty\}$ is bounded on compact intervals, and is free of discontinuities of the second kind, i.e., admits left-hand limits everywhere on $(0, \infty)$.

3.7. Problem: Let N_t be a Poisson process with intensity λ.

(a) For any $c > 0$,

$$\varlimsup_{t \to \infty} P[\sup_{0 \leq s \leq t} (N_s - \lambda s) \geq c\sqrt{\lambda t}] \leq \frac{1}{c\sqrt{2\pi}}$$

(b) For any $c > 0$,

$$\varlimsup_{t \to \infty} P[\inf_{0 \leq s \leq t} (N_s - \lambda s) \leq - c\sqrt{\lambda t}] \leq \frac{1}{c\sqrt{2\pi}} .$$

(c) For $0 < \sigma < \tau$, we have

$$E[\sup_{\sigma \leq t \leq \tau} (\frac{N_t}{t} - \lambda)^2] \leq \frac{4\tau\lambda}{\sigma^2} .$$

3.7′ Remark: From Problem 3.7 (a) and (b), we see that for each $c > 0$, there exists $T_c > 0$ such that

$$P[|\frac{N_t}{t} - \lambda| \geq c\sqrt{\frac{\lambda}{t}}] \leq \frac{3}{c\sqrt{2\pi}}, \forall t \geq T_c .$$

From this we can conclude the "weak law of large number" for Poisson processes: $\frac{N_t}{t} \to \lambda$, in probability as $t \to \infty$. In fact, by choosing $\sigma = 2^n$ and $\tau = 2^{n+1}$ in Problem 3.7 (c) and using Chebyshev's inequality, one can show

$$P[\sup_{2^n \leq t \leq 2^{n+1}} |\frac{N_t}{t} - \lambda| \geq \epsilon] \leq \frac{8\lambda}{\epsilon^2 2^n}$$

for every $n \geq 1$, $\epsilon > 0$. Then by a Borel-Cantelli argument (see Chung [1], Theorems 4.2.1, 4.2.2), we obtain the "strong law of large

all the P-negligible events in $\mathfrak{F}$.

3.11. Theorem: Let $\{X_t, \mathfrak{F}_t; 0 \leq t < \infty\}$ be a submartingale, and assume the filtration $\{\mathfrak{F}_t\}$ satisfies the usual conditions. Then the process $X = \{X_t; 0 \leq t < \infty\}$ has a right-continuous modification if and only if the function $t \mapsto EX_t$ from $[0, \infty)$ to R is right-continuous. If this right-continous modification exists, it can be chosen so as to be adapted to $\{\mathfrak{F}_t\}$, hence a submartingale with respect to $\{\mathfrak{F}_t\}$. $\square$

The proof of Theorem 3.11 requires the following proposition, which we prove first.

3.12 Proposition: Let $\{X_t, \mathfrak{F}_t; 0 \leq t < \infty\}$ be a submartingale. We have the following:

(i) The limits $X_{t+}(\omega) \triangleq \lim_{\substack{s \downarrow t \\ s \in Q}} X_s(\omega)$, $X_{t-}(\omega) \triangleq \lim_{\substack{s \uparrow t \\ s \in Q}} X_s(\omega)$

exist almost surely, for every $t \geq 0$ (respectively, $t > 0$).

(ii) The limits in (i) satisfy
$$E(X_{t+}|\mathfrak{F}_t) \geq X_t \quad \text{a.s. } P, \quad \forall \, t \geq 0.$$
$$E(X_t|\mathfrak{F}_{t-}) \geq X_{t-} \quad \text{a.s. } P, \quad \forall \, t > 0.$$

(iii) $\{X_{t+}, \mathfrak{F}_{t+}; 0 \leq t < \infty\}$ is a submartingale.

Proof: (i) We wish to imitate the proof of (v), Theorem 3.6, but because we have not assumed right-continuity of sample paths, we

as well as the P. Lévy Theorem 9.4.8 in Chung [1], help us identify this limit as $X_{t-} - E(X_t | \mathfrak{F}_{t-})$, which is thus shown to be non-positive.

(iii) Now we take two monotone decreasing sequences $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ of rational numbers, with $0 \leq s < s_n < t < t_n$ holding for every $n \geq 1$ and $\lim_{n \to \infty} s_n = s$, $\lim_{n \to \infty} t_n = t$. For fixed $n \geq 1$ and arbitrary ϵ in $(0, s_n - s)$, the submartingale property yields $\int_A X_{s_n} dP \leq \int_A X_{t_n} dP$, for every event A in $\mathfrak{F}_{s+\epsilon}$, and therefore for every A in $\mathfrak{F}_{s+} = \bigcap_{\epsilon > 0} \mathfrak{F}_{s+\epsilon}$. By the uniform integrability of both sequences $\{X_{s_n}\}_{n=1}^{\infty}$, $\{X_{t_n}\}_{n=1}^{\infty}$, we conclude that $\int_A X_{s+} dP \leq \int_A X_{t+} dP$, $\forall A \in \mathfrak{F}_{s+}$.

$\square$

Proof of Theorem 3.11:

Let X_{t+} be as in Proposition 3.12. Since $\{\mathfrak{F}_t\}$ is a right-continuous filtration and $\mathfrak{F}_0$ contains all P-negligible events of $\mathfrak{F}$, X_{t+} is $\mathfrak{F}_t$-measurable. Proposition 3.12 (ii) implies $X_{t+} \geq X_t$ a.s. P, for every $t \geq 0$. Thus, the (right-continuous!) process $\{X_{t+}; 0 \leq t < \infty\}$ is a modification of the process $\{X_t; 0 \leq t < \infty\}$ if and only if $E X_{t+} = E X_t$ for every $t \geq 0$. But the uniform integrability of $\{X_{t_n}\}_{n=1}^{\infty}$ with arbitrary sequence $t_n \downarrow t$, not necessarily through Q (Problem 3.8), yields $E(X_{t+}) = \lim_{n \to \infty} E(X_{t_n})$, and the stated condition amounts to right-continuity of the function $t \mapsto E(X_t)$.

Conversely, if $\{\tilde{X}_t; t \geq 0\}$ is a right-continuous modification of $\{X_t; t \geq 0\}$, then $E(\tilde{X}_t) = E(X_t)$ holds for every $t \geq 0$; besides,

$$E|X_t| = 2E(X_t^+) - E(X_t) \leq 2C - EX_0$$

shows that the assumption $\sup\limits_{t \geq 0} E(X_t^+) < \infty$ is equivalent to the apparently stronger one $\sup\limits_{t \geq 0} E|X_t| < \infty$, which in turn forces the integrability of X_∞, by Fatou's Lemma. $\qquad\qquad\square$

3.14 Problem: Let $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a right-continuous non-negative supermartingale; then $X_\infty(\omega) = \lim\limits_{t \to \infty} X_t(\omega)$ exists for P-a.e. $\omega \in \Omega$, and $\{X_t, \mathcal{F}_t; 0 \leq t \leq \infty\}$ is a supermartingale.

3.15 Definition: A right-continuous nonnegative supermartingale $\{Z_t, \mathcal{F}_t; 0 \leq t < \infty\}$ with $\lim\limits_{t \to \infty} E(Z_t) = 0$ is called a <u>potential</u>. $\qquad\square$

Problem 3.14 guarantees that a potential $\{Z_t, \mathcal{F}_t; 0 \leq t < \infty\}$ has a last element Z_∞, and $Z_\infty = 0$ a.s. P.

3.16 Problem: Suppose that the filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions. Then every right-continuous, uniformly integrable supermartingale $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ admits the <u>Riesz decomposition</u> $X_t = M_t + Z_t$, a.s. P, as the sum of a right-continuous, uniformly integrable martingale $\{M_t, \mathcal{F}_t; 0 \leq t < \infty\}$ and a potential $\{Z_t, \mathcal{F}_t; 0 \leq t < \infty\}$.

3.17 Problem: The following three conditions are equivalent for a right-continuous submartingale $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$:

(a) it is a uniformly integrable family of random variables;

What can happen if one samples a martingale at random, instead of fixed, times? For instance, if X_t represents the fortune, at time t, of an indefatiguable gambler (who plays continuously!) engaged in a "fair" game, can he hope to improve his expected fortune by judicious choice of the time-to-quit? If no clairvoyance into the future is allowed (in other words, if our gambler is restricted to quit at stopping times), and if there is any justice in the world, the answer should be "no". Doob's Optional Sampling Theorem tells us under what conditions we can expect this to be true.

3.20 Theorem: Optional Sampling

Let $\{X_t, \mathcal{F}_t; 0 \leq t \leq \infty\}$ be a right-continuous submartingale with a last element X_∞, and let $S \leq T$ be two optional times of the filtration $\{\mathcal{F}_t\}$. We have

$$E(X_T | \mathcal{F}_{S+}) \geq X_S, \quad \text{a.s. } P.$$

If S is a stopping time, then $\mathcal{F}_S$ can replace $\mathcal{F}_{S+}$ above. In particular, $EX_T \geq EX_0$, and for a martingale with a last element, we have $EX_T = EX_0$.

Proof: Consider the sequence of random times

$$S_n(\omega) = \begin{cases} S(\omega) & \text{if } S(\omega) = +\infty \\ \dfrac{k}{2^n} & \text{if } \dfrac{k-1}{2^n} \leq S(\omega) < \dfrac{k}{2^n}, \end{cases}$$

and the similarly defined sequences $\{T_n\}$. These were shown in Problem 2.24 to be stopping times. For every fixed integer $n \geq 1$, both S_n and T_n take on a countable number of values and we

3.22 Problem: Suppose that $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a right-continuous submartingale and $S \leq T$ are stopping times of $\{\mathcal{F}_t\}$. Then

(i) $\{X_{T \wedge t}, \mathcal{F}_t; 0 \leq t < \infty\}$ is a submartingale;

(ii) $E[X_{T \wedge t} | \mathcal{F}_S] \geq X_{S \wedge t}$ a.s. P, for every $t \geq 0$.

3.23 Problem: A submartingale of constant expectation, i.e., with $E(X_t) = E(X_0)$ for every $t \geq 0$, is a martingale.

3.24 Problem: A process $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ with $E|X_t| < \infty$, $0 \leq t < \infty$, is a submartingale, if and only if for every pair $S \leq T$ of bounded stopping times of the filtration $\{\mathcal{F}_t\}$ we have:

$$E(X_T) \geq E(X_S), \quad \text{a.s. P.}$$

3.25 Problem: Let $Z = \{Z_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a continuous, nonnegative martingale with $Z_\infty \triangleq \lim_{t \to \infty} Z_t = 0$, a.s. P. Then for every $s \geq 0$, $b > 0$:

(i) $P[\sup_{t > s} Z_t \geq b | \mathcal{F}_s] = \frac{1}{b} Z_s$, on $\{Z_s < b\}$.

(ii) $P[\sup_{t \geq s} Z_t \geq b] = P[Z_s \geq b] + \frac{1}{b} E[Z_s 1_{\{Z_s < b\}}]$.

1.4 THE DOOB-MEYER DECOMPOSITION

<u>4.1 Definition</u>: Consider a probability space $(\Omega, \mathfrak{F}, P)$ and a random sequence $\{A_n\}_{n=0}^{\infty}$ adapted to the discrete filtration $\{\mathfrak{F}_n\}_{n=0}^{\infty}$. The sequence is called <u>increasing</u>, if for P - a.e. $\omega \in \Omega$ we have $0 = A_0(\omega) \leq A_1(\omega) \leq \ldots,$ and $E(A_n) < \infty$ holds for every $n \geq 1$.

An increasing sequence is called <u>integrable</u> if $E(A_\infty) < \infty$ where $A_\infty = \lim_{n \to \infty} A_n$. An arbitrary random sequence is called <u>predict-</u> <u>able</u> for the filtration $\{\mathfrak{F}_n\}_{n=0}^{\infty}$, if for every $n \geq 1$ the random variable A_n is $\mathfrak{F}_{n-1}$ - measurable. Note that if $A = \{A_n, \mathfrak{F}_n;$ $n=0,1,\ldots\}$ is predictable with $E|A_n| < \infty$ for every n, and if $\{M_n, \mathfrak{F}_n; n=0,1,\ldots\}$ is a bounded martingale, then the <u>martingale</u> <u>transform</u> of A by M defined by

$$Y_0 = 0,$$

(4.1)

$$Y_n = \sum_{k=1}^{n} A_k(M_k - M_{k-1}), \quad n \geq 1,$$

is itself a martingale. This martingale transform is the discrete-time version of the stochastic integral with respect to a martingale, defined in Chapter 3. A fundamental property of such integrals is that they are martingales when parameterized by their upper limit of integration. □

Let us recall from Chung [1], Theorem 9.3.2 and Exercise 9.3.9, that any submartingale $\{X_n, \mathfrak{F}_n; n=0,1,\ldots\}$ admits the <u>Doob decomposi-</u>

__tion__ $X_n = M_n + A_n$ as the summation of a martingale $\{M_n, \mathfrak{F}_n\}$
and an increasing sequence $\{A_n, \mathfrak{F}_n\}$. It suffices for this to take
$A_0 = 0$ and $A_{n+1} = A_n - X_n + E(X_{n+1}|\mathfrak{F}_n) = \sum_{k=0}^{n}[E(X_{k+1}|\mathfrak{F}_k) - X_k]$, for
$n \geq 0$. This increasing sequence is actually predictable, and with
this proviso the Doob decomposition of a submartingale is unique.

We shall try in this section to extend the Doob decomposition
to suitable continuous-time submartingales. In order to motivate
the developments, let us discuss the concept of predictability for
stochastic sequences in some further detail.

__4.2 Definition:__ An increasing sequence $\{A_n, \mathfrak{F}_n; n=0, 1, \ldots\}$ is
called __natural__ if for every bounded martingale $\{M_n, \mathfrak{F}_n; n=0,1,\ldots\}$
we have

(4.2) $E(M_n A_n) = E \sum_{k=1}^{n} M_{k-1}(A_k - A_{k-1})$, $\forall \, n \geq 1$.

A simple rewrite of (4.1) shows that an increasing sequence
A is natural if and only if the martingale transform $Y = \{Y_n\}_{n=0}^{\infty}$
of A by every bounded martingale M satisfies $EY_n = 0$, $n \geq 0$.
It is clear then from our discussion of martingale transforms that
every predictable increasing sequence is natural. We now prove
the equivalence of these two concepts.

__4.3 Proposition:__ An increasing random sequence A is predictable
if and only if it is natural.

Proof: It remains only to show that a natural increasing sequence is predictable. Suppose that A is natural and M is a bounded martingale. With $\{Y_n\}_{n=0}^{\infty}$ defined by (4.1), we have

$$E[A_n(M_n-M_{n-1})] = EY_n-EY_{n-1} = 0, \quad n \geq 1.$$

It follows that

$$(4.3) \quad E[M_n\{A_n-E(A_n|\mathfrak{F}_{n-1})\}] =$$

$$= E[(M_n-M_{n-1})A_n] + E[M_{n-1}\{A_n-E(A_n|\mathfrak{F}_{n-1})\}]$$

$$- E[(M_n-M_{n-1}) E(A_n|\mathfrak{F}_{n-1})] = 0$$

for every $n \geq 1$. Let us take an arbitrary but fixed integer $n \geq 1$, and show that the random variable A_n is $\mathfrak{F}_{n-1}$ - measurable. Consider (4.3) for this fixed integer, and let the martingale M be given by

$$M_k = \begin{cases} \text{sgn}[A_n-E(A_n|\mathfrak{F}_{n-1})], & k = n, \\ M_n, & k > n, \\ E(M_n|\mathfrak{F}_k), & k = 0,1,\ldots,n. \end{cases}$$

We obtain $E|A_n-E(A_n|\mathfrak{F}_{n-1})| = 0$, whence the desired conclusion. $\square$

From now on we shall revert to our filtration $\{\mathfrak{F}_t\}$ parametrized by $t \in [0,\infty)$ on the probability space $(\Omega,\mathfrak{F},P)$. Let us consider a process $A = \{A_t; 0 \leq t < \infty\}$ adapted to $\{\mathfrak{F}_t\}$. By analogy with Definitions 4.1 and 4.2, we have the following:

4.4 <u>Definition</u>: A process A as above is called <u>increasing</u> if

for P - a.e. $\omega \in \Omega$ we have $A_0(\omega) = 0$, and $t \mapsto A_t(\omega)$ is

a nondecreasing, right-continuous function, and $E(A_t) < \infty$

holds for $0 \leq t < \infty$.

An increasing process is called <u>integrable</u> if $E(A_\infty) < \infty$,

where $A_\infty = \lim_{t \to \infty} A_t$; an arbitrary process A adapted to the filtra-

tion $\{\mathfrak{F}_t\}$ is called <u>predictable</u> with respect to $\{\mathfrak{F}_t\}$ if A_t is

$\mathfrak{F}_{t-}$ - measurable for every $0 \leq t < \infty$.

4.5 <u>Definition</u>: An increasing process A is called <u>natural</u> if

for every bounded, right-continuous martingale $\{M_t, \mathfrak{F}_t; \ 0 \leq t < \infty\}$

we have

$$(4.4) \qquad E \int_{(0,t]} M_s dA_s = E \int_{(0,t]} M_{s-} dA_s. \qquad \qquad \square$$

Clearly, any increasing and continuous process is both pre-

dictable and natural. It can be shown that every natural increas-

ing process is predictable (Theorem 3.10 in Liptser & Shiryaev

[13]). Rather than dealing with this thorny issue, we will not

use the concept of predictability for continuous-time processes,

although our proof of the existence of a "Doob decomposition" for

continuous-time processes does rely on the equivalence proved

in Proposition 4.3 for discrete-time processes.

4.6 <u>Remark on notation</u>: If A is an increasing and X. a measurable

process, then with $\omega \in \Omega$ fixed the sample path $\{X_t(\omega); \ 0 \leq t < \infty\}$

is a measurable function from $[0, \infty)$ into $\mathbb{R}$. It follows that the Lebesgue-Stieltjes integral

$$I_t(\omega) \triangleq \int_{(0, t]} X_s(\omega) dA_s(\omega)$$

is well-defined; in particular, if X is progressively measurable (e.g., right-continuous and adapted), then the right-continuous process $\{I_t;\ 0 \leq t < \infty\}$ with $I_0 = 0$ is also progressively measurable.

4.7 Lemma: In Definition 4.5, condition (4.4) is equivalent to

$$(4.4)' \qquad E(M_t A_t) = E\int_{(0, t]} M_{s-} dA_s.$$

Proof: Consider a partition $\Pi = \{t_0, t_1, \ldots, t_n\}$ of $[0, t]$, with $0 = t_0 \leq t_1 \leq \ldots \leq t_n = t$, and define

$$M_s^\Pi = \sum_{k=1}^{n} M_{t_k} 1_{(t_{k-1}, t_k]}(s).$$

The martingale property of M yields

$$E\int_{(0, t]} M_s^\Pi dA_s = E \sum_{k=1}^{n} M_{t_k}(A_{t_k} - A_{t_{k-1}}) = E\left[\sum_{k=1}^{n} M_{t_k} A_{t_k} - \sum_{k=1}^{n-1} M_{t_{k+1}} A_{t_k} \right]$$

$$= EM_t A_t + E \sum_{k=1}^{n} A_{t_k}(M_{t_{k+1}} - M_{t_k}) = E(M_t A_t).$$

Now let $\|\Pi\| \triangleq \max_{1 \leq k \leq n} (t_k - t_{k-1}) \to 0$, so $M_s^\Pi \to M_s$, and use the Bounded Convergence Theorem for Lebesgue-Stieltjes integration to obtain

(4.5) $E(M_t\, A_t) = \int\limits_{(0,t]} M_s\, dA_s .$ □

The following concept is a strengthening of the notion of uniform integrability for submartingales.

4.8 <u>Definition</u>: Let us consider the class $\mathcal{S}(\mathcal{S}_a)$ of all stopping times T of the filtration $\{\mathcal{F}_t\}$ which satisfy $P(T<\infty) = 1$ (respectively, $P(T\le a) = 1$ for a given finite number $a>0$). The right-continuous submartingale $\{X_t, \mathcal{F}_t;\ 0\le t<\infty\}$ is said to be <u>of class D</u>, if the family $\{X_T\}_{T\in\mathcal{S}}$ is uniformly integrable; <u>of class DL,</u> if the family $\{X_T\}_{T\in\mathcal{S}_a}$ is uniformly integrable, for every $0<a<\infty$.   □

4.9 <u>Problem</u>: Suppose $X = \{X_t, \mathcal{F}_t;\ 0\le t<\infty\}$ is a submartingale. Show that under any one of the following, conditions, X is of class DL.

(a) $X_t \ge 0$ a.s. for every $t\ge 0$.

.(b) X has the special form

(4.6) $X_t = M_t + A_t,$ $0\le t<\infty$

suggested by the Doob decomposition, where $\{M_t, \mathcal{F}_t;\ 0\le t<\infty\}$ is a martingale and $\{A_t, \mathcal{F}_t;\ 0\le t<\infty\}$ is an increasing process.

(c) X is a martingale.

Show also that if X is a uniformly integrable martingale, then it is of class D.

The celebrated theorem which follows asserts that membership in DL is also a sufficient condition for the decomposition of the semimartingale X in the form (4.6).

4.10 Theorem: Doob-Meyer decomposition.

Let $\{\mathfrak{F}_t\}$ satisfy the usual conditions (Definition 3.10). If the right-continuous submartingale $X = \{X_t, \mathfrak{F}_t; 0 \le t < \infty\}$ is of class DL, then it admits the decomposition (4.6) as the summation of a right-continuous martingale $M = \{M_t, \mathfrak{F}_t; 0 \le t < \infty\}$ and an increasing process $A = \{A_t, \mathfrak{F}_t; 0 \le t < \infty\}$.

The latter can be taken to be natural; under this further condition the decomposition (4.6) is unique (up to indistinguishability). Further, if X is of class D, then M is a uniformly integrable martingale, and A is integrable. $\square$

Proof: For uniqueness, let us assume that X admits both decom-positions $X_t = M'_t + A'_t = M''_t + A''_t$, where M' and M'' are martingales and A', A'' are natural increasing processes. Then $\{B_t \triangleq A'_t - A''_t = M''_t - M'_t, \mathfrak{F}_t; 0 \le t < \infty\}$ is a martingale (of bounded variation), and for every bounded and right-continuous martingale $\{\xi_t, \mathfrak{F}_t\}$ we have

$$E[\xi_t(A'_t - A''_t)] = E \int_{(0,t]} \xi_{s-} dB_s = \lim_{n \to \infty} E \sum_{j=1}^{m_n} \xi_{t_{j-1}^{(n)}} [B_{t_j^{(n)}} - B_{t_{j-1}^{(n)}}],$$

where $\Pi_n = \{t_0^{(n)}, \ldots, t_{m_n}^{(n)}\}$, $n \ge 1$ is a sequence of partitions of $[0,t]$ with $|\Pi_n| = \max_{1 \le j \le m_n} |t_j^{(n)} - t_{j-1}^{(n)}|$ converging to zero as $n \to \infty$.

But now

$$E[\xi_{t_{j-1}^{(n)}}(B_{t_j^{(n)}} - B_{t_{j-1}^{(n)}})] = 0, \quad \text{and thus} \quad E[\xi_t(A'_t - A''_t)] = 0.$$

For an arbitrary bounded random variable ξ, we can select $\{\xi_t, \mathfrak{F}_t\}$ to be a right-continuous modification of $\{E[\xi|\mathfrak{F}_t], \mathfrak{F}_t\}$ (Theorem 3.11); we obtain $E[\xi(A'_t - A''_t)] = 0$ and therefore $P(A'_t = A''_t) = 1$, for every $t \geq 0$. The right-continuity of A' and A'' now gives us their indistinguishability.

For the <u>existence</u> of the decomposition (4.6) on $[0, \infty)$, with X of class DL, it suffices to establish it on every finite interval $[0, a]$; by uniqueness, we can then extend the construction to the entire of $[0, \infty)$. Thus, for fixed $0 < a < \infty$, let us select a right-continuous modification of the nonpositive submartingale

$$Y_t \triangleq X_t - E[X_a|\mathfrak{F}_t], \quad 0 \leq t \leq a.$$

Let us consider the partitions $\Pi_n = \{t_0^{(n)}, t_1^{(n)}, \ldots, t_{2^n}^{(n)}\}$ of the interval $[0, a]$ of the form $t_j^{(n)} = \frac{j}{2^n} a$, $j = 0, 1, \ldots, 2^n$. For every $n \geq 1$, we have the Doob decomposition

$$Y_{t_j^{(n)}} = M_{t_j^{(n)}}^{(n)} + A_{t_j^{(n)}}^{(n)}, \quad j = 0, 1, \ldots, 2^n$$

where the predictable increasing sequence $A^{(n)}$ is given by

$$A_{t_j^{(n)}}^{(n)} = A_{t_{j-1}^{(n)}}^{(n)} + E[Y_{t_j^{(n)}} - Y_{t_{j-1}^{(n)}}|\mathfrak{F}_{t_{j-1}^{(n)}}]$$

$$= \sum_{k=0}^{j-1} E[Y_{t_{k+1}^{(n)}}(n) - Y_{t_k^{(n)}}(n) | \mathfrak{F}_{t_k^{(n)}}] \ , \quad j=1,\ldots,2^n.$$

We also notice that

$$(4.7) \qquad Y_{t_j^{(n)}}(n) = A_{t_j^{(n)}}^{(n)} - E[A_a^{(n)} | \mathfrak{F}_{t_j^{(n)}}] \ , \quad j=0,1,\ldots,2^n.$$

We now show that <u>the sequence</u> $\{A_a^{(n)}\}_{n=1}^{\infty}$ <u>is uniformly</u> <u>integrable.</u> With $\lambda > 0$, we define the random times

$$T_\lambda^{(n)} = \begin{cases} \min\{t_{j-1}^{(n)}; \ A_{t_j^{(n)}}^{(n)} > \lambda \quad \text{for some} \ j, \ 1 \le j \le 2^n\}, \\[2mm] a, \quad \text{if the above set is empty.} \end{cases}$$

We have $\{T_\lambda^{(n)} \le t_{j-1}^{(n)}\} = \{A_{t_j^{(n)}}^{(n)} > \lambda\} \in \mathfrak{F}_{t_{j-1}^{(n)}}$ for $j=1,\ldots,2^n$, and

$\{T_\lambda^{(n)} < a\} = \{A_a^{(n)} > \lambda\}$. Therefore, $T_\lambda^{(n)} \in \mathbf{S}_a$. On each set

$\{T_\lambda^{(n)} = t_j^{(n)}\}$, we have $E[A_a^{(n)} | \mathfrak{F}_{t_j^{(n)}}] = E[A_a^{(n)} | \mathfrak{F}_{T_\lambda^{(n)}}]$, so (4.7)

implies

$$(4.8) \qquad Y_{T_\lambda^{(n)}}(n) = A_{T_\lambda^{(n)}}^{(n)} - E[A_a^{(n)} | \mathfrak{F}_{T_\lambda^{(n)}}] \le \lambda - E[A_a^{(n)} | \mathfrak{F}_{T_\lambda^{(n)}}]$$

on $\{T_\lambda^{(n)} < a\}$. Thus

$$(4.9) \qquad \int_{\{A_a^{(n)} > \lambda\}} A_a^{(n)} dP \le \lambda P[T_\lambda^{(n)} < a] - \int_{\{T_\lambda^{(n)} < a\}} Y_{T_\lambda^{(n)}}(n) dP.$$

Replacing λ by $\frac{\lambda}{2}$ in (4.8) and integrating the equality over the $\mathcal{F}_{T_{\lambda/2}^{(n)}}$- measurable set $\{T_{\frac{\lambda}{2}}^{(n)} < a\}$, we obtain

$$- \int_{\{T_{\frac{\lambda}{2}}^{(n)}<a\}} Y_{T_{\lambda/2}^{(n)}} \, dP = \int_{\{T_{\frac{\lambda}{2}}^{(n)}<a\}} (A_a^{(n)} - A_{T_{\lambda/2}^{(n)}}^{(n)}) \, dP$$

$$\geq \int_{\{T_{\lambda}^{(n)}<a\}} (A_a^{(n)} - A_{T_{\lambda/2}^{(n)}}^{(n)}) \, dP \geq \frac{\lambda}{2} \, P[T_{\lambda}^{(n)}<a],$$

and thus (4.9) leads to

$$(4.10) \qquad \int_{\{A_a^{(n)}>\lambda\}} A_a^{(n)} \, dP \leq -2 \int_{\{T_{\lambda/2}^{(n)}<a\}} Y_{T_{\lambda}^{(n)}} \, dP - \int_{\{T_{\lambda}^{(n)}<a\}} Y_{T_{\lambda}^{(n)}} \, dP.$$

The family $\{X_T\}_{T\in\mathcal{S}_a}$ is uniformly integrable by assumption, and thus so is $\{Y_T\}_{T\in\mathcal{S}_a}$. But

$$P[T_{\lambda}^{(n)}<a] = P[A_a^{(n)}>\lambda] \leq \frac{E(A_a^{(n)})}{\lambda} = -\frac{E(Y_0)}{\lambda} \, ,$$

so

$$\sup_{n\geq 1} P[T_{\lambda}^{(n)}<a] \to 0 \quad \text{as} \quad \lambda \to \infty \, .$$

Since the sequence $\{Y_{T_c^{(n)}}\}_{n=1}^{\infty}$ is uniformly integrable for every $c>0$, it follows from (4.10) that the sequence $\{A_a^{(n)}\}_{n=0}^{\infty}$ is also uniformly integrable.

By the Dunford-Pettis compactness criterion (Meyer [16], p. 20 or Dunford & Schwartz [6], p. 294), uniform integrability of the sequence $\{A_a^{(n)}\}_{n=1}^{\infty}$ guarantees the existence of an integrable random variable A_a, as well as of a subsequence $\{A_a^{(n_k)}\}_{k=1}^{\infty}$ which converges to A_a weakly in L^1:

$$\lim_{k \to \infty} E(\xi A_a^{(n_k)}) = E(\xi A_a)$$

for every bounded random variable ξ. To simplify typography we shall assume henceforth that the above subsequence has been relabelled, and we shall denote it henceforth by $\{A_a^{(n)}\}_{n=1}^{\infty}$. By analogy with (4.7), we define the process $\{A_t, \mathfrak{F}_t\}$ as a right-continuous modification of

$$(4.11) \qquad A_t = Y_t + E(A_a | \mathfrak{F}_t); \quad 0 \le t \le a.$$

<u>4.11 Problem</u>: Show that if $\{A^{(n)}\}_{n=1}^{\infty}$ is a sequence of integrable random variables on a probability space $(\Omega, \mathfrak{F}, P)$ which converges weakly in L^1 to an integrable random variable A, then for each σ-field $\mathscr{G} \subset \mathfrak{F}$, the sequence $E[A^{(n)}|\mathscr{G}]$ converges to $E[A|\mathscr{G}]$ weakly in L^1.

Let $\Pi = \bigcup_{n=1}^{\infty} \Pi_n$. For $t \in \Pi$, we have from Problem 4.11 and a comparison of (4.7) and (4.11) that $\lim_{n \to \infty} E(\xi A_t^{(n)}) = E(\xi A_t)$ for every bounded random variable ξ. For $s, t \in \Pi$ with $0 \le s < t \le a$, and any bounded and nonnegative random variable ξ, we have $E[\xi(A_t - A_s)] = \lim_{n \to \infty} E[\xi(A_t^{(n)} - A_s^{(n)})] \ge 0$, and by selecting $\xi = 1_{\{A_s > A_t\}}$ we get $A_s \le A_t$, a.s. P. Because Π is countable, for

for a.e. $\omega \in \Omega$ the function $t \mapsto A_t(\omega)$ is nondecreasing on Π, and right-continuity shows that it is nondecreasing on $[0,a]$ as well. It is trivially seen that $A_0 = 0$, a.s. P. Further, for any bounded and right-continuous martingale $\{\xi_t, \mathcal{F}_t\}$, we have from (4.2) and Proposition 4.3:

$$E(\xi_a A_a^{(n)}) = E \sum_{j=1}^{2^n} \xi_{t_{j-1}^{(n)}} [A_{t_j^{(n)}}^{(n)} - A_{t_{j-1}^{(n)}}^{(n)}]$$

$$= E \sum_{j=1}^{2^n} \xi_{t_{j-1}^{(n)}} [Y_{t_j^{(n)}} - Y_{t_{j-1}^{(n)}}]$$

$$= E \sum_{j=1}^{2^n} \xi_{t_{j-1}^{(n)}} [A_{t_j^{(n)}} - A_{t_{j-1}^{(n)}}],$$

where we are making use of the fact that both sequences $\{A_t - Y_t, \mathcal{F}_t\}$ and $\{A_t^{(n)} - Y_t, \mathcal{F}_t\}$, for $t \in \Pi_n$, are martingales. Letting $n \to \infty$ one obtains by virtue of (4.5):

$$E \int_{(0,a]} \xi_s dA_s = E \int_{(0,a]} \xi_{s-} dA_s,$$

as well as: $E \int_{(0,t]} \xi_s dA_s = E \int_{(0,t]} \xi_{s-} dA_s, \quad \forall \, t \in [0,a]$, if one remembers that $\{\xi_{s \wedge t}, \mathcal{F}_s; 0 \le s \le a\}$ is also a (bounded) martingale (cf. Problem 3.22). Therefore, the process A defined in (4.11) is natural increasing, and (4.6) follows with $M_t = E[X_a - A_a | \mathcal{F}_t]$, $0 \le t \le a$.

Finally, if the submartingale X is of class D, it is uniformly integrable, hence it possesses a last element X_∞ to which it

converges both in L^1 and almost surely as $t \to \infty$ (Problem 3.17). The reader will have no difficulty repeating the above argument, with $a = \infty$, and observing that $E(A_\infty) < \infty$. □

Much of this book is devoted to the presentation of Brownian motion as the typical continuous martingale. To develop this theme, we must specialize the Doob-Meyer result just proved to continuous submartingales, where we discover that continuity and a bit more implies that both processes in the decomposition turn out to also be continuous. This fact allows us to conclude that the quadratic variation process for a continuous martingale (Section 1.5) is itself continuous.

4.12 Definition: A submartingale $\{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ is called <u>regular</u> if for every $a > 0$ and every nondecreasing sequence of stopping times $\{T_n\}_{n=1}^\infty \subseteq S_a$ with $T = \lim_{n \to \infty} T_n$, we have $\lim_{n \to \infty} E(X_{T_n}) = E(X_T)$. □

4.13 Remark: It can be verified easily that a continuous, nonnegative submartingale is regular.

4.14 Theorem: Suppose that $X = \{X_t; 0 \le t < \infty\}$ is a submartingale of class DL with respect to the filtration $\{\mathcal{F}_t\}$, which satisfies the usual conditions, and let $A = \{A_t; 0 \le t < \infty\}$ be the natural increasing process in the Doob-Meyer decomposition (4.6). The process A is continuous if and only if X is regular.

Proof: Continuity of A yields the regularity of X quite easily by appealing to the Optional Sampling Theorem for bounded stopping times (Problem 3.21 (i)).

Conversely, let us suppose that X is regular; then for any sequence $\{T_n\}_{n=1}^{\infty}$ as in Definition 4.12, we have by Optional Sampling: $\lim_{n\to\infty} E(A_{T_n}) = \lim_{n\to\infty} E(X_{T_n}) - E(M_T) = E(A_T)$, and therefore $A_{T_n} \uparrow A_T$ a.s. P as $n\to\infty$. Now let us consider the same sequence $\{\pi_n\}_{n=1}^{\infty}$ of partitions of the interval $[0,a]$ as in the proof of Theorem 4.10, and select a number $\lambda > 0$. For each interval $(t_j^{(n)}, t_{j+1}^{(n)})$, $j=0,1,\ldots,2^n-1$ we consider a right-continuous modification of the martingale

$$\xi_t^{(n)} = E[\lambda \wedge A_{t_{j+1}^{(n)}} | \mathcal{F}_t], \quad t_j < t \leq t_{j+1}^{(n)} .$$

This is possible by virtue of Theorem 3.11. The resulting process $\{\xi_t^{(n)}; 0 \leq t \leq a\}$ is right-continuous on $(0,a)$ except possibly at the points of the partition, and dominates the increasing process $\{\lambda \wedge A_t; 0 \leq t \leq a\}$; in particular, the two processes agree a.s. at the points $t_1^{(n)}, \ldots, t_{2^n}^{(n)}$. Because A is a natural increasing process, we have from (4.4)

$$E \int_{(t_j^{(n)}, t_{j+1}^{(n)}]} \xi_s^{(n)} dA_s = E \int_{(t_j^{(n)}, t_{j+1}^{(n)}]} \xi_{s-}^{(n)} dA_s; \quad j=0,1,\ldots,2^n-1$$

and by summing over j, we obtain

$$(4.12) \quad E \int_{(0,t]} \xi_s^{(n)} dA_s = E \int_{(0,t]} \xi_{s-}^{(n)} dA_s,$$

for any $0 \leq t \leq a$. Now the process

$$\eta_t^{(n)} = \begin{cases} \xi_{t+}^{(n)} - (\lambda \wedge A_t), & 0 \leq t < a, \\ 0 & , \quad t = a, \end{cases}$$

is right-continuous and adapted to $\{\mathcal{F}_t\}$; therefore, for any $\epsilon > 0$ the random time

$$T_n(\epsilon) = \begin{cases} \inf\{0 \leq t \leq a; \; \eta_t^{(n)} > \epsilon\} = \inf\{0 \leq t \leq a; \; \xi_t^{(n)} - (\lambda \wedge A_t) > \epsilon\}, \\ a \qquad\qquad , \text{ if } \{\ldots\} = \emptyset, \end{cases}$$

is an optional time of the right-continuous filtration $\{\mathcal{F}_t\}$, hence a stopping time in S_a (cf. Problem 2.5 and Proposition 2.3). Further, defining for each $n \geq 1$ the function $\varphi_n(.): [0,a] \to \Pi_n$ by $\varphi_n(t) = t_{j+1}^{(n)}; \; t_j^{(n)} < t \leq t_{j+1}^{(n)}$, we have

$$\varphi_n(T_n(\epsilon)) \in S_a.$$

Because $\xi^{(n)}$ is increasing in n, the limit $T_\epsilon = \lim_{n \to \infty} T_n(\epsilon)$ exists a.s., is a stopping time in S_a, and we also have

$$T_\epsilon = \lim_{n \to \infty} \varphi_n(T_n(\epsilon)) \quad \text{a.s. } P.$$

By Optional Sampling we obtain now

$$E[\xi_{T_n(\epsilon)}^{(n)}] = \sum_{j=1}^{2^n - 1} E[E(\lambda \wedge A_{t_{j+1}^{(n)}} | \mathcal{F}_{T_n(\epsilon)}) \, 1_{\{t_j^{(n)} < T_n(\epsilon) \leq t_{j+1}^{(n)}\}}]$$

$$= E[\lambda \wedge A_{\varphi_n(T_n(\epsilon))}],$$

and therefore

$$E[(\lambda_{\wedge}A_{\omega_n}(T_n(\epsilon))) - (\lambda_{\wedge}A_{T_n(\epsilon)})] = E[\xi^{(n)}_{T_n}(\epsilon) - (\lambda_{\wedge}A_{T_n(\epsilon)})] =$$

$$E[1_{\{T_n(\epsilon)<a\}}(\xi^{(n)}_{T_n}(\epsilon) - (\lambda_{\wedge}A_{T_n(\epsilon)}))] \geq \epsilon P[T_n(\epsilon) < a].$$

We employ now the regularity of A to conclude that for every $\epsilon>0$,

$$P[Q_n>\epsilon] = P[T_n(\epsilon)<a] \leq \frac{1}{\epsilon} E[(\lambda_{\wedge}A_{\omega_n}(T_n(\epsilon))) - (\lambda_{\wedge}A_{T_n(\epsilon)})] \to 0$$

as $n\to\infty$, where $Q_n \triangleq \sup_{0 \leq t \leq a} |\xi^{(n)}_t - (\lambda_{\wedge}A_t)|$. Therefore, this last

sequence of random variables converges to zero in probability , and

hence also almost surely along a (relabelled) subsequence. We

apply this observation to (4.12), along with the Monotone Convergence

Theorem for Lebesgue-Stieltjes integration, to obtain

$$E \int_{(0,t]} (\lambda_{\wedge}A_s)dA_s = E \int_{(0,t]} (\lambda_{\wedge}A_{s-})dA_s, \quad 0 \leq t < \infty,$$

which yields the continuity of the path $t \to \lambda_{\wedge}A_t(\omega)$ for every

$\lambda>0$, and hence the continuity of $t \to A_t(\omega)$ for P - a.e. $\omega\in\Omega$. $\square$

1.5 CONTINUOUS, SQUARE- INTEGRABLE MARTINGALES

In order to properly appreciate Brownian motion, one must understand the role it plays as the canonical example of various classes of processes. One such class is that of continuous, square-integrable martingales. Throughout this section, we have a fixed filtration $\{\mathcal{F}_t\}$ on a probability space $(\Omega, \mathcal{F}, P)$, which satisfies the usual conditions (Definition 3.10).

5.1 Definition: Let $X = \{X_t, \mathcal{F}_t;\ 0 \le t < \infty\}$ be a right-continuous martingale. We say that X is <u>square-integrable</u> if $EX_t^2 < \infty$ for every $t \ge 0$. If, in addition, $X_0 = 0$ a.s., we write $X \in \mathcal{M}_2$ (or $X \in \mathcal{M}_2^c$, if X is also continuous).

5.2 Remark: Although we have defined $\mathcal{M}_2^c$ so that its members have every sample path continuous, the results which follow are also true if we assume only that P-almost every sample path is continuous.

For any $X \in \mathcal{M}_2$, we have that $X^2 = \{X_t^2, \mathcal{F}_t;\ 0 \le t < \infty\}$ is a nonnegative submartingale (Proposition 3.5), hence of class DL, and so X^2 has a (unique) Doob-Meyer decomposition (Theorem 4.9):

$$X_t^2 = M_t + A_t;\quad 0 \le t < \infty$$

where $M = \{M_t, \mathcal{F}_t;\ 0 \le t < \infty\}$ is a martingale and $A = \{A_t, \mathcal{F}_t;\ 0 \le t < \infty\}$ is a natural increasing process. We normalize these processes, so that $M_0 = A_0 = 0$, a.s. P. If $X \in \mathcal{M}_2^c$, then A and M are continous (Theorem 4.14 and Remark 4.13); recall Definitions 4.4 and 4.5 for the terms "increasing" and "natural".

5.3 Definition: For $X \in \mathcal{M}_2$, we define the quadratic variation of

X to be the process $\langle X \rangle_t \triangleq A_t$, where A is the natural

increasing process in the Doob-Meyer decomposition of X^2.

In other words, $\langle X \rangle$ is that unique (up to indistinguish-

ability) adapted, natural increasing process, for which

$\langle X \rangle_0 = 0$ a.s. and $X^2 - \langle X \rangle$ is a martingale.

5.4 Example: Let $\{N_t, \mathfrak{F}_t; 0 \le t < \infty\}$ be a Poisson process (Definition

3.3) with associated martingale $M_t = N_t - \lambda t$ (Problem 3.4;

we take $\mathfrak{F}_t = \mathfrak{F}_t^N = \mathfrak{F}_t^M$). It is easy to verify that $M \in \mathcal{M}^2$,

and $\langle M \rangle_t = \lambda t$.

If we take two elements X, Y of $\mathcal{M}_2$, then both processes

$(X+Y)^2 - \langle X+Y \rangle$ and $(X-Y)^2 - \langle X-Y \rangle$ are martingales, and therefore

so is their difference $4XY - [\langle X+Y \rangle - \langle X-Y \rangle]$.

5.5 Definition: For any two martingales X, Y in $\mathcal{M}_2$, we define

their cross-variation process $\langle X, Y \rangle$ by

$$\langle X, Y \rangle_t \triangleq \frac{1}{4}[\langle X+Y \rangle_t - \langle X-Y \rangle_t]; \quad 0 \le t < \infty,$$

and observe that $XY - \langle X, Y \rangle$ is a martingale. Two elements

X, Y of $\mathcal{M}_2$ are called orthogonal, if $\langle X, Y \rangle_t = 0$, a.s. P,

holds for every $0 \le t < \infty$.

□

5.6 Remark: In view of the identities

$$E[(X_t-X_s)(Y_t-Y_s)|\mathfrak{F}_s] = E[X_tY_t - X_sY_s|\mathfrak{F}_s]$$

$$= E[<X,Y>_t - <X,Y>_s|\mathfrak{F}_s],$$

valid P - a.s. for every $0 \le s < t < \infty$, orthogonality of X,Y
in $\mathfrak{m}_2$ is equivalent to the statements "XY is a martingale"
or "the increments of X and Y over [s,t] are conditionally
uncorrelated, given $\mathfrak{F}_s$".

5.7 Problem: Show that $<\cdot,\cdot>$ is a bilinear form on $\mathfrak{m}_2$, i.e.,
for any members X,Y,Z of $\mathfrak{m}_2$ and real numbers α,β, we
have

(i) $<\alpha X + \beta Y, Z> = \alpha <X,Z> + \beta <Y,Z>$,

(ii) $<X,Y> = <Y,X>$.

The use of the term "quadratic variation" in Definition 5.3
may appear to be unfounded. Indeed, a more conventional use of
this term is the following. Let $X = \{X_t; 0 \le t < \infty\}$ be a process,
fix $t > 0$, and let $\Pi = \{t_0, t_1, \ldots, t_m\}$, with $0 = t_0 \le t_1 \le t_2 \le \ldots \le t_m = t$,
be a partition of [0,t]. Define the p-th variation (p>0) of X
over the partition Π to be

$$V_t^{(p)}(\Pi) = \sum_{k=1}^{m} |X_{t_k} - X_{t_{k-1}}|^p.$$

Now define the mesh of partition Π as $\|\Pi\| = \max_{1 \leq k \leq m} |t_k - t_{k-1}|$, and choose a sequence of partitions $\{\Pi_n\}_{n=1}^{\infty}$ of $[0,t]$ for which $\lim_{n \to \infty} \|\Pi_n\| = 0$. If $V_t^{(2)}(\Pi_n)$ converges in some sense as $n \to \infty$, the limit is entitled to be called the quadratic variation of X on $[0,t]$. Our justification of Definition 5.3 for <u>continuous</u> martingales (on which we shall concentrate from now on) is the the following result:

<u>5.8 Theorem</u>: Let X be in $\mathcal{M}_2^c$, and let $\{\Pi_n\}_{n=1}^{\infty}$ be a sequence of partitions of $[0,t]$ with $\lim_{n \to \infty} \|\Pi_n\| = 0$. Then $V_t^{(2)}(\Pi_n)$ converges in probability to $\langle X \rangle_t$.

The proof of Theorem 5.8 proceeds through two lemmas. The key fact employed here is that, when squaring sums of martingale increments and taking the expectation, one can neglect the cross-product terms. More precisely, if $X \in \mathcal{M}_2$ and $0 \leq s < t \leq u < v$, then

$$E[(X_v - X_u)(X_t - X_s)] = E\{E[X_v - X_u | \mathcal{F}_u](X_t - X_s)\} = 0.$$

We shall apply this fact to both martingales $X \in \mathcal{M}_2$ and $X^2 - \langle X \rangle$. In the latter case, we note that because

$$E[(X_v - X_u)^2 | \mathcal{F}_t] = E[X_v^2 - 2X_u E[X_v | \mathcal{F}_u] + X_u^2 | \mathcal{F}_t]$$

$$= E[X_v^2 - X_u^2 | \mathcal{F}_t] = E[\langle X \rangle_v - \langle X \rangle_u | \mathcal{F}_t],$$

the increment $X_v^2 - \langle X \rangle_v - (X_u^2 - \langle X \rangle_u)$ may be replaced by $(X_v - X_u)^2 - (\langle X_v \rangle - \langle X \rangle_u)$, and the expectation of products of such terms over different intervals is still zero.

5.9 Lemma: Let $X \in \mathcal{M}_2$ satisfy $|X_s| \leq K < \infty$ for all $s \in [0, t]$. Let $\Pi = \{t_0, t_1, \ldots t_m\}$ with $0 = t_0 \leq t_1 \leq \ldots \leq t_m = t$, be a partition of $[0, t]$. Then $E[V_t^{(2)}(\Pi)]^2 \leq 48 \, K^4$.

Proof: Using the martingale property, we have for $0 \leq k \leq m-1$,

$$E[\sum_{j=k+1}^{m} (X_t - X_{t_{j-1}})^2 | \mathcal{F}_{t_k}] = E[\{\sum_{j=k+1}^{m} (X_{t_j} - X_{t_{j-1}})\}^2 | \mathcal{F}_{t_k}]$$

$$= E[(X_{t_m} - X_{t_k})^2 | \mathcal{F}_{t_k}] \leq 4K^2$$

so

(5.1) $$E[\sum_{k=1}^{m-1} \sum_{j=k+1}^{m} (X_{t_j} - X_{t_{j-1}})^2 (X_{t_k} - X_{t_{k-1}})^2]$$

$$= E[\sum_{k=1}^{m-1} (X_{t_k} - X_{t_{k-1}})^2 \sum_{j=k+1}^{m} E[(X_{t_j} - X_{t_{j-1}})^2 | \mathcal{F}_{t_k}]]$$

$$\leq 4K^2 \, E[\sum_{k=1}^{m-1} (X_{t_k} - X_{t_{k-1}})^2]$$

$$\leq 16K^4.$$

We also have

(5.2) $$E[\sum_{k=1}^{m} (X_{t_k} - X_{t_{k-1}})^4] \leq 4K^2 \, E \sum_{k=1}^{m} (X_{t_k} - X_{t_{k-1}})^2$$

$$\leq 16 \, K^4.$$

Inequalities (5.1) and (5.2) imply

$$E[V_t^{(2)}(\Pi)]^2 = E[\sum_{k=1}^{m}(X_{t_k}-X_{t_{k-1}})^4]$$

$$+ 2E[\sum_{k=1}^{m-1}\sum_{j=k+1}^{m}(X_{t_j}-X_{t_{j-1}})^2(X_{t_k}-X_{t_{k-1}})^2]$$

$$\leq 48K^4.$$

$\square$

Lemma 5.10: Let $X \in \mathcal{M}_2^c$ satisfy $|X_s| \leq K < \infty$ a.s. P for all $s \in [0,t]$. Let $\{\Pi_n\}_{n=1}^{\infty}$ be a sequence of partitions of

$[0,t]$ with $\lim_{n\to\infty}\|\Pi_n\| = 0$. Then

$$\lim_{n\to\infty} E\ V_t^{(4)}(\Pi_n) = 0.$$

Proof: For any partition Π as before, Hölder's inequality implies

$$V_t^{(4)}(\Pi) \leq V_t^{(2)}(\Pi) \cdot \max_{1 \leq k \leq m}(X_{t_k}-X_{t_{k-1}})^2$$

and

$$EV_t^{(4)}(\Pi) \leq (E[V_t^{(2)}(\Pi)]^2)^{\frac{1}{2}}(E[\max_{1 \leq k \leq m}(X_{t_k}-X_{t_{k-1}})^4])^{\frac{1}{2}}.$$

As the mesh approaches zero, the first factor on the right-hand side remains bounded and the second term approaches zero by the bounded convergence theorem.

$\square$

Proof of Theorem 5.8 :

We consider first the case that $|X_s| \leq K < \infty$ for all $s \in [0,t]$.
For any partition $\Pi = \{t_0, t_1, \ldots, t_m\}$ as above we may
write (see the discussion preceding Lemma 5.9):

$$E(V_t^{(2)}(\Pi) - \langle X \rangle_t)^2 = E[\sum_{k=1}^{m} \{(X_{t_k} - X_{t_{k-1}})^2 - (\langle X \rangle_{t_k} - \langle X \rangle_{t_{k-1}})\}]^2$$

$$= \sum_{k=1}^{m} E[(X_{t_k} - X_{t_{k-1}})^2 - (\langle X \rangle_{t_k} - \langle X \rangle_{t_{k-1}})]^2$$

$$\leq 2 \sum_{k=1}^{m} E[(X_{t_k} - X_{t_{k-1}})^4 + (\langle X \rangle_{t_k} - \langle X \rangle_{t_{k-1}})^2]$$

$$\leq 2EV_t^{(4)}(\Pi) + 2 E[\langle X \rangle_t \cdot \max_{1 \leq k \leq m} \{\langle X \rangle_{t_k} - \langle X \rangle_{t_{k-1}}\}].$$

As the mesh of Π approaches zero, the first term on the right-
hand side of this inequality converges to zero because of Lemma
5.10; so does the second term as well, by the bounded convergence
theorem. Convergence in L^2 implies convergence in probability,
so this proves the theorem for martingales which are uniformly
bounded.

Now suppose $X \in \mathcal{M}_2^c$ is not necessarily bounded. We
use the technique of localization to reduce this case to the one
already studied. Let us define a sequence of stopping times (Problem
2.6) for $n = 1, 2, \ldots$ by

$$T_n = \begin{cases} \inf\{t \geq 0; |X_t| \geq n \text{ or } \langle X \rangle_t \geq n\} \\ \infty, \text{ if } \{\ldots\} = \emptyset. \end{cases}$$

Now $X_t^{(n)} \triangleq X_{t \wedge T_n}$ is a bounded martingale relative to the filtration $\{\mathfrak{F}_t\}$ (Problem 3.22), and likewise, $\{X_{t \wedge T_n}^2 - \langle X \rangle_{t \wedge T_n}, \mathfrak{F}_t; 0 \leq t < \infty\}$ is a martingale. From the uniqueness of the Doob-Meyer decomposition, we see that

$$\langle X^{(n)} \rangle_t = \langle X \rangle_{t \wedge T_n}.$$

Therefore, for partitions Π of $[0,t]$, we have

$$\lim_{\|\Pi\| \to 0} E[\sum_{k=1}^{m} (X_{t_k \wedge T_n} - X_{t_{k-1} \wedge T_n})^2 - \langle X \rangle_{t \wedge T_n}]^2 = 0.$$

Since $T_n \uparrow \infty$ a.s., we have for any fixed t that $\lim_{n \to \infty} P[T_n < t] = 0$. These facts can be used to prove the desired convergence of $V_t^{(2)}(\Pi)$ to $\langle X \rangle_t$ in probability. $\quad\square$

5.11 Problem: Let $\{X_t, \mathfrak{F}_t; 0 \leq t < \infty\}$ be a continuous process with the property that for each fixed $t > 0$ and for some $p > 0$,

$$\lim_{\|\Pi\| \to 0} V_t^{(p)}(\Pi) = L_t \quad \text{(in probability)},$$

where L_t is a random variable taking values in $[0, \infty)$ a.s. Show that for $q > p$, $\lim_{\|\Pi\| \to 0} V_t^{(q)}(\Pi) = 0$ (in probability), and for $0 < q < p$, $\lim_{\|\Pi\| \to 0} V_t^q(\Pi) \to \infty$ (in probability) on the set $\{L_t > 0\}$.

5.12 Problem: Let X be in $\mathcal{m}_2^c$. Show that if for some $t>0$, we have $\langle X\rangle_t = 0$ a.s., then $X_s = 0$, $0 \leq s \leq t$, a.s. □

The conclusion to be drawn from Theorem 5.8 and Problems 5.11 and 5.12 is that for continuous, square-integrable martingales, quadratic variation is the "right" variation to study. All variations of higher order are zero, and, except in trivial cases where the martingale is a.s. constant on an initial interval, all variations of lower order are infinite with positive probability. Thus, the sample paths of continuous, square-integrable martingales are quite different from "ordinary" continuous functions. Being of unbounded first variation, they cannot be differentiable, nor is it possible to define integrals of the form $\int_0^t Y_s(\omega) dX_s(\omega)$ with respect to $X \in \mathcal{m}_2^c$ in a pathwise (i.e., for every or P-almost every $\omega \in \Omega$), Lebesgue-Stieltjes sense. We shall return to this problem of the definition of stochastic integrals in Chapter 3, where we shall give Ito's construction and change-of-variable formula; the latter is the counterpart of the chain rule from classical calculus, adapted to account for the unbounded first, but bounded second variation of such processes.

It is also worth noting that for $X \in \mathcal{m}_2^c$, the process $\langle X\rangle$, being monotone, is its own first variation process and has quadratic variation zero. Thus, an integral of the form $\int Y_t d\langle X\rangle_t$ is defined in a pathwise, Lebesgue-Stieltjes sense.

We discuss now the cross-variation between two <u>continuous,</u> square-integrable martingales.

5.13 Theorem: Let $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ and $Y = \{Y_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be members of $\mathcal{M}_2^c$. There is a unique (up to indistinguishability) $\{\mathcal{F}_t\}$-adapted, continuous process of bounded variation $\{A_t, \mathcal{F}_t; 0 \leq t < \infty\}$ satisfying $A_0 = 0$ a.s. P, such that $\{X_t Y_t - A_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a martingale. This process is given by the cross-variation $\langle X, Y \rangle$ of Definition 3.4.

Proof: Clearly, $A = \langle X, Y \rangle$ enjoys the stated properties (continuity is a consequence of Theorem 4.14 and Remark 4.13). This shows existence of A. To prove uniqueness, suppose there exists another process B satisfying the conditions imposed on A. Then

$$M \triangleq (XY - A) - (XY - B) = B - A$$

is a continuous martingale with finite first variation. If we define

$$T_n = \inf\{t \geq 0: |M_t| = n\},$$

then $\{M_t^{(n)} \triangleq M_{t \wedge T_n}, \mathcal{F}_t; 0 \leq t < \infty\}$ is a continuous, bounded (hence square-integrable) martingale, with finite first variation on every interval $[0, t]$. It follows from Theorem 5.8 and Problem 5.11 that (cf. proof of Theorem 5.8):

$$\langle M \rangle_{t \wedge T_n} = \langle M^{(n)} \rangle_t = 0 \quad \text{a.s.,} \quad t \geq 0.$$

Problem 5.12 shows that $M^{(n)} \equiv 0$ a.s., and since $T_n \uparrow \infty$ as $n \to \infty$, we conclude that $M \equiv 0$, a.s. P. $\qquad \square$

5.14 Problem: Show that for $X, Y \in \mathcal{M}_2^c$ and $\Pi = \{t_0, t_1, \ldots, t_m\}$ a partition of $[0, t]$,

$$\lim_{\|\Pi\| \to 0} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}}) = \langle X, Y \rangle_t \quad \text{(in probability)}.$$

$\square$

Twice in this section we have used the technique of localization, once in the proof of Theorem 5.8 to extend a result about bounded martingales to square-integrable ones, and again in the proof of Theorem 5.11 to apply a result about square-integrable martingales to a continuous martingale which was not necessarily square-integrable. The next definitions and problem develop this idea formally.

5.15 Definition: Let $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a (continuous) process with $X_0 = 0$ a.s. If there exists a sequence $\{T_n\}_{n=1}^{\infty}$ such that $X^{(n)} \triangleq \{X_t^{(n)} \triangleq X_{t \wedge T_n}, \mathcal{F}_t; 0 \leq t < \infty\}$ is a martingale for each n, and if $T_n \uparrow \infty$ a.s., then we say that X is a (continuous) local martingale and write $X \in \mathcal{M}^{loc}$ (respectively, $X \in \mathcal{M}^{c, loc}$ if X is continuous).

Remark: Every martingale is a local martingale (cf. Problem 3.22), but the converse is not true. We shall encounter in Problem 3.4.12 a continuous process X with $E|X_t| < \infty$ for every $t \geq 0$, which is a local martingale but not a martingale. However, every bounded local martingale is a martingale.

The reader will verify easily that a nonnegative local martingale is a supermartingale, and that

$$\mathcal{M}_2^c \subsetneqq \mathcal{M}^{c,loc}.$$

5.16 Problem: Let X, Y be in $\mathcal{M}^{c,loc}$. Then there is a unique (up to indistinguishability) adapted, continuous process of bounded variation $\langle X, Y \rangle$ satisfying $\langle X, Y \rangle_0 = 0$ a.s. P, such that $XY - \langle X, Y \rangle \in \mathcal{M}^{c,loc}$. If $X = Y$, we write $\langle X \rangle = \langle X, X \rangle$, and this process is nondecreasing.

5.17 Definition: We call the process $\langle X, Y \rangle$ of Problem 5.14 the cross-variation of X and Y, in accordance with Definition 5.5. We call $\langle X \rangle$ the quadratic variation of X. □

We shall show in Theorem 3.2.6 that one-dimensional Brownian motion $\{B_t, \mathcal{F}_t; 0 \le t < \infty\}$ is the unique member of $\mathcal{M}^{c,loc}$ whose quadratic variation at time t is t, i.e., $B_t^2 - t$ is a martingale. We shall also show that d-dimensional Brownian motion $\{(B_t^{(1)}, \ldots, B_t^{(d)}), \mathcal{F}_t; 0 \le t < \infty\}$ is characterized by the condition

$$\langle B^{(i)}, B^{(j)} \rangle_t = \delta_{ij} t, \quad t \ge 0,$$

where δ_{ij} is the Kronecker delta.

5.18 Problem: Suppose $X \in \mathcal{m}_2^c$ has stationary, independent increments. Then $\langle X \rangle_t = t(EX_1^2)$, $t \geq 0$.

5.19 Remark: The reader can employ the localization technique used in the solution of Problem 5.16 to establish the following extension of Problem 5.12: If $X \in \mathcal{m}^{c,loc}$, and for some $t > 0$ we have $\langle X \rangle_t = 0$ a.s. P, then $P[X_s = 0, \ \forall \ 0 \leq s \leq t] = 1$.

We close this section by imposing a metric structure on $\mathcal{m}_2$, and discussing the nature of both $\mathcal{m}_2$ and its subspace $\mathcal{m}_2^c$ under this metric.

5.20 Definition: For any $X \in \mathcal{m}_2$ and $0 \leq t < \infty$, we define

$$\|X\|_t \triangleq \sqrt{E(X_t^2)}.$$

We also set: $\qquad \|X\| \triangleq \sum_{n=1}^{\infty} \frac{\|X\|_n \wedge 1}{2^n}.$ $\qquad \square$

Let us observe that the function $t \to \|X\|_t$ on $[0, \infty)$ is nondecreasing, because X^2 is a submartingale. Further, $\|X-Y\|$ is a pseudo-metric on $\mathcal{m}_2$, which becomes a metric if we identify indistinguishable processes. Indeed, suppose that for $X, Y \in \mathcal{m}_2$ we have $\|X-Y\| = 0$; this implies $X_n = Y_n$ a.s. P, for every $n \geq 1$, and thus $X_t = E(X_n | \mathcal{F}_t) = E(Y_n | \mathcal{F}_t) = Y_t$ a.s. P, for every $0 \leq t \leq n$. Since X and Y are right-continuous, they are indistinguishable (Problem 1.5).

<u>5.21 Proposition</u>:   Under the above metric, $\mathcal{M}_2$ is a complete

metric space, and $\mathcal{M}_2^c$ a closed subspace of $\mathcal{M}_2$.

<u>Proof</u>: Let us consider a Cauchy sequence $\{X^{(n)}\}_{n=1}^{\infty} \subseteq \mathcal{M}_2$:

$\lim_{n,m\to\infty} \|X^{(n)} - X^{(m)}\| = 0$. For any $\epsilon > 0$, $T > 0$ we have by the first

submartingale inequality (Theorem 3.6):

$$P[\sup_{0 \le t \le T} |X_t^{(n)} - X_t^{(m)}| \ge \epsilon] \le \frac{1}{\epsilon^2} E|X_T^{(n)} - X_T^{(m)}|^2 =$$

$$= \frac{1}{\epsilon^2} \|X^{(n)} - X^{(m)}\|_T \to 0$$

as $n, m \to \infty$. We deduce that there exists a process $X = \{X_t; 0 \le t < \infty\}$

such that: $\sup_{0 \le t \le T} |X_t^{(n)} - X_t| \to 0$ as $n \to \infty$ in probability,

as well as   almost surely along an appropriate subsequence $\{n_k\}$.

It follows that this process is adapted to $\{\mathcal{F}_t\}$, and we have

$E(X_t^2) < \infty$, as well as $\lim_{n\to\infty} E|X_t^{(n)} - X_t|^2 = 0$, for every $0 \le t < \infty$.

Furthermore, the sequences $\{X_t^{(n)}\}_{n=1}^{\infty}$, $\{X_s^{(n)}\}_{n=1}^{\infty}$ with $0 \le s < t < \infty$

are uniformly integrable, because $\sup_{n \ge 1} E(X_t^{(n)})^2 < \infty$.   Therefore,

$E[1_A X_t^{(n)}] = E[1_A X_s^{(n)}]$ implies $E[1_A X_t] = E[1_A X_s]$ for every

$A \in \mathcal{F}_s$, and X is seen to be a martingale; we can choose a right-

continuous modification so that $X \in \mathcal{M}_2$. If $\{X^{(n)}\}_{n=1}^{\infty}$ is a sequence

in $\mathcal{M}_c^2$, then X is continuous, as the (a.s.) uniform limit of

continuous processes.

<u>5.22 Problem</u>: Let $M = \{M_t, \mathcal{F}_t; 0 \le t < \infty\}$ be a martingale in $\mathcal{M}_2$,

and assume that its quadratic variation process $\langle M \rangle$

is integrable: $E\langle M\rangle_\infty < \infty$. Then:

(i)   the martingale M and the submartingale M^2 are both uniformly integrable; in particular, $M_\infty = \lim\limits_{t\to\infty} M_t$ exists a.s. P, and $EM_\infty^2 = E\langle M\rangle_\infty$.

(ii) $Z_t = E(M_\infty|\mathcal{F}_t) - M_t$; $t\geq 0$ is a potential.

1.6: SOLUTIONS TO PROBLEMS

<u>1.5 Solution</u>: If Q is the set of rational numbers in $[0,\infty)$,
then the event $A = \underset{s \in Q}{\cup} \{\omega;\ X_s(\omega) \neq Y_s(\omega)\}$ has zero proability.
Besides,

$$\{\omega;\ X_t(\omega) \neq Y_t(\omega),\ \text{ for some } t \geq 0\} \subseteq A,$$

by right-continuity of the processes. The result follows.

<u>1.7 Solution</u>: Let A_n be the event that X has a jump of size
greater than $\frac{1}{n}$ on $[0,t_0)$. Then $A = \underset{n=1}{\overset{\infty}{\cup}} A_n$, so it suffices
to prove $A_n \in \mathfrak{F}_{t_0}^X$. Letting Q be the set of rational numbers
in $[0,\infty)$, we have

$$A_n = \{\mathbf{\forall}\ m \geq 1,\ \exists q_1, q_2 \in Q \cap [0,t_0) \text{ with } |q_1 - q_2| < \tfrac{1}{m} \text{ and}$$

$$|X_{q_1} - X_{q_2}| > \tfrac{1}{n}\}$$

$$= \underset{\substack{m=1}}{\overset{\infty}{\cap}}\ \underset{\substack{q_1, q_2 \in Q \cap [0,t_0) \\ |q_1 - q_2| < \frac{1}{m}}}{\cup} \{|X_{q_1} - X_{q_2}| > \tfrac{1}{n}\} \in \mathfrak{F}_t^X.$$

<u>1.8 Solution</u>: We first construct an example with $A \notin \mathfrak{F}_{t_0}^X$. The
collection of sets of the form $\{(X_{t_1}, X_{t_2}, \ldots) \in B\}$ where
$B \in \mathfrak{B}(\mathbb{R}^d) \otimes \mathfrak{B}(\mathbb{R}^d) \otimes \ldots$ and $0 \leq t_1 < t_2 < \ldots \leq t_0$ forms a σ-field
and each such set is in $\mathfrak{F}_{t_0}^X$. Choose $\Omega = [0,2)$, $\mathfrak{F} = \mathfrak{B}([0,2))$,
and for $F \in \mathfrak{F}$, let

$$P(F) = \lambda(F \cap [0,1]),$$

where λ is Lebesgue measure. For $\omega\in[0,1]$, define $X_t(\omega) = 0$, $\forall$ $t\geq 0$; and for $\omega\in(1,2)$, define $X_t(\omega) = 0$ if $t\neq\omega$, $X_\omega(\omega) = 1$. Choose $t_0=2$. If $A\in\mathcal{F}_{t_0}^X$, then for some $B\in\mathcal{B}(R)\otimes\mathcal{B}(R)\otimes\ldots$ and some $0\leq t_1<t_2<\ldots\leq 2$, we have $A=\{(X_{t_1},X_{t_2},\ldots)\in B\}$. Choose $\bar{t}\in(1,2)$, $\bar{t}\notin\{t_1,t_2,\ldots\}$. Since $\omega = \bar{t}$ is not in A and $X_{t_k}(\bar{t}) = 0$, $k=1,2,\ldots$, we see that $(0,0,\ldots)\notin B$. Since $X_{t_k}(\omega) = 0$, $k=1,2,\ldots$, for all $\omega\in[0,1]$, we conclude that $[0,1] \cap A = \emptyset$, which contradicts the definition of A and the construction of X.

We next show that if $\mathcal{F}_{t_0}^X \subset \mathcal{F}_{t_0}$ and $\mathcal{F}_{t_0}$ is complete, then $A\in\mathcal{F}_{t_0}$. Let $N \subset \Omega$ be the set on which X is not right-continuous, and let

$$\tilde{N} = \{\omega\in N;\ X \text{ is continuous on } [0,t_0)\}.$$

Then

$$A = [(\bigcup_{n=1}^{\infty} A_n) \cap N^c] \setminus \tilde{N},,$$

where

$$A_n = \bigcap_{m=1}^{\infty} \bigcup_{\substack{q_1,q_2\in Q\cap[0,t] \\ |q_1-q_2|<\frac{1}{m}}} \{|X_{q_1}-X_{q_2}| > \tfrac{1}{n}\}.$$

1.10 Solution:

Set $\quad A_{k,n} = \bigcap_{m=1}^{\infty} \bigcup_{\substack{q_1,q_2\in Q\cap[0,t_0+\frac{1}{k}) \\ |q_1-q_2|<\frac{1}{m}}} \{|X_{q_1}-X_{q_2}| > \tfrac{1}{n}\},$

so

$$A_k \triangleq \bigcap_{n=1}^{\infty} A_{k,n}^c \in \mathcal{F}_{t_0+\frac{1}{k}}.$$

is the event that X is continuous on $[0, t_0 + \frac{1}{k})$. Since

$A = \bigcap\limits_{k=K}^{\infty} A_k$ for any positive integer K, we have

$A \in \bigcap\limits_{k=1}^{\infty} \mathcal{F}_{t_0 + \frac{1}{k}} = \mathcal{F}_{t_0+}.$

1.16 Solution: $X_T(\omega)$ is the composition of the two measurable

mappings

$\omega \mapsto (T(\omega), \omega): (\Omega, \mathcal{F}) \to ([0, \infty) \times \Omega, \, B([0, \infty)) \otimes \mathcal{F})$ and

$(t, \omega) \mapsto X_t(\omega): ([0, \infty) \times \Omega, \, \mathcal{B}([0, \infty)) \otimes \mathcal{F}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)).$

2.2 Solution: Let $t_0 = T(\omega)$, and let $A = \{T \leq t_0\}$. Since $\omega \in A$, $A \in \mathcal{F}_{t_0}^X$, and $X_t(\omega) = X_t(\omega')$, $t \in [0, t_0] \cap [0, \infty)$, we have $\omega' \in A$. (See the characterization of $\mathcal{F}_{t_0}^X$ in Solution 1.8.) Therefore $T(\omega') \leq T(\omega)$. Reversing the rôles of ω and ω', we can now argue that since $X_t(\omega) = X_t(\omega')$ for all $t \in [0, T(\omega')] \cap [0, \infty)$, we have $T(\omega) \leq T(\omega')$.

2.5 Solution: Try to argue the validity of the identity:

$\{H_\Gamma < t\} = \bigcup\limits_{\substack{s \in Q \\ 0 \leq s < t}} \{X_s \in \Gamma\}$, for any $t > 0$. The inclusion $\supseteq$ is

obvious, even for sets which are not open. Use right-continuity, and the fact that Γ is open, to go the other way.

2.6 Solution: (Wentzell [19]): For $x \in \mathbb{R}^d$, let $\rho(x, \Gamma) = \inf\{\|x - y\|; \, y \in \Gamma\}$, and consider the nested sequence of open neighborhoods of Γ given by $\Gamma_n = \{x \in \mathbb{R}^d; \, \rho(x, \Gamma) < \frac{1}{n}\}$. By virtue of Problem 2.5,

the times $T_n \triangleq H_{\Gamma_n}$; $n \geq 1$, are optional. They form a non-decreasing sequence, dominated by $H = H_\Gamma$, with limit $T \triangleq \lim_{n \to \infty} T_n \leq H$, and we have the following dichotomy:

On $\{H = 0\}$: $T_n = 0$, $\forall n \geq 1$.

On $\{H > 0\}$: there exists an integer $k = k(\omega) \geq 1$ such that

$$T_n = 0; \quad \forall \ 1 \leq n < k, \quad \text{and} \quad 0 < T_n < T_{n+1} < H; \quad \forall \ n \geq k.$$

We shall show that $T = H$, and for this it suffices to establish: $T \geq H$ on $\{H > 0, T < \infty\}$.

On the indicated event we have, by continuity of the sample paths of X: $X_T = \lim_{n \to \infty} X_{T_n}$ and $X_{T_m} \in \partial \Gamma_m \subseteq \Gamma_n$; $\forall \ m > n \geq k$. Now we can let $m \to \infty$, to obtain $X_T \in \Gamma_n$; $\forall \ n \geq k$, and thus $X_T \in \bigcap_{n=1}^{\infty} \Gamma_n = \Gamma$. We conclude with the desired result $H \leq T$. The conclusion follows now from $\{H \leq t\} = \bigcap_{n=1}^{\infty} \{T_n < t\}$, valid for $t > 0$, and $\{H = 0\} = \{X_0 \in \Gamma\}$.

<u>2.9 Solution</u>: Optionality of $T+S$ follows from Corollary to Proposition 2. and Lemma 2.8, or directly from: $\{T+S < t\} = \bigcup_{\substack{r \in Q \\ 0 \leq r \leq t}} \{T < r, \ S < t-r\}$.

For the rest, use again the decomposition in the proof of Lemma 2.8, just a little bit more subtly!

<u>2.16 Solution</u>: For any event $A \in \mathfrak{F}_T$, and any $t \geq 0$, we have

$$A \cap \{T \leq S\} \cap \{T \wedge S \leq t\} = A \cap \{T \leq S\} \cap \{T \leq t\} \in \mathfrak{F}_t, \quad \text{because the event}$$

$\{T \le S\}$ is in $\mathfrak{F}_T$ (Lemma 2.15). Therefore, $A \cap \{T \le S\} \in \mathfrak{F}_{T \wedge S}$ and

$$\int_A 1_{\{T \le S\}} E(Z | \mathfrak{F}_{T \wedge S}) dP = \int_{A \cap \{T \le S\}} Z dP = \int_{A \cap \{T \le S\}} E(Z | \mathfrak{F}_T) dP =$$

$$\int_A 1_{\{T \le S\}} E(Z | \mathfrak{F}_T) dP, \quad \text{so (i) follows.}$$

For claim (ii) we conclude from (i) that

$$1_{\{T \le S\}} E[E(Z | \mathfrak{F}_T) | \mathfrak{F}_S] = E[1_{\{T \le S\}} E(Z | \mathfrak{F}_T) | \mathfrak{F}_S]$$

$$= E[1_{\{T \le S\}} E(Z | \mathfrak{F}_{S \wedge T}) | \mathfrak{F}_S]$$

$$= 1_{\{T \le S\}} E[E(Z | \mathfrak{F}_{S \wedge T}) | \mathfrak{F}_S]$$

$$= 1_{\{T \le S\}} E[Z | \mathfrak{F}_{S \wedge T}],$$

which proves the desired result on the set $\{T \le S\}$. Interchanging the rôles of S and T and replacing Z by $E(Z | \mathfrak{F}_T)$, we can also conclude from (i) that

$$1_{\{S < T\}} E[E(Z | \mathfrak{F}_T) | \mathfrak{F}_S] = 1_{\{S < T\}} E[E(Z | \mathfrak{F}_T) | \mathfrak{F}_{S \wedge T}]$$

$$= 1_{\{S < T\}} E[Z | \mathfrak{F}_{S \wedge T}].$$

2.18 Solution: By assumption, the mappings

$$(s, \omega) \mapsto (s, X_s(\omega)) : ([0,t] \times \Omega, \mathfrak{B}([0,t]) \otimes \mathfrak{F}_t) \to ([0,t] \times \mathbb{R}^d,$$

$$\mathfrak{B}([0,t]) \otimes \mathfrak{B}(\mathbb{R}^d))$$

and $(s,x) \mapsto f(s,x): ([0,t] \times \mathbb{R}^d, \mathcal{B}([0,t]) \otimes \mathcal{B}(\mathbb{R}^d)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

are measurable, and then so is the composite mapping

$(s,\omega) \mapsto f(s, X_s(\omega)): ([0,t] \times \Omega, \mathcal{B}([0,t]) \otimes \mathcal{F}_t) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

The Fubini theorem yields $\mathcal{F}_t$-measurability of the random variable Y_t, and so the process Y is seen to be progressively measurable with respect to $\{\mathcal{F}_t\}$, since it is adapted and has continuous paths (Proposition 1.13). The $\mathcal{F}_T$-measurability of Y_T now follows from Proposition 2.17.

2.21 <u>Solution</u>: We only discuss the second claim, following Chung [2]. For any $A \in \mathcal{F}_{S+}$, we have

$$A = \left(\bigcup_{r \in \mathbb{Q}} [A \cap \{S < r < T\}] \right) \cup [A \cap \{S = \infty\}] \cup [A \cap \{T = 0\}].$$

Now $A \cap \{S < r < T\} = A \cap \{S < r\} \cap \{T > r\}$ is an event in $\mathcal{F}_T$, as is easily verified, because $A \cap \{S < r\} \in \mathcal{F}_r$. On the other hand, $A \cap \{S = \infty\} = [A \cap \{S = \infty\}] \cap \{T = \infty\}$ is seen to be in $\mathcal{F}_T$, since $A \cap \{S = \infty\} \in \mathcal{F}_\infty$. Finally, $A \cap \{T = 0\} = [A \cap \{S = 0\}] \cap \{T = 0\} \in \mathcal{F}_T$, because $A \cap \{S = 0\} \in \mathcal{F}_{0+}$. It follows that $A \in \mathcal{F}_T$.

2.22 <u>Solution</u>: T is an optional time, by Lemma 2.10, and so $\mathcal{F}_{T+}$ is defined and contained in $\mathcal{F}_{T_n+}$ for every $n \geq 1$. Therefore, $\mathcal{F}_{T+} \subseteq \bigcap_{n=1}^{\infty} \mathcal{F}_{T_n+}$. To go the other way, consider an event A such that $A \cap \{T_n < t\} \in \mathcal{F}_t$, for every $n \geq 1$ and $t \geq 0$. Obviously then, $A \cap \{T < t\} = A \cap \left(\bigcup_{n=1}^{\infty} \{T_n < t\} \right) = \bigcup_{n=1}^{\infty} (A \cap \{T_n < t\}) \in \mathcal{F}_t$, and thus $A \in \mathcal{F}_{T+}$. The second claim is justified similarly, using Problem 2.21.

2.23 Solution: Because $\{T_n = \frac{k}{2^n}\} = \{T < \frac{k}{2^n}\} \setminus \{T < \frac{k-1}{2^n}\}$ is an event

in $\mathfrak{F}_{\frac{k}{2^n}}$, we have

$$\{T_n \leq t\} = \bigcup_{\substack{k,n \geq 1 \\ k \leq t 2^n}} \{T_n = \frac{k}{2^n}\} \in \mathfrak{F}_t, \qquad \forall\ t \geq 0.$$

On the other hand, for $\Lambda \in \mathfrak{F}_{T+}$ we have

$$\Lambda \cap \{T_n = \frac{k}{2^n}\} = (\Lambda \cap \{T < \frac{k}{2^n}\}) \setminus (\Lambda \cap \{T < \frac{k-1}{2^n}\}) \in \mathfrak{F}_{\frac{k}{2^n}}.$$

3.2 Solution

(i)   Fix $s \geq 0$ and a nonnegative integer n. Consider the "trace" σ-field $\mathscr{G}$ of all sets obtained by intersecting the members of $\mathfrak{F}_s^N$ with the set $\{N_s = n\}$. Consider also the similar trace σ-field $\mathcal{H}$ of $\sigma(T_1, \ldots, T_n)$ on $\{N_s = n\}$. A generating family for $\mathscr{G}$ is the collection of sets of the form $\{N_{t_1} \leq n_1, \ldots, N_{t_k} \leq n_k, N_s = n\}$, where $0 \leq t_1 \leq \ldots \leq t_k \leq s$, and each such set is a member of $\mathcal{H}$. A generating family for $\mathcal{H}$ is the collection of all sets of the form $\{S_1 \leq t_1, \ldots, S_{n-1} \leq t_{n-1}, N_t = n\}$, where $0 \leq t_1 \leq \ldots \leq t_k \leq s$, and each such set is a member of $\mathscr{G}$. It follows that $\mathscr{G} = \mathcal{H}$.

For $\widetilde{A} \in \mathfrak{F}_s^N$ and $A \triangleq \widetilde{A} \cap \{N_s = n\}$, we have $A \in \mathcal{H} \subset \sigma(T_1, \ldots, T_n)$, so T_{n+1} is indeed independent of $(S_n, 1_A)$. It follows that the pair of random variables (T_{n+1}, S_n), when restricted to A, induces on $(\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2))$ the measure

$$P[T_{n+1} \in d\tau] \cdot P[S_n \in d\sigma; A],$$

where

$$P[T_{n+1} \in d\tau] = \lambda e^{-\lambda \tau} d\tau, \quad \tau \geq 0,$$

and $P[S_n \in d\sigma; A]$ is the measure defined by

$$\int_B P[S_n \in d\sigma; A] = P[S_n \in B; A], \quad \forall B \in \mathscr{B}(\mathbb{R}).$$

We may now compute

$$P[T_{n+1} + S_n > t, S_n \leq s, A]$$

$$= \int_{\sigma=0}^{s} \int_{\tau=t-\sigma}^{\infty} P[T_{n+1} \in d\tau] \, P[S_n \in d\sigma; A]$$

$$= e^{-\lambda t} \int_{\sigma = 0}^{s} e^{\lambda \sigma} P[S_n \epsilon d\sigma; A],$$

$$P[T_{n+1} + S_n > s, \ S_n \leq s, \ A] = e^{-\lambda s} \int_{\sigma = 0}^{s} e^{\lambda \sigma} P[S_n \epsilon d\sigma; A],$$

and if $P(A) > 0$, then

$$P[S_{n+1} > t \mid N_s = n, \ \widetilde{A}] = \frac{P[S_{n+1} > t, \ N_s = n, \ \widetilde{A}]}{P[S_{n+1} > s, \ N_s = n, \ \widetilde{A}]}$$

$$= \frac{P[T_{n+1} + S_n > t, \ S_n \leq s, \ A]}{P[T_{n+1} + S_n > s, \ S_n \leq s, \ A]} = e^{-\lambda(t-s)}.$$

From this, we may conclude that whenever $\widetilde{A} \epsilon \mathcal{F}_s^N$ and $P(\widetilde{A}) > 0$, then

$$P[S_{N_s+1} > t \mid \widetilde{A}] = \frac{\sum\limits_{n=0}^{\infty} P[S_{n+1} > t, \ N_s = n, \ \widetilde{A}]}{\sum\limits_{n=0}^{\infty} P[N_s = n, \ \widetilde{A}]}$$

$$= e^{-\lambda(t-s)}.$$

Therefore, for any $\widetilde{A} \epsilon \mathcal{F}_s^N$, whether or not $P(\widetilde{A}) > 0$, we have

$$P[\widetilde{A} \cap \{ S_{N_s+1} > t\}] = e^{-\lambda(t-s)} P(\widetilde{A}),$$

and (i) is proved.

(ii)   For $0 \leq s < t$, N_{t-s} is a Borel function ω of the inter-arrival times $T_1, T_2, \ldots$. With the same function ω, we have

$$N_t - N_s = \varphi(S_{N_s+1} - s, \; T_{N_s+2}, \; T_{N_s+3}, \; \ldots).$$

Thus, to prove that $N_t - N_s$ is independent of $\mathfrak{F}_s^N$, it suffices to prove that for arbitrary positive integer m, and for $t_1, t_2, \ldots, t_m$ in $[0, \infty)$,

$$P[S_{N_s+1} - s > t_1, \; T_{N_s+2} > t_2, \ldots, T_{N_s+m} > t_m \,|\, \mathfrak{F}_s^N]$$

is constant. We shall in fact show that this expression equals $P[T_1 > t_1, \; T_2 > t_2, \; \ldots, \; T_m > t_m]$, so the distribution of $N_t - N_s$ is the same as that of N_{t-s}.

We compute as follows:

$$P[S_{N_s+1} - s > t_1, T_{N_s+2} > t_2, \ldots, T_{N_s+m} > t_m \,|\, \mathfrak{F}_s^N]$$

$$= \sum_{n=0}^{\infty} 1_{\{N_s = n\}} \; P[S_{N_s+1} - s > t_1, T_{N_s+2} > t_2, \ldots, T_{N_s+m} > t_m \,|\, \mathfrak{F}_s^N]$$

$$= \sum_{n=0}^{\infty} 1_{\{N_s = n\}} \; P[S_{n+1} > t_1 + s, \; T_{n+2} > t_2, \ldots, T_{n+m} > t_m \,|\, \mathfrak{F}_s^N].$$

On the set $\{N_s = n\}$, the σ-fields $\mathfrak{F}_s^N$ and $\sigma(T_{n+2}, \ldots, T_{n+m})$ are independent (i.e., the trace σ-fields are independent). The random variable S_{n+1} is not independent of $\mathfrak{F}_s^N$ on $\{N_s = n\}$, but its conditional distribution was computed in (i). It follows that

$$\sum_{n=0}^{\infty} 1_{\{N_s = n\}} \, P[S_{n+1} > t_1+s, \ T_{n+2} > t_2, \ldots, T_{n+m} > t_m | \mathfrak{F}_s^N]$$

$$= e^{-\lambda(t_2 + \ldots + t_m)} \, P[S_{N_t+1} > t_1+s | \mathfrak{F}_s^N]$$

$$= e^{-\lambda(t_1 + t_2 + \ldots + t_m)}$$

$$= P[T_1 > t_1, \ T_2 > t_2, \ldots, T_m > t_m].$$

(iii) In light of (ii), it suffices to prove that N_t is Poisson with mean λt. A standard computation reveals that S_n has a gamma distribution with parameters (n, λ), i.e.,

$$P[S_n \in ds] = \frac{\lambda^n s^{n-1}}{(n-1)!} \cdot e^{-\lambda s} \, ds; \quad s > 0.$$

It is then easy to see that since $P[N_t \leq n] = P[S_n \geq t]$, we must have

$$P[N_t = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \, .$$

3.7 Solution

Let $\{M_t; \ t \geq 0\}$ be the martingale $\{N_t - \lambda t; \ t \geq 0\}$. We have

$$E \, M_{n/\lambda}^+ = \sum_{k=n+1}^{\infty} (k-n) \frac{n^k}{k!} e^{-n}$$

$$= \sum_{k=n+1}^{\infty} \frac{n^k}{(k-1)!} e^{-n} - \sum_{k=n+2}^{\infty} \frac{n^k}{(k-1)!} e^{-n}$$

$$= \frac{n^{n+1}}{n!} e^{-n}$$

According to Stirling's approximation, this implies

$$\lim_{n \to \infty} \frac{EM^+_{n/\lambda}}{\sqrt{n}} = \frac{1}{\sqrt{2\pi}} \; .$$

Now $\{M^+_t, \; t \geq 0\}$ is a submartingale (Proposition 3.5), so for $\frac{n-1}{\lambda} \leq t \leq \frac{n}{\lambda}$, we have $EM^+_{\frac{n-1}{\lambda}} \leq EM^+_t \leq EM^+_{\frac{n}{\lambda}}$, and thus

$$\frac{1}{\sqrt{n-1}} \; EM^+_{\frac{n-1}{\lambda}} \; \sqrt{\frac{n-1}{n}} \leq \frac{EM^+_t}{\sqrt{\lambda t}} \leq \frac{EM^+_{n/\lambda}}{\sqrt{n}} \; \sqrt{\frac{n}{n-1}} \; .$$

Upon letting $n \to \infty$, we conclude that

$$\lim_{t \to \infty} \frac{EM^+_t}{\sqrt{\lambda t}} = \frac{1}{\sqrt{2\pi}}$$

From Theorem 3.6 (i), we have

$$P[\sup_{0 \leq s \leq t} (N_s - \lambda s) \geq c\sqrt{\lambda t}] \leq \frac{EM^+_t}{c\sqrt{\lambda t}} \; ,$$

and (a) is proved. Part (b) follows in the same way from Theorem 3.6 (ii). We obtain (c) by applying Theorem 3.6 (iv) to the submartingale $\{|M_t| ; \; t \geq 0\}$.

Indeed,

$$E[\sup_{\sigma \leq t \leq \tau} (\frac{N_t}{t} - \lambda)^2] \leq \frac{1}{\sigma^2} E[\sup_{0 \leq t \leq \tau} M^2_t]$$

$$= \frac{1}{\sigma^2} E[\sup_{0 \leq t \leq \tau} |M_t|]^2$$

$$\leq \frac{4}{\sigma^2} EM^2_\tau = \frac{4\tau\lambda}{\sigma^2} \; .$$

3.8 Solution:

Thanks to the Jensen inequality (as in Proposition 3.5) we have that $\{X_n^+, \mathfrak{F}_n; n \geq 1\}$ is also a backwards submartingale, and so with $\lambda > 0$: $\lambda \cdot P[|X_n| > \lambda] \leq E|X_n| = -E(X_n) + 2E(X_n^+) \leq -\ell + 2E(X_1^+) < \infty$. It follows that $\sup_{n \geq 1} P[|X_n| > \lambda]$ converges to zero as $\lambda \to \infty$, and by the submartingale property:

$$\int_{\{X_n^+ > \lambda\}} X_n^+ \, dP \leq \int_{\{X_n^+ > \lambda\}} X_1^+ \, dP \leq \int_{\{|X_n| > \lambda\}} X_1^+ \, .$$

Therefore, $\{X_n^+\}_{n=1}^{\infty}$ is a uniformly integrable sequence. On the other hand,

$$0 \geq \int_{\{X_n < -\lambda\}} X_n \, dP = E(X_n) - \int_{\{X_n \geq -\lambda\}} X_n \, dP \geq E(X_n) - \int_{\{X_n \geq -\lambda\}} X_m \, dP$$

$$= E(X_n) - E(X_m) + \int_{\{X_n < -\lambda\}} X_m \, dP, \quad \text{for } n > m.$$

Given $\epsilon > 0$, we can certainly choose m so large that $0 \leq E(X_m) - E(X_n) \leq \frac{\epsilon}{2}$ holds for every $n > m$, and for that m we select $\lambda > 0$ in such a way that

$$\sup_{n > m} \int_{\{X_n < -\lambda\}} |X_m| \, dP < \frac{\epsilon}{2} \, .$$

Consequently, for these choices of m and λ we have:

$$\sup_{n > m} \int_{\{X_n^- > \lambda\}} X_n^- \, dP < \epsilon, \quad \text{and thus } \{X_n^-\}_{n=1}^{\infty} \text{ is also}$$

uniformly integrable.

3.14 Solution:

The existence and integrability of the limit follow from Theorem 3.13 applied to the nonpositive submartingale $\{-X_t, \mathfrak{F}_t ; 0 \leq t < \infty\}$. It remains to show

$$\int_A X_\infty \, dP \leq \int_A X_s \, dP, \qquad \forall \, A \in \mathfrak{F}_s$$

for an arbitrary $0 \leq s < \infty$. But we have $E[1_A X_t] \leq E[1_A X_s]$ for every $t > s$, and now the result follows from Fatou's Lemma.

3.16 Solution:

Uniform integrability implies that $\sup_{t \geq 0} E|X_t| < \infty$, so Theorem 3.7 gives the existence and integrability of $X_\infty = \lim_{t \to \infty} X_t$, and Theorem 3.11 guarantees the existence of a right-continuous modification M_t of the uniformly integrable martingale $E(X_\infty | \mathfrak{F}_t)$. Finally, observe that $Z_t \triangleq X_t - M_t$; $t \geq 0$ is a right-continuous, nonnegative (by Problem 3.14) supermartingale, with $\lim_{t \to \infty} E(Z_t) = \lim_{t \to \infty} E(X_t) - E(X_\infty) = 0$ (by uniform integrability).

3.17 Solution:

Exactly as in Theorem 9.4.5, Chung [1].

3.18 Solution:

Exactly as in Theorem 9.4.6, Chung [1].

3.19 Solution

(i) Choose $0 \leq s < t$. Recall from Problem 3.2 (b) that $N_t - N_s$ is independent of $\mathfrak{F}_t^N$. Therefore,

$$E[X_t | \mathfrak{F}_s^N] = E[X_s \exp\{N_t - N_s - \lambda(t-s)(e-1)\} | \mathfrak{F}_s^N]$$

$$= X_s \exp\{-\lambda(t-s)(e-1)\} E[\exp(N_t - N_s)] = X_s.$$

(ii) No. Since $X_t \geq 0$, Problem 3.14 implies X_t converges to a limit X_∞ a.s. From Problem 3.7 (a), we have that for each $c > 0$, there exists $T_c > 0$ such that

$$P[N_t - \lambda t \geq c\sqrt{\lambda t}] \leq \frac{2}{c\sqrt{2\pi}}, \forall \, t \geq T_c.$$

It follows that

$$P[X_t \geq \exp(c\sqrt{\lambda t} - \lambda t(e-2))] \leq \frac{2}{c\sqrt{2\pi}}, \quad \forall \, t \geq T_c,$$

so $X_t \to 0$ in probability and $X_\infty = 0$ a.s. But $EX_t = 1$, $0 \leq t < \infty$, and $EX_\infty = 0$, so $\{X_t, \mathfrak{F}_t^N; 0 \leq t \leq \infty\}$ is not a martingale (cf. Problem 3.18 (d)).

3.21 Solution

(i) Repeat verbatim the proof of Theorem 3.13, except that now you can refer to the "discrete" optional sampling Theorem 9.3.4 in Chung [1] for bounded stopping times.

(ii) The submartingale has a last element $X_\infty = E[Y | \mathfrak{F}_\infty]$. Theorem 3.20 thus applies.

3.22 Solution

(i)   We have to establish, for every $0 \le s < t < \infty$.

(*) $$E[X_{T \wedge t} | \mathfrak{F}_s] \ge X_{T \wedge s} \; ; \qquad \text{a.s. } P.$$

From the optional sampling theorem applied to the bounded stopping times $T \wedge s \le T \wedge t$, we have (Problem 3.21 (i)): $E(X_{T \wedge t} | \mathfrak{F}_{T \wedge s}] \ge X_{T \wedge s}$ a.s. P. But from Problem 2.16(i): $E[X_{T \wedge t} | \mathfrak{F}_{T \wedge s}] = E[X_{T \wedge t} | \mathfrak{F}_s]$, a.s. P on $\{T \ge s\}$, and so (*) is seen to hold on this event.

On the other hand, we have trivially $E[X_{T \wedge t} | \mathfrak{F}_s] = X_{T \wedge s}$, a.s. P on $\{T < s\}$.

(ii)   The proof is similar.

3.23 Solution:

With $0 \le s < t < \infty$, suppose that the event $A = \{E(X_t | \mathfrak{F}_s) > X_s\}$ has positive probability.   We have

$$E(X_t) \equiv E[E(X_t | \mathfrak{F}_s)] = E[1_A E(X_t | \mathfrak{F}_s) + 1_{A^c} E(X_t | \mathfrak{F}_s)],$$

as well as $E(X_t | \mathfrak{F}_s) \ge X_s$ a.s. on Ω.   The assumption $P(A) > 0$ thus leads to $E(X_t) > E(X_s)$, which contradicts the premise of the proposition.

3.24 Solution: Necessity of the above condition follows from the version of the optional sampling theorem for bounded stopping times (Problem 3.21 (i)).   For sufficiency, consider $0 \le s < t < \infty$, $A \in \mathfrak{F}_s$ and define the stopping time $S(\omega) \triangleq s 1_A(\omega) + t 1_{A^c}(\omega)$.   The condition $E(X_t) \ge E(X_S)$ a.s. P is tantamount to the submartingale property $E[X_t 1_A] \ge E[X_s 1_A]$.

3.25 Solution: (Robbins & Siegmund (1970)): With the stopping time

$$T = \begin{cases} \inf\{t \geq s;\ Z_t = b\} \\ + \infty, \quad \text{if} \quad \{\dots\} = \emptyset\ , \end{cases}$$

the process $\{Z_{T \wedge t}, \mathfrak{F}_t;\ 0 \leq t < \infty\}$ is a martingale (Problem 3.22(i)). It follows that for every $A \in \mathfrak{F}_s$, $t \geq s$:

$$\int_{A \cap \{Z_s < b\}} Z_s\, dP = \int_{A \cap \{Z_s < b\}} Z_{T \wedge t}\, dP = b \cdot P[A \cap \{Z_s < b,\ T \leq t\}] +$$

$$\int_{A \cap \{Z_s < b\}} Z_t\, 1_{\{T > t\}}\, dP\ .$$

The integrand $Z_t\, 1_{\{T > t\}}$ is dominated by b, and converges to zero as $t \to \infty$ by assumption; it develops then from the dominated convergence theorem that

$$\int_{A \cap \{Z_s < b\}} Z_s\, dP = b \cdot P[A \cap \{Z_s < b, T < \infty\}] = b \int_{A \cap \{Z_s < b\}} P[T < \infty \mid \mathfrak{F}_s]\, dP,$$

establishing the first conclusion. The second follows readily.

4.9 Solution: According to the Optional Sampling Theorem 3.20 as extended in Problem 3.21 (i), we have

$$\int\limits_{\{X_T > \lambda\}} X_T \, dP \leq \int\limits_{\{X_T > \lambda\}} X_a \, dP \qquad \forall \, T \in S_a,$$

and from Theorem 3.6 (i), $P\{X_T > \lambda\}$ approaches zero uniformly in T as $\lambda \downarrow 0$. This proves (a). Applying this same argument to the nonnegative submartingale M and observing that $A_T \leq A_a$ for $T \in S_a$, we obtain (b). Part (c) is a special case of (b) with $A \equiv 0$. If X is uniformly integrable, the optional sampling theorem and Problem 3.18 imply

$$\int\limits_{\{X_T > \lambda\}} X_T \, dP \leq \int\limits_{\{X_T > \lambda\}} X_\infty \, dP \,, \qquad \forall \, T \in S.$$

4.11 Solution: Let ξ be a bounded, $\mathcal{F}$-measurable, random variable. We have

$$E\{\xi \, E[A^{(n)}|\mathcal{G}]\} = E\{E[\xi|\mathcal{G}] \, E[A^{(n)}|\mathcal{G}]\}$$

$$= E\{E[E[\xi|\mathcal{G}] \, A^{(n)}|\mathcal{G}]\}$$

$$= E\{E[\xi|\mathcal{G}] \, A^{(n)}\} \,,$$

which converges to $E\{E[\xi|\mathcal{G}] \cdot A\} = E\{\xi E[A|\mathcal{G}]\}$.

5.7 Solution: (i) It is easily verified that

$$(\alpha X + \beta Y) Z - \alpha \langle X, Z \rangle - \beta \langle Y, Z \rangle$$

is a martingale.

5.11 Solution: Let $\Pi = \{t_0, \ldots, t_m\}$, with $0 = t_0 \leq t_1 \leq \ldots \leq t_m$, be a partition of $[0, t]$. For $q > p$, we have

$$V_t^{(q)}(\Pi) \leq V_t^{(p)}(\Pi) \cdot \max_{1 \leq k \leq m} |X_{t_k} - X_{t_{k-1}}|^{q-p}.$$

The first term on the right-hand side has a finite limit in probability, and the second term converges to zero in probability. Therefore, the product converges to zero in probability. **For** $0 < q < p$ and a sequence of partitions $\{\Pi_n\}_{n=1}^{\infty}$ with $\|\Pi_n\| \to 0$, the sequence $\{V_t^{(q)}(\Pi_n)\}_{n=1}^{\infty}$ must be unbounded on the set $\{L_t > 0\}$, for otherwise the argument just given (but with the roles of p and q interchanged) would show that $V_t^{(p)}(\Pi_n) \to 0$ in probability on this set. Since every such sequence $\{V_t^{(q)}(\Pi_n)\}_{n=1}^{\infty}$ is unbounded, we have $\lim_{\|\Pi\| \to 0} V_t^q(\Pi) = \infty$ (in probability) on $\{L_t > 0\}$.

5.12 Solution: Since $\langle X \rangle$ is nondecreasing, $\langle X \rangle_t = 0$ implies $\langle X \rangle_s = 0$ for $0 \leq s \leq t$. For each $s \in [0, t]$,

$$0 = E[X_s^2 - \langle X \rangle_s] = E(X_s^2) ,$$

which implies that $X_s = 0$ a.s. Since X is continuous, we must have that P-almost every sample path is identically zero on $[0, t]$.

5.14 Solution: Write

$$(X_{t_k} - X_{t_{k-1}}) \, (Y_{t_k} - Y_{t_{k-1}})$$

$$= \frac{1}{4} \, [(X_{t_k} + Y_{t_k}) - (X_{t_{k-1}} + Y_{t_{k-1}})]^2$$

$$- \frac{1}{4} \, [(X_{t_k} - Y_{t_k}) - (X_{t_{k-1}} - Y_{t_{k-1}})]^2 \, ,$$

and use Theorem 5.8 and the properties of $A = \langle X, Y \rangle$ in Theorem 5.13.

5.16 Solution: There are sequences $\{S_n\}$, $\{T_n\}$ of stopping times such that $S_n \uparrow \infty$, $T_n \uparrow \infty$ and $X_t^{(n)} \triangleq X_{t \wedge S_n}$, $Y_t^{(n)} \triangleq Y_{t \wedge T_n}$ are $\{\mathfrak{F}_t\}$ - martingales. Define

$$R_n \triangleq S_n \wedge T_n \wedge \inf\{t \geq 0 : |X_t| = n \text{ or } |Y_t| = n\},$$

and set $\widetilde{X}_t^{(n)} = X_{t \wedge R_n}$, $\widetilde{Y}_t^{(n)} = Y_{t \wedge R_n}$. Note that $R_n \uparrow \infty$ a.s. Since $\widetilde{X}_t^{(n)} = X_{t \wedge R_n}^{(n)}$, and likewise for $\widetilde{Y}^{(n)}$, these processes are also $\{\mathfrak{F}_t\}$ - martingales (Problem 3.22), and are in $\mathfrak{m}_2^c$ because they are bounded. For $m > n$, $\widetilde{X}_t^{(n)} = \widetilde{X}_{t \wedge R_n}^{(m)}$ and so

$$(\widetilde{X}_t^{(n)})^2 - \langle \widetilde{X}^{(m)} \rangle_{t \wedge R_n} = (\widetilde{X}_{t \wedge R_n}^{(m)})^2 - \langle \widetilde{X}^{(m)} \rangle_{t \wedge R_n}$$

is a martingale. This implies $\langle \widetilde{X}^{(n)} \rangle_t = \langle \widetilde{X}^{(m)} \rangle_{t \wedge R_n}$. We can thus define $\langle X \rangle_t = \langle \widetilde{X}^{(n)} \rangle_t$ whenever $t \leq R_n$ and be assured that $\langle X \rangle_t$ is well-defined. The process $\langle X \rangle$ is adapted,

continuous, nondecreasing, and satisfies $\langle X \rangle_0 = 0$ a.s. Furthermore,

$$X^2_{t \wedge R_n} - \langle X \rangle_{t \wedge R_n} = (\tilde{X}^{(n)}_t)^2 - \langle \tilde{X}^{(n)} \rangle_t$$

is a martingale for each n, so $X^2 - \langle X \rangle \in \mathcal{m}^{c,loc}$. As in Theorem 5.13, we may now take

$$\langle X, Y \rangle = \frac{1}{4}[\langle X+Y \rangle - \langle X-Y \rangle] .$$

As for the question of uniqueness, suppose both A and B satisfy the conditions required of $\langle X, Y \rangle$. Then $M \triangleq XY-A$ and $N \triangleq XY - B$ are in $\mathcal{m}^{c,loc}$, so just as before we can construct a sequence $\{R_n\}$ of stopping times with $R_n \uparrow \infty$ such that $M^{(n)}_t \triangleq M_{t \wedge R_n}$ and $N^{(n)}_t \triangleq N_{t \wedge R_n}$ are in $\mathcal{m}^c_2$. Consequently $M^{(n)}_t - N^{(n)}_t = B^{(n)}_t - A^{(n)}_t \in \mathcal{m}^c_2$, and being of bounded variation this process must be identically zero (see the proof of Theorem 5.13). It follows that $A = B$.

5.18 Solution: Let $\lambda = EX^2_1$. The martingale property implies

$$EX^2_{1/n} = \frac{1}{n} \sum_{k=1}^{n} E(X_{\frac{k}{n}} - X_{\frac{k-1}{n}})^2 = \frac{1}{n} E[\sum_{k=1}^{n} (X_{\frac{k}{n}} - X_{\frac{k-1}{n}})]^2 = \frac{\lambda}{n} .$$

Similarly, we can show $EX^2_{k/n} = \frac{k\lambda}{n}$ for all positive integers k and n. Since both EX^2_t and $\langle X \rangle_t$ are nondecreasing functions of t, we have $EX^2_t = \lambda t$, $t \geq 0$. We now show that $X^2_t - \lambda t$ is a martingale. For $0 \leq s < t$,

$$E[X_t^2 - \lambda t \,|\, \mathcal{F}_s] = E[((X_t - X_s) + X_s)^2 - \lambda t \,|\, \mathcal{F}_s] =$$

$$E[(X_t - X_s)^2 - \lambda t \,|\, \mathcal{F}_s] + X_s^2 = EX_{t-s}^2 - \lambda t + X_s^2 = X_s^2 - \lambda s.$$

<u>5..22 Solution</u>:   From $E(M_t^2) = E\langle M \rangle_t \leqslant E\langle M \rangle_\infty < \infty$ we obtain

$\sup_{t \geqslant 0} E(M_t^2) < \infty$,   which implies the uniform integrability of

M (Chung [1], Exercise 4.5.8). From Problem 1.3.18 we have

that $M_\infty = \lim_{t \to \infty} M_t$ exists a.s. P, and that $E(M_\infty \,|\, \mathcal{F}_t) = M_t$

holds a.s. P, for every $t \geqslant 0$. Fatou's lemma now yields

$$E(M_\infty^2) = E(\lim_{t \to \infty} M_t^2) \leqslant \lim_{t \to \infty} E(M_t^2) = \lim_{t \to \infty} E\langle M \rangle_t = E\langle M \rangle_\infty < \infty ,$$

and Jensen's inequality: $M_t^2 \leqslant E(M_\infty^2 \,|\, \mathcal{F}_t)$,   a.s. P,   for every

$t \geqslant 0$. It follows that the submartingale M^2 has a last

element, i.e., that $\{M_t^2, \mathcal{F}_t; \ 0 \leqslant t \leqslant \infty\}$ is a submartingale;

besides, we have $E(M_t^2) \leqslant E(M_\infty^2)$ whence $E\langle M \rangle_\infty \leqslant E(M_\infty^2)$.

Therefore, $\lim_{t \to \infty} E(M_t^2) = E(M_\infty^2) = E\langle M \rangle_\infty \cdot$ and so, by Problem

1.3.17, the submartingale M^2 is uniformly integrable.

Finally, $Z_t = E(M_\infty^2 \,|\, \mathcal{F}_t) - M_t^2$ is now seen to be a (right-con-

tinuous, by appropriate choice of modification) nonnegative

supermartingale, with $E(Z_t) = E(M_\infty^2) - E(M_t^2)$ converging to

zero as $t \to \infty$.

1.7 : NOTES

<u>Sections 1.1, 1.2</u>: These two sections could have been lumped together under the rubric "Fields, Optionality and Measurability" after the manner of article [3] by Chung & Doob. Although slightly dated, this article still makes excellent reading. Good accounts of this material in book form have been written by Meyer [16; Chapter IV], Dellacherie [4; Chapter III and to a lesser extent Chapter IV], and Chung [2; Chapter 1]. These sources provide material on the classification of stopping times as "predictable", "accessible" and "totally inaccessible", as well as corresponding notions of measurability for stochastic processes, which we need not broach here.

A new notion of "sameness" between two stochastic processes, called "synonimity", has been introduced by Aldous. It was expounded in a recent paper by Hoover [10] and was found to be useful in the study of martingales.

<u>Section 1.3</u>: The term "martingale" was introduced in Probability Theory by J. Ville in his 1939 book "Étude critique de la notion du collectif". The concept had been created by P. Lévy back in 1934, in an attempt to extend the Kolmogorov inequality and the law of large numbers beyond the case of independence. Lévy's 0-1 law (Theorem 9.4.8 and Corollary in Chung [1]) is the first martingale convergence theorem. The general theory, as we know it today, sprang fully armed from the forehead of J.L. Doob [5]. For the foundations of the discrete-parameter case there is perhaps no better source than the relevant

sections in Chapter 9 of Chung [1] that we have already mentioned; fuller accounts are Neveu [17] and Hall & Heyde [9]. Other books, which contain material on the continuous-parameter case, include Meyer [16; Chapter V, VI], Liptser & Shiryaev [13; Chapter 2, 3] and Elliott [7; Chapters 3, 4].

Section 1.4: Theorem 4.10 is due to P.A. Meyer [14, 15]; its proof was later simplified by K.M. Rao [18]. Our account of this theorem, as well as that of Theorem 4.14, follows closely Ikeda & Watanabe [11].

Section 1.5: The study of square-integrable martingales began with Fisk [8] and continued with the seminal article [12] by Kunita & Watanabe. Theorem 5.4 is due to Fisk [8].

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CHAPTER 2

BROWNIAN MOTION

CHAPTER 2

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2.1 INTRODUCTION.

"Brownian movement" was the name given to the irregular movement of pollen, suspended in water, observed by the botanist Robert Brown in 1828. This random movement, now attributed to the buffeting of the pollen by water molecules, results in a dispersal or "diffusion" of the pollen in the water. The range of application of Brownian motion as defined here goes far beyond a study of microscopic particles in suspension and includes modelling of stock prices, modelling of thermal noise in electrical circuits, modelling of certain limiting behaviour in queueing and inventory systems, and modelling of random perturbations in a variety of other physical, economic, biological, and management systems. In addition, integration with respect to Brownian motion gives us a unifying representation for a large class of martingales and diffusion processes. Diffusion processes represented this way exhibit a rich connection with the theory of partial differential equations. In particular, to each such process there corresponds a second order parabolic equation which governs the transition probabilities of the process. The history of Brownian motion is discussed more extensively in Section 10.

Definition 1.1: A (<u>standard, one-dimensional</u>) <u>Brownian motion</u> is a continuous, adapted process $B = \{B_t, \mathcal{F}_t; \ 0 \leq t < \infty\}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ with the properties that $B_0 = 0$ a.s. and for $0 \leq s < t$, the increment $B_t - B_s$ is independent of $\mathcal{F}_s$ and is normally distributed with mean

zero and variance t-s. We shall speak sometimes of a

Brownian motion $B = \{B_t, \mathcal{F}_t; 0 \leq t \leq T\}$ on $[0,T]$, for some

$T > 0$, and the meaning of this terminology is apparent.

If B is a Brownian motion and $0 = t_0 < t_1 < \ldots < t_n$, then the

increments $\{B_{t_j} - B_{t_{j-1}}\}_{j=1}^n$ are independent and the distribution

of $B_{t_j} - B_{t_{j-1}}$ depends on t_j and t_{j-1} only through the difference

$t_j - t_{j-1}$; to wit, it is normal with mean zero and variance $t_j - t_{j-1}$.

We say that the process B has <u>stationary, independent increments.</u>

It is easily verified that B is a square integrable martingale and

$\langle B \rangle_t = t$, $t \geq 0$.

The filtration $\{\mathcal{F}_t\}$ is a part of the definition of Brownian

motion. However, if we are given $\{B_t; 0 \leq t < \infty\}$ but no filtration,

and if we know that B has stationary, independent increments and

that $B_t = B_t - B_0$ is normal with mean zero and variance t, then

$\{B_t, \mathcal{F}_t^B; 0 \leq t < \infty\}$ is easily seen to be a Brownian motion. Moreover,

if $\{\mathcal{F}_t\}$ is a "larger" filtration in the sense that $\mathcal{F}_t^B \subset \mathcal{F}_t$ for

$t \geq 0$, and if $B_t - B_s$ is independent of $\mathcal{F}_s$ whenever $0 \leq s < t$, then

$\{B_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is also a Brownian motion.

The first problem one encounters with Brownian motion is its

existence. One approach to this question is to write down what the

finite-dimensional distributions of this process must be (based

on the stationarity, independence, and normality of its increments),

and then construct a probability measure and a process on an

appropriate measurable space in such a way that we obtain the pre-

scribed finite-dimensional distributions. This direct approach is

the one most often used to construct a Markov process, but is rather

lengthy and technical; we spell it out in section 2. A more elegant

approach for Brownian motion, based on Hilbert space theory, is provided in Section 3; it is close in spirit to Wiener's [ ] original construction, which was modified by Lévy [ ] and later further simplified by Ciesielski [ ]. Sections 2 and 3 are independent of one another, and with the exception of Problem 2.9 and Remark 2.12, which are used in Chapter 5, the only result we need from these sections is the fact that Brownian motion exists. Section 4 provides yet another proof of the existence of this process, this time based on the idea of Brownian motion as the weak limit of a sequence of random walks. The properties of the space $C[0, \infty)$ developed in this section will be used extensively throughout the book.

Section 5 defines the Markov property, which is enjoyed by Brownian motion. Section 6 presents the strong Markov property, and, using a proof based on the optional sampling theorem for martingales, shows that Brownian motion is a strong Markov process. In Section 7 we discuss various choices of the filtration for Brownian motion. The central idea here is augmentation of the filtration generated by the process, in order to obtain a right-continuous filtration. Developing this material in the context of strong Markov processes requires no additional effort, and so we adopt this level of generality.

Sections 8 and 9 are devoted to properties of Brownian motion. In Section we compute distributions of a number of elementary Brownian functionals; among these are first passage times, last exit times, and time and level of the maximum over a fixed time-interval. Section 9 deals with almost sure properties of the

Brownian sample path. Here we discuss its growth as $t \to \infty$, its oscillations near $t = 0$ (law of the iterated logarithm), its nowhere differentiability and nowhere monotonicity, and the topological perfectness of the set of times when the sample path is at the origin.

2.2 FIRST CONSTRUCTION OF BROWNIAN MOTION

Let $R^{[0,\infty)}$ denote the set of all real-valued functions on $[0,\infty)$. An <u>n-dimensional cylinder set</u> in $R^{[0,\infty)}$ is a set of the form

$$(2.1) \qquad C \triangleq \{\omega \in R^{[0,\infty)}; \ (\omega(t_1),\ldots,\omega(t_n)) \in A\},$$

where $t_i \in [0,\infty)$, $i=1,\ldots,n$, and $A \in \mathcal{B}(R^n)$. Let $\mathcal{C}$ denote the field of all cylinder sets (of all finite dimensions) in $R^{[0,\infty)}$, and let $\mathcal{B}(R^{[0,\infty)})$ denote the smallest σ-field containing $\mathcal{C}$.

<u>2.1 Definition</u>: Let T be the set of finite sequences $\underset{\sim}{t} = (t_1,\ldots,t_n)$ of distinct, nonnegative numbers, where the length n of these sequences ranges over the set of positive integers. Suppose that for each $\underset{\sim}{t}$ of length n, we have a probability measure $Q_{\underset{\sim}{t}}$ on $(R^n, \mathcal{B}(R^n))$. Then the collection $\{Q_{\underset{\sim}{t}}\}_{\underset{\sim}{t} \in T}$ is called a <u>family of finite-dimensional distributions</u>. This family is said to be <u>consistent</u> provided that the following two conditions are satisfied:

   (a) if $\underset{\sim}{s} = (t_{i_1}, t_{i_2},\ldots, t_{i_n})$ is a permutation of $\underset{\sim}{t} = (t_1, t_2,\ldots, t_n)$, then for any $A_i \in \mathcal{B}(R)$, $i=1,\ldots,n$, we have

$$Q_{\underset{\sim}{t}}(A_1 \times A_2 \times \ldots \times A_n) = Q_{\underset{\sim}{s}}(A_{i_1} \times A_{i_2} \times \ldots \times A_{i_n});$$

(b) if $\underset{\sim}{t} = (t_1, t_2, \ldots, t_n)$ with $n \geq 1$, $\underset{\sim}{s} = (t_1, t_2, \ldots, t_{n-1})$,
and $A \in \mathcal{B}(\mathbb{R}^{n-1})$, then

$$Q_{\underset{\sim}{t}}(A \times \mathbb{R}) = Q_{\underset{\sim}{s}}(A).$$ □

If we have a probability measure P on $(\mathbb{R}^{[0, \infty)}, \mathcal{B}(\mathbb{R}^{[0, \infty)}))$,
then we can define a family of finite-dimensional distributions
by

(2.2) $Q_{\underset{\sim}{t}}(A) = P[\omega \in \mathbb{R}^{[0, \infty)}; (\omega(t_1), \ldots, \omega(t_n)) \in A],$

where $A \in \mathcal{B}(\mathbb{R}^n)$ and $\underset{\sim}{t} = (t_1, \ldots, t_n) \in T$. This family is easily
seen to be consistent. We are interested in the converse of this
fact, because it will enable us to construct a probability measure
P from the finite-dimensional distributions of Brownian motion.

2.2 Theorem: Daniell (1918), Kolmogorov (1933).

Let $\{Q_{\underset{\sim}{t}}\}$ be a consistent family of finite-dimensional dis-
tributions. Then there is a probability measure P on
$(\mathbb{R}^{[0, \infty)}, \mathcal{B}(\mathbb{R}^{[0, \infty)}))$, such that (2.2) holds for every $\underset{\sim}{t} \in T$.

Proof: We begin by defining a set function Q on the field of
cylinders $\mathcal{C}$. If C is given by (2.1) and $\underset{\sim}{t} = (t_1, t_2, \ldots, t_n) \in T$,
we set

(2.3) $Q(C) = Q_{\underset{\sim}{t}}(A), C \in \mathcal{C}.$

Such a definition is indeed possible because of the consistency of the family of finite-dimensional distributions.

<u>2.3 Problem</u>: The set function Q is well-defined and finitely additive on $\mathscr{B}(\mathbb{R}^{[0,\infty)})$, with $Q(\mathbb{R}^{[0,\infty)}) = 1$. $\qquad\square$

We now prove the countable additivity of Q on $\mathbb{C}$, and we can then draw on the Carathéodory Extension Theorem to assert the existence of the desired extension P of Q to $\mathscr{B}(\mathbb{R}^{[0,\infty)})$. Thus, suppose $\{B_k\}_{k=1}^{\infty}$ is a sequence of disjoint sets in $\mathbb{C}$ with $B \overset{\Delta}{=} \bigcup_{k=1}^{\infty} B_k$ also in $\mathbb{C}$. Let $C_m = B \backslash \bigcup_{k=1}^{m} B_k$, so

$$Q(B) = Q(C_m) + \sum_{k=1}^{m} Q(B_k).$$

Countable additivity will follow from

$$(2.4) \qquad \lim_{m \to \infty} Q(C_m) = 0.$$

Now $Q(C_m) = Q(C_{m+1}) + Q(B_{m+1}) \geq Q(C_{m+1})$, so the above limit exists. Assume that this limit is equal to $\epsilon > 0$, and note that $\bigcap_{m=1}^{\infty} C_m = \emptyset$.

From $\{C_m\}_{m=1}^{\infty}$ we may construct another sequence $\{D_m\}_{m=1}^{\infty}$ which has the properties: $D_1 \supseteq D_2 \supseteq \cdots, \bigcap_{m=1}^{\infty} D_m = \emptyset$, and $\lim_{m \to \infty} Q(D_m) = \epsilon > 0$. Furthermore, each D_m has the form

$$D_m = \{\omega \in R^{[0,\infty)}; \ (\omega(t_1),\ldots,\omega(t_m)) \in A_m\}$$

for some $A_m \in \mathcal{B}(R^m)$, and the finite sequence $\underset{\sim}{t}_m \triangleq (t_1,\ldots,t_m) \in T$ is an extension of the finite sequence $\underset{\sim}{t}_{m-1} \triangleq (t_1,\ldots,t_{m-1}) \in T$, $m \geq 2$.

This may be accomplished as follows. Each C_k has a form

$$C_k = \{\omega \in R^{[0,\infty)}; \ (\omega(t_1),\ldots,\omega(t_{m_k})) \in A_{m_k}\}; \ A_{m_k} \in \mathcal{B}(R^{m_k}),$$

where $\underset{\sim}{t}_{m_k} = (t_1,\ldots,t_{m_k}) \in T$. Since $C_{k+1} \subseteq C_k$, we can choose these representations so that $\underset{\sim}{t}_{m_{k+1}}$ is an extension of $\underset{\sim}{t}_{m_k}$, and $A_{m_{k+1}} \subseteq A_{m_k} \times R^{m_{k+1}-m_k}$. Define

$$D_1 = \{\omega; \ \omega(t_1) \in R\},\ldots,D_{m_1-1} = \{\omega: (\omega(t_1),\ldots,\omega(t_{m_1-1})) \in R^{m_1-1}\}$$

and $D_{m_1} = C_1$, as well as

$$D_{m_1+1} = \{\omega; \ (\omega(t_1),\ldots,\omega(t_{m_1}),\omega(t_{m_1+1})) \in A_{m_1} \times R\},\ldots,$$

$$D_{m_2-1} = \{\omega; \ (\omega(t_1),\ldots,\omega(t_{m_1}),\omega(t_{m_1+1}),\ldots,\omega(t_{m_2-1})) \in A_{m_1} \times R^{m_2-m_1-1}\}$$

and $D_{m_2} = C_2$. Continue this process, and note that by construction

$$\bigcap_{m=1}^{\infty} D_m = \bigcap_{m=1}^{\infty} C_m = \emptyset.$$

<u>2.4 Problem</u>: Let Q be a probability measure on $(R^n, \mathcal{B}(R^n))$. We say that $A \in \mathcal{B}(R^n)$ is <u>regular</u> if for every $\epsilon > 0$, there is a closed set F and an open set G such that $F \subseteq A \subseteq G$ and $Q(G \setminus F) < \epsilon$. Show that every set in $\mathcal{B}(R^n)$ is regular. (Hint: Show that the collection of regular sets is a σ-field containing all closed sets.)

According to Problem 2.4, there exists for each m a closed set $F_m \subseteq A_m$ such that $Q_{\underset{\sim}{t}_m}(A_m \backslash F_m) < \frac{\epsilon}{2^m}$. By intersecting F_m with a sufficiently large closed sphere centered at the origin, we obtain a compact set K_m such that, with

$$E_m \triangleq \{\omega \in \mathbb{R}^{[0,\infty)}; \ (\omega(t_1), \ldots, \omega(t_m)) \in K_m\},$$

we have $E_m \subseteq D_m$ and

$$Q(D_m \backslash E_m) = Q_{\underset{\sim}{t}_m}(A_m \backslash K_m) < \frac{\epsilon}{2^m}.$$

The sequence $\{E_m\}$ may fail to be nonincreasing, so we define

$$\widetilde{E}_m = \bigcap_{k=1}^{m} E_k,$$

and we have

$$\widetilde{E}_m = \{\omega \in \mathbb{R}^{[0,\infty)}; \ (\omega(t_1), \ldots, \omega(t_m)) \in \widetilde{K}_m\},$$

where

$$\widetilde{K}_m = (K_1 \times \mathbb{R}^{m-1}) \cap (K_2 \times \mathbb{R}^{m-2}) \cap \ldots \cap (K_{m-1} \times \mathbb{R}) \cap K_m,$$

which is compact. We can bound $Q_{\underset{\sim}{t}_m}(\widetilde{K}_m)$ away from zero, since

$$Q_{\underset{\sim}{t}_m}(\widetilde{K}_m) = Q(\widetilde{E}_m) = Q(D_m) - Q(D_m \backslash \widetilde{E}_m)$$

$$= Q(D_m) - Q\left(\bigcup_{k=1}^{m} (D_m \backslash E_k)\right)$$

$$\geq Q(D_m) - Q\left(\bigcup_{k=1}^{m} (D_k \backslash E_k)\right)$$

$$\geq \epsilon - \sum_{k=1}^{m} \frac{\epsilon}{2^k} > 0.$$

Therefore, $\widetilde{K}_m$ is nonempty for each m, and we can choose $(x_1^{(m)}, \ldots, x_m^{(m)}) \in \widetilde{K}_m$. Being contained in the compact set $\widetilde{K}_1$, the sequence $\{x_1^{(m)}\}_{m=1}^{\infty}$ must have a convergent subsequence $\{x_1^{(m_k)}\}_{k=1}^{\infty}$ with limit x_1. But $\{x_1^{(m_k)}, x_2^{(m_k)}\}_{k=2}^{\infty}$ is contained in $\widetilde{K}_2$, so it has a convergent subsequence with limit (x_1, x_2). Continuing this process, we can construct $(x_1, x_2, \ldots) \in R \times R \times \ldots$, such that $(x_1, \ldots, x_m) \in \widetilde{K}_m$ for each m. Consequently, the set

$$S = \{\omega \in R^{[0,\infty)}; \ \omega(t_i) = x_i, \quad i = 1, 2, \ldots\}$$

is contained in each $\widetilde{E}_m$, and hence in each D_m. This contradicts the fact that $\bigcap_{m=1}^{\infty} D_m = \emptyset$. We conclude that (2.4) holds. □

Our aim is to construct a probability measure P on $(\Omega, \mathcal{F})$ $\triangleq (R^{[0,\infty)}, \mathcal{B}(R^{[0,\infty)}))$ so that the process $B = \{B_t, \mathcal{F}_t^B; \ 0 \leq t < \infty\}$ defined by $B_t(\omega) \triangleq \omega(t)$, the so-called <u>coördinate mapping process</u>, is almost a standard, one-dimensional Brownian motion under P. We say "almost" because we leave aside the requirement of sample path continuity for the moment, and concentrate on the finite-dimensional distributions. Recalling the discussion following Definition 1.1, we see that whenever $0 = s_0 < s_1 < s_2 < \ldots < s_n$, the cumulative distribution function for $(B_{s_1}, \ldots, B_{s_n})$ must be

$$(2.5) \qquad F_{(s_1, \ldots, s_n)}(x_1, \ldots, x_n) =$$

$$= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} p(s_1; 0, y_1)\, p(s_2 - s_1; y_1, y_2) \cdots$$

$$\cdots p(s_n - s_{n-1}; y_{n-1}, y_n)\, dy_n \cdots dy_2\, dy_1$$

for $(x_1, \ldots, x_n) \in \mathbf{R}^n$, where p is the Gaussian kernel

$$(2.6) \qquad p(t; x, y) \triangleq \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(x-y)^2}{2t}}, \quad t > 0, \; x, y \in \mathbf{R}.$$

The reader can verify (and should, if he has never done so!) that (2.5) is equivalent to the statement that the increments $\{B_{s_j} - B_{s_{j-1}}\}_{j=1}^{n}$ are independent, and $B_{s_j} - B_{s_{j-1}}$ is normally distributed with mean zero and variance $s_j - s_{j-1}$.

Now let $\underset{\sim}{t} = (t_1, t_2, \ldots, t_n)$, where the t_j are not necessarily ordered but are distinct. Let the random vector $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ have the distribution determined by (2.5) (where the t_j must be ordered from smallest to largest to obtain $(s_1, \ldots, s_n)$ appearing in (2.5)). For $A \in \mathcal{B}(\mathbf{R}^n)$, let $Q_{\underset{\sim}{t}}(A)$ be the probability under this distribution that $(B_{t_1}, B_{t_2}, \ldots B_{t_n})$ is in A. This defines a family of finite-dimensional distributions $\{Q_{\underset{\sim}{t}}\}_{\underset{\sim}{t} \in T}$.

2.5 Problem: Show that the family $\{Q_{\underset{\sim}{t}}\}_{\underset{\sim}{t} \in T}$ defined above is consistent.

2.6 Corollary to Theorem 2.2:   There is a probability measure

P on $(R^{[0,\infty)}, \mathcal{B}(R^{[0,\infty)}))$,   under which the coördinate mapping

process

$$B_t(\omega) = \omega(t), \quad \omega \in R^{[0,\infty)}, \ t \geq 0,$$

has stationary, independent increments.   An increment $B_t - B_s$

where $0 \leq s < t$,   is normally distributed with mean zero and

variance $t-s$.

Our construction of Brownian motion would now be complete,

were it not for the fact that we have built the process on the

sample space $R^{[0,\infty)}$ of all real-valued functions on $[0,\infty)$ rather

than on the space $C[0,\infty)$ of continuous functions on this half-

line.   One might hope to overcome this difficulty by showing that

the probability measure P in Corollary 2.6 assigns measure one to

$C[0,\infty)$.   However, as the next problem shows, $C[0,\infty)$, is not in the

σ-field $\mathcal{B}(R^{[0,\infty)})$,   so $P(C[0,\infty))$ is not defined.   This failure

is a manifestation of the fact that the σ-field $\mathcal{B}(R^{[0,\infty)})$ is,

quite uncomfortably, "too small" for a space as big as $R^{[0,\infty)}$; no

set in $\mathcal{B}(R^{[0,\infty)})$ can have restrictions on uncountably many

coördinates.   In contrast to the space $C[0,\infty)$,   it is not possible

to determine a function in $R^{[0,\infty)}$ by specifying its values at only

countably many coördinates.   Consequently, the next theorem takes

a different approach, which is to construct a continuous modification

of the coördinate mapping process in Corollary 2.6.

2.7 <u>Problem</u>: Show that the only $\mathcal{B}(R^{[0,\infty)})$-measurable set contained in $C[0,\infty)$ is the empty set. (Hint: a typical set in $\mathcal{B}(R^{[0,\infty)})$ has the form

$$E = \{\omega \in R^{[0,\infty)}; \ (\omega(t_1), \ \omega(t_2), \ldots) \in A\},$$

where $A \in \mathcal{B}(R \times R \times \ldots)$.

2.8 <u>Theorem</u>: <u>Kolmogorov</u>   , Čentsov (1956).

Suppose that a process $X = \{X_t; \ 0 \le t \le T\}$ on a probability space $(\Omega, \mathcal{F}, P)$ satisfies the condition

(2.6) $E|X_t - X_s|^\alpha \le C|t-s|^{1+\beta}, \quad 0 \le s, t \le T,$

for some positive constants α, β and C. Then there exists a continuous modification $\tilde{X} = \{\tilde{X}_t; \ 0 \le t \le T\}$ of X, which is locally Hölder continuous with exponent γ for every $\gamma \in (0, \frac{\beta}{\alpha})$, i.e.,

(2.7) $P\left[\omega; \ \sup_{\substack{0 < t-s < h(\omega) \\ s,t \in [0,T]}} \frac{|\tilde{X}_t(\omega) - \tilde{X}_s(\omega)|}{|t-s|^\gamma} \le \delta\right] = 1,$

where $h(\omega)$ is an a.s. positive random variable and $\delta > 0$ is an appropriate constant.

<u>Proof</u>:   For notational simplicity, we take $T = 1$. Much of what follows is a consequence of the Čebyšev inequality. First, for any $\epsilon > 0$, we have

$$P[\,|X_t - X_s| \geq \epsilon\,] \;\leq\; \frac{E\,|X_t - X_s|^\alpha}{\epsilon^\alpha} \;\leq\; C\,\epsilon^{-\alpha}|t-s|^{1+\beta},$$

and so $X_s \to X_t$ in probability as $s \to t$. Secondly, setting $t = \frac{k}{2^n}$, $s = \frac{k-1}{2^n}$ and $\epsilon = 2^{-\gamma n}$ (where $0 < \gamma < \frac{\beta}{\alpha}$) in the above inequality, we obtain

$$P[\,|X_{k/2^n} - X_{(k-1)/2^n}| \geq 2^{-\gamma n}] \;\leq\; C\,2^{-n(1+\beta-\alpha\gamma)},$$

and consequently,

$$P[\,\max_{1 \leq k \leq 2^n} |X_{k/2^n} - X_{(k-1)/2^n}| \geq 2^{-\gamma n}]$$

$$\leq\; P[\,\bigcup_{k=1}^{2^n} |X_{k/2^n} - X_{(k-1)/2^n}| \geq 2^{-\gamma n}]$$

$$\leq\; C\,2^{-n(\beta-\alpha\gamma)}.$$

The last expression is the general term of a convergent series; by the Borel-Cantelli Lemma, there is a set $\Omega^* \in \mathcal{F}$ with $P(\Omega^*) = 1$ such that for each $\omega \in \Omega^*$,

$$(2.8) \qquad \max_{1 \leq k \leq 2^n} |X_{k/2^n}(\omega) - X_{(k-1)/2^n}(\omega)| < 2^{-\gamma n}, \;\; \forall\, n \geq n^*(\omega),$$

where $n^*(\omega)$ is a positive, integer-valued random variable.

For each integer $n \geq 1$, let us consider the partition $D_n = \{\frac{k}{2^n};\; k = 0, 1, \ldots, 2^n\}$ of $[0,1]$, and let $D = \bigcup_{n=1}^{\infty} D_n$ be the

set of dyadic rationals in $[0,1]$. We shall fix $\omega \in \Omega^*$, $n \geq n^*(\omega)$, and show that for every $m > n$, we have

(2.9) $\qquad |X_t(\omega) - X_s(\omega)| \leq 2 \sum_{j=n+1}^{m} 2^{-\gamma j}; \quad \forall \; t, s \in D_m, \; 0 < t-s < 2^{-n}.$

For $m = n+1$, we can only have $t = \dfrac{k}{2^m}$, $s = \dfrac{k-1}{2^m}$, and (2.9) follows from (2.8). Suppose (2.9) is valid for $m = n+1, \ldots, M-1$. Take $s < t$, $s, t \in D_M$, consider the numbers $t^1 = \max\{u \in D_{M-1}; u \leq t\}$ and $s^1 = \min\{u \in D_{M-1}; u \geq s\}$, and notice the relationships $s \leq s^1 \leq t^1 \leq t$, $s^1 - s \leq 2^{-M}$, $t - t^1 \leq 2^{-M}$. From (2.8) we have $|X_{s^1}(\omega) - X_s(\omega)| \leq 2^{-\gamma M}$, $|X_t(\omega) - X_{t^1}(\omega)| \leq 2^{-\gamma M}$, and from (2.9) with $m = M-1$,

$$|X_{t^1}(\omega) - X_{s^1}(\omega)| \leq 2 \sum_{j=n+1}^{M-1} 2^{-\gamma j}.$$

We obtain (2.9) for $m = M$.

We can show now that $\{X_t(\omega); t \in D\}$ is uniformly continuous in t for every $\omega \in \Omega^*$. For any numbers $s, t \in D$ with $0 < t-s < h(\omega) \triangleq 2^{-n^*(\omega)}$, we select $n \geq n^*(\omega)$ such that $2^{-(n+1)} \leq t-s < 2^{-n}$. We have from (2.9)

(2.10) $\qquad |X_t(\omega) - X_s(\omega)| \leq 2 \sum_{j=n+1}^{\infty} 2^{-\gamma j} \leq \delta |t-s|^{\gamma}, \; 0 < t-s < h(\omega),$

where $\delta = \dfrac{2}{1-2^{-\gamma}}$. This proves the desired uniform continuity.

We define $\widetilde{X}$ as follows. For $\omega \notin \Omega^*$, set $\widetilde{X}_t(\omega) = 0$, $0 \leq t \leq 1$. For $\omega \in \Omega^*$ and $t \in D$, set $\widetilde{X}_t(\omega) = X_t(\omega)$. For $\omega \in \Omega^*$ and $t \in [0,1] \cap D^c$, choose a sequence $\{s_n\}_{n=1}^{\infty} \subseteq D$ with $s_n \to t$. Uniform

continuity and the Cauchy criterion imply that $\{X_{s_n}(\omega)\}_{n=1}^{\infty}$ has a limit which depends on t but not on the particular sequence $\{s_n\}_{n=1}^{\infty} \subseteq D$ chosen to converge to t. We set $\widetilde{X}_t(\omega) = \lim_{s_n \to t} X_{s_n}(\omega)$. The resulting process $\widetilde{X}$ is thereby continuous; indeed, $\widetilde{X}$ satisfies (2.10), so (2.7) is established.

To see that $\widetilde{X}$ is a modification of X, observe that $\widetilde{X}_t = X_t$ a.s. for $t \in D$; for $t \in [0,1] \cap D^c$ and $\{s_n\} \subseteq D$ with $s_n \to t$, we have $X_{s_n} \to X_t$ in probability and $X_{s_n} \to \widetilde{X}_t$ a.s., so $\widetilde{X}_t = X_t$ a.s.
$\square$

<u>2.9 Problem</u>: A <u>random field</u> is a collection of random variables $\{X_t\}$, where t is chosen from a partially ordered set. Suppose $\{X_t; t \in [0,T]^d\}$, $d \geq 2$, is a random field satisfying

$$(2.11) \qquad E|X_t - X_s|^{\alpha} \leq C\|t-s\|^{d+\beta}$$

for some positive constants α, β and C. Show that the conclusion of Theorem 2.8 holds with (2.7) replaced by

$$(2.12) \qquad P[\omega; \sup_{\substack{0 < \|t-s\| < h(\omega) \\ s,t \in [0,T]^d}} \frac{|\widetilde{X}_t(\omega) - \widetilde{X}_s(\omega)|}{\|t-s\|^{\gamma}} \leq \delta] = 1.$$

<u>2.10 Problem</u>: Show that if $B_t - B_s$, $0 \leq s < t$, is normally distributed with mean zero and variance $t-s$, then for each positive integer n, there is a positive constant C_n for which

$$E|B_t - B_s|^{2n} = C_n|t-s|^n.$$

2.11 Corollary to Theorem 2.8

There is a probability measure P on $(R^{[0,\infty)}, \mathcal{B}(R^{[0,\infty)}))$, and a stochastic process $W = \{W, \mathcal{F}_t^W; t \geq 0\}$ on the same space, such that under P, W is a Brownian motion.

Proof: According to Theorem 2.8 and Problem 2.10, there is for each $T > 0$ a modification W^T of the process B in Corollary 2.5 such that W^T is continuous on $[0, T]$. Let

$$\Omega_T = \{\omega: W_t^T(\omega) = B_t(\omega) \text{ for every rational } t \in [0, T]\},$$

so $P(\Omega_T) = 1$. On $\tilde{\Omega} \triangleq \bigcap_{T=1}^{\infty} \Omega_T$, we have for positive integers T_1 and T_2,

$$W_t^{T_1}(\omega) = W_t^{T_2}(\omega), \text{ for every rational } t \in [0, T_1 \wedge T_2].$$

Since both processes are continuous on $[0, T_1 \wedge T_2]$, we must have $W_t^{T_1}(\omega) = W_t^{T_2}(\omega)$ for every $t \in [0, T_1 \wedge T_2]$, $\omega \in \tilde{\Omega}$. Define $W_t(\omega)$ to be this common value. For $\omega \notin \tilde{\Omega}$, set $W_t(\omega) = 0$ for all $t \geq 0$. $\square$

2.12 Remark:
Actually, for P-a.e. $\omega \in R^{[0,\infty)}$, the Brownian sample path $\{W_t(\omega); 0 \leq t < \infty\}$ is locally Hölder continuous with exponent γ, for every $\gamma \in (0, 1/2)$. This is a consequence of Theorem 2.8 and Problem 2.10.

2.3   SECOND CONSTRUCTION OF BROWNIAN MOTION

If $\{B_t, \mathfrak{F}_t; t \geq 0\}$ is a Brownian motion and $0 \leq s < t$, then conditioned on $B_s = x_1$ and $B_t = x_3$, the random variable $B_{\frac{s+t}{2}}$ is normal with mean $\mu \triangleq \dfrac{x_1 + x_3}{2}$ and variance $\sigma^2 \triangleq \dfrac{t-s}{4}$. To verify this, observe that the known distribution and independence of the increments B_s, $B_{\frac{s+t}{2}} - B_s$, and $B_t - B_{\frac{s+t}{2}}$ results in a joint density

$$P[B_s \in dx_1, \ B_{\frac{s+t}{2}} \in dx_2, \ B_t \in dx_3]$$

$$= \frac{1}{\sqrt{2\pi s}} \frac{1}{\sqrt{\pi(t-s)}} \frac{1}{\sqrt{\pi(t-s)}} e^{-\frac{x_1^2}{2s}} e^{-\frac{(x_2-x_1)^2}{t-s}} e^{-\frac{(x_3-x_2)^2}{t-s}} dx_1 dx_2 dx_3$$

$$= \frac{1}{\sqrt{2\pi s}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{x_1^2}{2s}} e^{-\frac{(x_3-x_1)^2}{2(t-s)}} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_2-\mu)^2}{2\sigma^2}} dx_1 dx_2 dx_3.$$

Dividing by

$$P[B_s \in dx_1, B_t \in dx_3] = \frac{1}{\sqrt{2\pi s}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{x_1^2}{2s}} e^{-\frac{(x_3-x_1)^2}{2(t-s)}} dx_1 dx_2,$$

we obtain

$$P[B_{\frac{t+s}{2}} \in dx_2 \mid B_s = x_1, B_t = x_3] = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_2-\mu)^2}{2\sigma^2}} dx_2.$$

The simple form of this conditional distribution of $B_{\frac{t+s}{2}}$ suggests that we can construct Brownian motion on some finite time-interval, say $[0,1]$, by interpolation. Once we have completed the construction

on [0,1], a simple "patching together" of a sequence of such Brownian motions will result in a Brownian motion defined for all $t \geq 0$.

To carry out this program, we begin with a countable collection $\{\xi_k^{(n)}; k \in I(n), n=0,1,\ldots\}$ of independent, standard (zero mean and unit variance), normal random variables on a probability space $(\Omega, \mathfrak{F}, P)$. Here $I(n)$ is the set of odd integers between 0 and 2^n, i.e., $I(0) = \{1\}$, $I(1) = \{1\}$, $I(2) = \{1,3\}$, etc. For each $n \geq 0$, we define a process $B^{(n)} = \{B_t^{(n)}; 0 \leq t \leq 1\}$ by interpolating linearly between these points. For $n \geq 1$, $B_{\frac{k}{2^{n-1}}}^{(n)}$ will agree with $B_{\frac{k}{2^{n-1}}}^{(n-1)}$, $k=0,1,\ldots,2^{n-1}$. Thus, for each value of n, we need only specify $B_{\frac{k}{2^n}}^{(n)}$ for $k \in I(n)$. We set

$$B_0^{(0)} = 0, \quad B_1^{(0)} = \xi_1^{(0)}.$$

If the values of $B_{\frac{k}{2^{n-1}}}^{(n-1)}$, $k=0,1,\ldots,2^{n-1}$ have been specified (so $B_t^{(n-1)}$ is defined for $0 \leq t \leq 1$ by piecewise-linear interpolation) and $k \in I(n)$, we denote $s = \frac{k-1}{2^n}$, $t = \frac{k+1}{2^n}$, $\mu = \frac{1}{2}(B_s^{(n-1)} + B_t^{(n-1)})$, and $\sigma^2 = \frac{t-s}{4} = \frac{1}{2^{n+1}}$ and set

$$B_{\frac{k}{2^n}}^{(n)} = B_{\frac{t+s}{2}}^{(n)} = \mu + \sigma \, \xi_k^{(n)}.$$

We shall show that, almost surely, $B_t^{(n)}$ converges uniformly in t to a continuous function B_t, and $\{B_t, \mathfrak{F}_t^B; 0 \leq t \leq 1\}$ is a Brownian

motion.

Our first step is to give a more convenient representation for the processes $B^{(n)}$, $n=0,1,\dots$. We define the <u>Haar functions</u> by $H_1^{(0)}(t) = 1$, $0 \leq t \leq 1$, and for $n \geq 1$, $k \in I(n)$,

$$H_k^{(n)}(t) = \begin{cases} 2^{\frac{n-1}{2}}, & \frac{k-1}{2^n} \leq t < \frac{k}{2^n}, \\[2mm] -2^{\frac{n-1}{2}}, & \frac{k}{2^n} \leq t < \frac{k+1}{2^n}, \\[2mm] 0, & \text{otherwise.} \end{cases}$$

We define the <u>Schauder functions</u> by

$$S_k^{(n)}(t) = \int_0^t H_k^{(n)}(u)\,du, \quad 0 \leq t \leq 1, \ n \geq 0, \ k \in I(n).$$

Note that $S_1^{(0)}(t) = t$, and for $n \geq 1$ the graphs of $S_k^{(n)}$ are little tents of height $2^{-\frac{(n+1)}{2}}$ centered at $\frac{k}{2^n}$ and nonoverlapping for different values of $k \in I(n)$. It is clear that

$$B_t^{(0)} = \xi_1^{(0)} S_1^{(0)}(t),$$

and by induction on n, it is easily verified that

$$(3.1) \qquad B_t^{(n)}(\omega) = \sum_{m=0}^{n} \sum_{k \in I(m)} \xi_k^{(m)}(\omega) \, S_k^{(m)}(t), \quad 0 \leq t \leq 1, \ n \geq 0.$$

<u>Lemma 3.1</u> As $n \to \infty$, the sequence of processes $\{B_t^{(n)}(\omega); 0 \leq t \leq 1\}$, $n \geq 1$, given by (3.1) converges uniformly in t to a continuous process $\{B_t(\omega); 0 \leq t \leq 1\}$, for a.e. $\omega \in \Omega$.

Proof: Define $b_n = \max\limits_{k \in I(n)} |\xi_k^{(n)}|$. For $x > 0$

$$P[|\xi_k^{(n)}| > x] = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-\frac{u^2}{2}} du$$

$$(3.1)' \qquad \leq \sqrt{\frac{2}{\pi}} \int_x^\infty \frac{u}{x} e^{-\frac{u^2}{2}} du$$

$$= \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{x^2}{2}}}{x} \;,$$

which gives

$$P[b_n > n] = P[\bigcup_{k \in I_n} \{|\xi_k^{(n)}| > n\}]$$

$$\leq 2^n \, P[|\xi_1^{(n)}| > n]$$

$$\leq \sqrt{\frac{2}{\pi}} \frac{2^n e^{-\frac{n^2}{2}}}{n} \;, \quad n \geq 1.$$

But $\sum\limits_{n=1}^\infty \dfrac{2^n e^{-\frac{n^2}{2}}}{n} < \infty$, so the Borel-Cantelli lemma implies that there is a set $\widetilde{\Omega}$ with $P(\widetilde{\Omega}) = 1$ such that for each $\omega \in \widetilde{\Omega}$ there is an integer $n(\omega)$ satisfying $b_n(\omega) \leq n$ for all $n \geq n(\omega)$. But then

$$\sum_{n=n(\omega)}^\infty \sum_{k \in I(n)} |\xi_k^{(n)} \, S_k^{(n)}(t)| \leq \sum_{n=n(\omega)}^\infty n \, 2^{-\frac{n+1}{2}} < \infty \;;$$

so for $\omega \in \widetilde{\Omega}$, $B_t^{(n)}(\omega)$ converges uniformly in t to a limit $B_t(\omega)$. Continuity of $\{B_t(\omega); 0 \leq t \leq 1\}$ follows from the uniformity

of the convergence. □

Under the inner product $\langle f, g \rangle = \int_0^1 f(t) \, g(t) dt$, $L^2[0,1]$ is a Hilbert space, and the Haar functions $\{H_k^{(n)}; k \in I(n), n \geq 0\}$ form a complete, orthonormal system (see, e.g., Kaczmarz-Steinhaus (1951)). The Parseval equality

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \sum_{k \in I(n)} \langle f, H_k^{(n)} \rangle \langle g, H_k^{(n)} \rangle,$$

applied to $f = 1_{[0,t]}$ and $g = 1_{[0,s]}$ yields

$$(3.2) \qquad \sum_{n=1}^{\infty} \sum_{k \in I(n)} S_k^{(n)}(t) \, S_k^{(n)}(s) = \min(s,t); \quad 0 \leq s, t \leq 1.$$

Theorem 3.2: With $\{B_t^{(n)}\}_{n=1}^{\infty}$ defined by (3.1) and $B_t = \lim_{n \to \infty} B_t^{(n)}$, the process $\{B_t, \mathcal{F}_t^B; 0 \leq t \leq 1\}$ is a Brownian motion on $[0,1]$.

Proof: It suffices to prove that, for $0 = t_0 < t_1 < \ldots < t_n \leq 1$, the increments $\{B_{t_j} - B_{t_{j-1}}\}_{j=1}^n$ are independent, normally distributed, with mean zero and variance $t_j - t_{j-1}$. We prove this by showing that for $\lambda_j \in R$, $j = 1, \ldots, n$,

$$(3.3) \qquad E[\exp\{i \sum_{j=1}^n \lambda_j (B_{t_j} - B_{t_{j-1}})\}] = \prod_{j=1}^n \exp\{-\frac{1}{2} \lambda_j^2 (t_j - t_{j-1})\}.$$

Set $\lambda_{n+1} = 0$. Using the independence and standard normality of the random variables $\{\xi_k^{(n)}\}$, we have from (3.1)

$$E[\exp\{-i \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_j)\, B_{t_j}^{(M)}\}]$$

$$= E[\exp\{-i \sum_{m=0}^{M} \sum_{k \in I(m)} \xi_k^{(m)} \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_j) S_k^{(m)}(t_j)\}]$$

$$= \prod_{m=0}^{M} \prod_{k \in I(m)} E[\exp\{-i\, \xi_k^{(m)} \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_j) S_k^{(m)}(t_j)\}]$$

$$= \prod_{m=0}^{M} \prod_{k \in I(m)} \exp[-\tfrac{1}{2}\{\sum_{j=1}^{n} (\lambda_{j+1} - \lambda_j) S_k^{(m)}(t_j)\}^2]$$

$$= \exp[-\tfrac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} (\lambda_{j+1} - \lambda_j)(\lambda_{i+1} - \lambda_i) \cdot$$

$$\cdot \sum_{m=0}^{M} \sum_{k \in I(m)} S_k^{(m)}(t_i) S_k^{(m)}(t_j)].$$

Letting $M \to \infty$ and using (3.2), we obtain

$$E[\exp\{i \sum_{j=1}^{n} \lambda_j (B_{t_j} - B_{t_{j-1}})\}]$$

$$= E[\exp\{-i \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_j) B_{t_j}\}]$$

$$= \exp\{-\sum_{j=1}^{n-1} \sum_{i=j+1}^{n} (\lambda_{j+1} - \lambda_j)(\lambda_{i+1} - \lambda_i) t_j - \tfrac{1}{2} \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_j)^2 t_j\}$$

$$= \exp\{-\sum_{j=1}^{n-1} (\lambda_{j+1} - \lambda_j)(-\lambda_{j+1}) t_j - \tfrac{1}{2} \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_j)^2 t_j\}$$

$$= \exp\left\{\frac{1}{2} \sum_{j=1}^{n-1} (\lambda_{j+1}^2 - \lambda_j^2) t_j - \frac{1}{2} \lambda_n^2 t_n\right\}$$

$$= \prod_{j=1}^{n} \exp\left\{-\frac{1}{2} \lambda_j^2 (t_j - t_{j-1})\right\}. \qquad \qquad \Box$$

<u>3.3 Problem</u>: Prove Theorem 3.2 without resort to the Parseval

identity (3.2), by completing the following steps.

(a) The increments $\left\{B_{\frac{k}{2^n}}^{(n)} - B_{\frac{k-1}{2^n}}^{(n)}\right\}_{k=1}^{2^n}$ are independent, normal

random variables with mean zero and variance $\frac{1}{2^n}$.

(b) If $0 = t_0 < t_1 < \ldots < t_n \leq 1$ and each t_j is a dyadic rational,
then the increments $\left\{B_{t_j} - B_{t_{j-1}}\right\}_{j=1}^{n}$ are independent,
normal random variables with mean zero and variance
$(t_j - t_{j-1})$.

(c) The assertion in (b) holds even if $\{t_j\}_{j=1}^{n}$ is not
contained in the set of dyadic rationals.

<u>3.4 Corollary</u>: There is a probability space $(\Omega, \mathfrak{F}, P)$ and a stochas-
tic process $B = \{B_t, \mathfrak{F}_t^B; 0 \leq t < \infty\}$ on it, such that B is a
standard, one-dimensional Brownian motion.

<u>Proof</u>:   According to Theorem 3.2, there is a sequence $(\Omega_n, \mathfrak{F}_n, P_n)$,
n=1,2,... of probability spaces together with a Brownian motion
$\{X_t^{(n)}; 0 \leq t \leq 1\}$ on each space.   Let $\Omega = \Omega_1 \times \Omega_2 \times \ldots, \quad \mathfrak{F} = \mathfrak{F}_1 \otimes \mathfrak{F}_2 \otimes \ldots,$

and $P = P_1 \times P_2 \times \ldots$. Define B on Ω recursively by

$$B_t = X_t^{(1)}, \quad 0 \leq t \leq 1,$$

$$B_t = B_n + X_{t-n}^{(n+1)}, \quad n \leq t \leq n+1.$$

This process is clearly continuous, and the increments are easily seen to be independent and normal with zero mean and the proper variances.

$\square$

2.4: THE SPACE C[0, ∞), WEAK CONVERGENCE, AND WIENER MEASURE

The sample spaces for the Brownian motions we built in Sections 2 and 3 were, respectively, the space $R^{[0, \infty)}$ of all real-valued function on $[0, \infty)$ and a space Ω rich enough to carry a countable collection of independent, standard normal random variables. The "canonical" space for Brownian motion, the one most convenient for many future developments, is $C[0, \infty)$, the space of all continuous, real-valued functions on $[0, \infty)$ with metric

$$(4.1) \qquad \rho(\omega_1, \omega_2) \triangleq \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0 \leq t \leq n} (|\omega_1(t) - \omega_2(t)| \wedge 1).$$

In this section, we show how to construct a measure, called <u>Wiener measure</u>, on this space so that the coördinate mapping process is Brownian motion. This construction is given as the proof of Theorem 4.16 (Donsker's Theorem), and involves the notion of weak convergence of random walks to Brownian motion.

<u>4.1 Problem</u>: Show that ρ defined by (4.1) is a metric on $C[0, \infty)$ and, under ρ, $C[0, \infty)$ is a complete, separable, metric space.

4.2 <u>Problem</u>:　　Let $C(C_t)$ be the collection of finite-dimensional cylinder sets of the form

(2.1)´　$C = \{\omega \in C[0, \infty); (\omega(t_1), \ldots, \omega(t_n)) \in A\};$ $n \geq 1,$ $A \in \mathcal{B}(R^n),$

where, for all $i = 1, \ldots, n,$ $t_i \in [0, \infty)$ (respectively, $t_i \in [0, t]$). Denote by $\mathcal{G}(\mathcal{G}_t)$ the smallest σ-field containing $C(C_t)$.

　　Show that $\mathcal{G} = \mathcal{B}(C[0, \infty)),$ the Borel σ-field generated by the open sets in $C[0, \infty),$ and that $\mathcal{G}_t = \varphi_t^{-1}(\mathcal{B}(C[0, \infty)))$ $\overset{\Delta}{=} \mathcal{B}_t(C[0, \infty)),$ where $\varphi_t: C[0, \infty) \to C[0, \infty)$ is the mapping $(\varphi_t \omega)(s) = \omega(t \wedge s);$ $0 \leq s < \infty.$

<u>4.3 Definition:</u> Let (S,ρ) be a metric space with Borel σ-field $\mathfrak{B}(S)$. Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of probability measures $(S,\mathfrak{B}(S))$, and let P be another measure on this space. We say that $\{P_n\}_{n=1}^{\infty}$ <u>converges weakly to</u> P and write $P_n \xrightarrow{w} P$, if and only if

$$\lim_{n \to \infty} \int_S f(s) \, dP_n(s) = \int_S f(s) \, dP(s)$$

for every bounded, continuous, real-valued function f on S.

It follows, in particular, that the <u>weak limit</u> P is a probability measure, and that it is unique.

Whenever X is a random variable on a probability space $(\Omega, \mathfrak{F}, P)$ with values in $(S, \mathfrak{B}(S))$, i.e., the function $X: \Omega \to S$ is $\mathfrak{F}/\mathfrak{B}(S)$ - measurable, then X induces a probability measure PX^{-1} on $(S, \mathfrak{B}(S))$ by

$$PX^{-1}(B) = P\{\omega \in \Omega; \, X(\omega) \in B\}, \quad B \in \mathfrak{B}(S).$$

<u>4.4 Definition:</u> Let $\{(\Omega_n, \mathfrak{F}_n, P_n)\}_{n=1}^{\infty}$ be a sequence of probability spaces, and on each of them consider a random variable X_n with values in the metric space (S, ρ). Let $(\Omega, \mathfrak{F}, P)$ be another probability space, on which a random variable X with values in (S, ρ) is given. We say that $\{X_n\}_{n=1}^{\infty}$ <u>converges to X in distribution</u>, and write $X_n \xrightarrow{\mathfrak{D}} X$, if and only if the sequence of measures induced on $(S, \mathfrak{B}(S))$ by $\{X_n\}_{n=1}^{\infty}$ converges weakly to the measure induced by X.

Equivalently, $X_n \overset{\mathfrak{D}}{\longrightarrow} X$ if and only if

$$\lim_{n \to \infty} E_n \; f(X_n) = E \; f(X)$$

for every bounded, continuous, real-valued function f on S, where E_n and E denote expectations with respect to P_n and P, respectively.

Recall that if S in Definition 4.4 is R^d, then $X_n \overset{\mathfrak{D}}{\longrightarrow} X$ if and only if the sequence of characteristic functions $\varphi_n(u) \triangleq E_n \exp\{i(u, X_n)\}$ converges to $\varphi(u) \triangleq E \exp\{i(u, X)\}$, for every $u \in R^d$. This is the so-called Cramér-Wold device (Theorem 7.7 in Billingsley [1968]).

The most important example of convergence in distribution is that provided by the Central Limit Theorem. In the Lindeberg-Lévy form used here, the theorem asserts that if $\{\xi_n\}_{n=1}^{\infty}$ is a sequence of independent, identically distributed random variables with mean zero and variance σ^2, then $\{S_n\}$ defined by

$$S_n = \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{n} \xi_k$$

converges in distribution to a standard normal random variable. It is this fact which dictates that, properly normalized, a sequence of random walks will converge in distribution to Brownian motion.

4.5 Problem: Suppose $\{X_n\}_{n=1}^\infty$ is a sequence of random variables taking values in (S_1, ρ_1) and converging in distribution to X. Suppose (S_2, ρ_2) is another metric space, and $\varphi: S_1 \to S_2$ is continuous. Show that $Y_n \stackrel{\Delta}{=} \varphi(X_n)$ converges in distribution to $Y \stackrel{\Delta}{=} \varphi(X)$.

4.6 Definition: Let (S, ρ) be a metric space and let Π be a family of probability measures on $(S, \mathscr{B}(S))$. We say that Π is relatively compact if every sequence of elements of Π contains a weakly convergent subsequence. We say that Π is tight if for every $\epsilon > 0$, there exists a compact set $K \subseteq S$ such that $P(K) \geq 1 - \epsilon$, for every $P \in \Pi$. If $\{X_\alpha\}_{\alpha \in A}$ is a family of random variables taking values in S, we say that this family is relatively compact or tight if the family of induced measures $\{P X_\alpha^{-1}\}_{\alpha \in A}$ has the appropriate property.

The following theorem is stated without proof; its special case $S = \mathbb{R}$ is used to prove the central limit theorem. In the form provided here, a proof can be found in several sources, for instance Billingsley [1968], pp. 35-40, or Parthasarathy [1967], pp. 47-49.

4.7 Theorem: Prohorov (1956)

Let Π be a family of probability measures on a complete, separable metric space S. This family is relatively compact if and only if it is tight.

We are interested in the case $S = C[0, \infty)$. For this case, we shall provide a characterization of tightness (Theorem 4.10). To do so, we define for each $\omega \in C[0, \infty)$, $T > 0$, and $\delta > 0$ the _modulus of continuity_ on $[0, T]$:

$$m^T(\omega, \delta) \triangleq \max_{\substack{|s-t| \leq \delta \\ 0 \leq s, t \leq T}} |\omega(s) - \omega(t)|.$$

4.8 Problem: Show that $m^T(\omega, \delta)$ is continuous in $\omega \in C[0, \infty)$

under the ρ metric, is nondecreasing in δ, and

$\lim\limits_{\delta \downarrow 0} m^T(\omega, \delta) = 0$ for each $\omega \in C[0, \infty)$.

We shall need the following version of the Arzelà-Ascoli Theorem.

4.9 Theorem: A set $A \subseteq C[0, \infty)$ has compact closure if and only

if

(4.2) $\quad \sup\limits_{\omega \in A} |\omega(0)| < \infty,$

and for each $T > 0$,

(4.3) $\quad \lim\limits_{\delta \downarrow 0} \sup\limits_{\omega \in A} m^T(\omega, \delta) = 0.$

Proof:

Assume that the closure of A, denoted by $\bar{A}$, is compact. Since $\bar{A}$ is contained in the union of the open sets

$$G_n = \{\omega \in C[0,\infty); \ |\omega(0)| < n\}, \quad n=1,2,\ldots$$

it must be contained in some particular G_n, and (4.2) follows. For $\epsilon > 0$, let $K_\delta = \{\omega \in \bar{A}; \ m^T(\omega, \delta) \geq \epsilon\}$. Each K_δ is closed (Problem 4.8) and is contained in $\bar{A}$, so each K_δ is compact. Problem 4.8 implies $\bigcap_{\delta > 0} K_\delta = \emptyset$, so for some $\delta(\epsilon) > 0$, we have $K_{\delta(\epsilon)} = \emptyset$. This proves (4.3).

We now assume (4.2), (4.3) and prove compactness of $\bar{A}$. Since $C[0,\infty)$ is a metric space, it suffices to prove that every sequence $\{\omega_n\}_{n=1}^\infty \subseteq A$ has a convergent subsequence. We fix $T > 0$ and note that for some $\delta_1 > 0$, we have $m^T(\omega, \delta_1) \leq 1$ for each $\omega \in A$; so for fixed integer $m \geq 1$ and $(m-1)\delta_1 < t \leq m\delta_1 \leq T$, we have from (4.3):

$$|\omega(t)| \leq |\omega(0)| + \sum_{k=1}^{m-1} |\omega(k\delta_1) - \omega((k-1)\delta_1)| + |\omega(t) - \omega((m-1)\delta_1)|$$

$$\leq |\omega(0)| + m.$$

It follows that for each $r \in Q$, the set of nonnegative rationals, $\{\omega_n(r)\}_{n=1}^\infty$ is bounded. Let $\{r_0, r_1, r_2, \ldots\}$ be an enumeration of Q. Then choose $\{\omega_n^{(0)}\}_{n=1}^\infty$, a subsequence of $\{\omega_n\}_{n=1}^\infty$ with $\omega_n^{(0)}(r_0)$ converging to a limit denoted $\omega(r_0)$. From $\{\omega_n^{(0)}\}_{n=1}^\infty$, choose a further subsequence $\{\omega_n^{(1)}\}_{n=1}^\infty$ such that $\omega_n^{(1)}(r_1)$

converges to a limit $\omega(r_1)$. Continue this process, and then let $\{\tilde{\omega}_n\}_{n=1}^{\infty} = \{\omega_n^{(n)}\}_{n=1}^{\infty}$ be the "diagonal sequence". We have $\tilde{\omega}_n(r) \to \omega(r)$ for each $r \in Q$.

Let us note from (4.3) that for each $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $|\tilde{\omega}_n(s) - \tilde{\omega}_n(t)| \leq \epsilon$ whenever $0 \leq s, t \leq T$ and $|s-t| \leq \delta(\epsilon)$. The same inequality therefore holds for ω when we impose the additional condition $s, t \in Q$. It follows that ω is uniformly continuous on $[0,T] \cap Q$ and so has an extension to a continuous function, also called ω, on $[0,T]$; furthermore, $|\omega(s) - \omega(t)| \leq \epsilon$ whenever $0 \leq s, t \leq T$ and $|s-t| \leq \delta(\epsilon)$. For n sufficiently large, we have that whenever $t \in [0,T]$, there is some $r_k \in Q$ with $k \leq n$ and $|t-r_k| \leq \delta(\epsilon)$. For sufficiently large $M \geq n$, we have $|\tilde{\omega}_m(r_j) - \omega(r_j)| \leq \epsilon$ for all $j = 0, 1, \ldots, n$ and $m \geq M$. Consequently,

$$|\tilde{\omega}_m(t) - \omega(t)| \leq |\tilde{\omega}_m(t) - \tilde{\omega}_m(r_k)| + |\tilde{\omega}_m(r_k) - \omega(r_k)|$$
$$+ |\omega(r_k) - \omega(t)|$$

$$\leq 3\epsilon, \qquad \forall\, m \geq M, \; 0 \leq t \leq T.$$

We can make this argument for any $T > 0$, so $\{\tilde{\omega}_n\}_{n=1}^{\infty}$ converges uniformly on bounded intervals to the function $\omega \in C[0,\infty)$. $\square$

4.10 Theorem: A sequence $\{P_n\}_{n=1}^{\infty}$ of probability measures on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ is tight if and only if

$$(4.4) \qquad \lim_{\lambda \uparrow \infty} \sup_{n \geq 1} P_n[\omega; |\omega(0)| > \lambda] = 0,$$

and for each positive T and ϵ,

$$(4.5) \qquad \lim_{\delta \downarrow 0} \sup_{n \geq 1} P_n[\omega; \; m^T(\omega, \delta) > \epsilon] = 0.$$

Proof:

Suppose first that $\{P_n\}_{n=1}^{\infty}$ is tight. Given $\eta > 0$, there is a compact set K with $P_n(K) \geq 1 - \eta$, for every $n \geq 1$. According to Theorem 4.9, for sufficiently large $\lambda > 0$, we have $|\omega(0)| \leq \lambda$ for all $\omega \in K$; this proves (4.4). According to the same theorem, if T and ϵ are also given, then there exists δ_0 such that $m^T(\omega, \delta) \leq \epsilon$ for $0 < \delta < \delta_0$ and $\omega \in K$. This gives us (4.5).

Let us now assume (4.4) and (4.5). Given a positive integer T and $\eta > 0$, we choose $\lambda > 0$ so that

$$\sup_{n \geq 1} P_n[\omega; \; |\omega(0)| > \lambda] \leq \eta/2^{T+1} \; .$$

We choose $\delta_k > 0$, $k = 1, 2, \ldots$ such that

$$\sup_{n \geq 1} P_n[\omega; \; m^T(\omega, \delta_k) > \tfrac{1}{k}] \leq \eta/2^{T+k+1} \; .$$

Define

$$A_T = \{\omega; \; |\omega(0)| \leq \lambda, \; m^T(\omega, \delta_k) \leq \tfrac{1}{k} \; , \; k = 1, 2, \ldots\},$$

$$A = \bigcap_{T=1}^{\infty} A_T,$$

so $P_n(A_T) \geq 1 - \sum_{k=0}^{\infty} \eta/2^{T+k+1} = 1 - \eta/2^T$ and $P_n(A) \geq 1 - \eta$, for every $n \geq 1$. By Theorem 4.9, $\bar{A}$ is compact, so $\{P_n\}_{n=1}^{\infty}$ is tight. $\qquad \square$

4.10′ **Problem:** Let $\{X^{(m)}\}_{m=1}^{\infty}$ be a sequence of continuous stochastic processes $X^{(m)} = \{X_t^{(m)}; \ 0 \le t < \infty\}$ on $(\Omega, \mathfrak{F}, P)$, satisfying the following conditions:

(i) $\quad \sup_{m \ge 1} E|X_0^{(m)}|^{\nu} \triangleq M < \infty,$

(ii) $\quad \sup_{m \ge 1} E|X_t^{(m)} - X_s^{(m)}|^{\alpha} \le C_T|t-s|^{1+\beta}; \quad \forall \ T > 0 \quad \text{and} \quad 0 \le s, t \le T$

for some positive constants α, β, ν (universal) and C_T (depending on $T > 0$).

Show that the probability measures $P_m \triangleq P(X^{(m)})^{-1}; \ m \ge 1$ induced by these processes on $(C_{[0, \infty)}, \ \mathcal{B}(C_{[0, \infty)}))$ form a tight sequence.

(Hint: Follow the technique of proof in the Kolmogorov-Čentsov Theorem 2.8, to verify the conditions (4.4), (4.5) of Theorem 4.10).

Suppose X is a continuous process on some $(\Omega, \mathfrak{F}, P)$. For each ω, the function $t \mapsto X_t(\omega)$ is a member of $C[0, \infty)$, which we denote $X(\omega)$. Since $\mathfrak{B}(C[0, \infty))$ is generated by the one-dimensional cylinder sets and $X_t(\cdot)$ is $\mathfrak{F}$-measurable for each fixed t, the random object $X: \Omega \to C[0, \infty)$ is $\mathfrak{F}/\mathfrak{B}(C[0, \infty))$ - measurable. Thus, if $\{X^{(n)}\}_{n=1}^{\infty}$ is a sequence of continuous processes (with each $X^{(n)}$ defined on a perhaps distinct probability space $(\Omega_n, \mathfrak{F}_n, P_n)$), we can ask if $X^{(n)} \overset{\mathfrak{D}}{\longrightarrow} X$ in the sense of Definition 4.4. We can also ask if the finite-dimensional distributions of $\{X^{(n)}\}_{n=1}^{\infty}$ converge to those of X, i.e., if

$$(X_{t_1}^{(n)}, X_{t_2}^{(n)}, \ldots, X_{t_d}^{(n)}) \overset{\mathfrak{D}}{\longrightarrow} (X_{t_1}, X_{t_2}, \ldots, X_{t_d}).$$

The latter question is considerably easier to answer than the former, since the convergence in distribution of finite-dimensional random vectors can be resolved by studying characteristic functions.

For any finite subset $\{t_1, \ldots, t_d\}$ of $[0, \infty)$, let us define the <u>projection mapping</u> $\pi_{t_1, \ldots, t_2}: C[0, \infty) \to \mathbb{R}^d$ as

$$\pi_{t_1, \ldots, t_d}(\omega) = (\omega(t_1), \ldots, \omega(t_d)).$$

If the function $f: \mathbb{R}^d \to \mathbb{R}$ is bounded and continuous, then the composite mapping $f \circ \pi_{t_1, \ldots, t_d}: C[0, \infty) \to \mathbb{R}$ enjoys the same properties; thus, $X^{(n)} \underset{n \to \infty}{\overset{\mathfrak{D}}{\longrightarrow}} X$ implies

$$\lim_{n \to \infty} E_n f(X_{t_1}^{(n)}, \ldots, X_{t_d}^{(n)}) = \lim_{n \to \infty} E_n(f \circ \pi_{t_1}, \ldots, t_d)(X^{(n)})$$

$$= E(f \circ \pi_{t_1}, \ldots, t_d)(X) = E f(X_{t_1}, \ldots, X_{t_d}).$$

In other words, if the sequence of processes $\{X^{(n)}\}_{n=1}^{\infty}$ converges in distribution to the process X, then all finite-dimensional distributions converge as well. The converse holds in the presence of tightness (Theorem 4.12), but not in general; this failure is illustrated by the following example.

4.11 Problem: With probability one, let

$$X_t^{(n)} = \begin{cases} nt & , \quad 0 \leq t \leq \frac{1}{2n} \\ 1-nt & , \quad \frac{1}{2n} \leq t \leq \frac{1}{n} \\ 0 & , \quad t \geq \frac{1}{n} \end{cases},$$

and let $X_t = 0$, $t \geq 0$. Show that all finite-dimensional distributions of $X^{(n)}$ converge weakly to the corresponding finite-dimensional distributions of X, but the sequence of processes $\{X^{(n)}\}_{n=1}^{\infty}$ does not converge in distribution to the process X.

4.12 Theorem: Let $\{X^{(n)}\}_{n=1}^{\infty}$ be a tight sequence of continuous processes with the property that, whenever $0 \leq t_1 < \ldots < t_d < \infty$, then the sequence of random vectors $\{(X_{t_1}^{(n)}, \ldots, X_{t_d}^{(n)})\}_{n=1}^{\infty}$

converges in distribution. Let P_n be the measure induced on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ by $X^{(n)}$. Then $\{P_n\}_{n=1}^{\infty}$ converges weakly to a measure P, under which the coördinate mapping process $W_t(\omega) \triangleq \omega(t)$ on $C[0,\infty)$ satisfies

$$(X_{t_1}^{(n)}, \ldots, X_{t_d}^{(n)}) \xrightarrow{\mathcal{D}} (W_{t_1}, \ldots, W_{t_d}), \quad 0 \leq t_1 < \ldots < t_d < \infty, \quad d \geq 1.$$

Proof:

Every subsequence $\{\tilde{X}^{(n)}\}$ of $\{X^{(n)}\}$ is tight, and so has a further subsequence $\{\hat{X}^{(n)}\}$ such that the measures induced on $C[0,\infty)$ by $\{\hat{X}^{(n)}\}$ converge weakly to a probability measure P by the Prohorov Theorem 4.7. If a different subsequence $\{\check{X}^{(n)}\}$ induces measures on $C[0,\infty)$ converging to a probability measure Q, then P and Q must have the same finite-dimensional distributions, i.e.,

$$P[\omega \in C[0,\infty); (\omega(t_1), \ldots, \omega(t_d)) \in A]$$

$$= Q[\omega \in C[0,\infty); (\omega(t_1), \ldots, \omega(t_d)) \in A],$$

$$0 \leq t_1 < t_2 < \ldots < t_d, \quad A \in \mathcal{B}(\mathbb{R}^d), \quad d \geq 1.$$

This means $P = Q$.

Suppose the sequence of measures $\{P_n\}_{n=1}^{\infty}$ induced by $\{X^{(n)}\}_{n=1}^{\infty}$ did not converge weakly to P. Then there must be a bounded, continuous function $f : C[0,\infty) \to \mathbb{R}$ such that $\lim_{n \to \infty} \int f(\omega) P_n(d\omega)$ does not

exist, or else this limit exists but is different from $\int f(\omega) \, P(d\omega)$. In either case, we can choose a subsequence $\{\widetilde{P}_n\}_{n=1}^{\infty}$ for which $\lim\limits_{n \to \infty} \int f(\omega) \, \widetilde{P}_n \, (d\omega)$ exists but is different from $\int f(\omega) \, P(d\omega)$. This subsequence can have no further subsequence $\{\hat{P}_n\}_{n=1}^{\infty}$ with $\hat{P}_n \xrightarrow{w} P$, and this violates the conclusion of the previous paragraph.

$\square$

We shall need the following result.

<u>4.13 Problem</u>: Let $\{X^{(n)}\}_{n=1}^{\infty}$, $\{Y^{(n)}\}_{n=1}^{\infty}$ and X be random variables with values in the metric space (S, ρ); we assume that for each $n \geq 1$, $X^{(n)}$ and $Y^{(n)}$ are defined on the same probability space. If $X^{(n)} \xrightarrow{\mathscr{D}} X$ and $|X^{(n)} - Y^{(n)}| \to 0$ in probability, as $n \to \infty$, then $Y^{(n)} \xrightarrow{\mathscr{D}} X$ as $n \to \infty$.

Let us consider now a sequence $\{\xi_j\}_{j=1}^{\infty}$ of independent, identically distributed random variables with mean zero and variance σ^2, $0 < \sigma^2 < \infty$, as well as the sequence of partial sums $S_0 = 0$, $S_k = \sum\limits_{j=1}^{k} \xi_j$, $k \geq 1$. A continuous-time process $Y = \{Y_t; \, t \geq 0\}$ can be obtained from the sequence $\{S_k\}_{k=0}^{\infty}$ by linear interpolation, i.e.,

$$(4.6) \qquad Y_t = S_{[t]} + (t - [t])\xi_{[t]+1}, \quad t \geq 0,$$

where $[t]$ denotes the greatest integer less than or equal to t. Scaling appropriately both time and space, we obtain from Y a sequence of processes $\{X^{(n)}\}$:
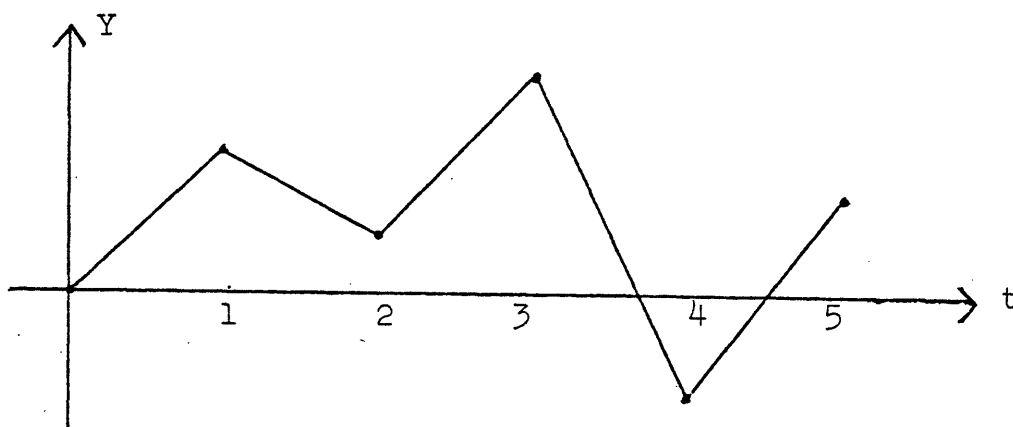
(4.7) $X_t^{(n)} = \dfrac{1}{\sigma\sqrt{n}}\, Y_{nt}$, $t \geq 0$.

Note that with $s = \dfrac{k}{n}$ and $t = \dfrac{k+1}{n}$, the increment $X_t^{(n)} - X_s^{(n)} = \dfrac{1}{\sigma\sqrt{n}}\, \xi_{k+1}$ is independent of $\mathcal{F}_s^{X^{(n)}} = \sigma(\xi_1, \ldots, \xi_k)$.

Furthermore, $X_t^{(n)} - X_s^{(n)}$ has zero mean and variance $t-s$. This suggests than $\{X_t^{(n)};\ t \geq 0\}$ is approximately a Brownian motion. We now show that, even though the random variables ξ_j are not necessarily normal, the Central Limit Theorem dictates that the limiting distributions of the increments of $X^{(n)}$ are normal.

<u>4.14 Theorem</u>: With $\{X^{(n)}\}$ defined by (4.7) and $0 \leq t_1 < \ldots < t_d$, we have

$$(X_{t_1}^{(n)}, \ldots, X_{t_d}^{(n)}) \xrightarrow{\ \mathcal{D}\ } (B_{t_1}, \ldots, B_{t_d}) \quad \text{as } n \to \infty,$$

where $\{B_t, \mathcal{F}_t^B; t \geq 0\}$ is a standard, one-dimensional Brownian motion.

Proof:

We take the case $d=2$; the other cases differ from this one only by being notationally more cumbersome. Set $s=t_1$, $t=t_2$. We wish to show

$$(X_s^{(n)}, X_t^{(n)}) \xrightarrow{\mathcal{D}} (B_s, B_t).$$

Since

$$|X_t^{(n)} - \frac{1}{\sigma\sqrt{n}} S_{[tn]}| \leq \frac{1}{\sigma\sqrt{n}} |\xi_{[tn]+1}|,$$

we have by the Čebyšev inequality,

$$P[|X_t^{(n)} - \frac{1}{\sigma\sqrt{n}} S_{[tn]}| > \epsilon] \leq \frac{1}{\epsilon^2 n} \to 0$$

as $n \to \infty$. It is clear then that

$$|(X_s^{(n)}, X_t^{(n)}) - \frac{1}{\sigma\sqrt{n}}(S_{[sn]}, S_{[tn]})| \to 0 \quad \text{in probability},$$

so, by Problem 4.13, it suffices to show:

$$\frac{1}{\sigma\sqrt{n}} (S_{[sn]}, S_{[tn]}) \xrightarrow{\mathcal{D}} (B_s, B_t).$$

From Problem 4.5 we see that this is equivalent to proving

$$\frac{1}{\sigma\sqrt{n}} \left(\sum_{j=1}^{[sn]} \xi_j, \sum_{j=[sn]+1}^{[tn]} \xi_j \right) \xrightarrow{\mathcal{D}} (B(s), B(t)-B(s)).$$

The independence of the random variables $\{\xi_j\}_{j=1}^{\infty}$ implies

$$(4.8) \qquad \lim_{n \to \infty} E[\exp\{\frac{iu}{\sigma\sqrt{n}} \sum_{j=1}^{[sn]} \xi_j + \frac{iv}{\sigma\sqrt{n}} \sum_{j=[sn]+1}^{[tn]} \xi_j\}] =$$

$$= \lim_{n \to \infty} E[\exp\{\frac{iu}{\sigma\sqrt{n}} \sum_{j=1}^{[sn]} \xi_j\}] \cdot \lim_{n \to \infty} E[\exp\{\frac{iv}{\sigma\sqrt{n}} \sum_{j=[sn]+1}^{[tn]} \xi_j\}],$$

provided both limits on the right-hand side exist. We deal with

$\lim_{n \to \infty} E[\exp\{\frac{iu}{\sigma\sqrt{n}} \sum_{j=1}^{[sn]} \xi_j\}]$;   the other can be treated similarly. Since

$$\left| \frac{1}{\sigma\sqrt{n}} \sum_{j=1}^{[sn]} \xi_j - \frac{\sqrt{s}}{\sigma\sqrt{[sn]}} \sum_{j=1}^{[sn]} \xi_j \right| \to 0 \quad \text{in probability},$$

and, by the Central Limit Theorem, $\frac{\sqrt{s}}{\sigma\sqrt{[sn]}} \sum_{j=1}^{[sn]} \xi_j$ converges in

distribution to a normal random variable with mean zero and

variance s, we have

$$\lim_{n \to \infty} E[\exp\{\frac{iu}{\sigma\sqrt{n}} \sum_{j=1}^{[sn]} \xi_j\}] = e^{-\frac{1}{2} u^2 s}.$$

Similarly,

$$\lim_{n \to \infty} E[\exp\{\frac{iv}{\sigma\sqrt{n}} \sum_{j=[sn]+1}^{[tn]} \xi_j\}] = e^{-\frac{1}{2} v^2 (t-s)}.$$

Substitution of these last two equations into (4.8) completes the

proof.

$\square$

The following two lemmas will enable us to prove tightness in Donsker's Theorem.

4.15 Lemma: Set $S_k = \sum_{j=1}^{k} \xi_j$, where $\{\xi_j\}_{j=1}^{\infty}$ is a sequence of independent, identically distributed random variables, with mean zero and variance σ^2, $0 < \sigma^2 < \infty$. Then, for any $\epsilon > 0$,

$$\lim_{\delta \downarrow 0} \overline{\lim_{n \to \infty}} \frac{1}{\delta} P[\max_{1 \le j \le [n\delta]+1} |S_j| > \epsilon \sigma \sqrt{n}] = 0.$$

Proof:

By the Central Limit Theorem, we have for each $\delta > 0$ that

$\dfrac{1}{\sigma \sqrt{[n\delta]+1}} S_{[n\delta]+1}$ converges in distribution to a standard

normal random variable Z. But $\left| \dfrac{1}{\sigma\sqrt{n\delta}} S_{[n\delta]+1} - \dfrac{1}{\sigma\sqrt{[n\delta]+1}} S_{[n\delta]+1} \right| \to 0$

in probability, so $\dfrac{1}{\sigma\sqrt{n\delta}} S_{[n\delta]+1} \overset{\mathcal{D}}{\longrightarrow} Z$. Fix $\lambda > 0$ and let

$\{\varphi_i\}_{i=1}^{\infty}$ be a sequence of bounded, continuous functions on R

with $\varphi_i \downarrow 1_{(-\infty, -\lambda] \cup [\lambda, \infty)}$. We have for each i,

$$\overline{\lim_{n \to \infty}} P[|S_{[n\delta]+1}| \ge \lambda \sigma \sqrt{n\delta}] \le \lim_{n \to \infty} E\varphi_i(\frac{1}{\sigma\sqrt{n\delta}} S_{[n\delta]+1})$$

$$= E\varphi_i(Z).$$

Let $i \to \infty$ to conclude

$$(4.9) \qquad \overline{\lim_{n \to \infty}} P[|S_{[n\delta]+1}| \ge \lambda \sigma \sqrt{n\delta}] \le P[|Z| \ge \lambda]$$

$$\le \frac{1}{\lambda^3} E|Z|^3, \quad \lambda > 0.$$

We now define $\tau = \min\{j \geq 1; |S_j| > \epsilon \sigma \sqrt{n}\}$. With $0 < \delta < \epsilon^2/2$, we have

$$(4.10) \quad P[\max_{0 \leq j \leq [n\delta]+1} |S_j| > \epsilon \sigma \sqrt{n}]$$

$$\leq P[|S_{[n\delta]+1}| \geq \sigma \sqrt{n}(\epsilon - \sqrt{2\delta})]$$

$$+ \sum_{j=1}^{[n\delta]} P[|S_{[n\delta]+1}| < \sigma \sqrt{n}(\epsilon - \sqrt{2\delta}) | \tau = j] \, P[\tau = j].$$

But if $\tau = j$, then $|S_{[n\delta]+1}| < \sigma \sqrt{n}(\epsilon - \sqrt{2\delta})$ implies

$|S_j - S_{[n\delta]+1}| > \sigma \sqrt{2n\delta}$. By the Čebyšev inequality, the probability of this event is bounded above by

$$\frac{1}{2n\delta\sigma^2} E[(S_j - S_{[n\delta]+1})^2 | \tau = j] = \frac{1}{2n\delta\sigma^2} E(\sum_{i=j+1}^{[n\delta]+1} \xi_i^2) \leq \frac{1}{2}, \, 1 \leq j \leq [n\delta].$$

Returning to (4.10), we may now write

$$P[\max_{0 \leq j \leq [n\delta]+1} |S_j| > \epsilon \sigma \sqrt{n}]$$

$$\leq P[|S_{[n\delta]+1}| \geq \sigma \sqrt{n}(\epsilon - \sqrt{2\delta})] + \frac{1}{2} P[\tau \leq [n\delta]]$$

$$\leq P[|S_{[n\delta]+1}| \geq \sigma \sqrt{n}(\epsilon - \sqrt{2\delta})]$$

$$+ \frac{1}{2} P[\max_{0 \leq j \leq [n\delta]+1} |S_j| > \epsilon \sigma \sqrt{n}],$$

from which follows

$$P[\max_{0 \le j \le [n\delta]+1} |S_j| > \epsilon \sigma \sqrt{n}] \le 2P[|S_{[n\delta]+1}| \ge \sigma \sqrt{n}(\epsilon - \sqrt{2\delta})].$$

Setting $\lambda = (\epsilon - \sqrt{2\delta})/\sqrt{\delta}$ in (4.9), we see that

$$\overline{\lim_{n \to \infty}} \frac{1}{\delta} P[\max_{0 \le j \le [n\delta]+1} |S_j| > \epsilon \sigma \sqrt{n}] \le \frac{2\sqrt{\delta}}{(\epsilon - \sqrt{2\delta})^3} E|Z|^3,$$

and letting $\delta \downarrow 0$ we obtain the desired result. $\square$

4.16 Lemma: Under the assumptions of Lemma 4.15, we have for any $T > 0$,

$$\lim_{\delta \downarrow 0} \overline{\lim_{n \to \infty}} P[\max_{\substack{1 \le j \le [n\delta]+1 \\ 0 \le k \le [nT]+1}} |S_{j+k} - S_k| > \epsilon \sigma \sqrt{n}] = 0.$$

Proof:

For $0 < \delta \le T$, let $m = m(\delta) \ge 2$ be the unique integer satisfying $\frac{T}{m} < \delta \le \frac{T}{m-1}$. Since

$$\lim_{n \to \infty} \frac{[nT]+1}{[n\delta]+1} = \frac{T}{\delta} < m,$$

we have $[nT]+1 < ([n\delta]+1)m$ for sufficiently large n. For such a large n, suppose $|S_{j+k} - S_k| > \epsilon \sigma \sqrt{n}$ for some k, $0 \le k \le [nT]+1$ and some j, $1 \le j \le [n\delta]+1$. There is a unique integer

p, $0 \le p \le m-1$, such that

$$([n\delta]+1)p \le k < ([n\delta]+1)(p+1).$$

There are two possibilities for $k+j$. One possibility is that

$$([n\delta]+1) \ p \le k+j \le ([n\delta]+1) \ (p+1),$$

in which case either $|S_k - S_{([n\delta]+1)p}| > \frac{1}{3} \epsilon \sigma\sqrt{n}$ or else

$|S_{k+j} - S_{([n\delta]+1)p}| > \frac{1}{3} \epsilon \sigma\sqrt{n}$. The other possibility is that

$$([n\delta]+1)(p+1) < k+j < ([n\delta]+1)(p+2),$$

in which case either $|S_k - S_{([n\delta]+1)p}| > \frac{1}{3} \epsilon \sigma\sqrt{n}$,

$$|S_{([n\delta]+1)p} - S_{([n\delta]+1)(p+1)}| > \frac{1}{3} \epsilon \sigma\sqrt{n} , \quad \text{or}$$

$|S_{([n\delta]+1)(p+1)} - S_{k+j}| > \frac{1}{3} \epsilon \sigma\sqrt{n}$. In conclusion,

we see that

$$\{ \max_{\substack{1 \le j \le [n\delta]+1 \\ 0 \le k \le [nT]+1}} |S_{j+k} - S_k| > \epsilon \sigma\sqrt{n}\}$$

$$\subseteq \bigcup_{p=0}^{m-1} \{ \max_{1 \le j \le [n\delta]+1} |S_{j+p([n\delta]+1)} - S_{p([n\delta]+1)}| > \frac{1}{3} \epsilon \sigma\sqrt{n}\}.$$

But

$$P\left[\max_{1\leq j\leq[n\delta]+1}|S_{j+p([n\delta]+1)}-S_{p([n\delta]+1)}| > \frac{1}{3}\,\epsilon\,\sigma\sqrt{n}\right]$$

$$= P\left[\max_{1\leq j\leq[n\delta]+1}|S_j| > \frac{1}{3}\,\epsilon\,\sigma\sqrt{n}\right],$$

so

$$P\left[\max_{\substack{1\leq j\leq[n\delta]+1\\0\leq k\leq[nT]+1}}|S_{j+k}-S_k| > \epsilon\,\sigma\sqrt{n}\right]$$

$$\leq\ m\quad P\left[\max_{1\leq j\leq[n\delta]+1}|S_j| > \frac{1}{3}\,\epsilon\,\sigma\sqrt{n}\right].$$

Since $m \leq \frac{T}{\delta} + 1$, we obtain the desired conclusion from Lemma 4.15.

$\square$

We are now in a position to establish the main result of this section, namely the convergence in distribution of the sequence of normalized random walks in (4.7) to Brownian motion.

4.17 Theorem: <u>Donsker (1951)</u>.

Let $(\Omega,\mathfrak{F},P)$ be a probability space on which is given a sequence $\{\xi_j\}_{j=1}^{\infty}$ of independent, identically distributed random variables with mean zero and variance σ^2, $0<\sigma^2<\infty$. Define $X^{(n)} = \{X_t^{(n)}; t\geq 0\}$ by (4.7), and let P_n be the measure induced by $X^{(n)}$ on $(C[0,\infty), \mathfrak{B}(C[0,\infty)))$. Then $\{P_n\}_{n=1}^{\infty}$ converges weakly to a measure P, under which the coördinate mapping process $W_t(\omega) \overset{\Delta}{=} \omega(t)$ on $C[0,\infty)$ is a standard, one-dimensional Brownian motion.

Proof:

This result is a special case of Theorem 4.12, and, in light of Theorem 4.14, it remains only to prove that $\{X^{(n)}\}_{n=1}^{\infty}$ is tight. For this we use Theorem 4.10, and, since $X_{0}^{(n)} = 0$ a.s. for every n, we need only show that for each positive T and ϵ,

$$\lim_{\delta \downarrow 0} \sup_{n \geq 1} P[\max_{\substack{|s-t| \leq \delta \\ 0 \leq s,t \leq T}} |X_{s}^{(n)} - X_{t}^{(n)}| > \epsilon] = 0.$$

We may replace $\sup_{n \geq 1}$ in this expression by $\overline{\lim}_{n \to \infty}$, since for a finite number of integers n we can make $P[\max_{\substack{|s-t| \leq \delta \\ 0 \leq s,t \leq T}} |X_{s}^{(n)} - X_{t}^{(n)}| > \epsilon]$ as small as we choose, by reducing δ. Now

$$P[\max_{\substack{|s-t| \leq \delta \\ 0 \leq s,t \leq T}} |X_{s}^{(n)} - X_{t}^{(n)}| > \epsilon] = P[\max_{\substack{|s-t| \leq n\delta \\ 0 \leq |s-t| \leq nT}} |Y_{s} - Y_{t}| > \epsilon \sigma \sqrt{n}],$$

and

$$\max_{\substack{|s-t| \leq n\delta \\ 0 \leq s,t \leq nT}} |Y_{s} - Y_{t}| \leq \max_{\substack{|s-t| \leq [n\delta]+1 \\ 0 \leq s,t \leq [nT]+1}} |Y_{s} - Y_{t}|$$

$$\leq \max_{\substack{1 \leq j \leq [n\delta]+1 \\ 0 \leq k \leq [nT]+1}} |S_{j+k} - S_{k}|,$$

where the last inequality follows from the fact that Y is piecewise linear and changes slope only at integer values of t. Tightness follows from Lemma 4.16.

$\square$

4.18 Definition: The probability measure P on $(C[0, \infty)$,
$\mathcal{B}(C[0, \infty)))$,   under which the coördinate mapping process
$W_t(\omega) \triangleq \omega(t)$, $0 \leq t < \infty$, is a standard, one-dimensional Brownian
motion, is called <u>Wiener measure</u>.

                                                                             □

4.19 Remark:

    A standard, one-dimensional, Brownian motion defined on any
probability space can be thought of as a random variable with values
in $C[0, \infty)$; regarded this way, Brownian motion induces the Wiener
measure on $(C[0, \infty)$, $\mathcal{B}(C[0, \infty)))$.   For this reason, we call
$(C[0, \infty)$, $\mathcal{B}(C[0, \infty)))$, P),   where P is Wiener measure, the <u>canonical</u>
<u>probability space</u> for Brownian motion.

2.5: THE MARKOV PROPERTY

In this section we define the notion of a d-dimensional
Markov process and cite d-dimensional Brownian motion as an example.
There are a number of equivalent statements of the Markov property,
and we spend some time developing them.

5.1 __Definition__: Let d be a positive integer and μ a probability
measure on $(\mathbb{R}^d, \mathbb{B}(\mathbb{R}^d))$. Let $B = \{B_t, \mathcal{F}_t; \ t \geq 0\}$ be an adapted,
d-dimensional process on some $(\Omega, \mathcal{F}, P)$, with components
$B_t^{(1)}, \ldots, B_t^{(d)}$. Define $\widetilde{B}_t = (\widetilde{B}_t^{(1)}, \ldots \widetilde{B}_t^{(d)}) = B_t - B_0$. The
process $B = \{B_t, \mathcal{F}_t; \ t \geq 0\}$ is a __d-dimensional Brownian motion__
__with initial distribution__ μ, if and only if

(i) $P[B_0 \in \Gamma] = \mu(\Gamma), \ \forall \ \Gamma \in \mathbb{B}(\mathbb{R}^d)$;

(ii)   For each $i = 1, \ldots, d$, the process $\{\widetilde{B}_t^{(i)}, \mathcal{F}_t; \ t \geq 0\}$ is
        a standard, one-dimensional Brownian motion; and

(iii)   The processes $\widetilde{B}^{(i)}$, $i = 1, \ldots, d$ are independent of one
        another and are also independent of $\mathcal{F}_0$; i.e. the
        σ-fields $\mathcal{F}_\infty^{\widetilde{B}^{(1)}}, \ldots, \mathcal{F}_\infty^{\widetilde{B}^{(d)}}$ and $\mathcal{F}_0$ are independent.

If μ assigns measure one to some singleton $\{x\}$, we
say that B is a __d-dimensional Brownian motion starting at x.__

Here is one way to construct a d-dimensional Brownian motion with initial distribution μ. Let $X(\omega_0) = \omega_0$ be the identity random variable on $(R^d, \mathscr{B}(R^d), \mu)$, and for $i=1,\ldots,d$, let $\widetilde{B}^{(i)} = \{\widetilde{B}_t^{(i)}, \widetilde{\mathscr{F}}_t^{B^{(i)}}; t \geq 0\}$ be a standard, one-dimensional, Brownian motion on some $(\Omega^{(i)}, \mathscr{F}^{(i)}, P^{(i)})$. On the product space

$$(R^d \times \Omega^{(1)} \times \ldots \times \Omega^{(d)}, \mathscr{B}(R^d) \otimes \mathscr{F}^{(1)} \otimes \ldots \otimes \mathscr{F}^{(d)}, \mu \times P^{(1)} \times \ldots \times P^{(d)}),$$

define

$$B_t(\omega) = X(\omega_0) + (\widetilde{B}_t^{(1)}(\omega_1), \ldots, \widetilde{B}_t^{(d)}(\omega_d)),$$

and set $\mathscr{F}_t = \mathscr{F}_t^B$. Then $B = \{B_t, \mathscr{F}_t; t \geq 0\}$ is the desired object.

There is a second construction of d-dimensional Brownian motion with initial distribution μ, a construction which motivates the concept of <u>Markov family</u> to be introduced in this section. Let $P^{(i)}$, $i=1,\ldots,d$, be d copies of Wiener measure on $(C[0,\infty), \mathscr{B}(C[0,\infty)))$. Then $P^0 \triangleq P^{(1)} \times \ldots \times P^{(d)}$ is a measure, called <u>d-dimensional Wiener measure</u>, on $(C[0,\infty)^d, \mathscr{B}(C[0,\infty)^d))$. Under P^0, the coördinate mapping process $B_t(\omega) \triangleq \omega(t)$ together with the filtration $\{\mathscr{F}_t^B\}$ is a d-dimensional Brownian motion starting at the origin. For $x \in R^d$, we define the probability measure P^x on $(C[0,\infty)^d, \mathscr{B}(C[0,\infty)^d))$ by

$$(5.1) \qquad P^x(F) = P^0(F-x), \quad F \in \mathscr{B}(C[0,\infty)^d),$$

where $F-x = \{\omega \in C[0,\infty)^d; \omega(\cdot) + x \in F\}$. Under P^x, $B \triangleq \{B_t, \mathscr{F}_t^B; t \geq 0\}$ is a d-dimensional Brownian motion starting at x. Finally, for

μ a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, we define P^μ on $(C[0,\infty)^d, \mathcal{B}(C[0,\infty)^d))$ by

$$(5.2) \qquad P^\mu(F) = \int_{\mathbb{R}^d} P^x(F) \, \mu(dx), \quad F \in \mathcal{B}(C[0,\infty)^d).$$

Problem 5.1″ shows that such a definition is possible. The solution of this problem, as well as the proof of several other results in this and the next section, can be conveniently based on the Dynkin System Theorem (cf. Ash [1972], p. 169), which we now state for future reference.

5.1 Definition: Let $\mathcal{D}$ be a collection of subsets of a set Ω. Then $\mathcal{D}$ is a <u>Dynkin System</u> if and only if the following conditions hold:

(i) $\Omega \in \mathcal{D}$;

(ii) If $A, B \in \mathcal{D}$ and $B \subseteq A$, then $A \setminus B \in \mathcal{D}$;

(iii) If $\{A_n\}_{n=1}^\infty \subseteq \mathcal{D}$ and $A_1 \subseteq A_2 \subseteq \cdots$, then $\bigcup_{n=1}^\infty A_n \in \mathcal{D}$.

5.1′ Dynkin System Theorem: Let $\mathcal{C}$ be a collection of subsets of Ω which is closed under pairwise intersection. If $\mathcal{D}$ is a Dynkin system containing $\mathcal{C}$, then $\mathcal{D}$ also contains the σ-field generated by $\mathcal{C}$.

5.1″ Problem: Show that for each $F \in \mathcal{B}(C[0,\infty)^d)$, the mapping $x \mapsto P^x(F)$ is $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}([0,1])$ - measurable.

5.2 Proposition: The coördinate mapping process

$B = \{B_t, \mathcal{F}_t^B;\ t \geq 0\}$ on $(C[0,\infty)^d,\ \mathcal{B}(C[0,\infty)^d),\ P^\mu)$ is a d-dimensional Brownian motion with initial distribution μ.

Proof:

We verify (i) - (iii) of Definition 5.1. With $F = \{\omega:\ \omega(0) \in \Gamma\}$, we have

$$P^x(F) = P^0(F-x) = 1_\Gamma(x)$$

and (i) follows directly from (5.2). Let $\tilde{B}_t = B_t - B_0$. For $F \in \mathcal{B}(C[0,\infty)^d)$, (5.1) implies

$$P^x[\tilde{B}. \in F] = P^x[B. \in F+x] = P^0[B. \in F],$$

so under any P^x, $\tilde{B}.$ induces d-dimensional Wiener measure on $(C[0,\infty)^d,\ \mathcal{B}(C[0,\infty)^d))$. It thus must also induce this measure under P^μ, and (ii) is proved. Finally, for $\Gamma \in \mathcal{B}(R^d)$, $F \in \mathcal{B}(C[0,\infty)^d)$, we have

$$P^\mu[B_0 \in \Gamma,\ \tilde{B}. \in F] = \int_{R^d} P^x[B_0 \in \Gamma,\ \tilde{B}. \in F]\ \mu(dx)$$

$$= \int_{R^d} 1_\Gamma(x)\ P^0[B. \in F]\ \mu(dx) = \mu(\Gamma)\ P^0[B. \in F] = P^\mu[B_0 \in \Gamma]\ P^\mu[\tilde{B}. \in F],$$

so $\mathcal{F}_0^B$ is independent of $\mathcal{F}_\infty^{\tilde{B}}$ under P^μ. The independence of $\mathcal{F}_\infty^{\tilde{B}^{(1)}}, \ldots, \mathcal{F}_\infty^{\tilde{B}^{(d)}}$ is a consequence of the product form of d-dimensional Wiener measure. This proves (iii). □

5.3 Definition: Given a metric space (S, ρ), we denote by

$\overline{\mathcal{B}(S)}^{\mu}$ the completion of the Borel σ-field $\mathcal{B}(S)$ (generated by the open sets) with respect to the finite measure μ on $(S, \mathcal{B}(S))$. The universal σ-field is $\mathcal{U}(S) \triangleq \bigcap_{\mu} \overline{\mathcal{B}(S)}^{\mu}$, where the intersection is over all finite measures (or, equivalently, all probability measures) μ. A $\mathcal{U}(S)/\mathcal{B}(\mathbb{R})$ - measurable, real-valued function is said to be __universally measurable__.

5.3´ Problem: Let (S, ρ) be a metric space and let f be a real-valued function defined on S. Show that f is universally measurable if and only if for every finite measure μ on $(S, \mathcal{B}(S))$, there is a Borel-measurable function $g_{\mu}: S \to \mathbb{R}$ such that $\mu\{x \in S;\ f(x) \neq g_{\mu}(x)\} = 0$.

5.4 Definition: A __d-dimensional Brownian family__ is an adapted, d-dimensional process $B = \{B_t, \mathcal{F}_t;\ t \geq 0\}$ on a measurable space $(\Omega, \mathcal{F})$, and a family of probability measures $\{P^x\}_{x \in \mathbb{R}^d}$,

such that:

(i)   For each $F \in \mathfrak{F}$, the mapping $x \to P^x(F)$ is universally

       measurable;

(ii)   For each $x \in \mathbb{R}^d$, $P^x[B_0 = x] = 1$;

(iii) Under any P^x, the process B is a d-dimensional

       Brownian motion starting at x. $\square$

We have already seen how to construct a family of probability

measures $\{P^x\}$ on the canonical space $(C[0, \infty)^d$, $\mathfrak{B}(C[0, \infty))^d)$ so

that the coördinate mapping process, relative to the filtration

it generates, is a Brownian motion starting at x under any P^x.

With $\mathfrak{F} = \mathfrak{B}(C[0, \infty)^d)$, Problem 5.1 shows that the universal

measurability requirement (i) of Definition 5.4 is satisfied.

Indeed, for this canonical example of a d-dimensional Brownian

family, the mapping $x \to P^x(F)$ is actually Borel-measurable for

each $F \in \mathfrak{F}$. The reason we formulate Definition 5.4 with the

weaker measurability condition is to allow expansion of $\mathfrak{F}$ to a

larger σ-field. See Remark 7.14. $\square$

Suppose $0 \leq s < t$, and we observe a Brownian motion with initial

distribution μ up to time s. In particular, we see the value of

B_s, which we call y. Conditioned on these observations, what is

the probability that B_t is in some set $\Gamma \in \mathfrak{B}(\mathbb{R}^d)$? Now

$B_t = (B_t - B_s) + B_s$, and the increment $B_t - B_s$ is independent of

the observations up to time s and is distributed just like

B_{t-s} under P^0. On the other hand, B_s does depend on the

observations; indeed, we are conditioning on $B_s = y$. It follows that the sum $(B_t - B_s) + B_s$ is distributed like B_{t-s} under P^y. Two things then become clear. First, knowledge of the whole past up to time s provides no more useful information about B_t than knowing the value of B_s; in other words,

$$(5.3) \qquad P^\mu[B_t \in \Gamma \,|\, \mathfrak{F}_s] = P^\mu[B_t \in \Gamma \,|\, B_s], \quad 0 \leq s < t, \ \Gamma \in \mathcal{B}(\mathbb{R}^d).$$

Secondly, conditioned on $B_s = y$, B_t is distributed like B_{t-s} under P^y; i.e.,

$$(5.4) \qquad P^\mu[B_t \in \Gamma \,|\, B_s = y] = P^y[B_{t-s} \in \Gamma], \quad 0 \leq s < t, \ \Gamma \in \mathcal{B}(\mathbb{R}^d).$$

5.5 Problem: Make the above discussion rigorous by proving the following. If X and Y are d-dimensional random vectors on $(\Omega, \mathfrak{F}, P)$, $\mathcal{G}$ is a sub-σ-field of $\mathfrak{F}$, X is independent of $\mathcal{G}$ and Y is $\mathcal{G}$-measurable, then for every $\Gamma \in \mathcal{B}(\mathbb{R}^d)$:

$$(5.5) \qquad P[X + Y \in \Gamma \,|\, \mathcal{G}] = P[X + Y \in \Gamma \,|\, Y], \quad \text{a.s. } P;$$

$$(5.6) \qquad P[X + Y \in \Gamma \,|\, Y = y] = P[X + y \in \Gamma], \quad \text{for } PY^{-1} \text{ - a.e. } y \in \mathbb{R}^d.$$

Here, PY^{-1} is the probability measure induced on $\mathbb{R}^d$ by Y.

5.6 **Definition:** Let d be a positive integer and μ a probability measure on $(R^d, \mathcal{B}(R^d))$. An adapted, d-dimensional process $X = \{X_t, \mathcal{F}_t; t \geq 0\}$ on some probability space $(\Omega, \mathcal{F}, P^\mu)$ is said to be a <u>Markov process with initial distribution μ</u> if and only if

(i) $P^\mu[X_0 \in \Gamma] = \mu(\Gamma)$, $\forall \Gamma \in \mathcal{B}(R^d)$;

(ii) For $s, t \geq 0$ and $\Gamma \in \mathcal{B}(R^d)$,

$$P^\mu[X_{t+s} \in \Gamma | \mathcal{F}_s] = P^\mu[X_{t+s} \in \Gamma | X_s], \quad P^\mu - a.s.$$

Our experience with Brownian motion indicates that it is notationally and conceptually helpful to have a whole family of probability measures rather than just one. Toward this end, we define the concept of a Markov family.

5.7 **Definition:** Let d be a positive integer. A d-dimensional <u>Markov family</u> is an adapted process $X = \{X_t, \mathcal{F}_t; t \geq 0\}$ on some $(\Omega, \mathcal{F})$, together with a family of probability measures $\{P^x\}_{x \in R^d}$ on $(\Omega, \mathcal{F})$, such that:

(a) For each $F \in \mathcal{F}$, the mapping $x \to P^x(F)$ is universally measurable;

(b) $P^x[X_0 = x] = 1$, $\forall x \in R^d$;

(c) For $x \in R^d$, $s, t \geq 0$ and $\Gamma \in \mathcal{B}(R^d)$,

$$P^x[X_{t+s} \in \Gamma | \mathcal{F}_s] = P^x[X_{t+s} \in \Gamma | X_s], \quad P^x - a.s ;$$

(d) For $x \in R^d$, $s, t \geq 0$ and $\Gamma \in \mathcal{B}(R^d)$,

$$P^X[X_{t+s} \in \Gamma | X_s = y] = P^y[X_t \in \Gamma], \quad P^X X_s^{-1} - \text{a.e. } y,$$

where $P^X X_s^{-1}$ is the measure induced on R^d by X_s under P^X.

The following statement is a consequence of Problem 5.5 and the discussion preceding it.

5.8 Theorem: A d-dimensional Brownian motion is a Markov process.

A d-dimensional Brownian family is a Markov family.

The Markov   property, encapsulated by conditions (c) and (d) of Definition 5.7, can be reformulated in several equivalent ways. Some of these formulations amount to incorporating (c) and (d) into a single condition; others replace the evaluation of X at the single time s+t by its evaluation at multiple times after s. The bulk of the remainder of this section presents those formulations of the Markov   property which we shall find most convenient in the sequel.

Given an adapted process $X = \{X_t, \mathcal{F}_t; t \geq 0\}$ on $(\Omega, \mathcal{F})$ and given a family of probability measures $\{P^X\}_{x \in R^d}$ such that condition (a) of Definition 5.7 is satisfied, we can define a collection of operators $\{U_t\}_{t \geq 0}$ which map bounded, Borel measurable, real-valued functions on R^d into bounded, universally measurable, real-valued functions on the same space. These are

defined by

(5.7) $(U_t f)(x) \triangleq E^x f(X_t).$

In the case where f is the indicator of $\Gamma \in \mathcal{B}(R^d)$, we have $E^x f(X_t) = P^x[X_t \in \Gamma]$, and universal measurability of $U_t f$ follows directly from Definition 5.7 (a); for an arbitrary, Borel measurable function f, the universal measurability of $U_t f$ is then a consequence of the Bounded Convergence Theorem.

<u>5.9 Proposition</u>: Conditions (c) and (d) of Definition 5.7 can

be replaced by:

(e) For $x \in R^d$, $s,t \geq 0$ and $\Gamma \in \mathcal{B}(R^d)$,

$$P^x[X_{s+t} \in \Gamma | \mathfrak{F}_s] = (U_t 1_\Gamma)(X_s), \quad P^x - \text{a.s.}$$

<u>Proof</u>:

First, let us assume that (c), (d) hold. We have from the latter:

$$P^x[X_{t+s} \in \Gamma | X_s = y] = (U_t 1_\Gamma)(y), \quad \text{for } P^x X_s^{-1} - \text{a.e. } y \in R^d.$$

If the function $\alpha(y) \triangleq (U_t 1_\Gamma)(y): R^d \to [0,1]$ were $\mathcal{B}(R^d)$ - measurable, as is the case for Brownian motion, we would then be able to conclude that, for all $x \in R^d$, $s \geq 0$:

$$P^x[X_{t+s} \in \Gamma | X_s] = \alpha(X_s), \quad \text{a.s. } P^x,$$

and from condition (c): $P^X[X_{t+s} \in \Gamma \mid \mathcal{F}_s] = \alpha(X_s)$, a.s. P^X, which would then establish (e).

However, we only know that $U_t 1_\Gamma(\cdot)$ is universally measurable. This means (from Problem 5.3′) that, for given $s, t \geq 0$, $x \in R^d$, there exists a Borel-measurable function $g: R^d \to [0,1]$ such that

(5.7)′ $\quad (U_t 1_\Gamma)(y) = g(y)$, for $P^X X_s^{-1}$ - a.e. $y \in R^d$,

or

(5.7)″ $\quad (U_t 1_\Gamma)(X_s) = g(X_s)$, a.s. P^X.

One can then repeat the preceding argument with g replacing the function α.

Secondly, let us assume that (e) holds; then for any given $s, t \geq 0$ and $x \in R^d$, (5.7)″ gives

(5.7)‴ $\quad P^X[X_{t+s} \in \Gamma \mid \mathcal{F}_s] = g(X_s)$, a.s. P^X.

It follows that $P^X[X_{t+s} \in \Gamma \mid \mathcal{F}_s]$ has a $\sigma(X_s)$ - measurable version, and this establishes (c). From the latter and (5.7)‴ we conclude

$$P^X[X_{t+s} \in \Gamma \mid X_s = y] = g(y); \text{ for } P^X X_s^{-1} \text{ - a.e. } y \in R^d,$$

and this in turn yields (d) thanks to (5.7)′. $\qquad \square$

For given $\omega\in\Omega$, $s\geq 0$, we denote by $X_{s+.}(\omega)$ the function $t \to X_{s+t}(\omega)$. Thus, $X_{s+.}$ is a measurable mapping from $(\Omega,\mathcal{F})$ into $((\mathbb{R}^d)^{[0,\infty)}, \mathcal{B}((\mathbb{R}^d)^{[0,\infty)}))$, the space of all $\mathbb{R}^d$ - valued functions on $[0,\infty)$ equipped with the smallest σ-field containing all finite-dimensional cylinder sets.

5.10 Proposition: For a Markov family $X,(\Omega,\mathcal{F})$, $\{P^x\}_{x\in\mathbb{R}^d}$, we have:

(c') For $x\in\mathbb{R}^d$, $s\geq 0$ and $F\in\mathcal{B}((\mathbb{R}^d)^{[0,\infty)})$,

$$P^x[X_{s+.}\in F|\mathcal{F}_s] = P^x[X_{s+.}\in F|X_s], \quad P^x - \text{a.s};$$

(d') For $x\in\mathbb{R}^d$, $s\geq 0$ and $F\in\mathcal{B}((\mathbb{R}^d)^{[0,\infty)})$,

$$P^x[X_{s+.}\in F|X_s = y] = P^y[X.\in F], \quad P^x X_s^{-1} - \text{a.e.y}.$$

Note: If $\Gamma\in\mathcal{B}(\mathbb{R}^d)$ and $F = \{\omega\in(\mathbb{R}^d)^{[0,\infty)}; \omega(t)\in\Gamma\}$, for fixed $t\geq 0$, then (c') and (d') reduce to (c) and (d), respectively, of Definition 5.7.

Proof of Proposition 5.10:

The collection of all sets $F\in\mathcal{B}((\mathbb{R}^d)^{[0,\infty)})$ for which (c') and (d') hold forms a Dynkin system; so by Theorem 5.1′, it suffices to prove (c') and (d') for finite-dimensional cylinder sets of the form

$$F = \{\omega\in\mathbb{R}^{[0,\infty)}; \omega(t_0)\in\Gamma_0,\ldots,\omega(t_{n-1})\in\Gamma_{n-1},\omega(t_n)\in\Gamma_n\},$$

where $0 = t_0<t_1<\ldots<t_n$, $\Gamma_i\in\mathcal{B}(\mathbb{R}^d)$, $i=0,1,\ldots,n$, and $n\geq 0$. For such an F, condition (c') becomes

(5.8) $P^X[X_s \epsilon \Gamma_0, \ldots, X_{s+t_{n-1}} \epsilon \Gamma_{n-1}, X_{s+t_n} \epsilon \Gamma_n | \mathcal{F}_s]$

$$= P^X[X_s \epsilon \Gamma_0, \ldots, X_{s+t_{n-1}} \epsilon \Gamma_{n-1}, X_{s+t_n} \epsilon \Gamma_n | X_s], \quad P^X - a.s.$$

We prove this statement by induction on n. For $n=0$, it is obvious. Assume it true for $n-1$. A consequence of this assumption is that for any bounded, Borel measurable $\varphi: R^{dn} \to R$,

(5.9) $E^X[\varphi(X_s, \ldots, X_{s+t_{n-1}}) | \mathcal{F}_s] = E^X[\varphi(X_s, \ldots, X_{s+t_{n-1}}) | X_s], \quad P^X - a.s.$

Now (c) implies that

$$P^X[X_s \epsilon \Gamma_0, \ldots, X_{s+t_{n-1}} \epsilon \Gamma_{n-1}, X_{s+t_n} \epsilon \Gamma_n | \mathcal{F}_s]$$

$$= E^X[1_{\{X_s \epsilon \Gamma_0, \ldots, X_{s+t_{n-1}} \epsilon \Gamma_{n-1}\}} P^X[X_{s+t_n} \epsilon \Gamma_n | \mathcal{F}_{s+t_{n-1}}] | \mathcal{F}_s]$$

$$= E^X[1_{\{X_s \epsilon \Gamma_0, \ldots, X_{s+t_{n-1}} \epsilon \Gamma_{n-1}\}} P^X[X_{s+t_n} \epsilon \Gamma_n | X_{s+t_{n-1}}] | \mathcal{F}_s].$$

As in the proof of Proposition 5.9, we see that the universal measurability assumption (a) of Definition 5.7 yields the existence of a Borel-measurable function $g: R^d \to [0,1]$, such that $P^X[X_{s+t_n} \epsilon \Gamma | X_{s+t_{n-1}}] = g(X_{s+t_{n-1}})$, a.s. P^X. Setting $\varphi(x_0, \ldots, x_{n-1}) \triangleq 1_{\Gamma_0}(x_0) \cdots 1_{\Gamma_{n-1}}(x_{n-1}) \cdot g(x_{n-1})$, we can use (5.9) to replace $\mathcal{F}_s$ by $\sigma(X_s)$ in this last expression, and then, reversing the previous steps, we obtain (5.8). The proof of (d') is similar, although notationally more complex.   □

It happens sometimes, for a given process $X = \{X_t, \mathcal{F}_t; \ t \geq 0\}$ on a measurable space $(\Omega, \mathcal{F})$, that one can construct a family of so-called <u>shift-operators</u> $\theta_s : \Omega \to \Omega$, $s \geq 0$, such that each θ_s is $\mathcal{F}/\mathcal{F}$ measurable and

$$(5.10) \qquad X_{s+t}(\omega) = X_t(\theta_s \omega); \quad \forall \ \omega \in \Omega, \ s, t \geq 0.$$

The most obvious examples occur when Ω is $(\mathbb{R}^d)^{[0, \infty)}$, the space of all $\mathbb{R}^d$ - valued functions on $[0, \infty)$, or Ω is $C[0, \infty)^d$, the space of all continuous, $\mathbb{R}^d$ - valued functions, $\mathcal{F}$ is the smallest σ-field containing all finite-dimensional cylinder sets, and X is the coördinate mapping process $X_t(\omega) = \omega(t)$. We can then define $\theta_s \omega = \omega(s+\cdot)$, i.e.,

$$(5.11) \qquad (\theta_s \omega)(t) = \omega(s+t), \quad t \geq 0.$$
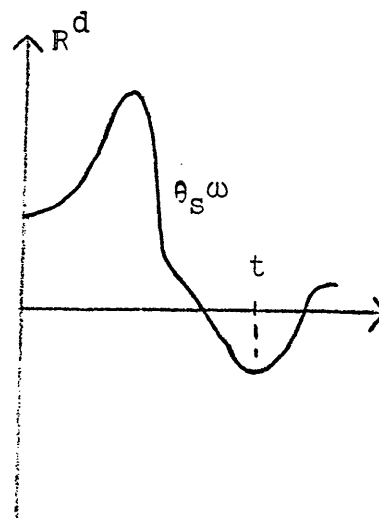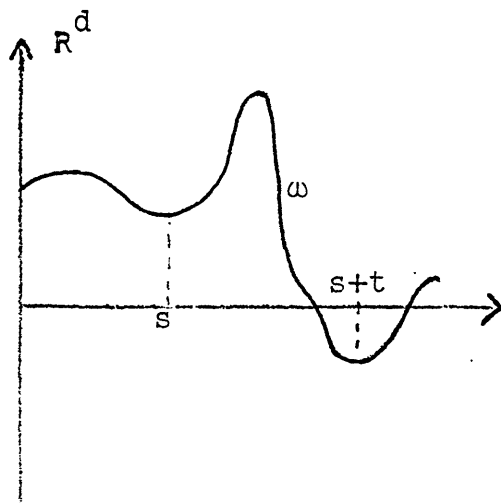
When the shift operators exist, then the function $X_{s+\cdot}(\omega)$ appearing in (c') and (d') is none other than $X_{\cdot}(\theta_s\omega)$, so $\{X_{s+\cdot}\epsilon F\} = \theta_s^{-1}\{X_{\cdot}\epsilon F\}$. As F ranges over $\mathcal{B}((\mathbb{R}^d)^{[0,\infty)})$, $\{X_{\cdot}\epsilon F\}$ ranges over $\mathcal{F}_\infty^X$. Thus, (c') and (d') can be reformulated as:

(c")   For $F\epsilon\mathcal{F}_\infty^X$ and $s\geq 0$,

$$P^X[\theta_s^{-1}F|\mathcal{F}_s] = P^X[\theta_s^{-1}F|X_s], \quad P^X - \text{a.s.}$$

(d")   For $F\epsilon\mathcal{F}_\infty^X$ and $s\geq 0$,

$$P^X[\theta_s^{-1}F|X_s = y] = P^y[F], \quad P^X X_s^{-1} - \text{a.e. } y.$$

In a manner analogous to what was achieved in Proposition 5.9, we can capture both (c") and (d") in the condition

(e")   For $F\epsilon\mathcal{F}_\infty^X$ and $s\geq 0$,

$$P^X[\theta_s^{-1}F|\mathcal{F}_s] = P^{X_s}(F), \quad P_X - \text{a.s.}$$

Since (e") is often given as the primary defining property for a Markov family, we state a result about its equivalence to our definition.

5.11 Theorem: Let $X = \{X_t, \mathcal{F}_t; t\geq 0\}$ be an adapted process on a measurable space $(\Omega, \mathcal{F})$, let $\{P_x\}_{x\in\mathbb{R}^d}$ be a family of probability measures on $(\Omega, \mathcal{F})$, and let $\{\theta_s\}_{s\geq 0}$ be a family of $\mathcal{F}/\mathcal{F}$ - measurable shift operators satisfying (5.10). Then

X, $(\Omega, \mathfrak{F})$, $\{P_x\}_{x \in R^d}$ is a Markov family if and only if (a), (b) and (e") hold.

5.12 **Problem**: Suppose that X, $(\Omega, \mathfrak{F})$, $\{P_x\}_{x \in R^d}$ is a Markov family with shift operators $\{\theta_s\}_{s \geq 0}$. Use (c") to show that:

(c''') For $x \in R^d$, $s \geq 0$, $G \in \mathfrak{F}_s$ and $F \in \mathfrak{F}_\infty^X$,

$$P^x[G \cap \theta_s^{-1}F | X_s] = P^x[G | X_s] \, P^x[\theta_s^{-1}F | X_s], \quad P^x \text{ - a.s.}$$

We may interpret this equation as saying the "past" G and the "future" $\theta_s^{-1}F$ are conditionally independent, given the "present" X_s.

Conversely, show that (c''') implies (c"). □

We close this section with two additional examples of a Markov family.

5.13 **Problem**: Suppose $X = \{X_t, \mathfrak{F}_t; t \geq 0\}$, $(\Omega, \mathfrak{F})$, $\{P^x\}_{x \in R^d}$ is a Markov family and $\varphi: [0, \infty) \to R^d$ and $\Psi: [0, \infty) \to L(R^d, R^d)$, the space of linear transformations from R^d to R^d, are given (nonrandom) functions with $\varphi_0 = 0$ and Ψ_t nonsingular for every $t \geq 0$. Set $Y_t = \varphi_t + \Psi_t \cdot X_t$. Then $Y = \{Y_t, \mathfrak{F}_t; t \geq 0\}$, $(\Omega, \mathfrak{F})$, $\{P^x\}_{x \in R^d}$ is also a Markov family.

5.14 **Definition**: Let $B = \{B_t, \mathfrak{F}_t; t \geq 0\}$, $(\Omega, \mathfrak{F})$, $\{P^x\}_{x \in R^d}$ be a d-dimensional Brownian family. If $\mu \in R^d$ and $\sigma \in L(R^d, R^d)$ are constant and σ is nonsingular, then with $Y_t \triangleq \mu t + \sigma B_t$,

we say $Y = \{Y_t, \mathcal{F}_t; t \geq 0\}$, $(\Omega, \mathcal{F})$, $\{P^x\}_{x \in \mathbb{R}^d}$ is a <u>d-dimensional</u> <u>Brownian family with drift</u> μ <u>and diffusion coefficient</u> σ.

This family is Markov. We may weaken the assumptions on the drift and diffusion coefficients considerably, allowing them to depend on both time and the location of the transformed process, and still obtain a Markov family. This is the subject of Chapter 5 on Stochastic Differential Equations.

<u>5.15 Definition</u>: A <u>Poisson family with intensity</u> $\lambda > 0$ is a process $N = \{N_t, \mathcal{F}_t; t \geq 0\}$ on a measurable space $(\Omega, \mathcal{F})$ and a family of probability measures $\{P^x\}_{x \in \mathbb{R}}$, such that

(i) For each $E \in \mathcal{F}$, the mapping $x \to P^x(E)$ is universally measurable;

(ii) For each $x \in \mathbb{R}$, $P^x[N_0 = x] = 1$;

(iii) Under any P^x, the process $\{\tilde{N}_t = N_t - N_0, \mathcal{F}_t: t \geq 0\}$ is a Poisson process with intensity λ and is independent of $\mathcal{F}_0$, i.e., $\mathcal{F}_\infty^{\tilde{N}}$ and $\mathcal{F}_0$ are independent.

<u>5.16 Problem</u>: Show that a Poisson family with intensity $\lambda > 0$ is a Markov family.

Standard, one-dimensional Brownian motion is both a martingale and a Markov process. There are many examples of Markov processes, such as Brownian motion with nonzero drift and the Poisson processes,

which are not martingales. There are also martingales which do
not enjoy the Markov property. We leave the construction of such
an example as a problem.

5.17 Problem: Construct a martingale which is not a Markov process.

2.6: THE STRONG MARKOV PROPERTY AND THE REFLECTION PRINCIPLE

Part of the appeal of Brownian motion lies in the fact that the distribution of certain of its functionals can be obtained in closed form. Perhaps the most fundamental of these functionals is the passage time T_b to a level $b \in R$, defined by

$$(6.1) \qquad T_b(\omega) = \begin{cases} \inf\{t \geq 0; \; B_t(\omega) = b\} \\ \infty, \quad \text{if} \quad \{\dots\} = \emptyset. \end{cases}$$

We recall that a passage time for a continuous process is a stopping time (Problem 1.2.6).

We shall first obtain the probability density function of T_b by a heuristic argument, based on the so-called reflection principle of Désiré André (Lévy [1948], p. 293). A rigorous presentation of this argument requires use of the strong Markov property for Brownian motion. Accordingly, after some motivational discussion, we define the concept of a strong Markov family, and prove that any Brownian family is strongly Markovian. This will allow us to place the heuristic argument on firm mathematical ground.

Here is the argument of Désiré André. Let $\{B_t, \mathcal{F}_t; \; 0 \leq t < \infty\}$ be a standard, one-dimensional Brownian motion on $(\Omega, \mathcal{F}, P^0)$. For $b > 0$, we have

$$P^0[T_b < t] = P^0[T_b < t, B_t > b] + P^0[T_b < t, B_t < b].$$

Now $P^0[T_b < t, B_t > b] = P^0[B_t > b]$. On the other hand, if $T_b < t$ and $B_t < b$, then sometime before time t the Brownian path reached

level b, and then in the remaining time it travelled from b
to a point c less than b. Because of the symmetry with respect
to b of a Brownian motion starting at b, the "probability" of
doing this is the same as the "probability" of travelling from b
to the point 2b-c. The heuristic rationale here is that, for
every path which crosses level b and is found at time t at a
point below b, there is a "shadow path" (see figure) obtained
from reflection about the level b which exceeds this level at
time t, and these two paths have the same "probability". Of
course, the actual probability for the occurrence of any particular
path is zero, so this argument is only heuristic. Nevertheless,
it leads us to the equation

$$P^0[T_b < t, B_t < b] = P^0[T_b < t, B_t > b] = P^0[B_t > b],$$
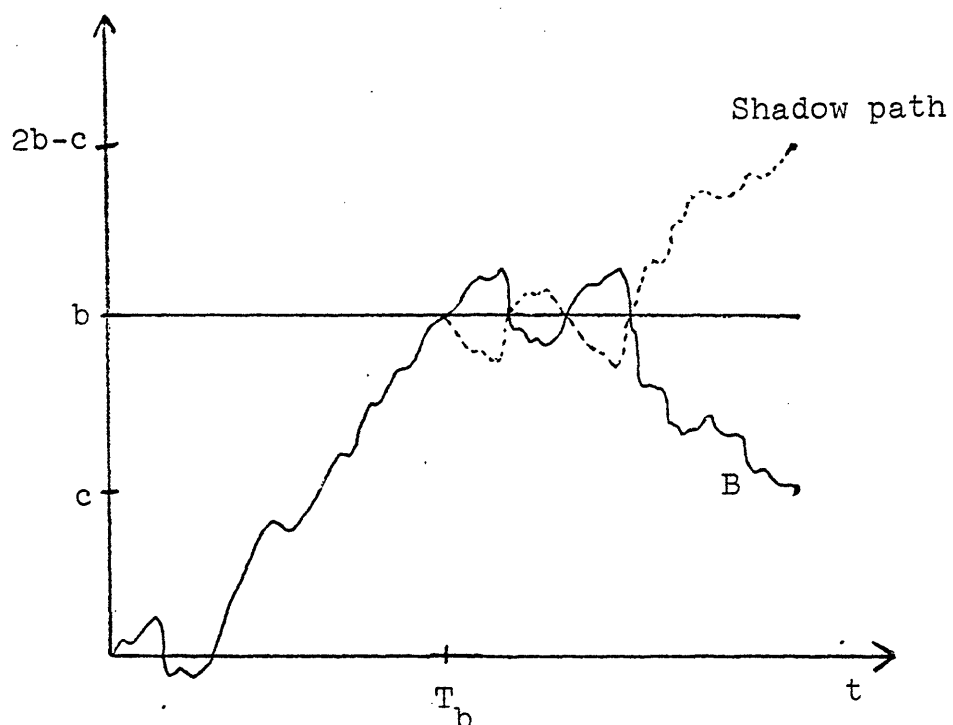
which then yields

$$(6.2) \qquad P^0[T_b < t] = 2P^0[B_t > b] = \sqrt{\frac{2}{\pi}} \int_{bt^{-\frac{1}{2}}}^{\infty} e^{-\frac{x^2}{2}} \, dx.$$

Differentiating with respect to t, we obtain the density of the passage time

$$(6.3) \qquad P^0[T_b \in dt] = \frac{|b|}{\sqrt{2\pi t^3}} e^{-\frac{b^2}{2t}} \, dt; \quad t > 0.$$

The above reasoning is based on the assumption that Brownian motion "starts afresh" (in the terminology of Itô & McKean [1974]) at the stopping time T_b, i.e., that the process $\{B_{t+T_b} - B_{T_b}; \ 0 \leq t < \infty\}$ is Brownian motion, independent of the σ-field $\mathcal{F}_{T_b}$. If T_b were replaced by a nonnegative constant, it would not be hard to show this; if T_b were replaced by an arbitrary random time, the statement would be false (cf. Problem 6.1 below). The fact, that this "starting afresh" actually takes place at stopping times like T_b, is a consequence of the <u>strong Markov property</u> for Brownian motion.

<u>6.1 Problem</u>: Let $\{B_t, \mathcal{F}_t; \ t \geq 0\}$ be a standard, one-dimensional Brownian motion. Give an example of a random time S with $P[0 \leq S < \infty] = 1$, such that with $W_t \triangleq B_{S+t} - B_S$, the process $W = \{W_t, \mathcal{F}_t^W; \ t \geq 0\}$ is <u>not</u> a Brownian motion.

6.2 <u>Definition</u>: Let d be a positive integer and μ a probability measure on $(R^d, \mathcal{B}(R^d))$. A progressively measurable, d-dimensional process $X = \{X_t, \mathcal{F}_t; t \geq 0\}$ on some $(\Omega, \mathcal{F}, P^\mu)$ is said to be a <u>strong Markov process with initial distribution μ</u> if and only if

(i) $\quad P^\mu[X_0 \in \Gamma] = \mu(\Gamma)$, $\forall \; \Gamma \in \mathcal{B}(R^d)$;

(ii) For any optional time S of $\{\mathcal{F}_t\}$, $t \geq 0$ and $\Gamma \in \mathcal{B}(R^d)$,

$$P^\mu[X_{S+t} \in \Gamma \mid \mathcal{F}_{S+}] = P^\mu[X_{S+t} \in \Gamma \mid X_S], \quad P^\mu \text{ - a.s.} \quad \text{on} \quad \{S < \infty\}.$$

6.3 <u>Definition</u>: Let d be a positive integer. A d-dimensional, <u>strong Markov family</u> is a progressively measurable process $X = \{X_t, \mathcal{F}_t; t \geq 0\}$ on some $(\Omega, \mathcal{F})$, together with a family of probability measure $\{P^x\}_{x \in R^d}$ on $(\Omega, \mathcal{F})$, such that:

(a) For each $F \in \mathcal{F}$, the mapping $x \mapsto P^x(F)$ is universally measurable;

(b) $P^x[X_0 = x] = 1$, $\forall \; x \in R^d$;

(c) For $x \in R^d$, $t \geq 0$, $\Gamma \in \mathcal{B}(R^d)$ and any optional time S of $\{\mathcal{F}_t\}$,

$$P^x[X_{S+t} \in \Gamma \mid \mathcal{F}_{S+}] = P^x[X_{S+t} \in \Gamma \mid X_S], \quad P^x \text{ - a.s.} \quad \text{on} \quad \{S < \infty\};$$

(d) For $x \in R^d$, $t \geq 0$, $\Gamma \in \mathcal{B}(R^d)$ and any optional time S of $\{\mathcal{F}_t\}$,

$$P^x[X_{S+t} \in \Gamma \mid X_S = y] = P^y[X_t \in \Gamma], \quad P^x X_S^{-1} \text{ - a.e. } y.$$

6.4 <u>Remark</u>: On the set $\{S = \infty\}$, X_{S+t} is undefined. Thus, the event $\{X_{S+t} \in \Gamma\}$ appearing in Definitions 6.2 and 6.3 is $\{X_{S+t} \in \Gamma, S < \infty\}$.

6.5 <u>Remark</u>: An optional time of $\{\mathfrak{F}_t\}$ is a stopping time of $\{\mathfrak{F}_{t+}\}$ (Corollary to Proposition 1.2.3). Because of the assumption of progressive measurability, the random variable X_S appearing in Definitions 6.2 and 6.3 is $\mathfrak{F}_{S+}$ - measurable (Proposition 1.2.17). Moreover, if S is a stopping time of $\{\mathfrak{F}_t\}$, then X_S is $\mathfrak{F}_S$ - measurable. In this case, we can take conditional expectations with respect to $\mathfrak{F}_S$ on both sides of (c) in Definition 6.3, to obtain:

$$P^x[X_{S+t} \in \Gamma | \mathfrak{F}_S] = P^x[X_{S+t} \in \Gamma | X_S], \quad P^x \text{ - a.s. on } \{S < \infty\}.$$

Setting S equal to a constant $s \geq 0$, we obtain condition (c) of Definition 5.7. Thus, every strong Markov family is a Markov family. Likewise, every strong Markov process is a Markov process. However, not every Markov family enjoys the strong Markov property; a counterexample to this effect, involving a progressively measurable process X, appears in Wentzell [1981], p. 161.

Whenever S is an optional time of $\{\mathfrak{F}_t\}$ and $u > 0$, then $S+u$ is a stopping time of $\{\mathfrak{F}_t\}$ (Problem 1.2.9). This fact can be used to replace the constant s in the proof of Proposition 5.10 by the optional time S, thereby obtaining the following result.

2.6.6

6.6 Proposition: For a strong Markov family $X = \{X_t, \mathcal{F}_t; t \geq 0\}$, $(\Omega, \mathcal{F})$, $\{P^x\}_{x \in R^d}$, we have:

(c') For $x \in R^d$, $F \in \mathcal{B}((R^d)^{[0, \infty)})$ and any optional time S of $\{\mathcal{F}_t\}$,

$$P^x[X_{S+.} \in F | \mathcal{F}_{S+}] = P^x[X_{S+.} \in F | X_S], \quad P^x\text{-a.s.} \quad \text{on} \quad \{S < \infty\};$$

(d') For $x \in R^d$, $F \in \mathcal{B}((R^d)^{[0, \infty)})$ and any optional time S of $\{\mathcal{F}_t\}$,

$$P^x[X_{S+.} \in F | X_S = y] = P^y[X. \in F], \quad P^x X_S^{-1} \text{ - a.e. } y .\qquad \square$$

Using the operators $\{U_t\}_{t \geq 0}$ in (5.7), conditions (c) and (d) of Definition 6.3 can be combined.

6.7 Proposition: Let $X = \{X_t, \mathcal{F}_t; t \geq 0\}$ be a progressively measurable process on $(\Omega, \mathcal{F})$, and let $\{P^x\}_{x \in R^d}$ be a family of probability measures satisfying (a) and (b) of Definition 6.3. Then X, $(\Omega, \mathcal{F})$, $\{P^x\}_{x \in R^d}$ is strong Markov if and only if for any $\{\mathcal{F}_t\}$ - optional time S and $t \geq 0$, one of the following holds:

(e) For any $\Gamma \in \mathcal{B}(R^d)$,

$$P^x[X_{S+t} \in \Gamma | \mathcal{F}_{S+}] = (U_t 1_\Gamma)(X_S), \quad P^x \text{ - a.s.} \quad \text{on} \quad \{S < \infty\};$$

(e') For any bounded, continuous $f: R^d \to R$,

$$E^X[f(X_{S+t}) \mid \mathfrak{F}_{S+}] = (U_t f)(X_S), \quad P^X - \text{a.s. on } \{S < \infty\}.$$

Proof:

The proof that (e) is equivalent to (c) and (d) is the same as the proof of the analogous equivalence for Markov families given in Proposition 5.9. Since any bounded, continuous, real-valued function on R^d is the pointwise limit of a bounded sequence of linear combinations of indicators of Borel sets, (e') follows from (e) and the Bounded Convergence Theorem. On the other hand, if (e') holds and $\Gamma \subseteq R^d$ is closed, then 1_Γ is the pointwise limit of $\{f_n\}_{n=1}^\infty$, where

$$f_n(x) = [1 - n \, \rho(x, \Gamma)] \vee 0,$$

$$\rho(x, \Gamma) = \inf\{\|x - y\|; \ y \in \Gamma\}.$$

Each f_n is bounded and continuous, so (e) holds for closed sets Γ. The collection of sets $\Gamma \in \mathcal{B}(R^d)$ for which (e) holds forms a Dynkin system, so, by Theorem 5.1', (e) holds for all $\Gamma \in \mathcal{B}(R^d)$.

$\square$

<u>6.7´ Remark:</u>

If $X = \{X_t, \mathfrak{F}_t; \ t \geq 0\}$, $(\Omega, \mathfrak{F})$, $\{P^X\}_{x \in R^d}$ is a strong Markov family and μ is a probability measure on $(R^d, \mathfrak{B}(R^d))$, we can define a probability measure P^μ by $P^\mu(F) \triangleq \int_{R^d} P^X(F) \, \mu(dx)$; $F \in \mathfrak{F}$, and then X on $(\Omega, \mathfrak{F}, P^\mu)$ is a strong Markov process with initial distribution μ. Condition (ii) of Definition 6.2 can be verified upon writing condition (e) in integrated form:

$$\int_F (U_t 1_\Gamma)(X_S) \, dP^X = P^X[X_{S+t} \in \Gamma, F]; \quad F \in \mathfrak{F}_{S+} \ ,$$

and then integrating both sides with respect to μ. Similarly, if X, $(\Omega, \mathfrak{F})$, $\{P^X\}_{x \in R^d}$ is a Markov family, then X on $(\Omega, \mathfrak{F}, P^\mu)$ is a Markov process with initial distribution μ.

□

It is often convenient to work with bounded optional times only. The following problem shows that stating the strong Markov property in terms of such optional times entails no loss of generality. We shall use this fact in our proof that Brownian families are strongly Markovian.

6.8 <u>Problem</u>:   Let S be an optional time of the filtration $\{\mathfrak{F}_t\}$ on some $(\Omega, \mathfrak{F}, P)$.

(i) Show that if Z_1 and Z_2 are integrable random variables, s is a positive constant, and $Z_1 = Z_2$ on $\{S < s\}$, then

$$E[Z_1 | \mathfrak{F}_{S+}] = E[Z_2 | \mathfrak{F}_{S+}], \quad \text{a.s.} \quad \text{on} \quad \{S < s\}.$$

(ii) Show under the conditions of (i) that

$$E[Z_1 | \mathfrak{F}_{S+}] = E[Z_2 | \mathfrak{F}_{(S \wedge s)+}], \quad \text{a.s.} \quad \text{on} \quad \{S < s\}.$$

(Hint: Use Problem 1.2.16 (i)).

(iii) Show that if (e) (or (e')) in Proposition 6.6 holds for every bounded, optional time S of $\{\mathfrak{F}_t\}$, then it holds for every optional time.

□

Conditions (e) and (e') are statements about the conditional distribution of X at a single time S+t after the optional time S. If there are shift operators $\{\theta_s\}_{s \geq 0}$ satisfying (5.10), then for any random time S we can define the <u>random shift</u> $\theta_S \colon \{S < \infty\} \to \Omega$ by

$$(\theta_S \, \omega)(t) = (\theta_s \, \omega)(t) \quad \text{on} \quad \{S = s\}.$$

In other words, θ_S is defined so that whenever $S(\omega) < \infty$, then

$$X_{S(\omega)+t}(\omega) = X_t(\theta_S(\omega)).$$

In particular, we have $\{X_{S+.} \in E\} = \theta_S^{-1}\{X. \in E\}$, and (c') and (d') are respectively equivalent to the statements:

(c") For $x \in R^d$, $F \in \mathcal{F}_\infty^X$ and any optional time S of $\{\mathcal{F}_t\}$,

$$P^x[\theta_S^{-1}F|\mathcal{F}_{S+}] = P^x[\theta_S^{-1}F|X_S], \quad P^x - \text{a.s.} \quad \text{on} \quad \{S<\infty\};$$

(d") For $x \in R^d$, $F \in \mathcal{F}_\infty^X$ and any optional time S of $\{\mathcal{F}_t\}$,

$$P^x[\theta_S^{-1}F|X_S = y] = P^y(F), \quad P^x X_S^{-1} - \text{a.e.} \; y.$$

Both (c") and (d") can be captured by the single condition:

(e") For $x \in R^d$, $F \in \mathcal{F}_\infty^X$ and any optional time S of $\{\mathcal{F}_t\}$,

$$P^x[\theta_S^{-1}F|\mathcal{F}_{S+}] = P^{X_S}(F), \quad P^x - \text{a.s.} \quad \text{on} \quad \{S<\infty\}.$$

Since (e") is often given as the primary defining property for a strong Markov family, we summarize this discussion with a theorem.

<u>6.9 Theorem</u>: Let $X = \{X_t, \mathcal{F}_t; t \geq 0\}$ be a progressively measurable process on $(\Omega, \mathcal{F})$, let $\{P^x\}_{x \in R^d}$ be a family of probability measures on $(\Omega, \mathcal{F})$, and let $\{\theta_S\}_{S \geq 0}$ be a family of $\mathcal{F}/\mathcal{F}$ - measurable shift operators satisfying (5.10). Then X, $(\Omega, \mathcal{F})$, $\{P^x\}_{x \in R^d}$ is a strong Markov family if and only if (a), (b) and (e") hold.

6.10 Problem: Show that (e") is equivalent to the following

condition:

(e"')   For any $x \in R^d$,   any bounded, $\mathcal{F}_\infty^X$ - measurable random

variable Y,   and any optional time S of $\{\mathcal{F}_t\}$,   we

have

$$E^X[Y \circ \theta_S | \mathcal{F}_{S+}] = E^{X_S}(Y), \quad P^X - \text{a.s.} \quad \text{on} \quad \{S < \infty\}.$$

Note: If we write this equation with the arguments filled in,

it becomes

$$E^X[Y \circ \theta_S | \mathcal{F}_{S+}](\omega) = \int_\Omega Y(\omega') \, P^{X_{S(\omega)}(\omega)}(d\omega'),$$

$$P^X - \text{a.e.} \quad \omega \quad \text{in} \quad \{S < \infty\},$$

where $(Y \circ \theta_S)(\omega'') \triangleq Y(\theta_{S(\omega'')}(\omega''))$.   □

We now begin the discussion on the strong Markov property of

Brownian motion.

6.11 Definition: Let X be a random variable on a probability

space $(\Omega, \mathcal{F}, P)$ taking values in a complete, separable metric

space $(S, \mathcal{B}(S))$. Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. A regular

conditional probability of X given $\mathcal{G}$ is a function

$Q: \Omega \times \mathcal{B}(S) \to [0,1]$ such that

(i)   for each $\omega \in \Omega$, $Q(\omega; \cdot)$ is a probability measure on
       $(S, \mathcal{B}(S))$,

(ii)   for each $E \in \mathcal{B}(S)$, the mapping $\omega \to Q(\omega; E)$ is $\mathcal{G}$ -
       measurable, and

(iii) for each $E \in \mathcal{B}(S)$, $P[X \in E | \mathcal{G}](\omega) = Q(\omega; E)$, P - a.e. ω.

Under the conditions of Definition 6.11 on X,
$(\Omega, \mathcal{F}, P)$, $(S, \mathcal{B}(S))$ and $\mathcal{G}$, a regular conditional probability
for X given $\mathcal{G}$ exists (Ash [1972, pp. 264-265] or
Parthasarathy [1967, pp. 146-150]). One consequence of this
fact is that the conditional characteristic function of a
random vector can be used to determine its conditional dis-
tribution, in the manner outlined by the next lemma.

6.12 <u>Lemma</u>: Let X be a d-dimensional random vector on $(\Omega, \mathcal{F}, P)$.
Suppose $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$ and suppose that for each
$\omega \in \Omega$, there is a function $\varphi(\omega; \cdot): R^d \to \mathbb{C}$ such that for
each $u \in R^d$,

$$\varphi(\omega; u) = E[e^{i(u, X)} | \mathcal{G}](\omega), \quad P \text{ - a.e. } \omega.$$

If, for each ω, $\varphi(\omega; \cdot)$ is the characteristic function of
some probability measure P^ω on $(R^d, \mathcal{B}(R^d))$, i.e.,

$$\varphi(\omega; u) = \int_{R^d} e^{i(u, x)} P^\omega(dx),$$

then for each $\Gamma \in \mathcal{B}(R^d)$, we have

$$P[X \in \Gamma | \mathcal{G}](\omega) = P^\omega(\Gamma), \quad P \text{ - a.e. } \omega.$$

Proof:

Let Q be a regular conditional probability for X given $\mathcal{G}$, so for each fixed $u \in R^d$ we can build up from indicators to show that

(6.3)´ $\quad \varphi(\omega; u) = E[e^{i(u, X)} | \mathcal{G}](\omega) = \int_{R^d} e^{i(u, X)} Q(\omega; dx)$, P-a.e. ω.

The set of ω for which (6.3)´ fails may depend on u, but we can choose a countable, dense subset D of R^d and an event $\widetilde{\Omega} \in \mathcal{F}$ with $P(\widetilde{\Omega}) = 1$, so that (6.3)´ holds for every $\omega \in \widetilde{\Omega}$ and $u \in D$. Continuity in u of both sides of (6.3)´ allows us to conclude its validity for every $\omega \in \widetilde{\Omega}$ and $u \in R^d$. Since a measure is uniquely determined by its characteristic function, we must have $P^\omega = Q(\omega; \cdot)$ for P-a.e. ω, and the result follows. $\quad\square$

Recall that a d-dimensional random vector N has a <u>d-variate</u> <u>normal distribution</u> with mean $\mu \in R^d$ and covariance matrix $\Sigma \in R^d x R^d$, if and only if it has characteristic function

(6.4) $\quad E \, e^{i(u, N)} = e^{i(u, \mu) - \frac{1}{2}(u, \Sigma u)}$; $u \in R^d$.

Suppose $B = \{B_t, \mathcal{F}_t; t \geq 0\}$, $(\Omega, \mathcal{F})$, $\{P^x\}_{x \in R^d}$ is a d-dimensional Brownian family. Choose $u \in R^d$ and define the complex-valued process

$$M_t \triangleq \exp[i(u, B_t) + \frac{t}{2} \|u\|^2], \quad t \geq 0.$$

We denote the real and imaginary parts of this process by R_t and I_t, respectively.

6.13 Lemma: For each $x \in \mathbb{R}^d$, the processes $\{R_t, \mathcal{F}_t; t \geq 0\}$ and $\{I_t, \mathcal{F}_t; t \geq 0\}$ are martingales on $(\Omega, \mathcal{F}, P^x)$.

Proof:

For $0 \leq s < t$, we have

$$E_x[M_t \mid \mathcal{F}_s] = E_x[M_s \exp(i(u, B_t - B_s) + \frac{t-s}{2} \|u\|^2 \mid \mathcal{F}_s]$$

$$= M_s E_x[\exp(i(u, B_t - B_s) + \frac{t-s}{2} \|u\|^2]$$

$$= M_s,$$

where we have used the independence of $B_t - B_s$ and $\mathcal{F}_s$, as well as (6.4). Taking real and imaginary parts, we obtain the martingale property for $\{R_t, \mathcal{F}_t; t \geq 0\}$ and $\{I_t, \mathcal{F}_t; t \geq 0\}$. □

6.14 Theorem: A d-dimensional Brownian family is a strong Markov family. A d-dimensional Brownian motion is a strong Markov process.

Proof:

We verify that a Brownian family $B = \{B_t, \mathcal{F}_t; t \geq 0\}$, $(\Omega, \mathcal{F})$, $\{P^x\}_{x \in \mathbb{R}^d}$ satisfies condition (e) of Proposition 6.7. Thus, let S be an optional time of $\{\mathcal{F}_t\}$. In light of Problem 6.8, we may assume that S is bounded. Fix $x \in \mathbb{R}^d$. The Optional Sampling Theorem (Theorem 3.20 and Problem 3.21 (i)) applied to the martingales of Lemma 6.13 yields

$$E^X[\exp(i(u, B_{S+t}))|\mathcal{F}_{S+}](\omega)$$

$$= \exp[-\tfrac{1}{2}(S(\omega)+t)\ \|u\|^2]\ \cdot\ E^X[M_{S+t}|\mathcal{F}_{S+}](\omega)$$

$$= M_{S(\omega)}(\omega)\ \cdot\ \exp[-\tfrac{1}{2}(S(\omega)+t)\ \|u\|^2]$$

$$= \exp[i(u,\ B_{S(\omega)}(\omega)) - \tfrac{t}{2}\ \|u\|^2]\ ,\quad P^X - \text{a.e. } \omega.$$

Comparing this to (6.4), we see that the conditional distribution of B_{S+t}, given $\mathcal{F}_{S+}$, is normal with mean $B_{S(\omega)}(\omega)$ and covariance matrix $t\ I_d$. This proves (e). $\qquad\square$

We can carry this line of argument a bit farther, to obtain a related result.

6.15 Theorem: If S is an a.s. finite optional time of $\{\mathcal{F}_t\}$ for a d-dimensional Brownian motion $B = \{B_t, \mathcal{F}_t;\ t\geq 0\}$, then with $W_t \triangleq B_{S+t} - B_S$, the process $W = \{W_t, \mathcal{F}_t^W;\ t\geq 0\}$ is a standard, d-dimensional, Brownian motion, independent of $\mathcal{F}_{S+}$.

Proof:

We show that for $0\leq t_0 \leq \ldots \leq t_n$ and $u_1, \ldots, u_n \in R^d$,

$$(6.5) \qquad E[\exp(i\ \sum_{k=1}^{n}\ (u_k, W_{t_k} - W_{t_{k-1}}))|\mathcal{F}_{S+}]$$

$$= \prod_{k=1}^{n}\ \exp[-\tfrac{1}{2}(t_k - t_{k-1})\ \|u_k\|^2]\ ,\quad P - \text{a.s.};$$

thus, according to Lemma 6.12 and (6.4), not only are the increments

$\{W_{t_k} - W_{t_{k-1}}\}_{k=1}^n$ independent normal random vectors with mean zero and covariance matrices $(t_k - t_{k-1}) I_d$, but they are also independent of the σ-field $\mathcal{F}_{S+}$. This substantiates the claim of the theorem.

We prove (6.5) for bounded, optional times S of $\{\mathcal{F}_t\}$; the argument given in Solution 6.8 can be used to extend this result to a.s. finite S. Assume (6.5) holds for some n, and choose $0 \le t_0 \le \ldots \le t_n \le t_{n+1}$. Applying the Optional Sampling Theorem to the martingales in Lemma 6.13 with $u = u_{n-1}$ and the optional time $S+t_n$, we have

$$(6.6) \qquad E\{\exp[i(u_{n+1}, W_{t_{n+1}} - W_{t_n})] \mid \mathcal{F}_{(S+t_n)+}\}$$

$$= \exp[-\tfrac{1}{2}(S+t_{n+1})\|u_{n+1}\|^2 - i(u_{n+1}, B_{S+t_n})] \, E[M_{S+t_{n+1}} \mid \mathcal{F}_{(S+t_n)+}]$$

$$= \exp[-\tfrac{1}{2}(t_{n+1} - t_n)\|u_{n+1}\|^2], \quad P - a.s.$$

Therefore,

$$E[\exp(i \sum_{k=1}^{n+1} (u_k, W_{t_k} - W_{t_{k-1}})) \mid \mathcal{F}_{S+}]$$

$$= E[\exp(i \sum_{k=1}^{n} (u_k, W_{t_k} - W_{t_{k-1}}))$$

$$\cdot E[\exp(i(u_{n+1}, W_{t_{n+1}} - W_{t_n})) \mid \mathcal{F}_{(S+t_n)+}\} \mid \mathcal{F}_{S+}]$$

$$= \exp[-\tfrac{1}{2}(t_{n+1} - t_n)\|u_{n+1}\|^2] \, E[\exp(i \sum_{k=1}^{n} (u_k, W_{t_k} - W_{t_{k-1}}))]$$

$$= \prod_{k=1}^{n+1} \exp[-\tfrac{1}{2}(t_k - t_{k-1})\|u_k\|^2], \quad P - a.s.,$$

which completes the induction step. The proof that (6.5) holds for $n=1$ is obtained by setting $t_n=0$ in (6.6). $\qquad\square$

To make rigorous the derivation of the passage time density with which we began this section, a slight extension of the strong Markov property for right-continuous families will be needed.

6.16 Proposition: Let $X = \{X_t, \mathfrak{F}_t; t \geq 0\}$, $(\Omega, \mathfrak{F})$, $\{P^x\}_{x \in \mathbb{R}^d}$ be a strong Markov family, and the process X be right-continuous. Let S be an optional time of $\{\mathfrak{F}_t\}$ and T an $\mathfrak{F}_{S+}$ - measurable random time satisfying $T(\omega) \geq S(\omega)$ for all $\omega \in \Omega$. Then, for any $x \in \mathbb{R}^d$ and any bounded, continuous $f: \mathbb{R}^d \to \mathbb{R}$,

$$(6.7) \qquad E^x[f(X_T) | \mathfrak{F}_{S+}](\omega) = (U_{T(\omega)-S(\omega)} f)(X_{S(\omega)}(\omega)),$$

$$\text{for } P^x \text{ - a.e. } \omega \in \{T < \infty\}.$$

Proof:

For $n \geq 1$, let

$$T_n = \begin{cases} S + \dfrac{1}{2^n}([\,2^n(T-S)\,]+1), & \text{if } T < \infty, \\[2mm] \infty, & \text{if } T = \infty, \end{cases}$$

so that $T_n = S + \dfrac{k}{2^n}$ when $\dfrac{k-1}{2^n} \leq T-S < \dfrac{k}{2^n}$. We have $T_n \downarrow T$ on $\{T < \infty\}$. From (e') we have for $k \geq 0$,

$$E^x[f(X_{S + \frac{k}{2^n}}) | \mathfrak{F}_{S+}] = (U_{\frac{k}{2^n}} f)(X_S), \quad P^x \text{ - a.s. on } \{S < \infty\},$$

and Problem 6.8 (i) then implies

$$E^X[f(X_{T_n})|\mathfrak{F}_{S+}](\omega) = (U_{T_n(\omega)-S(\omega)}f)(X_{S(\omega)}(\omega)), \quad P^X - \text{a.e.} \ \omega \in \{T < \infty\}.$$

The Bounded Convergence Theorem for conditional expectations and the right-continuity of X imply that the left-hand side converges to $E^X[f(X_T)|\mathfrak{F}_{S+}](\omega)$ as $n \to \infty$. Since $(U_t f)(y) = E^y f(X_t)$ is right-continuous in t for every $y \in \mathbb{R}^d$, the right-hand side converges to $(U_{T(\omega)-S(\omega)}f)(X_{S(\omega)}(\omega))$. $\qquad\square$

6.17 Corollary: Under the conditons of Proposition 6.16, (6.7) holds for every bounded, $\mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$ - measurable function f. In particular, for any $\Gamma \in \mathcal{B}(\mathbb{R}^d)$ we have

$$P^X[X_T \in \Gamma | \mathfrak{F}_{S+}](\omega) = (U_{T(\omega)-S(\omega)} \mathbf{1}_\Gamma)(X_{S(\omega)}(\omega)),$$

$$P^X - \text{a.e.} \ \omega \in \{T < \infty\}.$$

Proof:

Approximate the indicator of a closed set Γ by bounded, continuous functions as in the proof of Proposition 6.7. Then prove the result for any $\Gamma \in \mathcal{B}(\mathbb{R}^d)$, and extend to bounded, Borel-measurable functions. $\qquad\square$

6.18 Proposition: Let $\{B_t, \mathfrak{F}_t; \ t \geq 0\}$ be a standard, one-dimensional Brownian motion, and for $b \neq 0$, let T_b be the first passage time to b given by (6.1). Then T_b has density (6.3).

Proof:

Because $\{-B_t, \mathfrak{F}_t; t \geq 0\}$ is also a standard, one-dimensional Brownian motion, it suffices to consider the case $b > 0$. In Corollary 6.17 set $S = T_b$,

$$T = \begin{cases} t & \text{if } S < t, \\ \infty & \text{if } S \geq t, \end{cases}$$

and $\Gamma = (-\infty, b)$. On the set $\{T < \infty\} = \{S < t\}$, we have $B_{S(\omega)}(\omega) = b$ and $(U_{T(\omega)-S(\omega)} 1_\Gamma)(B_{S(\omega)}(\omega)) = \frac{1}{2}$.

Therefore,

$$P^0[T_b < t, \; B_t < b] = \int_{\{T_b < t\}} P^0[B_T \in \Gamma \mid \mathfrak{F}_{S+}] dP^0 = \tfrac{1}{2} P^0[T_b < t] .$$

Thus,

$$P^0[T_b < t] = P^0[T_b < t, \; B_t > b] + P^0[T_b < t, \; B_t < b]$$

$$= P^0[B_t > b] + \tfrac{1}{2} P^0[T_b < t],$$

and (6.2) is proved.

$\square$

2.7 BROWNIAN FILTRATIONS

The inquisitive reader may well have wondered why we have made a point of defining Brownian motion $B = \{B_t, \mathcal{F}_t; t \geq 0\}$ with a filtration $\{\mathcal{F}_t\}$ which is not necessarily the same as $\{\mathcal{F}_t^B\}$, the one generated by the process itself. One reason has to do with the fact that, although the filtration $\{\mathcal{F}_t^B\}$ is left-continuous, it fails to be right-continuous (Problem 7.1). Some of the developments in later chapters require either right or two-sided continuity of the filtration $\{\mathcal{F}_t\}$, and so in this section we construct filtrations with these properties.

Let us recall the basic definitions from section 1.1. For a filtration $\{\mathcal{F}_t; t \geq 0\}$ on the measurable space $(\Omega, \mathcal{F})$, we set

$$\mathcal{F}_{t+} = \bigcap_{\epsilon > 0} \mathcal{F}_{t+\epsilon} \quad \text{for} \quad t \geq 0, \quad \mathcal{F}_{t-} = \sigma\left(\bigcup_{s < t} \mathcal{F}_s\right) \quad \text{for} \quad t > 0,$$

$$\mathcal{F}_{0-} = \mathcal{F}_0 \quad \text{and} \quad \mathcal{F}_\infty = \sigma\left(\bigcup_{t \geq 0} \mathcal{F}_t\right).$$

We say that $\{\mathcal{F}_t\}$ is <u>right</u> (respectively, <u>left</u>) - <u>continuous</u> if $\mathcal{F}_{t+} = \mathcal{F}_t$ (respectively, $\mathcal{F}_{t-} = \mathcal{F}_t$) holds for every $0 \leq t < \infty$. When $X = \{X_t, \mathcal{F}_t^X; t \geq 0\}$ is a process on $(\Omega, \mathcal{F})$, then left-continuity of $\{\mathcal{F}_t^X\}$ at some fixed $t > 0$ can be interpreted to mean that X_t can be discovered by observing X_s, $0 \leq s < t$. Right-continuity means intuitively that if X_s has been observed for $0 \leq s \leq t$, then nothing more can be learned by peeking infinitesimally far into the future. We recall here that $\mathcal{F}_t^X = \sigma(X_s; 0 \leq s \leq t)$.

7.1 Problem: Let $\{X_t, \mathcal{F}_t^X; 0 \leq t < \infty\}$ be a d-dimensional process.

    (i) Show that the filtration $\{\mathcal{F}_{t+}^X\}$ is right-continuous.

    (ii) Show that if X is left-continuous, then the filtration

$\{\mathfrak{F}_t^X\}$ is left-continuous.

(iii) Show by example that, even if X is continuous, $\{\mathfrak{F}_t^X\}$ can fail to be right-continuous and $\{\mathfrak{F}_{t+}^X\}$ can fail to be left-continuous.

We shall need to develop the important notions of "completion" and "augmentation" of σ-fields, in the context of a strong Markov process $X = \{X_t, \mathfrak{F}_t^X; \ 0 \leq t < \infty\}$ with initial distribution μ on the space $(\Omega, \mathfrak{F}, P^\mu)$. We start by setting, for $0 \leq t \leq \infty$,

$$\eta_t^\mu \triangleq \{F \subseteq \Omega; \ \exists G \in \mathfrak{F}_t^X \ \text{with} \ F \subseteq G, \ P^\mu(G) = 0\} \ .$$

η_∞^μ will be called "the collection of $\underline{P^\mu\text{-null sets}}$", and denoted simply by η^μ.

<u>7.2 Definition</u>: For any $0 \leq t < \infty$, we define the <u>completion</u>

$$\overline{\mathfrak{F}_t^\mu} \triangleq \sigma(\mathfrak{F}_t^X \cup \eta_t^\mu), \quad \text{and the} \ \underline{\text{augmentation}}$$

$$\mathfrak{F}_t^\mu \triangleq \sigma(\mathfrak{F}_t^X \cup \eta^\mu)$$

of the σ-field $\mathfrak{F}_t^X$ under P^μ. For $t = \infty$ the two concepts agree, and we set simply

$$\mathfrak{F}^\mu \triangleq \sigma(\mathfrak{F}_\infty^X \cup \eta^\mu).$$

The <u>augmented filtration</u> $\{\mathfrak{F}_t^\mu\}$ possesses certain desirable properties, which will be used frequently in the sequel and are developed in the ensuing problems and propositions.

7.3 Problem: For any sub-σ-field $\mathcal{G}$ of $\mathcal{F}_\infty^X$, define

$$\mathcal{G}^\mu = \sigma(\mathcal{G} \cup \eta^\mu) \quad \text{and}$$

$$\mathcal{H} = \{F \subseteq \Omega; \exists G \in \mathcal{G} \text{ such that } F \triangle G \in \eta^\mu\}.$$

Show that $\mathcal{G}^\mu = \mathcal{H}$. We now extend P^μ by defining $P^\mu(F) \triangleq P^\mu(G)$ whenever $F \in \mathcal{G}^\mu$, and $G \in \mathcal{G}$ is chosen to satisfy $F \triangle G \in \eta^\mu$. Show that the probability space $(\Omega, \mathcal{G}^\mu, P^\mu)$ is __complete__:

$$F \in \mathcal{G}^\mu, \quad P^\mu(F) = 0, \quad D \subseteq F \Rightarrow D \in \mathcal{G}^\mu.$$

7.4 Problem: From Definition 7.2 we have $\mathcal{F}_t^\mu \subseteq \bar{\mathcal{F}}_t^\mu$, for every $0 \leq t < \infty$. Show by example that the inclusion can be strict:

$$\mathcal{F}_0^\mu \subset \bar{\mathcal{F}}_0^\mu.$$

7.5 Problem: Show that the σ-field $\mathcal{F}^\mu$ of Definition 7.2 agrees with

$$\mathcal{F}_\infty^\mu \triangleq \sigma\left(\bigcup_{t \geq 0} \mathcal{F}_t^\mu \right).$$

7.6 Problem: If the process X has left-continuous paths, then the filtration $\{\mathcal{F}_t^\mu\}$ is left-continuous. $\square$

We are ready now for the key result of this section.

7.7 Proposition: For a d-dimensional, strong Markov process $X = \{X_t, \mathcal{F}_t^X; t \geq 0\}$ with initial distribution μ, the augmented filtration $\{\mathcal{F}_t^\mu\}$ is right-continuous.

Proof:

Let $(\Omega, \mathfrak{F}^X_\infty, P^\mu)$ be the probability space on which X is defined. Fix $s \geq 0$ and consider the degenerate, $\{\mathfrak{F}^X_t\}$ - optional time $S = s$. The strong Markov property implies for $t > 0$ and $\Gamma \in \mathfrak{B}(\mathbb{R}^d)$,

$$P^\mu[X_t \in \Gamma \mid \mathfrak{F}^X_{s+}] = P^\mu[X_t \in \Gamma \mid X_s], \quad P^\mu \text{ - a.s.}$$

For $t > s$, we see then that $P^\mu[X_t \in \Gamma \mid \mathfrak{F}^X_{s+}]$ has an $\mathfrak{F}^X_s$-measurable version. For $t \leq s$, X_t is $\mathfrak{F}^X_s$-measurable, so again $P^\mu[X_t \in \Gamma \mid \mathfrak{F}^X_{s+}]$ has an $\mathfrak{F}^X_s$-measurable version. The collection of all sets $F \in \mathfrak{F}^X_\infty$ for which $P^\mu[F \mid \mathfrak{F}_{s+}]$ has an $\mathfrak{F}^X_s$-measurable version is a σ-field, and since $\mathfrak{F}^X_\infty$ is generated by sets of the form $\{X_t \in \Gamma\}$, we see that $P^\mu[F \mid \mathfrak{F}_{s+}]$ has an $\mathfrak{F}^X_s$-measurable version for every $F \in \mathfrak{F}^X_\infty$. But suppose $F \in \mathfrak{F}^X_{s+} \subseteq \mathfrak{F}^X_\infty$. Then

$$P^\mu[F \mid \mathfrak{F}^X_{s+}] = 1_F, \quad P^\mu \text{ - a.s.,}$$

so 1_F has an $\mathfrak{F}^X_s$-measurable version, which we call Y. Let $G = \{Y = 1\} \in \mathfrak{F}^X_s$. Since $F \triangle G \subseteq \{1_F \neq Y\} \in \mathfrak{N}^\mu$, we have $F \in \mathfrak{F}^\mu_s$. Therefore,

$$\mathfrak{F}^X_{s+} \subseteq \mathfrak{F}^\mu_s, \quad s \geq 0.$$

Suppose now that $F \in \mathfrak{F}^\mu_{s+}$. Then for each positive integer n, $F \in \mathfrak{F}^\mu_{s+1/n}$, so there exists $G_n \in \mathfrak{F}^X_{s+1/n}$ such that $F \triangle G_n \in \mathfrak{N}^\mu$. Set

$G \overset{\Delta}{=} \bigcap\limits_{m=1}^{\infty} \bigcup\limits_{n=m}^{\infty} G_n$, and since $G = \bigcap\limits_{m=M}^{\infty} \bigcup\limits_{n=m}^{\infty} G_n$ for any positive integer

M, we have $G \in \mathcal{F}_{s+}^X \subseteq \mathcal{F}_s^\mu$. To prove that $F \in \mathcal{F}_s^\mu$, it suffices to prove

that $F \bigtriangleup G \in \eta^\mu$. Now

$$G \setminus F \subseteq (\bigcup\limits_{n=1}^{\infty} G_n) \setminus F = \bigcup\limits_{n=1}^{\infty} (G_n \setminus F) \in \eta^\mu .$$

On the other hand

$$F \setminus G = F \cap (\bigcap\limits_{m=1}^{\infty} \bigcup\limits_{n=m}^{\infty} G_n)^C = F \cap (\bigcup\limits_{m=1}^{\infty} \bigcap\limits_{n=m}^{\infty} G_n^C)$$

$$= \bigcup\limits_{m=1}^{\infty} [F \cap (\bigcap\limits_{n=m}^{\infty} G_n^C)] \subseteq \bigcup\limits_{m=1}^{\infty} (F \cap G_m^C)$$

$$= \bigcup\limits_{m=1}^{\infty} (F \setminus G_m) \in \eta^\mu.$$

It follows that $F \in \mathcal{F}_s^\mu$, so $\mathcal{F}_{s+}^\mu \subseteq \mathcal{F}_s^\mu$ and right-continuity is

proved.

$\square$

7.8 Corollary: For a d-dimensional, left-continuous, strongly

Markov process $X = \{X_t, \mathcal{F}_t^X; t \geq 0\}$ with initial distribution

μ, the augmented filtration $\{\mathcal{F}_t^\mu\}$ is continuous.

7.9 Theorem: Let $B = \{B_t, \mathcal{F}_t^B; t \geq 0\}$ be a d-dimensional Brownian

motion with initial distribution μ on $(\Omega, \mathcal{F}_\infty^B, P^\mu)$. Relative

to the right-continuous filtration $\{\mathcal{F}_t^\mu\}$, $\{B_t, t \geq 0\}$ is still

a d-dimensional Brownian motion.

Proof:

Augmentation of σ-fields does not disturb any of the independence assumptions of Definition 5.1. $\square$

Since any d-dimensional Brownian motion is strongly Markov (Theorem 6.14), the augmentation of the filtration in Theorem 7.9 does not affect the strong Markov property. This raises the following general question. Suppose $\{X_t, \mathcal{F}_t^X; t \geq 0\}$ is a d-dimensional, strong Markov process with initial distribution μ on $(\Omega, \mathcal{F}_\infty^X, P^\mu)$. Is the process $\{X_t, \mathcal{F}_t^\mu; t \geq 0\}$ also strongly Markov? In other words, is it true, for every optional time S of $\{\mathcal{F}_t^\mu\}$, $t \geq 0$ and $\Gamma \in \mathcal{B}(\mathbb{R}^d)$, that

$$(7.1) \qquad P^\mu[X_{S+t} \in \Gamma | \mathcal{F}_{S+}^\mu] = P^\mu[X_{S+t} \in \Gamma | X_S], \qquad P^\mu \text{- a.s. on } \{S < \infty\}?$$

Although the answer to this question is affirmative, phrased in this generality, the question is not as important as it might appear. In each particular case, some technique must be used to prove that $\{X_t, \mathcal{F}_t^X; t \geq 0\}$ is strongly Markov in the first place, and this technique can usually be employed to establish the strong Markov property for $\{X_t, \mathcal{F}_t^\mu; t \geq 0\}$ as well. Theorems 7.9 and 6.14 exemplify this kind of argument for d-dimensional Brownian motion. The interested reader can work though the following series of problems, to verify that (7.1) is valid in the generality claimed. We shall make no subsequent use of them.

In Problems 7.10 - 7.13, $X = \{X_t, \mathcal{F}_t^X; \; 0 \leq t < \infty\}$ is a strong Markov process with initial distribution μ on $(\Omega, \mathcal{F}_\infty^X, P^\mu)$.

7.10 Problem: Show that any optional time S of $\{\mathcal{F}_t^\mu\}$ is also a stopping time of this filtration, and for each such S there exists an optional time T of $\{\mathcal{F}_t^X\}$ with $\{S \neq T\} \in \mathcal{N}^\mu$. Conclude that $\mathcal{F}_{S+}^\mu = \mathcal{F}_S^\mu = \mathcal{F}_T^\mu$, where $\mathcal{F}_T^\mu$ is defined to be the collection of sets $A \in \mathcal{F}^\mu$ satisfying $A \cap \{T \leq t\} \in \mathcal{F}_t^\mu$, $\forall \; 0 \leq t < \infty$.

7.11 Problem: Suppose that T is an optional time of $\{\mathcal{F}_t^X\}$. For fixed positive integer n, define

$$T_n = \begin{cases} T, & \text{on } \{T = \infty\} \\ \dfrac{k}{2^n}, & \text{on } \{\dfrac{k-1}{2^n} \leq T < \dfrac{k}{2^n}\} . \end{cases}$$

Show that T_n is a stopping time of $\{\mathcal{F}_t^X\}$, and $\mathcal{F}_T^\mu \subseteq \sigma(\mathcal{F}_{T_n}^X \cup \mathcal{N}^\mu)$. Conclude that $\mathcal{F}_T^\mu \subseteq \sigma(\mathcal{F}_{T+}^X \cup \mathcal{N}^\mu)$. (Hint: Use Problems 1.2.22 and 1.2.23).

7.12 Problem: Establish the following proposition: if for each $t \geq 0$, $\Gamma \in \mathcal{B}(R^d)$ and optional time T of $\{\mathcal{F}_t^X\}$, we have the strong Markov property

$$(7.2) \qquad P^\mu[X_{T+t} \in \Gamma \mid \mathcal{F}_{T+}^X] = P^\mu[X_{T+t} \in \Gamma \mid X_T], \quad P^\mu - \text{a.s. on } \{T < \infty\},$$

then (7.1) holds for every optional time S of $\{\mathcal{F}_t^\mu\}$.

This completes our discussion of the augmentation of the filtration generated by a strongly Markov process. At first glance, augmentation appears to be a rather artifical device, but in retrospect, it can be seen to be more useful and natural than merely completing each σ-field $\mathfrak{F}_t^X$ with respect to P^μ. It is more natural because it involves only one collection of P^μ-null sets, the collection we called η^μ, rather than a separate collection for each $t \geq 0$. It is more useful because completing each σ-field $\mathfrak{F}_t^X$ does not result in a right-continuous filtration, as the next problem demonstrates.

<u>7.13 Problem</u>: Let $\{B_t, t \geq 0\}$ be the coördinate mapping process on $(C[0,\infty), \mathfrak{B}(C[0,\infty)))$, and let P^0 be Wiener measure. Let $\bar{\mathfrak{F}}_t$ denote the completion of $\mathfrak{F}_t^B$ under P^0. Consider the set

$$F = \{\omega \in C[0,\infty); \omega \text{ is constant on } [0,\epsilon] \text{ for some } \epsilon > 0\}.$$

Show that: (i) $P^0(F) = 0$, (ii) $F \in \mathfrak{F}_{0+}^B$, and (iii) $F \notin \bar{\mathfrak{F}}_0$.

The difficulty with the filtration $\{\mathfrak{F}_t^\mu\}$, obtained for a strong Markov process with initial distribution μ, is its dependence on μ. In particular, such a filtration is inappropriate for a strong Markov family, where there is a continuum of initial conditions. We now construct a filtration which is well suited for this case.

Let $\{X_t, \mathfrak{F}_t^X; t \geq 0\}$, $(\Omega, \mathfrak{F}_\infty^X)$, $\{P^X\}_{x \in \mathbb{R}^d}$ be a d-dimensional, strong Markov family. For each probability measure μ on $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$, we define P^μ as in (5.2):

$$P^\mu(F) = \int_{\mathbb{R}^d} P^x(F) \, \mu(dx), \quad \forall \, F \in \mathcal{F}_\infty^X ,$$

and we construct the augmented filtration $\{\mathcal{F}_t^\mu\}$ as before. We define

(7.3) $\qquad \widetilde{\mathcal{F}}_t \triangleq \bigcap_\mu \mathcal{F}_t^\mu ,$

where the intersection is over all probability measures μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Note that $\mathcal{F}_t^X \subseteq \widetilde{\mathcal{F}}_t \subseteq \mathcal{F}_t^\mu$, $0 \leq t < \infty$ for any probability measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$; therefore, if $\{X_t, \mathcal{F}_t^X; \, t \geq 0\}$ and $\{X_t, \mathcal{F}_t^\mu; \, t \geq 0\}$ are both strongly Markovian under P^μ, then so is $\{X_t, \widetilde{\mathcal{F}}_t; \, t \geq 0\}$. Because the order of intersection is interchangeable and $\{\mathcal{F}_t^\mu\}$ is right-continuous, we have

$$\widetilde{\mathcal{F}}_{t+} = \bigcap_{s > t} \bigcap_\mu \mathcal{F}_s^\mu = \bigcap_\mu \bigcap_{s > t} \mathcal{F}_s^\mu = \bigcap_\mu \mathcal{F}_t^\mu = \widetilde{\mathcal{F}}_t .$$

Thus $\{\widetilde{\mathcal{F}}_t\}$ is also right-continuous.

7.14 Theorem: Let $B = \{B_t, \mathcal{F}_t^B; \, t \geq 0\}$, $(\Omega, \mathcal{F}_\infty^B)$, $\{P^x\}_{x \in \mathbb{R}^d}$ be a d-dimensional Brownian family. Then $\{B_t, \widetilde{\mathcal{F}}_t; \, t \geq 0\}$, $(\Omega, \widetilde{\mathcal{F}}_\infty)$, $\{P^x\}_{x \in \mathbb{R}^d}$ is also a Brownian family.

Proof:

It is easily verified that $\{B_t, \widetilde{\mathcal{F}}_t; \, t \geq 0\}$ is a d-dimensional Brownian motion starting at x. It remains only to establish the universal measurability of condition (i) of Definition 5.4. Fix $F \in \widetilde{\mathcal{F}}_\infty$. For each probability measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, we have $F \in \mathcal{F}^\mu$, so there is some $G \in \mathcal{F}_\infty^B$ with $F \triangle G \in \mathcal{N}^\mu$. Let $N \in \mathcal{F}_\infty^B$ satisfy

$F \triangle G \subseteq N$ and $P^\mu(N) = 0$. The functions $g(x) \triangleq P^X(G)$ and $n(x) \triangleq P^X(N)$ are universally measurable by assumption. Furthermore,

$$\int_{\mathbb{R}^d} n(x) \, \mu(dx) = P^\mu(N) = 0,$$

so $n = 0$, μ - a.e. The nonnegative functions $h_1(x) \triangleq P^X(F \backslash G)$ and $h_2(x) \triangleq P^X(G \backslash F)$ are dominated by n, so h_1 and h_2 are zero $\underline{\mu}$ - a.e., and hence h_1 and h_2 are measurable with respect to $\overline{\mathcal{B}(\mathbb{R}^d)}^\mu$, the completion of $\mathcal{B}(\mathbb{R}^d)$ under μ. Set $f(x) \triangleq P^X(F)$. We have

$$f(x) = g(x) + h_1(x) - h_2(x),$$

so f is also $\overline{\mathcal{B}(\mathbb{R}^d)}^\mu$ - measurable. This is true for every μ; thus, f is universally measurable. $\square$

$\underline{7.15 \text{ Remark}}$: In Theorem 7.14, even if the mapping $x \mapsto P^X(F)$ is Borel-measurable for each $F \in \mathcal{F}_\infty^B$ (c.f. Problem 5.1), we can conclude only its universal measurability for each $F \in \tilde{\mathcal{F}}_\infty$. This explains why Definition 5.4 was designed with a condition of universal rather than Borel measurability.

We close this section with a useful consequence of the results concerning augmentation.

7.16 Theorem: Blumenthal(1957) Zero-One Law

Let $\{B_t, \widetilde{\mathcal{F}}_t; t \geq 0\}$, $(\Omega, \mathcal{F})$, $\{P^x\}_{x \in \mathbb{R}^d}$ be a d-dimensional Brownian family, where $\widetilde{\mathcal{F}}_t$ is given by (7.3). If $F \in \widetilde{\mathcal{F}}_0$, then for each $x \in \mathbb{R}^d$ we have either $P^x(F) = 0$ or $P^x(F) = 1$.

Proof:

For $F \in \widetilde{\mathcal{F}}_0$ and each $x \in \mathbb{R}^d$, there exists $G \in \mathcal{F}_0^B$ such that $P^x(F \triangle G) = 0$. But G must have the form $G = \{B_0 \in \Gamma\}$ for some $\Gamma \in \mathcal{B}(\mathbb{R}^d)$, so

$$P^x(F) = P^x(G) = P^x\{B_0 \in \Gamma\} = 1_\Gamma(x),$$

which is either zero or one.

$\square$

7.17 Problem: Show that a standard, one-dimensional Brownian motion changes sign infinitely many times in any time-interval $[0, \epsilon]$, $\epsilon > 0$, with probability one.

7.18 Problem: Let $\{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a standard, one-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$, and define

$$S_b = \inf\{t \geq 0; W_t > b\}; \quad b \geq 0.$$

(i)   Show that for each $b \geq 0$, $P[T_b \neq S_b] = 0$.

(ii)   Show that if L is a finite, nonnegative random variable on $(\Omega, \mathcal{F}, P)$ which is independent of $\mathcal{F}_\infty^W$, then $\{\omega \in \Omega; T_{L(\omega)}(\omega) \neq S_{L(\omega)}(\omega)\} \in \mathcal{F}$ and $P[T_L \neq S_L] = 0$.

2.8: COMPUTATIONS BASED ON PASSAGE TIMES

In order to motivate the strong Markov property in §2.6, we derived the density for the first passage time of a one-dimensional Brownian motion from the origin to $b \neq 0$. In this section we obtain a number of distributions related to this one, including the distribution of <u>reflected Brownian motion, Brownian motion on [0,a] absorbed at the endpoints</u>, the <u>time and value of the maximum</u> of Brownian motion on a fixed time interval, and the <u>time of the last exit</u> of Brownian motion from the origin before a fixed time. While derivations of all of these distributions can be based on the strong Markov property and the reflection principle, we shall occasionally provide arguments based on the optional sampling theorem for martingales. The former method yields densities, whereas the latter yields Laplace transforms of densities. The reader should be acquainted with both methods.

Throughout this section, $\{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$, $(\Omega, \mathcal{F})$, $\{P_x\}_{x \in R}$ will be a one-dimensioal Brownian family. We recall from (6.1) the <u>passage times</u>

$$T_b = \inf\{t \geq 0; W_t = b\}; \quad b \in R,$$

and define the <u>running maximum</u> (or maximum-to-date)

$$(8.1) \qquad M_t = \max_{0 \leq s \leq t} W_s.$$

8.1 Proposition: We have for $t>0$:

$$(8.2) \qquad P^0[W_t \in da, \ M_t \in db] = \frac{2(2b-a)}{\sqrt{2\pi t^3}} \ \exp\{- \frac{(2b-a)^2}{2t}\}da \ db; \quad a \le b, \ b \ge 0.$$

Proof:

For $a \le b$, $b \ge 0$, the symmetry of Brownian motion implies that

$$(U_{t-s} \ 1_{(-\infty, a]})(b) := P^b[W_{t-s} \le a] = P^b[W_{t-s} \ge 2b-a]$$

$$=: (U_{t-s} \ 1_{[2b-a, \infty)})(b); \quad 0 \le s \le t.$$

Corollary 6.17 then yields

$$P^0[W_t \le a | \mathcal{F}_{T_b+}] = (U_{t-T_b} \ 1_{(-\infty, a]})(b)$$

$$= (U_{t-T_b} \ 1_{[2b-a, \infty)})(b)$$

$$= P^0[W_t \ge 2b-a | \mathcal{F}_{T_b+}], \quad \text{a.s. } P^0 \text{ on } \{T_b \le t\}.$$

Integrating both sides of this equation over $\{T_b < t\}$ and noting that $\{T_b < t\} = \{M_t > b\}$, a.s. P^0, we obtain

$$P^0[W_t \le a, \ M_t > b] = P^0[W_t \ge 2b-a, \ M_t > b]$$

$$= P^0[W_t \ge 2b-a] = \frac{1}{\sqrt{2\pi t}} \int_{2b-a}^{\infty} e^{-\frac{x^2}{2t}} \ dt.$$

Differentiation leads to (8.2).   □

8.2 Problem: Show that for $t>0$,

$$(8.3) \qquad P^0[M_t \in db] = P^0[|W_t| \in db] = P^0[M_t - W_t \in db]$$

$$= \frac{2}{\sqrt{2\pi t}}\ e^{-\frac{b^2}{2t}}\ db; \quad b>0.$$

8.3 Remark: From (8.3) we see that

$$(8.4) \qquad P^0[T_b \leq t] = P^0[M_t \geq b] = \frac{2}{\sqrt{2\pi}} \int_{\frac{b}{\sqrt{t}}}^{\infty} e^{-\frac{x^2}{2}}\ dx; \quad b>0.$$

By differentiation, we recover the passage time density (6.3):

$$(8.5) \qquad P^0[T_b \in dt] = \frac{b}{\sqrt{2\pi t^3}} e^{-\frac{b^2}{2t}}\ dt; \quad b>0,\ t>0.$$

For future reference, we note that this density has Laplace transform

$$(8.6) \qquad E^0 e^{-\alpha T_b} = e^{-b\sqrt{2\alpha}}; \qquad b>0,\ \alpha>0.$$

By letting $t\uparrow\infty$ in (8.4) or $\alpha\downarrow 0$ in (8.6), we see that $P^0[T_b<\infty] = 1$. It is clear from (8.5), however, that $E^0 T_b = \infty$.

8.4 Exercise: Derive (8.6) (and consequently (8.5)) by applying the optional sampling theorem to the $\{\mathfrak{F}_t\}$-martingale

$$(8.7) \qquad M_t = \exp\{\lambda W_t - \frac{1}{2}\lambda^2 t\}; \quad 0\leq t<\infty,$$

with $\lambda = \sqrt{2\alpha} > 0$.

8.5 Problem: Derive the transition density for <u>Brownian motion</u> <u>absorbed at the origin</u> $\{W_{t\wedge T_0},\ \mathfrak{F}_t;\ 0\leq t<\infty\}$, by verifying that

(8.8) $P^x[W_t \in dy, \; T_0 > t] = p_-(t; x,y)dy$

$\underline{\underline{\Delta}} \; [p(t; x,y) - p(t; x,-y)]dy; \; t>0, \; x,y>0.$

8.6 Problem: Show that under P^0, reflected Brownian motion
$|W| \underline{\underline{\Delta}} \; \{|W_t|, \mathfrak{F}_t; \; 0 \leq t < \infty\}$ is a Markov process with transition density

(8.9) $P^0[\; |W_{t+s}| \in dy \big| \; |W_t| = x] = p_+(s; x,y)dy$

$\underline{\underline{\Delta}} \; [p(s; x,y) + p(s; x,-y)]dy; \; s>0, t \geq 0 \; \text{and} \; x,y \geq 0.$

8.7 Problem: Define $Y_t \underline{\underline{\Delta}} M_t - W_t; \; 0 \leq t < \infty$.   Show that under P^0, the process $Y = \{Y_t, \mathfrak{F}_t; \; 0 \leq t < \infty\}$ is Markov and has transition density

(8.10) $P^0[Y_{t+s} \in dy | Y_t = x] = p_+(s; x,y)dy; \; s>0, t \geq 0 \; \text{and} \; x,y \geq 0.$

Conclude that under P^0 the processes $|W|$ and Y have the same finite-dimensional distributions.

The surprising equivalence in law of the processes Y and $|W|$ was observed by P. Lévy (1948), who employed it in his deep study of Brownian local time (c f. Chapter 6). The third process M appearing in (8.3) cannot be equivalent in law to Y and $|W|$ since the paths of M are nondecreasing, whereas those of Y and $|W|$ are not. Nonetheless, M will turn out to be an object of considerable interest because it is the local time at the origin of the reflected Brownian motion Y.

The following simple proposition will also be extremely helpful in our study of local time.

8.8 Proposition:

The process of passage times $T = \{T_a, \mathfrak{F}_{T_a+}; \; 0 \leq a < \infty\}$ has the property that, under P^0 and for $0 \leq a < b$, the increment $T_b - T_a$ is independent of $\mathfrak{F}_{T_a+}$ and has the density

$$P^0[T_b - T_a \in dt] = \frac{b-a}{\sqrt{2\pi t^3}} e^{-\frac{(b-a)^2}{2t}} \, dt; \quad 0 < t < \infty.$$

In particular,

$$(8.11) \qquad E^0[e^{-\alpha(T_b - T_a)} \mid \mathfrak{F}_{T_a+}] = e^{-(b-a)\sqrt{2\alpha}}; \quad \alpha > 0.$$

Proof:

This is a direct consequence of Theorem 6.15 and the fact that $T_b - T_a = \inf\{t \geq 0; \; W_{T_a+t} - W_{T_a} = b-a\}$. □

In Problem 8.5 we computed the transition density for Brownian motion absorbed at the origin. We now undertake the study of Brownian motion on $[0,a]$ absorbed at 0 and a; to wit, $\{W_{t \wedge T_0 \wedge T_a}, \mathfrak{F}_t; \; 0 \leq t < \infty\}$.

8.9 Proposition: Choose $0 < x < a$. Then

$$(8.12) \qquad P^x[W_t \in dy, \; T_0 \wedge T_a > t] = \sum_{n=-\infty}^{\infty} p_-(t; x, y+2na); \quad 0 < y < a, t > 0.$$

Proof:

We follow Dynkin and Yushkevich (1969). Set $\sigma_0 \triangleq 0$, $\tau_0 \triangleq T_0$, and define recursively $\sigma_n \triangleq \inf\{t \geq \tau_{n-1}; W_t = a\}$, $\tau_n = \inf\{t \geq \sigma_n; W_t = 0\}$; $n = 1, 2, \ldots$. We know that $P^x[\tau_0 < \infty] = 1$, and using Theorem 6.15 we can show by induction on n that $\sigma_n - \tau_{n-1}$ is the passage time of the standard Brownian motion $W_{\cdot + \tau_{n-1}} - W_{\tau_{n-1}}$ to a, $\tau_n - \sigma_n$ is the passage time of the standard Brownian motion $W_{\cdot + \sigma_n} - W_{\sigma_n}$ to $-a$, and the sequence of differences $\sigma_1 - \tau_0$, $\tau_1 - \sigma_1$, $\sigma_2 - \tau_1$, $\tau_2 - \sigma_2$, $\ldots$ consists of independent and identically distributed random variables with Laplace transform $e^{-a\sqrt{2\alpha}}$ (cf. (8.11)). It follows that $\tau_n - \tau_0$, being the sum of $2n$ such differences, has Laplace transform $e^{-2na\sqrt{2\alpha}}$, and so

$$P^x[\tau_n - \tau_0 \leq t] = P^0[T_{2na} \leq t].$$

We have then

(8.13) $\lim\limits_{n \to \infty} P^x[\tau_n \leq t] = 0; \quad 0 \leq t < \infty.$

For any $y \in (0, \infty)$, we have from Corollary 6.17 and the symmetry of Brownian motion that

$$P^x[W_t \geq y \mid \mathcal{F}_{\tau_n+}] = P^x[W_t \leq -y \mid \mathcal{F}_{\tau_n+}] \quad \text{on} \quad \{\tau_n \leq t\},$$

and so

(8.14) $P^x[W_t \geq y, \ \tau_n \leq t] = P^x[W_t \leq -y, \ \tau_n \leq t] = P^x[W_t \leq -y, \sigma_n \leq t]; \quad n \geq 0.$

Similarly, for $y \in (-\infty, a)$, we have

$$P^x[W_t \leq y \mid \mathcal{F}_{\sigma_n+}] = P^x[W_t \geq 2a - y \mid \mathcal{F}_{\sigma_n+}] \quad \text{on} \quad \{\sigma_n \leq t\},$$

and

(8.15) $P^X[W_t \le y, \ \sigma_n \le t] = P^X[W_t \ge 2a-y, \ \sigma_n \le t]$

$= P^X[W_t \ge 2a-y, \ \tau_{n-1} \le t]; \quad n \ge 1.$

We may apply (8.14) and (8.15) alternately and repeatedly to conclude that

$P^X[W_t \ge y, \ \tau_n \le t] = P^X[W_t \le -y-2na]; \quad 0 < y < a, \quad n \ge 0,$

$P^X[W_t \le y, \ \sigma_n \le t] = P^X[W_t \le y-2na]; \quad 0 < y < a, \quad n \ge 0.$

Differentiation with respect to y results in the formulas

(8.16) $P^X[W_t \in dy, \ \tau_n \le t] = p(t; \ x, -y-2na); \quad 0 < y < a, \quad n \ge 0,$

(8.17) $P^X[W_t \in dy, \ \sigma_n \le t] = p(t; \ x, y-2na); \quad 0 < y < a, \quad n \ge 0.$

Now set $\pi_0 = 0$, $\rho_0 = T_a$ and define recursively

$\pi_n = \inf\{t \ge \rho_{n-1}; \ W_t = 0\}, \ \rho_n = \inf\{t \ge \pi_n; \ W_t = a\}; \quad n = 1, 2, \dots .$

We may proceed as above to obtain the formulas

(8.18) $\lim_{n \to \infty} P^X[\rho_n \le t] = 0; \quad 0 \le t < \infty,$

(8.19) $P^X[W_t \in dy, \ \rho_n \le t] = p(t; \ x, -y+(2n+1)a); \quad 0 < y < a, \ n \ge 0,$

(8.20) $P^X[W_t \in dy, \ \pi_n \le t] = p(t; \ x, y+2na); \quad 0 < y < a, \ n \ge 0.$

It is easily verified by considering the cases $T_0 < T_a$ and $T_0 > T_a$ that $\tau_{n-1} \vee \rho_{n-1} = \sigma_n \wedge \pi_n$ and $\sigma_n \vee \pi_n = \tau_n \wedge \rho_n$; $n \geq 1$. Consequently,

$$(8.21) \qquad P^X[W_t \in dy, \ \tau_{n-1} \wedge \rho_{n-1} \leq t] = P^X[W_t \in dy, \ \tau_{n-1} \leq t]$$

$$+ P^X[W_t \in dy, \ \rho_{n-1} \leq t] - P^X[W_t \in dy, \ \sigma_n \wedge \pi_n \leq t],$$

and

$$(8.22) \qquad P^X[W_t \in dy, \ \sigma_n \wedge \pi_n \leq t] = P^X[W_t \in dy, \ \sigma_n \leq t] + P^X[W_t \in dy, \ \pi_n \leq t]$$

$$- P^X[W_t \in dy, \ \tau_n \wedge \rho_n \leq t].$$

Successive application of (8.21) and (8.22) yields for every integer $k \geq 1$:

$$(8.23) \qquad P^X[W_t \in dy, \ \tau_0 \wedge \rho_0 \leq t] = \sum_{n=1}^{k} \{P^X[W_t \in dy, \ \tau_{n-1} \leq t]$$

$$+ P^X[W_t \in dy, \ \rho_{n-1} \leq t] - P^X[W_t \in dy, \ \sigma_n \leq t] - P^X[W_t \in dy, \ \pi_n \leq t]\}$$

$$+ P^X[W_t \in dy, \ \tau_k \wedge \rho_k \leq t].$$

Now we let k tend to infinity in (8.23); because of (8.13), (8.18) the last term converges to zero, whereas using (8.16), (8.17) and (8.19), (8.20) the remaining terms give

$$P^X[W_t \in dy, \ T_0 \wedge T_a > t] = P^X[W_t \in dy] - P^X[W_t \in dy, \ \tau_0 \wedge \rho_0 \leq t]$$

$$= \sum_{n=-\infty}^{\infty} p_-(t; x, y+2na) \, dy; \quad 0 < y < a, \ t > 0. \qquad \square$$

8.10 Exercise: Show that

$$(8.24) \quad P^X[T_0 \wedge T_a \epsilon dt] = \frac{1}{\sqrt{2\pi t^3}} \sum_{n=-\infty}^{\infty} [(2na+x) \exp\{-\frac{(2na+x)^2}{2t}\}$$

$$+ (2na+a-x) \exp\{-\frac{(2na+a-x)^2}{2t}\}]dt; \quad t>0, 0<x<a.$$

It is now tempting to guess the decomposition of (8.24):

$$(8.25) \quad P^X[T_0 \epsilon dt, T_0<T_a] = \frac{1}{\sqrt{2\pi t^3}} \sum_{n=-\infty}^{\infty} (2na+x)\exp\{-\frac{(2na+x)^2}{2t}\}dt;$$

$$t>0, \quad 0<x<a,$$

$$(8.26) \quad P^X[T_a \epsilon dt, T_a<T_0] = \frac{1}{\sqrt{2\pi t^3}} \sum_{n=-\infty}^{\infty} (2na+a-x)\exp\{-\frac{(2na+a-x)^2}{2t}\}dt;$$

$$t>0, \quad 0<x<a.$$

Indeed, one can use the identity (8.6) to compute the Laplace transforms of the right-hand sides; then (8.25), (8.26) are seen to be equivalent to

$$(8.27) \quad E^X e^{-\alpha T_0} 1_{\{T_0<T_a\}} = \frac{\sinh((a-x)\sqrt{2\alpha})}{\sinh(a\sqrt{2\alpha})} ; \quad 0<x<a, \alpha>0,$$

$$(8.28) \quad E^X e^{-\alpha T_a} 1_{\{T_a<T_0\}} = \frac{\sinh(x\sqrt{2\alpha})}{\sinh(a\sqrt{2\alpha})} ; \quad 0<x<a, \alpha>0.$$

We leave the verification of these identities as a problem.   Note that by adding (8.27) and (8.28) we obtain the transform of (8.24):

$$(8.29) \quad E^X e^{-\alpha(T_0 \wedge T_a)} = \frac{\cosh((x-\frac{a}{2})\sqrt{2\alpha})}{\cosh(\frac{a}{2}\sqrt{2\alpha})}; \quad 0<x<a, \alpha>0.$$

This provides an independent verification of (8.24).

8.11 <u>Problem</u>: Derive the formulas (8.27), (8.28) by applying the optional sampling theorem to the martingale of (8.7).

8.12 <u>Problem</u>: Show that

$$P^x[T_0 < T_a] = \frac{a-x}{a}, \quad P^x[T_a < T_0] = \frac{x}{a}; \quad 0 \leq x \leq a, \quad a > 0.$$

8.13 <u>Problem</u>: Show that $E^x(T_0 \wedge T_a) = x(a-x); \quad 0 \leq x \leq a.$

Proposition 8.1 coupled with the Markov property enables one to compute distributions for a wide variety of Brownian functionals. We illustrate the method by computing the joint distribution of (W_t, M_t) and the last time at which W achieves its maximum over $[0,t]$.

8.14 <u>Proposition</u>: Define

(8.30) $\quad \theta_t \triangleq \sup\{0 \leq s \leq t; \ W_s = M_t\}.$

Then

(8.31) $\quad P^0[W_t \in da, \ M_t \in db, \ \theta_t \in ds]$

$$= \frac{b(b-a)}{\pi \sqrt{s^3 (t-s)^3}} \exp\{-\frac{b^2}{2s} - \frac{(b-a)^2}{2(t-s)}\} da \ db \ ds;$$

$$a \in R, b \geq a, \ b \geq 0, \ 0 < s < t.$$

<u>Proof</u>:

For $b \geq 0, \ \epsilon > \delta > 0, \ x \geq 0, \ a \leq b$ and $0 < s < t,$ we have

2.8.11

(8.32) $P^0[b < M_s \leqslant b+\delta, \ W_s \in b-dx, \ \max_{s \leqslant u \leqslant t} W_u \leqslant b, \ W_t \in da]$

$\leqslant P^0[b < M_t \leqslant b+\delta, \ \theta_t \leqslant s, \ W_s \in b-dx, \ W_t \in da]$

$\leqslant P^0[b < M_s \leqslant b+\delta, \ W_s \in b-dx, \ \max_{s \leqslant u \leqslant t} W_u \leqslant b+\epsilon, \ W_t \in da].$

Divide by δ and let $\delta \downarrow 0$, $\epsilon \downarrow 0$ (in that order). The upper and lower bounds in the above inequalities converge to the same limit, which is

(8.33) $P^0[M_t \in db, \ \theta_t \leqslant s, \ W_s \in b-dx, \ W_t \in da]$

$= P^0[M_s \in db, \ W_s \in b-dx, \ \max_{s \leqslant u \leqslant t} W_u \leqslant b, \ W_t \in da]$

$= P^0[M_s \in db, \ W_s \in b-dx] \cdot P^{b-x}[M_{t-s} \leqslant b, \ W_{t-s} \in da]$

$$= \frac{b+x}{\pi\sqrt{s^3(t-s)^3}}[\exp\{-\frac{(x+\mu_+)^2}{2\sigma^2} - \frac{(2b-a)^2}{2t}\}$$

$$- \exp\{-\frac{(x+\mu_-)^2}{2\sigma^2} - \frac{a^2}{2t}\}dx \ da \ db,$$

where we have used (8.3) and

$$\mu_\pm \triangleq \frac{b(t-s)\pm(a-b)s}{t}, \quad \sigma^2 \triangleq \frac{s(t-s)}{t}.$$

In terms of $\Phi(z) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}} dx$ we may now evaluate the integrals

$$\int_0^\infty (b+x)\exp\{-\frac{(b\pm\mu_\pm)^2}{2\sigma^2}\}dx = \sigma^2 e^{-\frac{\mu_\pm^2}{2\sigma^2}} + \frac{s}{t}(b\pm(b-a))\sigma\sqrt{2\pi} \cdot \Phi(-\frac{\mu_\pm}{\sigma}),$$

and so integrating out x in (8.33) and using the equality

(8.34) $\quad \dfrac{\mu_{\pm}^2}{2\sigma^2} + \dfrac{(b\pm(b-a))^2}{2t} = \dfrac{b^2}{2s} + \dfrac{(b-a)^2}{2(t-s)} \; ,$

we arrive at the formula

$$P^0[M_t \epsilon db, \; \theta_t \leq s, \; W_t \epsilon da]$$

$$= \frac{2}{\sqrt{2\pi t^3}}[\Phi(-\frac{\mu_+}{\sigma})(2b-a)\exp\{-\frac{(2b-a)^2}{2t}\}$$

$$- \Phi(-\frac{\mu_-}{\sigma}) \; a \; \exp\{-\frac{a^2}{2t}\}]da \; db.$$

Note that $\dfrac{\partial}{\partial s}(-\dfrac{\mu_{\pm}}{\sigma}) = \dfrac{1}{2\sigma}(\dfrac{b}{s} \pm \dfrac{b-a}{t-s})$, and so

$$\frac{\partial}{\partial s} P^0[M_t \epsilon db, \; \theta_t \leq s, \; W_t \epsilon da]$$

$$= \frac{b(b-a)}{\pi\sqrt{s^3(t-s)^3}}\exp\{-\frac{b^2}{2s} - \frac{(b-a)^2}{2(t-s)}\}da \; db \; ds. \qquad \square$$

8.15 Remark: If we define $\hat{\theta}_t \triangleq \inf\{0\leq s\leq t; \; W_s = M_t\}$ to be the first time W attains its maximum over $[0,t]$, then (8.32) is still valid when θ_t is replaced by $\hat{\theta}_t$. Thus, θ_t and $\hat{\theta}_t$ have the same distribution, and since $\hat{\theta}_t \leq \theta_t$, we see that $P^0[\hat{\theta}_t = \theta_t] = 1$. In other words, the time at which the maximum over $[0,t]$ is attained is almost surely unique.

8.16 Problem: Show that

$$P^0[M_t \epsilon db, \; \theta_t \epsilon ds] = \frac{b}{\pi\sqrt{s^3(t-s)}} e^{-\frac{b^2}{2s}} db \; ds; \quad b\geq 0, \; 0<s<t,$$

$$P^0[\theta_t \epsilon ds] = \frac{ds}{\pi\sqrt{s(t-s)}}; \quad 0<s<t,$$

$$P^0[M_t \in db | \theta_t = s] = \frac{b}{s} e^{-\frac{b^2}{2s}} db; \quad b \geq 0, \quad 0 < s < t.$$

In particular, the conditional density of M_t given θ_t does not depend on t. We say that θ_t obeys the <u>arc-sine law</u>, since

$$P^0[\theta_t \leq s] = \frac{2}{\pi} \arcsin \sqrt{\frac{s}{t}}; \quad 0 \leq s \leq t, \quad t > 0.$$

<u>8.17 Problem</u>: Define the time of last exit from the origin before t by

$$(8.35) \quad \gamma_t \triangleq \sup\{0 \leq s \leq t; W_s = 0\}.$$

Show that γ_t obeys the arc-sine law, i.e.,

$$P^0[\gamma_t \in ds] = \frac{ds}{\pi\sqrt{s(t-s)}}; \quad 0 < s < t.$$

(<u>Hint</u>: Use Problem 8.7).

<u>8.18 Exercise</u>: With γ_t defined as in (8.35), derive the quadrivariate density

$$P^0[W_t \in da, \ M_t \in db, \ \gamma_t \in ds, \ \theta_t \in du]$$

$$= \frac{-2ab^2}{(2\pi u(s-u)(t-s))^{3/2}} \exp\left\{-\frac{ub^2}{2u(s-u)} - \frac{a^2}{2(t-s)}\right\} da \ db \ ds \ du;$$
$$0 < u < s < t, \quad a < 0 < b.$$

2.9 THE BROWNIAN SAMPLE PATHS

We present in this section a detailed discussion of the basic "absolute" properties of Brownian motion, i.e., those properties which hold with probability one (also called "sample path" properties). These include characterizations of "bad" behaviour (nondifferentiability and lack of points of increase) as well as "good" behaviour (law of the iterated logarithm and Lévy modulus of continuity) of the Brownian paths. We also study the local maxima and the zero sets of these paths. We shall see in Section 3.4 that the sample paths of any continuous martingale can be obtained by running those of a Brownian motion according to a different, path-dependent clock. Thus, this study of Brownian motion has much to say about the sample path properties of much more general classes of processes, including continuous martingales and diffusions.

We start by collecting together, in Lemma 9.4, the fundamental "equivalence transformations" of Brownian motion. These will prove handy, both in this section and throughout the book; indeed, we made frequent use of symmetry in the previous section.

9.1 Definition: A real-valued stochastic process $X = \{X_t; 0 \leq t < \infty\}$ is called Gaussian if, for any integer $k \geq 1$ and real numbers $0 \leq t_1 < t_2 < \ldots < t_k < \infty$, the random vector $(X_{t_1}, X_{t_2}, \ldots, X_{t_k})$ has a k-variate normal distribution.

If X is a Gaussian process, then its finite-dimensional distributions are determined by its expectation function

$m(t) \triangleq EX_t$; $t \geq 0$, and its covariance function

$$\rho(s,t) \triangleq E[(X_s - m(s))(X_t - m(t))]; \quad s, t \geq 0.$$

If $m(t) \equiv 0$; $t \geq 0$, we say that X is a <u>zero-mean Gaussian</u> <u>process</u>.

9.2 <u>Remark</u>: Brownian motion is a zero-mean Gaussian process with covariance function

(9.1) $\rho(s,t) = s \wedge t$; $s, t \geq 0$.

Conversely, any zero-mean Gaussian process $X = \{X_t, \mathfrak{F}_t^X; 0 \leq t < \infty\}$ with a.s. continuous paths and covariance function given by (9.1) is a Brownian motion. See Definition 1.1.

Throughout this section, $W = \{W_t, \mathfrak{F}_t; 0 \leq t < \infty\}$ is a standard, one-dimensional Brownain motion on $(\Omega, \mathfrak{F}, P)$. In particular $W_0 = 0$, a.s.P. For fixed $\omega \in \Omega$, we denote by $W.(\omega)$ the sample path $t \mapsto W_t(\omega)$.

9.3 <u>Problem</u>   (Strong law of large numbers):

Show that

(9.2) $\lim\limits_{t \to \infty} \dfrac{W_t}{t} = 0$, a.s.

(<u>Hint</u>: Recall the analogous property for the Poisson process, Remark 1.3.7´).

9.4 **Lemma:** When $W = \{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a standard Brownian motion, so are the processes obtained from the following "equivalence transformations":

(i) <u>Scaling:</u> $X = \{X_t, \mathcal{F}_{ct}; 0 \leq t < \infty\}$ defined by

$$(9.3) \qquad X_t = \frac{1}{\sqrt{c}} W_{ct}; \quad 0 \leq t < \infty,$$

where $c > 0$;

(ii) <u>Time -inversion:</u> $Y = \{Y_t, \mathcal{F}_t^Y; 0 \leq t < \infty\}$ defined by

$$(9.4) \qquad Y_t = \begin{cases} t\, W_{\frac{1}{t}} & ; \ 0 < t < \infty; \\ 0 & ; \ t = 0; \end{cases}$$

(iii) <u>Time-reversal:</u> $Z = \{Z_t; \mathcal{F}_t^Z; 0 \leq t \leq T\}$ defined by

$$(9.5) \qquad Z_t = W_T - W_{T-t}; \quad 0 \leq t \leq T, \quad \text{for every fixed } T > 0;$$

(iv) <u>Symmetry:</u> $\quad -W = \{-W_t, \mathcal{F}_t; 0 \leq t < \infty\}$.

<u>Proof:</u>

We shall discuss only part (ii), the others being either similar or completely evident. The process Y of (9.4) is easily seen to have a.s. continuous paths; continuity at the origin is a corollary of Problem 9.3. On the other hand, Y is a zero-mean, Gaussian process with covariance function

$$E(Y_s Y_t) = st(\frac{1}{s} \wedge \frac{1}{t}) = s \wedge t; \quad s, t > 0$$

and the conclusion follows from Remark 9.2. $\qquad\qquad \square$

9.5 Problem: Show that the probability that Brownian motion

returns to the origin infinitely often is one.

We take up now the study of the <u>zero set of the Brownian path</u>. Define

(9.6) $\quad Z \triangleq \{(t,\omega) \in [0,\infty) \times \Omega; \quad W_t(\omega) = 0\}$,

and for fixed $\omega \in \Omega$, define the zero set of $W.(\omega)$:

(9.7) $\quad Z_\omega \triangleq \{0 \leq t < \infty; \quad W_t(\omega) = 0\}$.

9.6 Theorem: For P-a.e. $\omega \in \Omega$, the zero set Z_ω

(i) has Lebesgue measure zero,

(ii) is closed and unbounded,

(iii) has an accumulation point at $t=0$,

(iv) has no isolated point in $(0,\infty)$, and therefore

(v) is dense in itself.

Proof:

We start by observing that the set Z of (9.6) is in $\mathcal{B}[0,\infty) \otimes \mathcal{F}$, because W is a (progressively) measurable process. By Fubini's theorem,

$$E[\text{meas}(Z_\omega)] = (\text{meas} \times P)(Z) = \int_0^\infty P[W_t=0]dt = 0,$$

and therefore $\text{meas}(Z_\omega) = 0$ for P-a.e. $\omega \in \Omega$, proving (i); here and in the sequel, "meas" means "Lebesgue measure". On the other hand, for P-a.e. $\omega \in \Omega$ the mapping $t \mapsto W_t(\omega)$ is continuous, and Z_ω is the inverse image under this mapping of the closed

set $\{0\}$. Thus, for every such ω, the set Z_ω is closed, unbounded (Problem 9.5), and has an accumulation point at the origin $t=0$ (Problem 7.17).

For (iv), let us observe that $\{\omega\in\Omega;\ Z_\omega$ has an isolated point in $(0,\infty)\}$ can be written as

$$(9.8) \qquad \bigcup_{\substack{a,b\in Q \\ 0\leq a<b<\infty}} \{\omega\in\Omega;\ \text{there is exactly one}\ s\in(a,b)\ \text{with}\ W_s(\omega)=0\}$$

where Q is the set of rationals. Let us consider the family of almost surely finite optional times (Problem 1.2.5)

$$\beta_t \triangleq \inf\{s>t;\ W_s=0\}; \quad t\geq 0.$$

According to (iii) we have $\beta_0=0$, a.s. P; moreover,

$$\beta_{\beta_t(\omega)}(\omega) = \inf\{s>\beta_t(\omega);\ W_s(\omega) = 0\}$$

$$= \beta_t(\omega) + \inf\{s>0;\ W_{s+\beta_t(\omega)}(\omega)-W_{\beta_t(\omega)}(\omega) = 0\}$$

$$= \beta_t(\omega)$$

for P-a.e. $\omega\in\Omega$, because $\{W_{s+\beta_t}-W_{\beta_t};\ 0\leq s<\infty\}$ is a standard Brownian motion (Theorem 6.15). Therefore, for $0\leq a<b<\infty$,

$$\{\omega\in\Omega;\ \text{there is exactly one}\ s\in(a,b)\ \text{with}\ W_s(\omega) = 0\}$$

$$\subseteq \{\omega\in\Omega;\ \beta_a(\omega) < b\ \text{and}\ \beta_{\beta_a(\omega)}(\omega) > b\}$$

has probability zero, and the same is then true for the union (9.8). $\square$

9.7 Remark: From Theorem 9.6 and the strong Markov property

in the form of Theorem 6.5, we see that for every fixed

$b \in R$ and P-a.e. $\omega \in \Omega$, the level set

$$Z_\omega(b) \triangleq \{0 \leq t < \infty; \ W_t(\omega) = b\}$$

is closed, unbounded, of Lebesgue measure zero and dense in

itself.   □

The following Problem strengthens the result of Theorem

1.5.8 in the special case of Brownian motion.

9.8 Problem: Let $\{\Pi_n\}_{n=1}^\infty$ be a sequence of partitions of the

interval $[0,t]$ with $\lim_{n \to \infty} \|\Pi_n\| = 0$. Then the quadratic varia-

tions

$$V_t^{(2)}(\Pi_n) \triangleq \sum_{k=1}^{m_n} |W_{t_k^{(n)}} - W_{t_k^{(n)}}|^2$$

of the Brownian motion W over these partitions converge

to t in L^2, as $n \to \infty$. If, furthermore, the partitions

become so fine that

$$\sum_{n=1}^\infty \|\Pi_n\| < \infty$$

holds, the above convergence takes place also with probability

one.   □

As discussed in section 1.5, one can easily show using

Problem 9.8 that for almost every $\omega \in \Omega$, the sample path $W.(\omega)$

is of unbounded variation on every finite interval [0,t]. In the remainder of this section we describe just how oscillatory the Brownian path is.

9.9 Theorem: For almost every $\omega \in \Omega$,   the sample path $W.(\omega)$ is
   monotone in no interval.

Proof:

   If we denote by F the set of $\omega \in \Omega$ with the property that $W.(\omega)$ is monotone in some interval, we have

$$F = \bigcup_{\substack{s,t \in Q \\ 0 \leq s < t < \infty}} \{\omega \in \Omega; \; W.(\omega) \text{ is monotone on } [s,t]\},$$

since every nonempty interval includes one with rational endpoints. Therefore, it suffices to show that on any such interval, say on [0,1], the path $W.(\omega)$ is monotone for almost no ω. By virtue of the symmetry property (iv) of Lemma 9.4, it suffices then to show that the event

$$A \triangleq \{\omega \in \Omega; \; W.(\omega) \text{ is nondecreasing on } [0,1]\}$$

is in $\mathfrak{F}$ and has probability zero. But $A = \bigcap_{n=1}^{\infty} A_n$, where

$$A_n \triangleq \bigcap_{i=0}^{n-1} \{\omega \in \Omega; \; W_{\frac{i+1}{n}}(\omega) - W_{\frac{i}{n}}(\omega) \geq 0\} \in \mathfrak{F}$$

has probability $P(A_n) = \prod_{i=1}^{n-1} P[W_{\frac{i+1}{n}} - W_{\frac{i}{n}} \geq 0] = 2^{-n}$. Thus,

$$P(A) = \lim_{n \to \infty} P(A_n) = 0. \qquad \square$$

In order to proceed with our study of the Brownian sample paths, we need a few elementary notions and results concerning real-valued functions of one variable.

9.10 Definition: Let $f: [0, \infty) \to R$ be a given function. A number $t \geq 0$ is called

(i)   a point of increase of size δ, if for given $\delta > 0$ we have

$$\max_{(t-\delta)^+ \leq s \leq t} f(s) = f(t) = \min_{t \leq s \leq t+\delta} f(s);$$

(ii)   a point of increase, if it is a point of increase of size δ for some $\delta > 0$;

(iii)   a point of local maximum, if there exists a number $\delta > 0$ with $f(s) \leq f(t)$ valid for every $s \in [(t-\delta)^+, t+\delta]$; and

(iv)   a point of strict local maximum, if there exists a number $\delta > 0$ with $f(s) < f(t)$ valid for every $s \in [(t-\delta)^+, t+\delta] \setminus \{t\}$.

9.11 Problem: Let $f: [0, \infty) \to R$ be continuous.

(i)   Show that the set of points of strict local maximum for f is countable.

(ii)   If f is monotone on no interval, then the set of points of local maximum for f is dense in $[0, \infty)$.

9.12 Theorem: For almost every $\omega \in \Omega$, the set of points of local maximum for the Brownian path $W.(\omega)$ is countable and dense in $[0, \infty)$, and all local maxima are strict.

Proof:

Thanks to Theorem 9.9 and Problem 9.11, it suffices to show that the set

$$A = \{\omega \in \Omega; \text{ every local maximum of } W.(\omega) \text{ is strict}\}$$

includes an event of probability one. Indeed, A includes the (countable) intersection of events of the type

$$(9.9) \qquad A_{t_1, \ldots, t_4} \triangleq \{\omega \in \Omega; \max_{t_3 \leq t \leq t_4} W_t(\omega) - \max_{t_1 \leq t \leq t_2} W_t(\omega) \neq 0\},$$

taken over all quadruples (t_1, t_2, t_3, t_4) of rational numbers satisfying $0 \leq t_1 < t_2 < t_3 < t_4 < \infty$. Therefore, it remains to prove that for every such quadruple, the event in (9.9) has probability one. But the difference of the two random variables in (9.9) can be written as

$$(W_{t_3} - W_{t_2}) + \min_{t_1 \leq t \leq t_2} [W_{t_2}(\omega) - W_t(\omega)] + \max_{t_3 \leq t \leq t_4} [W_t(\omega) - W_{t_3}(\omega)],$$

and the three terms appearing in this sum are independent. Consequently,

$$P[A_{t_1, \ldots, t_4}] = \int_0^\infty \int_{-\infty}^0 P[W_{t_2} - W_{t_3} \neq x+y] \, P[\min_{t_1 \leq t \leq t_2} (W_{t_2} - W_t) \in dx]$$

$$\cdot P[\max_{t_3 \leq t \leq t_4} (W_t - W_{t_3}) \in dy] = 1$$

because $\quad P[W_{t_3} - W_{t_2} \neq x+y] = 1.$ $\qquad\qquad$ □

Let us now discuss the question of occurrence of points of increase on the Brownian path. We start by observing that the set

$$\Lambda = \{(t,\omega) \in [0,\infty) \times \Omega; \ t \ \text{is a point of increase of } W_.(\omega)\}$$

is product measurable: $\Lambda \in \mathcal{B}[0,\infty) \otimes \mathcal{F}$. Indeed, Λ can be written as the countable union $\Lambda = \bigcup_{m=1}^{\infty} \Lambda(m)$, with

$$\Lambda(m) \triangleq \{(t,\omega) \in [0,\infty) \times \Omega; \ \max_{(t-\frac{1}{m})^+ \leq s \leq t} W_s(\omega)$$

$$= W_t(\omega) = \min_{t \leq s \leq t+\frac{1}{m}} W_s(\omega)\},$$

and each $\Lambda(m)$ is in $\mathcal{B}[0,\infty) \otimes \mathcal{F}$. We denote the sections of Λ by

$$\Lambda_t \triangleq \{\omega \in \Omega; \ (t,\omega) \in \Lambda\}, \quad \Lambda_\omega \triangleq \{t \in [0,\infty); \ (t,\omega) \in \Lambda\},$$

and $\Lambda_t(m)$, $\Lambda_\omega(m)$ have a similar meaning. For $0 \leq t < \infty$,

$$P[\Lambda_t(m)] \leq P[W_{s+t} - W_t \geq 0; \ \forall s \in [0, \tfrac{1}{m}]] = 0$$

because $\{W_{s+t} - W_t; \ s \geq 0\}$ is a standard Brownian motion (Problem 7.17); now $\Lambda_t = \bigcup_{m=1}^{\infty} \Lambda_t(m)$ gives also

(9.10) $\quad P(\Lambda_t) = 0; \ 0 \leq t < \infty$

as well as

$$\int_{\Omega} \text{meas}(\Lambda_{\omega})dP = (\text{meas} \times P)(\Lambda) = \int_{0}^{\infty} P(\Lambda_t)dt = 0$$

from Fubini's theorem. It follows that $P[\omega\in\Omega; \text{meas}(\Lambda_{\omega}) = 0] = 1$. The question is whether this assertion can be strengthened to $P[\omega\in\Omega; \Lambda_{\omega} = \emptyset] = 1$, or equivalently

(9.11) $P[\omega\in\Omega;$ the path $W.(\omega)$ has no point of increase] = 1.

That the answer to this question turns out to be _affirmative_ is perhaps one of the most surprising aspects of Brownian sample path behaviour. We state this result here but defer the proof to Chapter 6.

9.13 Theorem: <u>Dvoretzky, Erdös and Kakutani (1961)</u>

Almost every Brownian sample path has no point of increase (or decrease); that is, (9.11) holds.   □

9.14 Remark: We have already seen that almost every Brownian path has a dense set of local maxima. If $T(\omega)$ is a local maximum for $W.(\omega)$, then one might imagine that by reflection (replacing $W_t(\omega)-W_{T(\omega)}(\omega)$ by $-(W_t(\omega)-W_{T(\omega)}(\omega))$ for $t\geq T(\omega)$), one could turn the point $T(\omega)$ into a point of increase for a new Brownian motion. Such an approach was used successfully at the beginning of Section 2.6 to derive the passage time distribution. Here, however, it fails completely. Of course, the results of Section 2.6 are inappropriate in this context because $T(\omega)$ is <u>not</u> a stopping time. Even if the filtration $\{\mathcal{F}_t\}$ is right-

continuous, so that $\{\omega \in \Omega; \; W.(\omega)$ has a local maximum at $t\}$ is in $\mathcal{F}_t$ for each $t \geq 0$, it is not possible to define a stopping time T for $\{\mathcal{F}_t\}$ such that $W.(\omega)$ has local maximum at $T(\omega)$ for all ω in some event of positive probability. In other words, one cannot specify in a "proper way" which of the numerous times of local maximum is to be selected. Indeed, if it were possible to do this, Theorem 9.13 would be violated.

9.15 Remark: It is quite possible that, for each fixed $t \geq 0$, a certain property holds almost surely, but then it fails to hold for all $t \geq 0$ simultaneously on an event whose probability is one (or even positive!). As an extreme and rather trivial example, consider that $P[\omega \in \Omega; \; W_t(\omega) \neq 1]=1$ holds for every $0 \leq t < \infty$, while

$P[\omega \in \Omega; \; W_t(\omega) \neq 1$, for every $t \in [0, \infty)] = 0$. The point here is that in order to pass from the consideration of fixed but arbitrary t to the consideration of all t simultaneously, it is usually necessary to reduce the latter consideration to that of a countable number of coördinates. This is precisely the problem which must be overcome in the passage from (9.10) to (9.11), and the proof of Theorem 9.13 in Dvoretzky, Erdös and Kakutani (1961) is demanding because of the difficulty of reducing the property of "being a point of increase" for all $t \geq 0$ to a description involving only countably many coördinates. We choose to give a completely different proof of Theorem 9.13 in Chapter 6 based on

the concept of local time. We do, however, illustrate the abovementioned technique by taking up a less demanding question, the nondifferentiability of the Brownian path.

9.16 Definition: For a continuous function $f: [0, \infty) \to R$,   we denote by

$$(9.12) \qquad D^{\pm} f(t) \; = \; \overline{\lim_{h \to 0\pm}} \; \frac{f(t+h) \; - \; f(t)}{h}$$

the <u>upper (right and left) Dini derivates</u> at t,   and by

$$(9.13) \qquad D_{\pm} f(t) \; = \; \underline{\lim_{h \to 0\pm}} \; \frac{f(t+h) \; - \; f(t)}{h}$$

the <u>lower (right and left) Dini derivates</u> at t. The function f is said to be <u>differentiable at t from the right</u> (respectively, <u>the left</u>), if $D^{+}f(t)$ and $D_{+}f(t)$ (respectively, $D^{-}f(t)$ and $D_{-}f(t)$) are finite numbers and equal. The function f is said to be <u>differentiable at t>0</u> if it is differentiable from both the right and the left and the four Dini derivates agree. At t=0,   differentiability is defined as differentiability from the right.

9.17 Problem: Show that

$$(9.14) \qquad P[\omega \in \Omega; \; D^{+}W_{t}(\omega) = \infty \quad \text{and} \quad D_{+}W_{t}(\omega) = -\infty] = 1; \quad 0 \leqslant t < \infty.$$

9.18 Theorem: <u>Paley, Wiener and Zygmund (1932)</u>
For almost every $\omega \in \Omega$,   the Brownian sample path W.(ω) is nowhere differentiable. More precisely, the set

(9.15)   {ω∈Ω; for each t∈[0,∞), either $D^+W_t(\omega)=\infty$ or $D_+W_t(\omega)=\infty$}

contains an event F∈𝔍 with P(F) = 1.

Remark: At every point t of local maximum for W.(ω) we have $D^+W_t \leq 0$, and at every point s of local minimum, $D_+W_s \geq 0$. Thus, the "or" in (9.15) cannot be replaced by "and".

Remark: We do not know whether the set in (9.15) belongs to $\mathcal{F}_\infty^W$.

Proof:

It is enough to consider the interval [0,1]. For fixed integers j≥1, k≥1, we define the set

(9.16) A_{jk} = {ω∈Ω; $|W_{t+h}(\omega)-W_t(\omega)| \leq jh$ for some t∈[0,1]

and all $h \in [0, \frac{1}{k}]$}.

Certainly we have

{ω∈Ω ; $-\infty < D_+W_t(\omega) \leq D^+W_t(\omega) < \infty$, for some t∈[0,1]} = $\bigcup\limits_{j=1}^{\infty} \bigcup\limits_{k=1}^{\infty} A_{jk}$,

and the proof of the theorem will be complete if we find, for each fixed j,k, an event C∈𝔍 with P(C) = 0 and $A_{jk} \subseteq C$.

Let us fix a sample path ω∈A_{jk}, i.e., suppose there exists a number t∈[0,1] with $|W_{t+h}(\omega)-W_t(\omega)| \leq jh$ for every $0 \leq h \leq \frac{1}{k}$.
Take an integer n≥4k. Then there exists an integer i, 1≤i≤n, such that $\frac{i-1}{n} \leq t \leq \frac{i}{n}$, and it is easily verified that we also have $\frac{i+\nu}{n} - t \leq \frac{\nu+1}{n} \leq \frac{1}{k}$ (ν=1,2,3). It follows that

$$|W_{\frac{i+1}{n}}(\omega)-W_{\frac{i}{n}}(\omega)| \leq |W_{\frac{i+1}{n}}(\omega)-W_t(\omega)| + |W_{\frac{i}{n}}(\omega)-W_t(\omega)| \leq \frac{2j}{n} + \frac{j}{n} = \frac{3j}{n} \ .$$

The crucial observation here is that the assumption ω∈A_{jk} provides information about the size of the Brownian increment, not only over

the interval $[\frac{i}{n}, \frac{i+1}{n}]$, but also over the neighbouring intervals $[\frac{i+1}{n}, \frac{i+2}{n}]$ and $[\frac{i+2}{n}, \frac{i+3}{n}]$. Indeed,

$$|W_{\frac{i+2}{n}}(\omega) - W_{\frac{i+1}{n}}(\omega)| \leq |W_{\frac{i+2}{n}} - W_t| + |W_{\frac{i+1}{n}} - W_t| \leq \frac{3j}{n} + \frac{2j}{n} = \frac{5j}{n},$$

$$|W_{\frac{i+3}{n}}(\omega) - W_{\frac{i+2}{n}}(\omega)| \leq |W_{\frac{i+3}{n}} - W_t| + |W_{\frac{i+2}{n}} - W_t| \leq \frac{4j}{n} + \frac{3j}{n} = \frac{7j}{n}.$$

Therefore, with

$$C_i^{(n)} \triangleq \bigcap_{\nu=1}^{3} \{\omega\in\Omega; |W_{\frac{i+\nu}{n}}(\omega) - W_{\frac{i+\nu-1}{n}}(\omega)| \leq \frac{2\nu+1}{n} j\},$$

we have observed that $A_{jk} \subseteq \bigcup_{i=1}^{n} C_i^{(n)}$ holds for every $n \geq 4k$. But now

$$\sqrt{n}(W_{\frac{i+\nu}{n}} - W_{\frac{i+\nu-1}{n}}) \triangleq Z_\nu; \quad \nu=1,2,3$$

are independent, standard normal random variables, and one can easily verify the bound $P[|Z_\nu| \leq \epsilon] \leq \epsilon$. It develops that

(9.17) $\quad P(C_i^{(n)}) \leq \frac{105\, j^3}{n^{3/2}}$; $\quad i=1,2,\ldots,n.$

We have $A_{jk} \subseteq C$ upon taking

(9.18) $\quad C \triangleq \bigcap_{n=4k}^{\infty} \bigcup_{i=1}^{n} C_i^{(n)} \in \mathcal{F},$

and (9.17) shows us that $P(C) \leq \inf_{n \geq 4k} P(\bigcup_{i=1}^{n} C_i^{(n)}) = 0.$ $\quad\square$

9.19 Problem: By modifying the above proof, establish the following stronger result: for almost every $\omega\in\Omega$, the Brownian path $W_\cdot(\omega)$ is nowhere Hölder continuous with

exponent $\gamma > \frac{1}{2}$. (Hint: By analogy with (9.16), consider

the sets

(9.19) $A_{jk} \triangleq \{\omega \in \Omega; |W_{t+h}(\omega) - W_t(\omega)| \leq jh^{\gamma}$ for some $t \in [0,1]$

                                          and all $h \in [0,\frac{1}{k}]\}$

and show that each A_{jk} is included in a P-null event).   □

Our next result is the celebrated "law of the iterated
logarithm", which describes the oscillations of Brownian motion
near $t=0$ and as $t \to \infty$. In preparation for the theorem, we
recall the following upper and lower bounds on the tail of the
normal distribution.

9.20 Problem: For every $x > 0$, we have

(9.20) $\dfrac{x}{1+x^2} e^{-\frac{x^2}{2}} \leq \displaystyle\int_x^{\infty} e^{-\frac{u^2}{2}} du \leq \dfrac{1}{x} e^{-\frac{x^2}{2}}$.

9.21 Theorem: __Law of the iterated logarithm__ (A. Hinčin (1933)).

For almost every $\omega \in \Omega$, we have

(i) $\overline{\lim}_{t \downarrow 0} \dfrac{W_t(\omega)}{\sqrt{2t \log \log \frac{1}{t}}} = 1$,   (ii) $\underline{\lim}_{t \downarrow 0} \dfrac{W_t(\omega)}{\sqrt{2t \log \log \frac{1}{t}}} = -1$,

(iii) $\overline{\lim}_{t \to \infty} \dfrac{W_t(\omega)}{\sqrt{2t \log \log t}} = 1$,   (iv) $\underline{\lim}_{t \to \infty} \dfrac{W_t(\omega)}{\sqrt{2t \log \log t}} = -1$.

Remark: By symmetry, property (ii) follows from (i), and by
time inversion, properties (iii) and (iv) follow from
(i) and (ii), respectively (cf. Lemma 9.4). Thus if suffices
to establish (i).

Proof:

The submartingale inequality (Theorem 1.3.6 (i)) applied to
the exponential martingale $\{M_t, \mathfrak{I}_t; \ 0 \leq t < \infty\}$ of (8.7) gives

(9.21) $\quad P[\max_{0 \leq s \leq t} (W_s - \frac{\lambda}{2} s) \geq \beta] = P[\max_{0 \leq s \leq t} M_s \geq e^{\lambda \beta}] \leq e^{-\lambda \beta}; \ \lambda > 0, \beta > 0.$

With the notation $h(t) \triangleq \sqrt{2t \log \log \frac{1}{t}}$ and fixed numbers θ, δ
in $(0,1)$, we choose $\lambda = (1+\delta) \theta^{-n} h(\theta^n)$, $\beta = \frac{1}{2} h(\theta^n)$, and
$t = \theta^n$ in (9.21), which becomes:

$$P[\max_{0 \leq s \leq \theta^n} (W_s - \frac{\lambda}{2} s) \geq \beta] \leq \frac{1}{(n \log \frac{1}{\theta})^{1+\delta}}; \quad n \geq 1.$$

The last expression is the general term of a convergent series;
by the Borel-Cantelli lemma, there exists an event $\Omega_{\theta\delta} \in \mathfrak{I}$ of
probability one and an integer-valued random variable $N_{\theta\delta}$, so
that for every $\omega \in \Omega_{\theta\delta}$ we have

$$\max_{0 \leq s \leq \theta^n} [W_s(\omega) - \frac{1+\delta}{2} s \theta^{-n} h(\theta^n)] < \frac{1}{2} h(\theta^n); \quad n \geq N_{\theta\delta}(\omega).$$

Thus, for every $t \in (\theta^{n+1}, \theta^n]$:

$$W_t(\omega) \leq \max_{0 \leq s \leq \theta^n} W_s \leq (1 + \frac{\delta}{2}) h(\theta^n) \leq (1 + \frac{\delta}{2}) \theta^{-\frac{1}{2}} h(t).$$

Therefore,

$$\sup_{\theta^{n+1} < t \leq \theta^n} \frac{W_t(\omega)}{h(t)} \leq (1 + \frac{\delta}{2}) \theta^{-\frac{1}{2}}; \quad n \geq N_{\theta\delta}(\omega),$$

holds for every $\omega \in \Omega_{\theta\delta}$, and letting $n \uparrow \infty$ we obtain

$$\overline{\lim_{t \downarrow 0}} \frac{W_t(\omega)}{h(t)} \le (1 + \frac{\delta}{2}) \theta^{-\frac{1}{2}} , \quad \text{a.s. P.}$$ By letting $\delta \downarrow 0$, $\theta \uparrow 1$ through the rationals, we deduce

$$(9.22) \qquad \overline{\lim_{t \downarrow 0}} \frac{W_t}{h(t)} \le 1 ; \qquad \text{a.s.P.}$$

In order to obtain an inequality in the opposite direction, we have to employ the second half of the Borel-Cantelli lemma, which relies on independence. We introduce the independent events

$$A_n = \{W_{\theta^n} - W_{\theta^{n+1}} \ge \sqrt{1-\theta}\, h(\theta^n)\}; \quad n=1,2,\ldots,$$

again for fixed $0 < \theta < 1$. Inequality (9.20) with $x = \sqrt{2 \log n + 2 \log \log \frac{1}{\theta}}$ provides lower bounds on the probabilities of these events:

$$P(A_n) = P[\frac{W_{\theta^n} - W_{\theta^{n+1}}}{\sqrt{\theta^n - \theta^{n+1}}} \ge x] \ge \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}\, (x+\frac{1}{x})} \ge \frac{\text{const.}}{n \log n} ; \quad n \ge 2.$$

Now the last expression is the general term of a divergent series, and the second half of the Borel-Cantelli lemma (Chung (1974), p. 76 or Ash (1972), p. 272) guarantees the existence of an event $\Omega_\theta \in \mathcal{F}$ with $P(\Omega_\theta) = 1$ such that, for every $\omega \in \Omega_\theta$ and $k \ge 1$, there exists an integer $m = m(k,\omega) \ge k$ with

$$(9.23) \qquad W_{\theta^m}(\omega) - W_{\theta^{m+1}}(\omega) \ge \sqrt{1-\theta}\, h(\theta^m).$$

On the other hand, (9.22) applied to the Brownian motion $-W$ shows that there exist an event $\Omega^* \in \mathcal{F}$ of probability one and

an integer-valued random variable N^*, so that for every $\omega \in \Omega^*$

$$(9.24) \qquad -W_{\theta^{n+1}}(\omega) \leq 2h(\theta^{n+1}) \leq 4\theta^{\frac{1}{2}} h(\theta^n); \quad n \geq N^*(\omega).$$

From (9.23) and (9.24) we conclude that, for every $\omega \in \Omega_\theta \cap \Omega^*$ and every integer $k \geq 1$, there exists an integer $m = m(k,\omega) \geq k \vee N^*(\omega)$ such that

$$\frac{W_{\theta^m}(\omega)}{h(\theta^m)} \geq \sqrt{1-\theta} - 4\sqrt{\theta}.$$

By letting $m \to \infty$, we conclude that $\overline{\lim}\limits_{t \downarrow 0} \dfrac{W_t}{h(t)} \geq \sqrt{1-\theta} - \sqrt{4\theta}$ holds a.s.P, and letting $\theta \downarrow 0$ through the rationals we obtain

$$\overline{\lim_{t \downarrow 0}} \frac{W_t}{h(t)} \geq 1; \quad a.s.P. \qquad\qquad \square$$

We observed in Remark 2.12 that almost every Brownian sample path is locally Hölder continuous with exponent γ for every $\gamma \in (0, \frac{1}{2})$, and we also saw in Problem 9.19 that Brownian paths are nowhere locally Hölder continuous for any exponent $\gamma > \frac{1}{2}$. The Law of the Iterated Logarithm applied to $\{W_{t+h}-W_h; 0 \leq h < \infty\}$ for fixed $t \geq 0$ gives

$$(9.25) \qquad \overline{\lim_{h \downarrow 0}} \frac{|W_{t+h}-W_t|}{\sqrt{h}} = \infty, \quad P - \text{almost surely.}$$

Thus a typical Brownian path cannot be "locally Hölder continuous with exponent $\gamma = \frac{1}{2}$" everywhere on $[0, \infty)$; however, one may not conclude from this that such a path has the abovementioned property nowhere on $[0, \infty)$; see Remark 9.15 and the Notes, section 11.

Another way to measure the oscillations of the Brownian path is to seek a modulus of continuity. A function $h(.)$ is called a _modulus of continuity for the function_ $f: [0,T] \to R$ if $0 \leq s < t \leq T$ and $|t-s| \leq \delta$ imply $|f(t)-f(s)| \leq h(\delta)$, for all sufficiently small positive δ. Because of the Law of the Iterated Logarithm, any modulus of continuity for Brownian motion on a bounded interval, say $[0,1]$, should be at least as large as $\sqrt{2\delta \log \log \frac{1}{\delta}}$, but because of the established local Hölder continuity it need not be any larger than a constant multiple of δ^γ, for any $\gamma \in (0,1/2)$. A remarkable result by P. Lévy (1937) asserts that with

$$(9.26) \qquad h(\delta) \underset{=}{\triangle} \sqrt{2\delta \log \frac{1}{\delta}} ; \quad \delta > 0,$$

$ch(\delta)$ is a modulus of continuity for almost every Brownian path on $[0,1]$ if $c > 1$, but is a modulus for almost no Brownian path on $[0,1]$ if $0 < c < 1$. We say that h in (9.26) is the _exact modulus of continuity_ of almost every Brownian path. The assertion just made is a straightforward consequence of the following theorem.

9.22 Theorem: Lévy modulus (1937)

With $h: (0,1] \to (0,\infty)$ given by (9.26), we have

$$(9.27) \qquad P[\overline{\lim_{\delta \downarrow 0}} \frac{1}{h(\delta)} \max_{\substack{0 \leq s < t \leq 1 \\ t-s \leq \delta}} |W_t - W_s| = 1] = 1.$$

Proof:

With $n \geq 1$, $0 < \theta < 1$, we have by the independence of increments and (9.20):

$$P[\max_{1\leq j\leq 2^n}|W_{\frac{j}{2^n}} - W_{\frac{j-1}{2^n}}|\leq (1-\theta)^{\frac{1}{2}} h(2^{-n})] = (1-\xi)^{2^n}\leq\exp(-\xi 2^n) ,$$

where $\xi \triangleq 2P[2^{n/2}W_{1/2^n} > (1-\theta)^{\frac{1}{2}} 2^{n/2} h(2^{-n})] \geq \dfrac{2}{\sqrt{2\pi}}\cdot\dfrac{e^{-\frac{x^2}{2}}}{x + \frac{1}{x}}$

and $x = \sqrt{(1-\theta) 2n \log 2}$. It develops easily that for $n\geq 1$ sufficiently large, we have $\xi \geq$ (const.) . $2^{-n(1-\theta)}$, and thus

$$P[\max_{1\leq j\leq 2^n}|W_{\frac{j}{2^n}} - W_{\frac{j-1}{2^n}}| \leq (1-\theta)^{\frac{1}{2}} h(2^{-n})] \leq \text{(const.)} . \exp(-2^{n\theta}).$$

By the Borel-Cantelli lemma, there exists an event $\Omega_\theta\in\mathcal{F}$ with $P(\Omega_\theta) = 1$ and an integer-valued random variables N_θ such that, for every $\omega\in\Omega_\theta$, we have

$$\frac{1}{h(2^{-n})} \max_{1\leq j\leq 2^n}|W_{\frac{j}{2^n}}(\omega)-W_{\frac{j-1}{2^n}}(\omega)| > \sqrt{1-\theta}; \quad n\geq N_\theta(\omega).$$

Consequently, we obtain

$$\overline{\lim_{\delta\downarrow 0}} \frac{1}{h(\delta)} \max_{\substack{0\leq s<t\leq 1\\ t-s\leq\delta}}|W_t-W_s| \geq \sqrt{1-\theta}$$

and by letting $\theta\downarrow 0$ along the rationals, we have

$$\overline{\lim_{\delta\downarrow 0}} \frac{1}{h(\delta)} \max_{\substack{0\leq s<t\leq 1\\ t-s\leq\delta}}|W_t-W_s| \geq 1, \quad \text{a.s. } P.$$

For the proof of the opposite inequality, which is much more demanding, we select $\theta \in (0,1)$ and $\epsilon > \frac{1+\theta}{1-\theta} - 1$, and observe the inequalities

$$(9.28) \quad P\left[\max_{\substack{1 \le i \le j \le 2^n \\ k=j-i \le 2^{n\theta}}} \frac{1}{h(k2^{-n})}\left|W_{\frac{j}{2^n}} - W_{\frac{i}{2^n}}\right| \ge 1+\epsilon\right]$$

$$\le \sum_{k=1}^{2^{n\theta}} P\left[\max_{0 \le i < i+k \le 2^n}\left|W_{\frac{k+i}{2^n}} - W_{\frac{i}{2^n}}\right| \ge (1+\epsilon)\, h\left(\frac{k}{2^n}\right)\right]$$

$$\le 2^n \sum_{k=1}^{2^{n\theta}} P\left[\frac{\left|W_{k/2^n}\right|}{\sqrt{k2^{-n}}} \ge (1+\epsilon)\sqrt{\log \frac{4^n}{2^n}}\,\right].$$

The probability in the last summand of (9.28) is bounded above, thanks to (9.20), by a constant multiple of $n^{-\frac{1}{2}}(k2^{-n})^{(1+\epsilon)^2}$,

and $\displaystyle\sum_{k=1}^{2^{n\theta}} k^{(1+\epsilon)^2} \le \int_0^{2^{n\theta}+1} x^{(1+\epsilon)^2}\, dx = \frac{(2^{n\theta}+1)^{(1+(1+\epsilon)^2)}}{1+(1+\epsilon)^2}$.

Therefore,

$$P\left[\max_{\substack{1 \le i \le j \le 2^n \\ k=j-i \le 2^{n\theta}}} \frac{\left|W_{j/2^n} - W_{i/2^n}\right|}{h\left(\frac{k}{2^n}\right)} \ge 1+\epsilon\right] \le \frac{\text{const.}}{\sqrt{n}}\, 2^{-\rho n},$$

with $\rho = (1-\theta)(1+\epsilon)^2 - (1+\theta)$, a positive constant by choice of ϵ. Again by the Borel-Cantelli lemma, we have the existence of an event $\Omega_\theta \in \mathscr{F}$ with $P(\Omega_\theta) = 1$, and of an integer-valued random variable N_θ such that

$$(9.29) \qquad \max_{\substack{1 \le i \le j \le 2^n \\ k=j-i \le 2^{n\theta}}} \frac{|W_{j/2^n}(\omega) - W_{i/2^n}(\omega)|}{h\left(\frac{k}{2^n}\right)} < 1+\epsilon; \quad n \ge N_\theta(\omega).$$

9.23 Problem: Consider the set $D = \bigcup_{n=1}^{\infty} D_n$ of dyadic rationals in $[0,1]$, with $D_n = \{k2^{-n}; k=0,1,\ldots,2^n\}$. For every $\omega \in \Omega_\theta$ and every $n \ge N_\theta(\omega)$, the inequality

$$(9.30) \qquad |W_t(\omega) - W_s(\omega)| \le (1+\epsilon) \left[2 \sum_{j=n+1}^{\infty} h(2^{-j}) + h(t-s) \right]$$

is valid for every pair (s,t) of dyadic rationals satisfying $0 < t-s < 2^{-n(1-\theta)}$.

(Hint: Proceed as in the proof of Theorem 2.8 and use the fact that $h(.)$ is strictly increasing on $(0,1]$.) $\qquad \Box$

Returning to the proof of Theorem 9.22, let us observe that if the dyadic rationals s,t in (9.30) are chosen to satisfy the stronger condition

$$(9.31) \qquad 2^{-(n+1)(1-\theta)} \le \delta \triangleq t-s < 2^{-n(1-\theta)},$$

then because

$$\sum_{j=n+1}^{\infty} h(2^{-j}) \le ch(2^{-n-1}) \le \frac{c}{\sqrt{1-\theta}} 2^{-\frac{1}{2}\theta(n+1)} h(\delta)$$

holds for an appropriate constant $c > 0$, we may conclude from (9.30) and the continuity of $W.(\omega)$ that for every $\omega \in \Omega_\theta$ and $n \ge N_\theta(\omega)$,

$$\frac{1}{h(\delta)} \max_{\substack{0 \leq s < t \leq 1 \\ t-s=\delta}} |W_t(\omega) - W_s(\omega)| \leq (1+\epsilon)[1+ \frac{2c}{1-\theta} 2^{-\frac{1}{2}\theta(n+1)}]$$

holds for all $\delta \in [2^{-(n+1)(1-\theta)}, 2^{-n(1-\theta)})$. Letting $n \to \infty$, we obtain

$$\overline{\lim_{\delta \downarrow 0}} \frac{1}{h(\delta)} \max_{\substack{0 \leq s < t \leq 1 \\ t-s=\delta}} |W_t(\omega) - W_s(\omega)| \leq 1+\epsilon ,$$

and because h is increasing, we may replace the condition $t-s=\delta$ by $t-s \leq \delta$ in the above expression. It remains only to let $\theta \downarrow 0$ (and hence simultaneously $\epsilon \downarrow 0$) along the rationals, to conclude that

$$\overline{\lim_{\delta \downarrow 0}} \frac{1}{h(\delta)} \max_{\substack{0 \leq s < t \leq 1 \\ t-s \leq \delta}} |W_t(\omega) - W_s(\omega)| \leq 1; \quad \text{a.s.P.}$$

The proof is complete. $\square$

Since $\mathfrak{F}_{T+}^X \subseteq \mathfrak{F}_{T+}^\mu = \mathfrak{F}_T^\mu = \mathfrak{F}_{S+}^\mu$, the integrand $P^\mu[X_{S+t}\epsilon\Gamma|\mathfrak{F}_{T+}^X]$ is $\mathfrak{F}_{S+}^\mu$-measurable. This justifies the first equality in (S.4). A similar justification can be given for the last equality. The second and fourth equalities are consequences of the fact that random variables which agree a.s. have the same conditional expectations. The remaining equality is (7.2), where we take account of the fact that $\{T<\infty\} \Delta \{S<\infty\}\epsilon\eta^\mu$.

7.13 Solution:

(i) Let $F_n = \{\omega\epsilon C[0,\infty);$ ω is constant on $[0, \frac{1}{n}]\}$. Since $F_n \subseteq \{\omega: B_{1/n}(\omega) = 0\}$, we have $P^0(F_n) = 0$, $\forall$ $n\geq 1$. But then $F = \bigcup_{n=1}^\infty F_n$ also has P^0-measure zero.

(ii) We have $F = \bigcup_{n=1}^\infty F_n$ for each positive integer m, so $F\epsilon\mathfrak{F}_{1/m}^B$, $\forall$ $m\geq 1$. It follows that $F\epsilon\mathfrak{F}_{0+}^B$.

(iii) If $F\epsilon\bar{\mathfrak{F}}_0$, then $F \subseteq G$ for some $G\epsilon\mathfrak{F}_0^B$ with $P^0(G) = 0$. Such a G has the form $G = \{\omega(0)\epsilon\Gamma\}$ for some $\Gamma\epsilon\mathscr{B}(\mathbb{R})$, and $P^0(G) = 0$ implies $0 \notin \Gamma$. But then the identically zero function, which is a member of F, is not in G. This contradiction shows that $F \notin \bar{\mathfrak{F}}_0$.

This example provides another solution to Problem 7.4.

PAGES 2.S.28 - 2.S.37 ARE MISSING FROM THE ORIGINAL
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now

$$C_i^{(n)} \triangleq \bigcap_{\nu=1}^{\ell} \{\omega \in \Omega;\ |W_{\frac{i+\nu}{n}}(\omega) - W_{\frac{i+\nu-1}{n}}(\omega)|\sqrt{n} \leq \frac{2j(\nu+1)^\gamma}{n^{\gamma-\frac{1}{2}}}\}$$

has probability bounded above by $[2j(\nu+1)^\gamma n^{-(\gamma-\frac{1}{2})}]^\ell$, and everything works as before provided $\ell(\gamma-\frac{1}{2}) > 1$. When $\gamma > \frac{1}{2}$, we can choose ℓ to satisfy this inequality.

9.20 Solution: An integration by parts gives

$$\int_x^\infty e^{-\frac{u^2}{2}}\, du = \frac{1}{x} e^{-\frac{x^2}{2}} - \int_x^\infty \frac{1}{u^2} e^{-\frac{u^2}{2}}\, du,$$

so

$$\frac{1}{x} e^{-\frac{x^2}{2}} = \int_x^\infty (1+\frac{1}{u^2}) e^{-\frac{u^2}{2}}\, du \leq (1+\frac{1}{x^2}) \int_x^\infty e^{-\frac{u^2}{2}}\, du.$$

The upper bound has already been observed in $(3.1)'$, and it is also implicit in the equality in the relation just above.

9.23 Solution: Certainly it suffices to show that for every $m > n \geq N_\theta(\omega)$, we have

$$(S.5) \qquad |W_t(\omega) - W_s(\omega)| \leq (1+\epsilon)[2 \sum_{j=n+1}^{m-1} h(2^{-j}) + h(t-s)]$$

valid for every $s, t \in D_m$ satisfying $0 < t-s < 2^{-n(1-\theta)}$. For $m = n+1$, $(S.5)$ follows from (9.29). Let us assume that $(S.5)$ holds for $m = n+1, \ldots, M-1$. With $s, t \in D_m$ and $0 < t-s < 2^{-n(1-\theta)}$, we consider, as in the proof of Theorem 2.8, the numbers

$t^1 = \max\{u \in D_{M-1}; u \leq t\}$ and $s^1 = \min\{u \in D_{M-1}; u \geq s\}$ and observe the relations $t - t^1 \leq 2^{-M}$, $s^1 - s \leq 2^{-M}$ and $0 \leq t^1 - s^1 \leq t - s < 2^{-n(1-\theta)}$. We have

$$|W_{t^1}(\omega) - W_{s^1}(\omega)| \leq (1+\epsilon)[2 \sum_{j=n+1}^{M-2} h(2^{-j}) + h(t^1 - s^1)]$$

by the induction assumption, and $|W_t(\omega) - W_{t^1}(\omega)| \leq (1+\epsilon)h(2^{-M})$ as well as $|W_s(\omega) - W_{s^1}(\omega)| \leq (1+\epsilon)h(2^{-M})$ because of (9.29). Since $h(t^1 - s^1) \leq h(t-s)$, we conclude that (S.5) holds with $m = M$.

2.11: NOTES

Section 2.1: The first quantitative work on Brownian motion is

due to Bachelier (1900), who was interested in stock price

fluctuations. Einstein (1905) derived the transition density

for Brownian motion from the molecular-kinetic theory of

heat. A rigorous mathematical treatment of Brownian motion

began with N. Wiener (1923, 1924), who provided the first

existence proof. The most profound work in this early period

is that of P. Lévy (1939, 1948); he introduced the construc-

tion by interpolation expounded in Section 2.3, studied in

detail the passage times and other related functionals

(Section 2.8), described in detail the so-called "fine

structure" of the typical sample path (Section 2.9), and

discovered the notion and properties of the "mesure du

voisinage" or "local time" (Section 3.6 and Chapter 6).

Most amazingly, he carried out this program without the

formal concepts and tools of filtrations, stopping times, or

the strong Markov property.

Section 2.2: The construction of a probability measure from a

consistent family of finite-dimensional distributions is

clearly explained in Kolmogorov (1933); Daniell (1918-19)

had constructed earlier an integral on a space of sequences.

The existence of a continuous modification under the con-

ditions of Theorem 2.8 was established by Kolmogorov

(published in Slutsky (1937)); Loève ((1960), p. 519) noticed that the same argument also provides local Hölder continuity with exponent γ for any $0<\gamma<\frac{\beta}{\alpha}$. For related results, see also Čentsov (1956.a). The extension to random fields as in Problem 2.9 was carried out by Čentsov (1956.b).

Section 2.3:   The Haar function construction of Brownian motion was originally carried out by P. Lévy (1948) and later simplified by Ciesielski (1961).

Section 2.4   is adapted from Billingsley (1968). The original proof of Theorem 4.17 is in Donsker (1951), but the one offered here is essentially due to Prohorov (1956).

Sections 2.5, 2.6:   The "Markov property" derives its name from A.A. Markov, whose own work (1906) was in discrete time and state space; in that context, of course, the "usual" and the "strong" Markov properties coincide. It was not immediately realized that the latter is actually stronger than the former; Ray ((1956), pp. 463-464) provides an example of a continuous Markov process which is not strongly Markov. It is rather amazing that a complete and rigorous statement about the strongly Markovian character of Brownian motion (Theorem 6.15) was proved only in 1956; see Hunt (1956).

A Markov family for which the function $x \mapsto E^x f(X_t)$ is continuous for any bounded, continuous $f: R^d \to R$ and

$t \in [0, \infty)$ is said to have the <u>Feller property</u>, and a right-continuous Markov family with the Feller property is strongly Markovian. Very readable introductions to Markov process theory can be found in Dynkin & Yushkevich (1969), Wentzell ((1982), Chapters 8-13) and Chung (1982), whilst more comprehensive treatments are those by Dynkin (1965) and Blumenthal & Getoor (1968). Markov processes with continuous sample paths receive very detailed treatments in the monographs by Itô & McKean (1974), Stroock & Varadhan (1979) and Knight (1981).

<u>Sections 2.8, 2.9</u>: The material here comes mostly from P. Lévy (1939, 1948). Section 1.4 in D. Freedman (1971) can be consulted for further information on the subject matter of Theorems 9.6, 9.9 and 9.12. Our discussion of the law of the iterated logarithm follows McKean (1969) and Williams (1979). Theorem 9.18 was strengthened by Dvoretzky (1963), who showed that there exists a universal constant $c > 0$ such that

$$P[\omega \in \Omega; \varliminf_{h \downarrow 0} \frac{|W_{t+h}(\omega) - W_t(\omega)|}{\sqrt{h}} \geq c, \ \forall \ t \in [0, \infty)] = 1.$$

For every $\omega \in \Omega$, $\mathbf{S}_\omega \triangleq \{t \in [0, \infty); \varliminf_{h \downarrow 0} \frac{|W_{t+h}(\omega) - W_t(\omega)|}{\sqrt{h}} < \infty\}$ has been called by Kahane (1976) <u>the set of slow points from the right for the path</u> $W_\cdot(\omega)$. Fubini's theorem applied to (9.25) shows that $\text{meas}(\mathbf{S}_\omega) = 0$ for P - a.e. $\omega \in \Omega$ but, for a typical path, $\mathbf{S}_\omega$ is far from being empty; in fact,

we have

$$P[\omega \in \Omega: \inf_{0 \le t < \infty} \varlimsup_{h \downarrow 0} \frac{|W_{t+h}(\omega) - W_t(\omega)|}{\sqrt{h}} = 1] = 1.$$

This is proved in B.Davis (1983), where we refer the interested reader for more information and references on this subject.

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CHAPTER 3

TABLE OF CONTENTS

3.1 INTRODUCTION

A tremendous range of problems in the natural, social and biological sciences came under the dominion of the theory of functions of a real variable when Newton and Leibnitz invented the calculus. The primary components of this invention were the use of differentation to describe rates of change, the use of integration to pass to the limit in approximating sums, and the fundamental theorem of calculus, which relates the two concepts and thereby makes the latter amenable to computation. All of this gave rise to the concept of ordinary differential equations, and it is the application of these equations to the modelling of real-world phenomena which reveals much of the power of calculus.

Stochastic calculus grew out of the need to assign meaning to ordinary differential equations involving continuous stochastic processes. Since the most important continuous process, Brownian motion, cannot be differentiated, stochastic calculus takes the track opposite to that of classical calculus: the stochastic integral is defined first, and then the stochastic differential is given meaning through the fundamental "theorem" of calculus. This "theorem" is really a definition in stochastic calculus, because the differential has no meaning apart from that assigned to it when it enters an integral. For this theory to achieve its full potential, it must have some simple rules for computation. These are contained in the change of variable formula (Itô's rule), which is the counterpart of the chain rule from classical calculus.

Stochastic calculus has an important additional feature not found in its classical counterpart, a feature based on the change of measure theorem of Girsanov. This result provides a device for solving stochastic differential equations driven by Brownian motion by changing the underlying probability measure, so that the process which was the driving Brownian motion becomes, under the new probability measure, the solution to the differential equation. This profound idea is first presented in Section 5, but it does not reach its culmination until the discussion of weak solutions of stochastic differential equations in Chapter 4. In some cases, this device is merely a convenient way of finding out the distribution of an already existent solution of a stochastic differential equation; in other cases it provides us with a proof of the existence of a solution when the more standard existence proofs fail. Although "optional" in the sense that stochastic calculus can (and did for 25 years) exist and be useful without it, the Girsanov theorem today plays such a central rôle in further developments of the subject that the reader would be remiss not to come to acquire a thorough understanding of this admittedly difficult concept. We make extensive use of it in Chapter 5.

We take up applications of the stochastic integral to problems of optimal stopping, optimal control, and filtering in Chapter 7.

3.2.1

3.2: CONSTRUCTION OF THE STOCHASTIC INTEGRAL

Let us consider a continuous, square-integrable martingale
$M = \{M_t, \mathfrak{F}_t; 0 \leq t < \infty\}$ on a probability space $(\Omega, \mathfrak{F}, P)$ equipped with
the filtration $\{\mathfrak{F}_t\}$, which will be assumed throughout this chapter
to satisfy the usual conditions of Definition 1.3.10. We have
shown in Section 2.7 how to obtain such a filtration for standard
Brownian motion. We assume $M_0 = 0$ a.s. P. Such a process $M \in \mathcal{M}_2^c$
is of unbounded variation on any finite interval $[0,T]$ (c.f.
Problems 1.5.9, 1.5.10 and the discussion following them), and
consequently integrals of the form

$$(2.1) \qquad I_T(X) = \int_0^T X_t(\omega) \; dM_t(\omega)$$

cannot be defined "pathwise" (i.e., for each $\omega \in \Omega$ separately)
as ordinary Lebesgue-Stieltjes integrals. Nevertheless, the
martingale M has a finite second (or quadratic) variation, given
by the continuous, increasing process $\langle M \rangle$; c.f. Theorem 1.5.6.
It is precisely this fact that allows one to proceed, in a highly
nontrivial yet straightforward manner, with the construction of the
stochastic integral (2.1) with respect to the continuous, square-
integrable martingale M, for an appropriate class of integrands
X. The construction is due to Itô [1942] for the special case
$M = W = $ Brownian motion, and to Kunita & Watanabe [1967] for general
$M \in \mathcal{M}_2^c$. We shall first confine ourselves to $M \in \mathcal{M}_2^c$, and denote by
$\langle M \rangle$ the unique (up to indistinguishability) adapted, continuous

and increasing process, such that $\{M_t^2 - \langle M \rangle_t, \; \mathfrak{F}_t; \; 0 \leq t < \infty\}$ is a martingale (c.f. Definition 1.5.3 and Theorem 1.5.11). The construction will then be extended to general continuous, local martingales M.

We now consider what kinds of integrands are appropriate for (2.1). We first define a measure μ_M on $([0, \infty) \times \Omega, \; \mathfrak{B}[0, \infty) \otimes \mathfrak{F})$ by setting

$$(2.1)' \qquad \mu_M(A) = E \int_0^\infty 1_A(t, \omega) \; d\langle M_t \rangle(\omega).$$

We will say that two measurable, adapted processes $X = \{X_t, \mathfrak{F}_t; \; 0 \leq t < \infty\}$ and $Y = \{Y_t, \mathfrak{F}_t; \; 0 \leq t < \infty\}$ are <u>equivalent</u> if

$$X_t(\omega) = Y_t(\omega); \quad \mu_M - \text{a.e.} \; (t, \omega).$$

This defines an equivalence relation. For a measurable, $\{\mathfrak{F}_t\}$ - adapted process X, we define

$$[X]_T^2 \triangleq E \int_0^T X_t^2 \; d\langle M \rangle_t,$$

provided that the right-hand side is finite. Then $[X]_T$ is the L^2-norm for X, regarded as a function of (t, ω) restricted to the space $[0, T] \times \Omega$, under the measure μ_M. We have $[X-Y]_T = 0$ for all $T > 0$ if and only if X and Y are equivalent. The stochastic integral will be defined in such a manner that $I(X)$ and $I(Y)$ will be indistinguishable:

$$P[I_T(X) = I_T(Y), \qquad \forall \ T \geq 0] = 1,$$

wherever X and Y are equivalent.

<u>2.1 Definition</u>:   Let $\mathcal{L}$ denote the set of equivalence classes of all measurable, $\{\mathcal{F}_t\}$ - adapted processes X, for which $[X]_T < \infty$ for all $T > 0$. We define a metric on $\mathcal{L}$ by $[X-Y]$, where

$$[X] \underset{=}{\triangle} \sum_{n=1}^{\infty} 2^{-n}(1 \wedge [X]_n).$$

Let $\mathcal{L}^*$ denote the set of equivalence classes of progressively measurable processes satisfying $[X]_T < \infty$ for all $T > 0$, and define a metric of $\mathcal{L}^*$ in the same way.

We shall follow the usual custom of not being too careful about the distinction between equivalence classes and the processes which are members of those equivalence classes. For example, we will have no qualms about saying "$\mathcal{L}^*$ consists of those processes in $\mathcal{L}$ which are progressively measurable".

Note that $\mathcal{L}$ (respectively, $\mathcal{L}^*$) contains all measurable, $\{\mathcal{F}_t\}$ - adapted (respectively, progressively measurable) processes. Both $\mathcal{L}$ and $\mathcal{L}^*$ depend on the martingale $M = \{M_t, \mathcal{F}_t; \ t \geq 0\}$. When we wish to indicate this dependence explicitly, we write $\mathcal{L}(M)$ and $\mathcal{L}^*(M)$.

When $\langle M \rangle_t(\omega)$ is an absolutely continuous function of t for P - a.e. ω, one is able to construct $\int_0^T X_t \, dM_t$ for all $X \in \mathcal{L}$ and all $T \geq 0$. In the absence of this condition on $\langle M \rangle$, we shall construct the stochastic integral for X in the slightly smaller class $\mathcal{L}^*$. In order to define the stochastic integral with respect to general martingales in $\mathcal{m}_2$ (possibly discontinuous, such as the compensated Poisson process), one has to select an even narrower class of integrands among the so-called <u>predictable</u> processes. This notion is a slight extension of left-continuity of the sample paths of the process; since we do not develop stochastic integration with respect to discontinuous martingales, we shall forego further discussion and send the interested reader to the literature (Kunita & Watanabe [1967], Liptser & Shiryayev [1977] Ikeda & Watanabe [1981], Elliott [1982], Chung & Williams [1983]).

Later in this section, we weaken the conditions that $M \in \mathcal{m}_2^c$ and $[X]_T^2 < \infty$, $\forall\, T \geq 0$, replacing them by $M \in \mathcal{m}^{c, loc}$ and

$$P[\int_0^T X_t^2 \, d\langle M \rangle_t < \infty] = 1, \quad \forall\, T \geq 0.$$

This is accomplished by localization.

We pause in our development of the stochastic integral to prove a lemma we will need in Section 4. For $0 < T < \infty$, let $\mathcal{L}_T^*$ denote the class of processes X in $\mathcal{L}^*$ for which $X_t(\omega) = 0$; $\forall\, t > T$, $\omega \in \Omega$. For $T = \infty$, $\mathcal{L}_T^*$ is defined as the class of processes $X \in \mathcal{L}^*$ for which $E \int_0^T X_t^2 \, d\langle M \rangle_t < \infty$ (a condition we already have for $T < \infty$, by virtue of membership in $\mathcal{L}^*$). A process $X \in \mathcal{L}_T^*$ can

be identified with one defined only for $t \in [0,T]$, $\omega \in \Omega$,

and so we can regard $\mathfrak{L}_T^*$ as a subspace of the Hilbert space

(2.1)'' $\quad \mathfrak{H}_T \triangleq \mathfrak{L}_2([0,T] \times \Omega, \mathfrak{B}[0,T] \otimes \mathfrak{F}_T, \mu_M).$

Here and below we replace $[0,T]$ by $[0,\infty)$ when $T=\infty$.

2.1' Lemma: For $0 < T \leqslant \infty$, $\mathfrak{L}_T^*$ is a closed subspace of $\mathfrak{H}_T$. In

particular, $\mathfrak{L}_T^*$ is complete under the norm

$$[X]_T = [E \int_0^T X_t^2 \, d\langle M \rangle_t]^{1/2}.$$

Proof:

Let $\{X^{(n)}\}_{n=1}^\infty$ be a convergent sequence in $\mathfrak{L}_T^*$ with limit

$X \in \mathfrak{H}_T$. We may extract a subsequence, also called $\{X^{(n)}\}_{n=1}^\infty$, for

which

$$\mu_M\{(t,\omega) \in [0,T] \times \Omega; \ \lim_{n \to \infty} X_t^{(n)}(\omega) = X_t(\omega)\} = 1.$$

By virtue of its membership in $\mathfrak{H}_T$, X is $\mathfrak{B}[0,T] \otimes \mathfrak{F}_T$ - measurable,

but it may not be progressively measurable. However, with

$$A \triangleq \{(t,\omega) \in [0,T] \times \Omega; \ \lim_{n \to \infty} X_t^{(n)}(\omega) \text{ exits in } \mathbb{R}\},$$

the process

$$Y_t(\omega) \triangleq \begin{cases} \lim_{n \to \infty} X_t^{(n)}(\omega); & (t,\omega) \in A \\ \\ 0 & ; \ (t,\omega) \notin A \end{cases}$$

is progressively measurable, belongs to $\mathfrak{L}_T^*$ and $\lim_{n \to \infty}[X^{(n)} - Y]_T = 0.$ $\square$

2.2 Definition: A process X is called <u>simple</u> if there exists

a strictly increasing sequence of real numbers $\{t_n\}_{n=0}^\infty$ with

$t_0 = 0$ and $\lim_{n \to \infty} t_n = \infty$, as well as a sequence of random variables $\{\xi\}_{n=1}^{\infty}$ with $\sup_{n \geq 0} |\xi_n(\omega)| \leq C < \infty$, for every $\omega \in \Omega$, such that ξ_n is $\mathfrak{F}_{t_n}$ - measurable for every $n \geq 0$ and

$$X_t(\omega) = \xi_0(\omega) 1_{\{0\}}(t) + \sum_{i=0}^{\infty} \xi_i(\omega) 1_{(t_i, t_{i+1}]}(t); \quad 0 \leq t < \infty, \ \omega \in \Omega.$$

The class of all simple processes will be denoted by $\mathcal{L}_0$. Note that, because members of $\mathcal{L}_0$ are progressively measurable and bounded, we have $\mathcal{L}_0 \subseteq \mathcal{L}^*(M) \subseteq \mathcal{L}(M)$. □

Our program for the construction of the stochastic integral (2.1) can now be outlined as follows: the integral is defined in the obvious way for $X \in \mathcal{L}_0$ as a <u>martingale transform</u>:

$$I_t(X) \triangleq \sum_{i=0}^{n-1} \xi_i(M_{t_{i+1}} - M_{t_i}) + \xi_n(M_t - M_{t_n})$$

(2.2)

$$= \sum_{i=0}^{\infty} \xi_i(M_{t \wedge t_{i+1}} - M_{t \wedge t_i}), \quad 0 \leq t < \infty,$$

where $n \geq 0$ is the unique integer for which $t_n \leq t < t_{n+1}$, and its properties are studied. The definition is then extended to integrands $X \in \mathcal{L}^*$ and $X \in \mathcal{L}$, thanks to the crucial results which show that elements of $\mathcal{L}^*$ and $\mathcal{L}$ can be approximated, in a suitable sense, by simple processes (Propositions 2.5 and 2.7).

<u>2.3 Lemma</u>: Let X be a bounded, measurable, $\{\mathfrak{F}_t\}$ - adapted process. Then there exists a sequence $\{X^{(m)}\}_{m=1}^{\infty}$ of simple processes such that

(2.3) $\quad \sup_{T > 0} \lim_{m \to \infty} E \int_0^T |X_t^{(m)} - X_t|^2 \, dt = 0.$

Proof:

We shall show how to construct, for each fixed $T > 0$, a sequence $\{X^{(m)}\}_{m=1}^{\infty}$ of bounded, simple processes so that

$$\lim_{n \to \infty} E \int_0^T |X_t^{(n,T)} - X_t|^2 \, dt = 0.$$

Thus, for each positive integer m, there is another integer n_m such that

$$E \int_0^m |X_t^{(n_m, m)} - X_t|^2 \, dt \leq \frac{1}{m},$$

and the sequence $\{X^{(n_m, m)}\}_{m=1}^{\infty}$ has the desired properties. Henceforth, T is a fixed, positive number.

We proceed in three steps.

(a) Suppose that X is <u>continuous</u>; then the sequence of simple processes

$$X_t^{(n)}(\omega) \triangleq X_0(\omega) 1_{\{0\}}(t) + \sum_{k=0}^{2^n - 1} X_{\frac{k}{2^n} T}(\omega) 1_{(\frac{k}{2^n} T, \frac{k+1}{2^n} T]}(t); \quad n \geq 1,$$

satisfies $\lim_{n \to \infty} E \int_0^T |X_t^{(n)} - X_t|^2 \, dt = 0$ by the bounded convergence theorem.

(b) Now suppose that X is <u>progressively measurable</u>; we consider the continuous, progressively measurable processes

$$(2.4) \qquad F_t(\omega) \triangleq \int_0^{t \wedge T} X_s(\omega)\,ds; \quad \widetilde{X}_t^{(m)} \triangleq m[F_t(\omega) - F_{(t-\frac{1}{m})+}(\omega)]; \quad m \geq 1,$$

for $t \geq 0$, $\omega \in \Omega$ (c.f. Problem 1.2.18). By virtue of step (a) above, there exists, for each $m \geq 1$, a sequence of simple processes $\{\widetilde{X}^{(m,n)}\}_{n=1}^{\infty}$ such that $\lim_{n \to \infty} E \int_0^T |\widetilde{X}_t^{(m,n)} - \widetilde{X}_t^{(m)}|^2 = 0$. Let us consider the $\mathcal{B}[0,T] \otimes \mathcal{F}_T$ - measurable product set

$$A \triangleq \{(t,\omega) \in [0,T] \times \Omega; \quad \lim_{m \to \infty} \widetilde{X}_t^m(\omega) = X_t(\omega)\}.$$

For each $\omega \in \Omega$, the cross-section

$$A_\omega \triangleq \{t \in [0,T]; \ (t,\omega) \in A\}$$

is $\mathcal{B}[0,T]$ - measurable and, according to the fundamental theorem of calculus, has Lebesgue measure zero. The bounded convergence theorem now gives $\lim_{m \to \infty} E \int_0^T |\widetilde{X}_t^{(m)} - X_t|^2\,dt = 0$, and so a sequence $\{\widetilde{X}^{(m,n_m)}\}$ of bounded, simple processes can be chosen for which

$$\lim_{m \to \infty} E \int_0^T |\widetilde{X}^{(m,n_m)} - X_t|^2\,dt = 0.$$

(c) Finally, let X be <u>measurable and adapted.</u> We cannot
guarantee immediately that the continuous process
$F = \{F_t;\ 0 \leqslant t < \infty\}$ in (2.4) is progressively measurable, because
we do not know whether it is adapted. We do know, however,
that the process X has a progressively measurable modification
Y (Proposition 1.1.12), and we now show that the progressively
measurable process $\{G_t \triangleq \int_0^{t \wedge T} Y_s ds,\ \mathcal{F}_t;\ 0 \leqslant t \leqslant T\}$ is a modifica-
tion of F.

Let λ denote Lebesgue measure. For the measurable
process $\eta_t(\omega) = 1_{\{X_t(\omega)\ \neq\ Y_t(\omega)\}};\ 0 \leqslant t \leqslant T,\ \omega \in \Omega,$ we have from
Fubini: $E\int_0^T \eta_t(\omega)dt = \int_0^T P[X_t(\omega) \neq Y_t(\omega)]dt = 0.$ Therefore,
$\int_0^T \eta_t(\omega)dt = 0$ for P - a.e. $\omega \in \Omega.$ Now $\{F_t \neq G_t\}$ is con-
tained in the event $\{\omega;\ \int_0^T \eta_t(\omega)dt > 0\},$ G_t is $\mathcal{F}_t$ - measur-
able, and, by assumption, $\mathcal{F}_t$ contains all subsets of P -
null events. Therefore, F_t is also $\mathcal{F}_t$ - measurable.
Adaptivity and continuity imply progressive measurability,
and we may now repeat verbatim the argument in (b).   □

2.4 <u>Problem</u>:   This problem outlines a method by which the use
of Proposition 1.1.12, a result not proved in this text, can
be avoided in part (c) of the proof of Lemma 2.3. Let X
be a bounded, measurable, $\{\mathcal{F}_t\}$ - adapted process. Let $0 < T < \infty$
be fixed. We wish to construct a sequence $\{X^{(k)}\}_{k=1}^{\infty}$ of
simple processes so that

(2.5) $\lim\limits_{k \to \infty} E \int_0^T |X_t^{(k)} - X_t|^2 \, dt = 0.$

To simplify notation, we set $X_t = 0$ for $t \le 0$. Let $\varphi_n : R \to \{j2^{-n}; j = \pm 1, \pm 2, \dots\}$ be given by

$$\varphi_n(t) = \frac{j-1}{2^n}, \quad \text{for} \quad \frac{j-1}{2^n} < t \le \frac{j}{2^n} .$$

(a)   Fix $s \ge 0$. Show that $t - \frac{1}{2^n} \le \varphi_n(t-s) + s < t$, and that

$$X_t^{(n,s)} = X_{\varphi_n(t-s)+s}, \quad \mathfrak{F}_t; \quad t \ge 0$$

is a simple, adapted process.

(b)   Show that $\lim\limits_{h \downarrow 0} E \int_0^T |X_t - X_{t-h}|^2 \, dt = 0.$

(c)   Use (a) and (b) to show that

$$\lim\limits_{n \to \infty} E \int_0^T \int_0^1 |X_t^{(n,s)} - X_t|^2 \, ds \, dt = 0.$$

(d)   Show that for some choice of $s \ge 0$ and some increasing sequence $\{n_k\}_{k=1}^\infty$ of integers, (2.5) holds with $X^{(k)} = X^{(n_k, s)}$.

This argument is adapted from Liptser & Shiryayev [1977].

2.5 Proposition: If the function $t \mapsto \langle M \rangle_t(\omega)$ is absolutely continuous for P - a.e. $\omega \in \Omega$, then $\mathcal{L}_o$ is dense in $\mathcal{L}$ with respect to the metric of Definition 2.1.

Proof:

If $X \in \mathcal{L}$ is bounded, then Lemma 2.3 guarantees the existence of a bounded sequence $\{X^{(m)}\}$ of simple processes satisfying (2.3). From these we extract a subsequence $\{X^{(m_k)}\}$, such that the set

$$\{(t,\omega) \in [0,\infty) \times \Omega; \lim_{k \to \infty} X_t^{(m_k)}(\omega) \neq X_t(\omega)\}$$

has $\lambda \times P$ - measure zero. The absolute continuity of $t \mapsto \langle M \rangle_t(\omega)$ and the bounded convergence theorem now imply $[X^{(m_k)} - X] \to 0$ as $k \to \infty$.

If $X \in \mathcal{L}$ is not necessarily bounded, we define

$$X_t^{(n)}(\omega) \triangleq X_t(\omega) 1_{\{|X_t(\omega)| \leq n\}}; \quad 0 \leq t < \infty, \quad \omega \in \Omega,$$

and thereby obtain a sequence of bounded processes in $\mathcal{L}$. The dominated convergence theorem implies

$$[X^{(n)} - X]_T^2 = E \int_0^T X_t^2 \, 1_{\{|X_t| > n\}} \, d\langle M \rangle_t \xrightarrow[n \to \infty]{} 0$$

for every $T > 0$, whence $\lim_{n \to \infty} [X^{(n)} - X] = 0$. Each $X^{(n)}$ can be approximated by bounded, simple processes, so X can be as well. $\square$

When $t \mapsto \langle M \rangle_t$ is not an absolutely continuous function of the time variable t, we simply choose a more convenient clock. We show how to do this in slightly greater generality than needed for the present application.

2.6 Lemma: Let $\{A_t; 0 \leq t < \infty\}$ be a continuous, increasing

(Definition 1.4.4) process adapted to the filtration of the

martingale $M = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$. If $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$

is a progressively measurable process satisfying

$$E\int_0^T X_t^2 \, dA_t < \infty$$

for each $T > 0$, then there exists a sequence $\{X^{(n)}\}_{n=1}^\infty$ of

simple processes such that

$$\sup_{T>0} \lim_{n \to \infty} E\int_0^T |X_t^{(n)} - X_t|^2 \, dA_t = 0.$$

Proof:

We may assume without loss of generality that X is bounded

(c.f. second paragraph in the proof of Proposition 2.5), i.e.,

(2.6) $|X_t(\omega)| \leq C < \infty; \quad t \geq 0, \ \omega \in \Omega.$

As in the proof of Lemma 2.3, it suffices to show how to construct,

for each fixed $T > 0$, a sequence $\{X^{(n)}\}_{n=1}^\infty$ of simple processes

for which

$$\lim_{n \to \infty} E\int_0^T |X_t^{(n)} - X_t|^2 \, dA_t = 0.$$

Henceforth, $T > 0$ is fixed, and we assume without loss of generality

that

(2.7) $X_t(\omega) = 0; \quad \forall \ t > T, \ \omega \in \Omega.$

We now describe the time change. Since $A_t(\omega) + t$ is strictly increasing in $t \geq 0$ for P - a.e. ω, there is a continuous, strictly increasing inverse function $T_s(\omega)$, defined for $s \geq 0$, such that

$$A_{T_s(\omega)}(\omega) + T_s(\omega) = s; \qquad \forall \ s \geq 0.$$

In particular, $T_s \leq s$ and $\{T_s \leq t\} = \{A_t + t \geq s\} \in \mathcal{F}_t$. Thus, for each $s \geq 0$, T_s is a bounded stopping time for $\{\mathcal{F}_t\}$. Taking s as our new time variable, we define a new filtration $\{\mathscr{S}_s\}$ by

$$\mathscr{S}_s = \mathcal{F}_{T_s}; \quad s \geq 0,$$

and the time-changed process

$$Y_s(\omega) = X_{T_s(\omega)}(\omega); \quad s \geq 0, \ \omega \in \Omega$$

which is adapted to $\{\mathscr{S}_s\}$, because of the progressive measurability of X (Proposition 1.2.17). Lemma 2.3 implies that, given any $\epsilon > 0$ and $R > 0$, there is a simple process $\{Y_s^\epsilon, \mathscr{S}_s; \ 0 \leq s < \infty\}$ for which

(2.8) $E \int_0^R |Y_s^\epsilon - Y_s|^2 \, ds < \epsilon/2.$

But from (2.6), (2.7) it develops that

$$E\int_0^\infty Y_s^2 \, ds = E\int_0^\infty 1_{\{T_s \leq T\}} X_{T_s}^2 \, ds$$

$$= E\int_0^{A_T+T} X_{T_s}^2 \, ds \leq C^2(EA_T+T) < \infty,$$

so by choosing R in (2.8) sufficiently large and setting $Y_s^\epsilon = 0$ for $s > R$, we can obtain

$$E\int_0^\infty |Y_s^\epsilon - Y_s|^2 \, ds < \epsilon.$$

Now Y_s^ϵ is simple, and because it vanishes for $s > R$, there is a finite partition $0 = s_0 < s_1 < \ldots < s_n \leq R$ with

$$Y_s^\epsilon(\omega) = \xi_0(\omega)1_{\{0\}}(s) + \sum_{j=1}^n \xi_{s_{j-1}}(\omega)1_{(s_{j-1}, s_j]}(s), \quad 0 \leq s < \infty,$$

where each ξ_{s_j} is measurable with respect to $\mathscr{G}_{s_j} = \mathfrak{F}_{T_{s_j}}$ and bounded in absolute value by a constant, say K. Reverting to the original clock, we observe that

$$X_t^\epsilon \triangleq Y_{t+A_t}^\epsilon = \xi_0 1_{\{0\}}(t) + \sum_{j=1}^n \xi_{T_{s_{j-1}}} 1_{(T_{s_{j-1}}, T_{s_j}]}(t), \quad 0 \leq t < \infty,$$

is measurable and adapted, because $\xi_{T_{s_{j-1}}}$ restricted to $\{T_{s_j} \leq t\}$ is $\mathfrak{F}_t$-measurable (Lemma 1.2.14). We have

$$E\int_0^T |X_t^\epsilon - X_t|^2 \, dA_t \leq E\int_0^T |X_t^\epsilon - X_t|^2 \, (dA_t + dt)$$

$$\leq E\int_0^\infty |Y_s^\epsilon - Y_s|^2 \, ds < \epsilon.$$

The proof is not yet complete because X^ϵ is not a simple process. To finish it off, we must show how to approximate

$$\eta_t(\omega) \triangleq \xi_{T_{s_{j-1}}}(\omega) \, 1_{(T_{s_{j-1}}(\omega), T_{s_j}(\omega)]}(t); \quad 0 \leq t < \infty, \quad \omega \in \Omega,$$

by simple processes. Recall that $T_{s_{j-1}} \leq T_{s_j} \leq s_j$ and simplify notation by taking $s_{j-1} = 1$, $s_j = 2$.

Set

$$T_i^{(m)}(\omega) = \sum_{k=1}^{2^{m+1}} \frac{k}{2^m} 1_{[\frac{k-1}{2^m}, \frac{k}{2^m})}(T_i(\omega)), \quad i = 1, 2$$

and define

$$\eta_t^{(m)}(\omega) \triangleq \xi_{T_1}(\omega) \, 1_{(T_1^{(m)}(\omega), T_2^{(m)}(\omega)]}(t) =$$

$$= \sum_{k=1}^{2^{m+1}} \xi_{T_1}(\omega) 1_{\{T_1 < \frac{k-1}{2^m} \leq T_2\}}(\omega) 1_{(\frac{k-1}{2^m}, \frac{k}{2^m}]}(t).$$

Because $\{T_1 < \frac{k-1}{2^m} \leq T_2\} \in \mathfrak{I}_{\frac{k-1}{2^m}}$ and ξ_{T_1} restricted to $\{T_1 < \frac{k-1}{2^m}\}$ is $\mathfrak{I}_{\frac{k-1}{2^m}}$ - measurable, $\eta^{(m)}$ is simple.

Furthermore,

$$E \int_0^\infty |\eta_t^{(m)} - \eta_t|^2 \, dA_t \leq K^2 [E(A_{T_2^{(m)}} - A_{T_2}) + E(A_{T_1^{(m)}} - A_{T_1})] \underset{m \to \infty}{\to} 0.$$

$\square$

<u>2.7 Proposition</u>: The set $\mathcal{L}_0$ of simple processes is dense

in $\mathcal{L}^*$ with respect to the metric of Definition 2.1.

<u>Proof</u>:

Take $A_t = \langle M \rangle_t$ in Lemma 2.6. $\qquad \square$

We have already defined the stochastic integral of a simple

process $X \in \mathcal{L}_0$ by the recipe (2.2). Let us list certain properties

of this integral: for $X, Y \in \mathcal{L}_0$ and $0 \leq s < t < \infty$, we have

(2.9) $\qquad I_0(X) = 0, \quad$ a.s. P

(2.10) $\qquad E[I_t(X) | \mathcal{F}_s] = I_s(X), \quad$ a.s. P

(2.11) $\qquad E(I_t(X))^2 = E\int_0^t X_u^2 \, d\langle M \rangle_u$

(2.12) $\qquad \| I(X) \| = [X]$

(2.13) $\qquad E[(I_t(X) - I_s(X))^2 | \mathcal{F}_s] = E[\int_s^t X_u^2 \, d\langle M \rangle_u | \mathcal{F}_s], \quad$ a.s. P

(2.14) $\qquad I(\alpha X + \beta Y) = \alpha I(X) + \beta I(Y); \quad \alpha, \beta \in R.$

Properties (2.9) and (2.14) are obvious. Property (2.10) follows

from the fact that for any $0 \leq s < t < \infty$ and any integer $i \geq 1$, we

have, in the notation of (2.2),

$$E[\xi_i(M_{t \wedge t_{i+1}} - M_{t \wedge t_i}) | \mathfrak{F}_s] = \xi_i(M_{s \wedge t_{i+1}} - M_{s \wedge t_i}), \quad \text{a.s. } P;$$

this can be verified separately for each of the three cases $s \leq t_i$, $t_i < s \leq t_{i+1}$ and $t_{i+1} < s$ by using the $\mathfrak{F}_{t_i}$-measurability of ξ_i. Thus, we see that $I(X) = \{I_t(X), \mathfrak{F}_t; 0 \leq t < \infty\}$ is a <u>continuous martingale</u>. With $0 \leq s < t < \infty$ and m and n chosen so that $t_{m-1} \leq s < t_m$ and $t_n \leq t < t_{n+1}$, we have (c.f. the discussion preceding Lemma 1.5.9)

$$(2.15) \quad E[(I_t(X) - I_s(X))^2 | \mathfrak{F}_s]$$

$$= E[\{\xi_{m-1}(M_{t_m} - M_s) + \sum_{i=m}^{n-1} \xi_i(M_{t_{i+1}} - M_{t_i}) + \xi_n(M_t - M_{t_n})\}^2 | \mathfrak{F}_s]$$

$$= E[\xi_{m-1}^2(M_{t_m} - M_s)^2 + \sum_{i=m}^{n-1} \xi_i^2(M_{t_{i+1}} - M_{t_i}) + \xi_n^2(M_t - M_{t_n})^2 | \mathfrak{F}_s]$$

$$= E[\xi_{m-1}^2(\langle M \rangle_{t_m} - \langle M \rangle_s) + \sum_{i=m}^{n-1} \xi_i^2(\langle M \rangle_{t_{i+1}} - \langle M \rangle_{t_i})$$

$$\qquad + \xi_n^2(\langle M \rangle_t - \langle M \rangle_{t_n}) | \mathfrak{F}_s]$$

$$= E[\int_s^t X_u^2 \, d\langle M \rangle_u | \mathfrak{F}_s].$$

This proves (2.13), and establishes the fact that the continuous martingale $I(X)$ is <u>square-integrable</u>: $I(X) \in \mathfrak{M}_2^c$, with quadratic variation

$$(2.16) \qquad <I(X)>_t = \int_0^t X_u^2 \, d<M>_u.$$

Setting $s=0$ and taking expectations in (2.13), we obtain (2.11), and (2.12) follows immediately, upon recalling Definition 1.5.18.

For $X \in \mathcal{L}^*$, Proposition 2.7 implies the existence of a sequence $\{X^{(n)}\}_{n=1}^{\infty} \subseteq \mathcal{L}_0$ such that $[X^{(n)} - X] \to 0$ as $n \to \infty$. It follows from (2.12) and (2.14) that

$$\|I(X^{(n)}) - I(X^{(m)})\| = \|I(X^{(n)} - X^{(m)})\| = [X^{(n)} - X^{(m)}] \to 0$$

as $n, m \to \infty$. In other words, $\{I(X^{(n)})\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{m}_2^c$. By Proposition 1.5.19, there exists a process $I(X) = \{I_t(X); 0 \le t < \infty\}$ in $\mathcal{m}_2^c$, defined modulo indistinguishability, such that $\|I(X^{(n)}) - I(X)\| \to 0$ as $n \to \infty$. Because it belongs to $\mathcal{m}_2^c$, $I(X)$ enjoys properties (2.9) and (2.10). For $0 \le s < t < \infty$, $\{I_s(X^{(n)})\}_{n=1}^{\infty}$ and $\{I_t(X^{(n)})\}_{n=1}^{\infty}$ converge in mean-square to $I_s(X)$ and $I_t(X)$, respectively; so for $A \in \mathcal{F}_s$, (2.13) applied to $\{X^{(n)}\}_{n=1}^{\infty}$ gives

$$(2.17) \qquad E[1_A (I_t(X) - I_s(X))^2] = \lim_{n \to \infty} E[1_A (I_t(X^{(n)}) - I_s(X^{(n)}))^2]$$

$$= \lim_{n \to \infty} E[1_A \int_s^t (X_u^{(n)})^2 \, d<M>_u] = E[1_A \int_s^t X_u^2 \, d<M>_u],$$

where the last equality follows from $\lim_{n \to \infty} [X^{(n)} - X]_t = 0$. This proves that $I(X)$ also satisfies (2.3) and, consequently, (2.11) and (2.12). Because X and M are progressively measurable,

$\int_s^t X_u^2 \, d\langle M\rangle_u$ is $\mathfrak{F}_t$-measurable for fixed $0\leq s<t<\infty$, and so (2.13) gives us (2.16). The validity of (2.14) for $X, Y \in \mathcal{L}^*$ also follows from its validity for processes in $\mathcal{L}_0$ upon passage to the limit.

The process $I(X)$ for $X \in \mathcal{L}^*$ is well-defined; if we have two sequences $\{X^{(n)}\}_{n=1}^{\infty}$ and $\{Y^{(n)}\}_{n=1}^{\infty}$ in $\mathcal{L}_0$ with the property $\lim_{n\to\infty} [X^{(n)}-X] = 0$, $\lim_{n\to\infty} [Y^{(n)}-X] = 0$, we can construct a third sequence $\{Z^{(n)}\}_{n=1}^{\infty}$ with this property, by setting $Z^{(2n-1)} = X^{(n)}$ and $Z^{(2n)} = Y^{(n)}$, for $n\geq 1$. The limit $I(X)$ of the sequence $\{I(Z^{(n)})\}_{n=1}^{\infty}$ in $\mathfrak{m}_2^c$ has to agree with the limits of both sequences, namely $\{I(X^{(n)})\}_{n=1}^{\infty}$ and $\{I(Y^{(n)})\}_{n=1}^{\infty}$.

2.8 Definition: For $X \in \mathcal{L}^*$, the <u>stochastic integral of X with</u> <u>respect to the martingale</u> $M \in \mathfrak{m}_2^c$ is the unique, square-integrable martingale $I(X) = \{I_t(X), \mathfrak{F}_t; 0\leq t<\infty\}$ which satisfies $\lim_{n\to\infty} \|I(X^{(n)}) - I(X)\| = 0$, for every sequence $\{X^{(n)}\}_{n=1}^{\infty} \subseteq \mathcal{L}_0$ with $\lim_{n\to\infty} [X^{(n)} - X] = 0$. We write

$$I_t(X) = \int_0^t X_s \, dM_s, \qquad t\geq 0.$$

2.9 Proposition: For $M \in \mathfrak{m}_2^c$ and $X \in \mathcal{L}^*$, the stochastic integral $I(X) = \{I_t(X), \mathfrak{F}_t; 0\leq t<\infty\}$ of X with respect to M satisfies (2.9) - (2.13), as well as (2.14) for every $Y \in \mathcal{L}^*$, and has quadratic variation process given by (2.16). Furthermore, for any two stopping times $S\leq T$ of the filtration $\{\mathfrak{F}_t\}$ and any number $t>0$, we have

(2.18) $E[I_{t\wedge T}(X)|\mathfrak{F}_S] = I_{t\wedge S}(X)$,   a.s. P.

With $X, Y \in \mathfrak{L}^*$, we have, a.s.P:

(2.19) $E[(I_{t\wedge T}(X) - I_{t\wedge S}(X))(I_{t\wedge T}(Y) - I_{t\wedge S}(Y))|\mathfrak{F}_S] =$

$$= E[\int_{t\wedge S}^{t\wedge T} X_u Y_u \, d\langle M\rangle_u |\mathfrak{F}_S],$$

and in particular, for any number s in $[0,t]$,

(2.20) $E[(I_t(X) - I_S(X))(I_t(Y) - I_S(Y))|\mathfrak{F}_S] = E[\int_s^t X_u Y_u \, d\langle M\rangle_u |\mathfrak{F}_S].$

Finally,

(2.21) $I_{t\wedge T}(X) = I_t(\widetilde{X})$ a.s.,

where $\widetilde{X}_t(\omega) \triangleq X_t(\omega) \, 1_{\{t \le T(\omega)\}}.$

Proof:

We have already proved (2.9) - (2.14) and (2.16). From (2.10) and the Optional Sampling Theorem (Problem 1.3.22(ii)), we obtain (2.18). The same result applied to the martingale $\{I_t^2(X) - \int_0^t X_u^2 \, d\langle M\rangle_u, \mathfrak{F}_t; t \ge 0\}$ provides the identities

$$E[(I_{t\wedge T}(X) - I_{t\wedge S}(X))^2|\mathfrak{F}_S] = E[I_{t\wedge T}^2(X) - I_{t\wedge S}^2(X)|\mathfrak{F}_S]$$

$$= E[\int_{t\wedge S}^{t\wedge T} X_u^2 \, d\langle M\rangle_u |\mathfrak{F}_S],$$ valid P-a.s.

Replacing X in this equation, first by $X + Y$ and then by $X - Y$ and subtracting the resulting equations, we obtain (2.19).

It remains to prove (2.21). We write

$$I_{t \wedge T}(X) - I_t(\widetilde{X}) = I_{t \wedge T}(X - \widetilde{X}) - [I_t(\widetilde{X}) - I_{t \wedge T}(\widetilde{X})].$$

Both $\{I_{t \wedge T}(X - \widetilde{X}), \mathfrak{F}_t; t \geq 0\}$ and $\{I_t(\widetilde{X}) - I_{t \wedge T}(\widetilde{X}), \mathfrak{F}_t; t \geq 0\}$ are in $\mathfrak{m}_2^c$; we show that they both have quadratic variation zero, and then appeal to Problem 1.5.12. Now relation (2.19) gives, for the first process,

$$E[(I_{t \wedge T}(X - \widetilde{X}) - I_{s \wedge T}(X - \widetilde{X}))^2 | \mathfrak{F}_s] = E[\int_{s \wedge T}^{t \wedge T} (X_u - \widetilde{X}_u)^2 \, d\langle M \rangle_u | \mathfrak{F}_s] = 0$$

a.s. P, and for the second:

$$E[(I_t(\widetilde{X}) - I_{t \wedge T}(\widetilde{X}))^2] = E[\int_{t \wedge T}^{t} \widetilde{X}_u^2 \, d\langle M \rangle_u] = 0.$$

Since this is the expectation of the quadratic variation of this process, we have the desired result. $\qquad\square$

2.10 <u>Remark</u>: If the sample paths $t \to \langle M \rangle_t(\omega)$ of the quadratic variation process $\langle M \rangle$ are absolutely continuous functions of t for P-a.e. ω, then Proposition 2.5 can be used in place of Proposition 2.7 to define $I(X)$ for every $X \in \mathfrak{L}$. We have $I(X) \in \mathfrak{m}_2^c$ and all the properties of Proposition 2.9 in this case. The only sticking point in the above arguments

under **these** conditions is the proof that the measurable process $F_t \triangleq \int_0^t X_s^2 \, d\langle M \rangle_s$ is $\{\mathfrak{F}_t\}$-adapted. To see that it is, we can choose Y, a progressively measurable modification of X (Proposition 1.1.12), and define the progressively measurable process $G_t \triangleq \int_0^t Y_s^2 \, d\langle M \rangle_s$. Following the proof of Lemma 2.3(c), we can then show that $P[F_t = G_t] = 1$ holds for every $t \geq 0$. Because G_t is $\mathfrak{F}_t$-measurable, and $\mathfrak{F}_t$ contains all P-negligible events in $\mathfrak{F}$ (the usual conditions!), F is easily seen to be adapted to $\{\mathfrak{F}_t\}$ and continuous, hence progressively measurable.

In the important case that M is standard Brownian motion with $\langle M \rangle_t = t$, the use of the unproved Proposition 1.1.12 can again be avoided. Problem 2.4 shows how to construct a sequence $\{X^{(k)}\}_{k=1}^\infty$ of bounded, simple processes so that (2.5) holds; in particular, for λ-almost every $t \in [0,T]$,

$$F_t \triangleq \int_0^t X_s^2 \, ds = \lim_{k \to \infty} \int_0^t (X_s^{(k)})^2 \, ds, \qquad \text{a.s. P.}$$

Since the right-hand side is $\mathfrak{F}_t$-measurable and $\mathfrak{F}_t$ contains all null events in $\mathfrak{F}$, the left-hand side is also $\mathfrak{F}_t$-measurable for λ-a.e.t. The continuity of the sample paths of $\{F_t; t \geq 0\}$ leads to the conclusion that this process is adapted to $\{\mathfrak{F}_t\}$.

We shall not continue to deal explicitly with the case of absolutely continuous $\langle M \rangle_t$ and $X \in \mathcal{L}$, but all results obtained for $X \in \mathcal{L}^*$ can be modified in the obvious way to account for this case. In later applications involving stochastic integrals with

respect to martingales whose quadratic variations are absolutely continuous, we shall require only measurability and adaptivity rather than progressive measurability of integrands. □

2.11 Problem: Let $W = \{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a standard, one-dimensional Brownian motion, and let T be a stopping time of $\{\mathcal{F}_t\}$ with $ET < \infty$. Prove the "Wald identities"

$$E(W_T) = 0, \quad E(W_T^2) = ET.$$

Warning: The Optional Sampling Theorem cannot be applied directly because W does not have a last element and T may not be bounded. The stopping time $t \wedge T$ is bounded for fixed $0 \leq t < \infty$, so $E(W_{t \wedge T}) = 0$, $E(W_{t \wedge T}^2) = E(t \wedge T)$, but it is not a priori evident that

(2.22) $$\lim_{t \to \infty} E(W_{t \wedge T}) = EW_T, \quad \lim_{t \to \infty} E(W_{t \wedge T}^2) = E(W_T^2).$$

2.12 Problem: Let W be as in Problem 2.11, let b be a real number, and let T_b be the first passage time to b (Definition 2.6.1). Use Problem 2.11 to show that for $b \neq 0$, we have $ET_b = \infty$. □

Suppose $M = \{M_t, \mathcal{F}_t; 0 \leq t < \infty\}$ and $N = \{N_t, \mathcal{F}_t; 0 \leq t < \infty\}$ are in $\mathcal{M}_2^c$, and take $X \in \mathcal{L}^*(M)$, $Y \in \mathcal{L}^*(N)$. Then $I_t^M(X) \triangleq \int_0^t X_s \, dM_s$,

$I_t^N(Y) \triangleq \int_0^t Y_s \, dM_s$ are also in $\mathcal{m}_2^c$ and, according to (2.16),

$$\langle I^M(X) \rangle_t = \int_0^t X_u^2 \, d\langle M \rangle_u, \quad \langle I^N(Y) \rangle_t = \int_0^t Y_u^2 \, d\langle N \rangle_u.$$

We now derive the cross variation formula

(2.23) $\quad \langle I^M(X), I^N(Y) \rangle_t = \int_0^t X_u Y_u \, d\langle M, N \rangle_u; \quad t \geq 0, \; P\text{-a.s.}$

If X and Y are simple, then it is straightforward to show by a computation similar to (2.15) that for $0 \leq s < t < \infty$,

(2.24) $\quad E[(I_t^M(X) - I_s^M(X))(I_t^N(Y) - I_s^N(Y)) | \mathcal{F}_s]$

$$= E[\int_s^t X_u Y_u \, d\langle M, N \rangle_u | \mathcal{F}_s]; \qquad P\text{-a.s.}$$

This is equivalent to (2.23). It remains to extend this result from simple processes to the case of $X \in \mathcal{L}^*(M)$, $Y \in \mathcal{L}^*(N)$. We carry out this extension in several stages.

2.13 Lemma: If $M, N \in \mathcal{m}_2^c$, $X \in \mathcal{L}^*(M)$ and $\{X^{(n)}\}_{n=1}^\infty \subseteq \mathcal{L}_0$ is such that for some $T > 0$,

$$\lim_{n \to \infty} \int_0^T |X_t^{(n)} - X_t|^2 \, d\langle M \rangle_t = 0; \qquad \text{a.s. } P,$$

then

$$\lim_{n \to \infty} \langle I(X^{(n)}), N \rangle_t = \langle I(X), N \rangle_t; \quad 0 \leq t < T, \qquad \text{a.s. } P.$$

Proof:

Problem 1.5.7 (iii) implies for $0 \leq t < T$,

$$|< I(X^{(n)}) - I(X), N>_t|^2 \leq < I(X^{(n)} - X)>_t <N>_t$$

$$\leq \int_0^T |X_u^{(n)} - X_u|^2 \, d<M>_u \cdot <N>_T.$$

$\square$

2.14 Lemma: If $M, N \in \mathcal{m}_2^C$ and $X \in \mathcal{f}^*(M)$, then

(2.25) $<I^M(X), N>_t = \int_0^t X_u \, d<M, N>_u; \quad t \geq 0,$ a.s. P.

Proof:

We consider first the case of bounded X. Let V_t be the total variation of $<M, N>$ on $[0, t]$. According to Lemma 2.6, there exists a bounded sequence $\{X^{(n)}\}_{n=1}^{\infty}$ of simple processes such that with $A_t \triangleq <M>_t + V_t$,

$$\sup_{T > 0} \lim_{n \to \infty} E \int_0^t |X_u^{(n)} - X_u|^2 \, dA_u = 0.$$

Consequently, for each $T > 0$, a subsequence $\{\tilde{X}^{(n)}\}_{n=1}^{\infty}$ can be extracted, for which

$$\lim_{n \to \infty} \int_0^T |\tilde{X}_u^{(n)} - X_u|^2 \, dA_u = 0, \quad \text{a.s. P.}$$

From (2.23) with $Y = 1$, we have

$$\langle I^M(\tilde{X}^{(n)}), \ N \rangle_t = \int_0^t \tilde{X}^{(n)} \, d\langle M, N \rangle_u; \quad t \geq 0, \qquad \text{a.s. } P,$$

and letting $n \to \infty$ we obtain (2.25) from Lemma 2.13 and the bounded convergence theorem.

If X is unbounded but nonnegative, we let

$$Y_t^{(n)}(\omega) = X_t(\omega) \wedge n; \qquad 0 \leq t < \infty, \qquad \omega \in \Omega.$$

We have just proved (2.25) when X is replaced by the bounded process $Y^{(n)}$, and we can now let $n \to \infty$, using Lemma 2.13 and the monotone convergence theorem to obtain (2.25) for X. Finally, for general $X \in \mathcal{L}^*(M)$, we consider separately $X_t^+(\omega) \triangleq X_t(\omega) \vee 0$ and $X_t^-(\omega) = (-X_t(\omega)) \vee 0$. $\qquad \square$

2.15 Proposition: If $M, N \in \mathcal{M}_2^c$, $X \in \mathcal{L}^*(M)$ and $Y \in \mathcal{L}^*(N)$, then the equivalent formulae (2.23) and (2.24) hold.

Proof:

Lemma 2.14 states that $d\langle M, I^N(Y) \rangle_u = Y_u \, d\langle M, N \rangle_u$. Replacing N in (2.25) by $I^N(Y)$, we have

$$\langle I^M(X), \ I^N(Y) \rangle_t = \int_0^t X_u \, d\langle M, I^N(Y) \rangle_u$$

$$= \int_0^t X_u Y_u \, d\langle M, N \rangle_u; \quad t \geq 0, \quad P - \text{a.s.} \qquad \square$$

2.16 Problem: Show that if $M, N \in \mathcal{M}_2^c$, $X \in \mathcal{L}^*(M)$ and $Y \in \mathcal{L}^*(N)$,
    then

$$\left| \int_0^t X_u Y_u \, d\langle M, N \rangle_u \right|^2 \leq \int_0^t X_u^2 \, d\langle M \rangle_u \cdot \int_0^t Y_u^2 \, d\langle N \rangle_u, \quad 0 \leq t < \infty; \quad P\text{-a.s.}$$

2.16* Problem: Let $M = \{M_t, \mathcal{F}_t; \; 0 \leq t < \infty\}$ and $N_t = \{N_t, \mathcal{F}_t; \; 0 \leq t < \infty\}$
    be in $\mathcal{M}_2^c$ and suppose $X \in \mathcal{L}_\infty^*(M)$, $Y \in \mathcal{L}_\infty^*(N)$.   Then the martin-
    gales $I^M(X)$, $I^N(Y)$ are uniformly integrable and so have last
    elements $I_\infty^M(X)$, $I_\infty^N(Y)$,   the cross variation $\langle I^M(X), I^N(Y) \rangle_t$
    converges almost surely as $t \to \infty$,   and

$$E[I_\infty^M(X) \, I_\infty^N(Y)] = E\langle I^M(X), \; I^N(Y) \rangle_\infty$$

$$= E\int_0^\infty X_t Y_t \, d\langle M, N \rangle_t.$$

In particular,

$$E\left(\int_0^\infty X_t \, dM_t \right)^2 = E\int_0^\infty X_t^2 \, d\langle M \rangle_t.$$

2.17 Proposition: Consider a martingale $M \in \mathcal{M}_2^c$ and a process
    $X \in \mathcal{L}^*(M)$. The stochastic integral $I^M(X)$ is the unique
    martingale $\Phi \in \mathcal{M}_2^c$ which satisfies

(2.26) $\langle \Phi, N \rangle_t = \int_0^t X_u \, d\langle M, N \rangle_u; \quad 0 \leq t < \infty, \qquad$ a.s. P,

    for every $N \in \mathcal{M}_2^c$.

Proof:

We already know from (2.25) that $\Phi = I^M(X)$ satisfies (2.26). For uniqueness, suppose Φ satisfies (2.26) for every $N \in \mathcal{m}_2^c$. Subtracting (2.25) from (2.26), we have

$$<\Phi - I^M(X), \ N>_t = 0; \qquad 0 \leq t < \infty, \qquad \text{a.s. } P.$$

Setting $N = \Phi - I^M(X)$, we see that the martingale $\Phi - I^M(X)$ has quadratic variation zero, so $\Phi = I^M(X)$.

$\square$

Proposition 2.17 characterizes the stochastic integral $I^M(X)$ in terms of the more familiar Lebesgue-Stieltjes integral appearing on the right-hand side of (2.26). Such a characterization is extremely useful, as the following corollaries illustrate.

2.18 Corollary: Suppose $M \in \mathcal{M}_2^c$, $X \in \mathcal{L}^*(M)$ and $N \triangleq I^M(X)$.
Suppose further that $Y \in \mathcal{L}^*(N)$. Then $XY \in \mathcal{L}^*(M)$ and
$I^N(Y) = I^M(XY)$.

Proof:

Because $\langle N \rangle_t = \int_0^t X_s^2 \, d\langle M \rangle_s$, we have

$$E \int_0^T X_t^2 Y_t^2 \, d\langle M \rangle_t = E \int_0^T Y_t^2 \, d\langle N \rangle_t < \infty$$

for all $T > 0$, so $XY \in \mathcal{L}^*(M)$. For any $\widetilde{N} \in \mathcal{M}_2^c$, (2.23) gives

$$d\langle N, \widetilde{N} \rangle_s = X_s \, d\langle M, \widetilde{N} \rangle_s,$$

so

$$\langle I^M(XY), \widetilde{N} \rangle_t = \int_0^t X_s Y_s \, d\langle M, \widetilde{N} \rangle_s$$

$$= \int_0^t Y_s \, d\langle N, \widetilde{N} \rangle_s = \langle I^N(Y), \widetilde{N} \rangle_t.$$

According to Proposition 2.17, $I^M(XY) = I^N(Y)$.

□

2.19 Corollary: Suppose $M, \widetilde{M} \in \mathcal{M}_2^c$, $X \in \mathcal{L}^*(M)$ and $\widetilde{X} \in \mathcal{L}^*(\widetilde{M})$,
and there exists a stopping time T of the common filtration
for these processes, such that for P - almost every ω,

$$X_t^{(\omega)} = \widetilde{X}_t^{(\omega)}, \quad M_t^{(\omega)} = \widetilde{M}_t^{(\omega)}; \quad 0 \leq t \leq T(\omega).$$

Then

$$I_t^M(X)(\omega) = I_t^{\widetilde{M}}(\widetilde{X})(\omega); \quad 0 \leq t \leq T(\omega), \qquad \text{for } P \text{ - a.e. } \omega.$$

Proof:

For any $N \in \mathcal{M}_2^c$, we have

$$\langle M - \tilde{M}, \; N \rangle_t = 0; \quad 0 \le t \le T,$$

and so (2.25) implies

$$\langle I^M(X) - I^{\tilde{M}}(\tilde{X}), \; N \rangle_t = 0; \quad 0 \le t \le T.$$

Setting $N = I^M(X) - I^{\tilde{M}}(\tilde{X})$ and using Problem 1.5.12, we obtain the desired result.

□

Corollary 2.19 shows that stochastic integrals are determined locally by the local values of the integrator and integrand. This fact allows us to broaden the classes of both integrators and integrands, a project which we now undertake.

Let $M = \{M_t, \mathcal{F}_t; \; 0 \le t < \infty\}$ be a continuous, local martingale on a probability space $(\Omega, \mathcal{F}, P)$ with $M_0 = 0$ a.s., i.e., $M \in \mathcal{M}^{c, loc}$ (Definition 1.5.13). Recall the standing assumption that $\{\mathcal{F}_t\}$ satisfies the usual conditions. We define an equivalence relation on the set of measurable, $\{\mathcal{F}_t\}$ - adapted processes just as we did in the paragraph preceding Definition 2.1.

2.20 Definition: We denote by $\mathcal{P}$ the collection of equivalence classes of all measurable, $\{\mathcal{F}_t\}$ - adapted processes $X = \{X_t, \mathcal{F}_t; \; 0 \le t < \infty\}$ satisfying

$$(2.27) \quad P\left[\int_0^T X_t^2 \, d\langle M \rangle_t < \infty\right] = 1, \quad \text{for every} \quad T \in [0, \infty).$$

We denote by $\mathcal{P}^*$ the collection of equivalence classes of all progressively measurable processes satisfying this condition.

Again, we shall abuse terminology by speaking of $\mathcal{P}$ and $\mathcal{P}^*$ as if they were classes of processes. As an example of such an abuse, we write $\mathcal{P}^* \subseteq \mathcal{P}$, and if M belongs to $\mathfrak{m}_2^c$, in which case both $\mathcal{L}$ and $\mathcal{L}^*$ are defined, we write $\mathcal{L} \subseteq \mathcal{P}$ and $\mathcal{L}^* \subseteq \mathcal{P}^*$.

We shall continue our development only for integrands in $\mathcal{P}^*$. If a.e. path $t \longmapsto \langle M \rangle_t(\omega)$ of the quadratic variation process $\langle M \rangle$ is an absolutely continuous function, we can choose integrands from the wider class $\mathcal{P}$. The reader will see how to accomplish this with the aid of Remark 2.10 once we finish the development for $\mathcal{P}^*$.

Because M is in $\mathfrak{m}^{c,loc}$, there is a nondecreasing sequence $\{S_n\}_{n=1}^\infty$ of stopping times of $\{\mathcal{F}_t\}$, such that $\lim\limits_{n\to\infty} S_n = \infty$ a.s. P, and $\{M_{t \wedge S_n}, \mathcal{F}_t; 0 \leq t < \infty\}$ is in $\mathfrak{m}_2^c$. For $x \in \mathcal{P}^*$, one constructs another sequence of stopping times by setting

$$
R_n(\omega) \triangleq \begin{cases} \inf\{0 \leq t \leq n; \int_0^t X_s^2(\omega)\, d\langle M \rangle_s(\omega) \geq n\} \\ n, \quad \text{if } \{\ldots\} = \emptyset . \end{cases}
$$

This is also a nondecreasing sequence and, because of (2.27), $\lim\limits_{n\to\infty} R_n = \infty$, a.s. P. Set

$$
T_n(\omega) = R_n(\omega) \wedge S_n(\omega),
$$

$$M_t^{(n)}(\omega) \triangleq M_{t \wedge T_n}(\omega), \quad X_t^{(n)}(\omega) = X_t^{(n)}1_{\{T_n(\omega) \geq t\}}; \quad 0 \leq t < \infty, \ \omega \in \Omega, n \geq 1.$$

Then $M^{(n)} \in \mathcal{M}_c^2$ and $X^{(n)} \in \mathcal{L}^*(M^{(n)})$, $n \geq 1$, so $I^{M^{(n)}}(X^{(n)})$ is defined. Corollary 2.19 implies that for $1 \leq n \leq m$,

$$I_t^{M^{(n)}}(X^{(n)}) = I_t^{M^{(m)}}(X^{(m)}), \quad 0 \leq t \leq T_n,$$

so we may define the stochastic integral as

$$(2.28) \quad I_t(X) \triangleq I_t^{M^{(n)}}(X^{(n)}) \quad \text{on} \quad \{0 \leq t \leq T_n\}.$$

This definition is consistent and determines a continuous process, which is also a local martingale.

2.21 Definition: For $M \in \mathcal{M}^{c,loc}$ and $X \in \mathcal{P}^*$, the <u>stochastic integral of X with respect to M</u> is the process $I(X) = \{I_t(X), \mathcal{F}_t; 0 \leq t < \infty\}$ in $\mathcal{M}^{c,loc}$ defined by (2.28). As before, we often write $\int_0^t X_s dM_s$ instead of $I_t(X)$.

When $M \in \mathcal{M}^{c,loc}$ and $X \in \mathcal{P}^*$, the integral $I(X)$ will not in general satisfy conditions (2.10) - (2.13), (2.18) - (2.20), or (2.24), which involve expectations at fixed times or unrestricted stopping times. However, <u>the sample path properties (2.9), (2.14), (2.16), (2.21) and (2.23)</u> are still valid and can be easily proved by localization. We have the following version of Proposition 2.17.

2.22 Proposition: Consider a local martingale $M \in \mathcal{M}^{c,loc}$ and a process $X \in P^*(M)$. The stochastic integral $I^M(X)$ is the unique martingale $\Phi \in \mathcal{M}^{c,loc}$ which satisfies (2.26) for every $N \in \mathcal{M}_2^c$ (or equivalently, for every $N \in \mathcal{M}^{c,loc}$).

2.23 Problem: Suppose $M, N \in \mathcal{M}^{c,loc}$ and $X \in P^*(M) \cap P^*(N)$. Show that for all $\alpha, \beta \in \mathbb{R}$ we have

$$I^{\alpha M + \beta N}(X) = \alpha I^M(X) + \beta I^N(X).$$

2.24 Problem: Let M be standard, one-dimensional Brownian motion and choose $X \in P$. Show that there exists a sequence of simple processes $\{X^{(n)}\}_{n=1}^{\infty}$ such that for every $T > 0$,

$$\lim_{n \to \infty} \int_0^T |X_t^{(n)} - X_t|^2 \, dt = 0$$

and

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T} |I_t(X^{(n)}) - I_t(X)| = 0$$

hold a.s. P.

2.25 Problem: Let $M = W$ be standard Brownian motion and $X \in P$. We define for $0 \leq s < t < \infty$

$$(2.29) \qquad \zeta_t^s(X) \triangleq \int_s^t X_u \, dW_u - \frac{1}{2} \int_s^t X_u^2 \, du; \quad \zeta_t(X) \triangleq \zeta_t^0(X).$$

The process $\{\exp \zeta_t(X), \mathcal{F}_t; \ 0 \leq t < \infty\}$ is a supermartingale; it is a martingale if $X \in \mathcal{L}_0$.

Can one characterize the class of processes $X \in P^*$, for which the "exponential supermartingale" $\{\exp \zeta_t(X), \mathcal{F}_t; 0 \leq t < \infty\}$ of the above Problem is in fact a martingale? This question is at the heart of the important result known as <u>Girsanov's theorem</u> (Theorem 5.1); we shall try to provide an answer in section 5.

<u>2.26 Problem</u>: Based on "first principles", i.e., on the definition only, compute the stochastic integral $\int_0^t W_s dW_s$ when W is a standard Brownian motion. The reader should consult the solution to this problem for discussion of alternate definitions of integration with respect to Brownian motion.

We know all too well that it is one thing to develop a theory of integration in some reasonable generality, and a completely different task to <u>compute</u> the integral in any specific case of interest. Indeed, one cannot be expected to repeat the (sometimes arduous) process which fortunately led to an anser in the preceding problem. Just as we develop a calculus for the Riemann integral, which provides us with tools necessary for more or less mechanical computations, we need a <u>stochastic calculus</u> for the Itô integral and its extensions. We take up this task in the next section.

3.3: THE CHANGE-OF-VARIABLE FORMULA

One of the most important tools in the study of stochastic processes of the martingale type is the change-of-variable formula, or "Itô's rule", as it is better known. It provides us with an integral-differential calculus for the sample paths of such processes.

Let us consider again a basic probability space $(\Omega, \mathcal{F}, P)$ with an associated filtration $\{\mathcal{F}_t\}$ which we always assume to satisfy the usual conditions.

3.1. Definition: A <u>continuous semimartingale</u> $X = \{X_t, \mathcal{F}_t; \ 0 \leq t < \infty\}$ is an adapted process which has the decomposition, P - a.s.,

(3.1) $$X_t = X_0 + M_t + B_t; \qquad 0 \leq t < \infty,$$

where $M = \{M_t, \mathcal{F}_t; \ 0 \leq t < \infty\} \in \mathfrak{m}^{c, loc}$ (Definition 1.5.15) and $B = \{B_t, \mathcal{F}_t; \ 0 \leq t < \infty\}$ is the difference of continuous, nondecreasing, adapted processes $\{A_t^{\pm}, \mathcal{F}_t; \ 0 \leq t < \infty\}$:

(3.2) $$B_t = A_t^+ - A_t^-; \qquad 0 \leq t < \infty,$$

with $A_0^{\pm} = 0$, P - a.s. We shall always assume that (3.2) is the minimal decomposition of B; in other words, A_t^+ is the positive variation of B on $[0,t]$ and A_t^- is the negative variation. The total variation of B on $[0,t]$ is then $\overset{\vee}{B}_t \triangleq A_t^+ + A_t^-$.

The following problem discusses the question of uniqueness for the decomposition (3.1) of a continuous semimartingale.

3.2. <u>Problem</u>: Let $X = \{X_t, \mathfrak{F}_t; \ 0 \leq t < \infty\}$ be a continuous semimartingale with decomposition (3.1). Suppose that X has another decomposition

$$X_t = X_0 + \widetilde{M}_t + \widetilde{B}_t; \qquad 0 \leq t < \infty,$$

where $\widetilde{M} \in \mathfrak{m}^{c,loc}$ and $\widetilde{B}$ is a continuous, adapted process which has finite total variation on each bounded interval $[0,t]$. Prove that P - a.s.,

$$M_t = \widetilde{M}_t, \qquad B_t = \widetilde{B}_t; \qquad 0 \leq t < \infty.$$

Itô's formula states that a "smooth" function of a continuous semimartingale is a continuous semimartingale, and provides us with its decomposition.

3.3. <u>Theorem</u>: <u>Itô (1951), Kunita & Watanabe (1967)</u>

Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^2 (continuous, with continuous first and second derivatives) and let $X = \{X_t, \mathfrak{F}_t; \ 0 \leq t < \infty\}$ be a semimartingale with decomposition (3.1). Then, P - a.s.,

$$(3.3) \qquad f(X_t) = f(X_0) + \int_0^t f'(X_s)dM_s + \int_0^t f'(X_s)dB_s$$

$$+ \frac{1}{2} \int_0^t f''(X_s)d\langle M \rangle_s, \qquad 0 \leq t < \infty.$$

3.4. <u>Remark</u>: For fixed ω and $t > 0$, the function $X_s(\omega)$ is bounded for $0 \leq s \leq t$, so $f'(X_s(\omega))$ is bounded on this interval. It follows that $\int_0^t f'(X_s)dM_s$ is defined as in the last section and this stochastic integral is a continuous, local martingale. The other two integrals in (3.3) are to be understood in the Lebesgue-Stieltjes sense, and so, as functions of the upper limit of integration, are of bounded variation. Thus, $\{f(X_t), \mathfrak{F}_t; \ 0 \leq t < \infty\}$ is a continuous semimartingale.

3.5. <u>Remark</u>: Equation (3.3) is often written in differential notation:

$$(3.3)' \qquad df(X_t) = f'(X_t)dM_t + f'(X_t)dB_t + \frac{1}{2}f''(X_t)d\langle M\rangle_t$$

$$= f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle M\rangle_t, \quad 0 \leq t < \infty.$$

This is the "chain-rule" for stochastic calculus.

<u>Proof of Theorem 3.3</u>:

The proof will be accomplished in several steps.

<u>Step 1: Localization</u>. In the notation of Definition 3.1 we introduce, for each $n \geq 1$, the stopping time

$$T_n = \begin{cases} 0; & \text{if } |X_0| \geq n, \\ \inf\{t \geq 0; \ |M_t| \geq n \text{ or } B_t \geq n \text{ or } \langle M\rangle_t \geq n\}; & \text{if } |X_0| < n, \\ \infty & ; \text{if } |X_0| < n \text{ and } \{\cdots\} = \emptyset. \end{cases}$$

The resulting sequence is nondecreasing with $\lim\limits_{n\to\infty} T_n = \infty$, P - a.s. Thus, if we can establish (3.3) for the stopped process $X_t^{(n)} \triangleq X_{t\wedge T_n}$; $t \geq 0$, then we obtain the desired result upon letting $n \to \infty$. We may assume, therefore, that $X_0(\omega)$ and the random functions $M_t(\omega)$, $\overset{\vee}{B}_t(\omega)$ and $\langle M\rangle_t(\omega)$ on $[0,\infty) \times \Omega$ are all bounded by a common constant K; in particular, M is then a bounded martingale. Under this assumption, we have $|X_t| \leq 3K$; $0 \leq t < \infty$, $\omega \in \Omega$, so the values of f outside $[-3K, 3K]$ are irrelevant. We assume without loss of generality that f has compact support, and so f, f' and f'' are bounded.

Step 2: Taylor expansion. Let us fix $t > 0$ and a partition $\Pi = \{t_0, t_1, \ldots, t_m\}$ of $[0,t]$, with $0 = t_0 < t_1 < \ldots < t_m = t$. A Taylor expansion yields

$$f(X_t) - f(X_0) = \sum_{k=1}^{m} \{f(X_{t_k}) - f(X_{t_{k-1}})\}$$

$$= \sum_{k=1}^{m} f'(X_{t_{k-1}})(X_{t_k} - X_{t_{k-1}}) + \frac{1}{2}\sum_{k=1}^{m} f''(\eta_k)(X_{t_k} - X_{t_{k-1}})^2,$$

where $\eta_k(\omega) = X_{t_{k-1}}(\omega) + \theta_k(\omega)(X_{t_k}(\omega) - X_{t_{k-1}}(\omega))$ for some appropriate $\theta_k(\omega)$ satisfying $0 \leq \theta_k(\omega) \leq 1, \omega \in \Omega$. We conclude that

(3.4) $$f(X_t) - f(X_0) = J_1(\Pi) + J_2(\Pi) + \frac{1}{2}J_3(\Pi),$$

where

$$J_1(\Pi) \triangleq \sum_{k=1}^{m} f'(X_{t_{k-1}})(B_{t_k} - B_{t_{k-1}}),$$

$$J_2(\Pi) \triangleq \sum_{k=1}^{m} f'(X_{t_{k-1}})(M_{t_k} - M_{t_{k-1}}),$$

$$J_3(\Pi) \triangleq \sum_{k=1}^{m} f''(\eta_k)(X_{t_k} - X_{t_{k-1}})^2.$$

It is easily seen that $J_1(\Pi)$ converges to the Lebesgue-Stieltjes integral $\int_0^t f'(X_s)dB_s$, a.s. P, as the mesh $\|\Pi\| = \max_{1 \leq k \leq m} |t_k - t_{k-1}|$ of the partition decreases to zero. On the other hand, the process

$$Y_s(\omega) \triangleq f'(X_s(\omega)); \quad 0 \leq s \leq t, \quad \omega \in \Omega,$$

is in $\mathcal{L}^*$ (adapted, continuous and bounded); we intend to approximate it by the simple process

$$Y_s^{\pi}(\omega) \triangleq f'(X_0(\omega))1_{\{0\}}(s) + \sum_{k=1}^{m} f'(X_{t_{k-1}}(\omega))1_{(t_{k-1},t_k]}(s).$$

Indeed, we have $E\,I^2(Y_t^{\pi} - Y_t) = E\int_0^t |Y_s^{\pi} - Y_s|^2 d\langle M\rangle_s \to 0$ as $\|\Pi\| \to 0$, by the bounded convergence theorem, and so

$$J_2(\Pi) = \int_0^t Y_s^{\pi}\, dM_s \xrightarrow[\|\Pi\| \to 0]{} \int_0^t Y_s\, dM_s$$

in quadratic mean.

Step 3: The quadratic variation term. $J_3(\Pi)$ can be written
as

$$J_3(\Pi) = J_4(\Pi) + J_5(\Pi) + J_6(\Pi),$$

where

$$J_4(\Pi) \triangleq \sum_{k=1}^{m} f''(\eta_k)(B_{t_k} - B_{t_{k-1}})^2,$$

$$J_5(\Pi) \triangleq 2\sum_{k=1}^{m} f''(\eta_k)(B_{t_k} - B_{t_{k-1}})(M_{t_k} - M_{t_{k-1}}),$$

$$J_6(\Pi) \triangleq \sum_{k=1}^{m} f''(\eta_k)(M_{t_k} - M_{t_{k-1}})^2.$$

Because B has total variation bounded by K, we have

$$|J_4(\Pi)| + |J_5(\Pi)| \leq 3K\|f''\|_\infty (\max_{1\leq k\leq m}|B_{t_k} - B_{t_{k-1}}| + \max_{1\leq k\leq m}|M_{t_k} - M_{t_{k-1}}|),$$

and, thanks to the continuity of the processes B and M,
this last term converges to zero almost surely as $\|\Pi\| \to 0$
(as well as in $L^1(\Omega, \mathscr{F}, P)$, because of the bounded
convergence theorem). As for $J_6(\Pi)$, we define

$$J_6^*(\Pi) \triangleq \sum_{k=1}^{m} f''(X_{t_{k-1}})(M_{t_k} - M_{t_{k-1}})^2$$

and observe

$$|J_6^*(\Pi) - J_6(\Pi)| \leq V_t^{(2)}(\Pi) \cdot \max_{1\leq k\leq m}|f''(\eta_k) - f''(X_{t_{k-1}})|,$$

where $V_t^{(2)}(\Pi)$ is as defined in Section 1.5. According to Lemma 1.5.9 and the Hölder inequality,

$$E|J_6^*(\Pi) - J_6(\Pi)| \leq \sqrt{48K^4} \sqrt{E(\max_{1\leq k\leq m} |f''(\eta_k) - f''(X_{t_{k-1}})|)^2},$$

and this is seen to converge to zero as $\|\Pi\| \to 0$ because of the continuity of the process X and the bounded convergence theorem. Thus, in order to establish the convergence of the quadratic variation term $J_3(\Pi)$ to the integral $\int_0^t f''(X_s)d\langle M\rangle_s$ in $L^1(\Omega,\mathfrak{F},P)$ as $\|\Pi\| \to 0$, it suffices to compare $J_6^*(\Pi)$ to the approximating sum

$$J_7(\Pi) \triangleq \sum_{k=1}^m f''(X_{t_{k-1}})(\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}}).$$

We have

$$E|J_6^*(\Pi) - J_7(\Pi)|^2 = E|\sum_{k=1}^m f''(X_{t_{k-1}})\{(M_{t_k} - M_{t_{k-1}})^2 - (\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})\}|^2$$

$$= E\left[\sum_{k=1}^m [f''(X_{t_{k-1}})]^2 \{(M_{t_k} - M_{t_{k-1}})^2 - (\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})\}^2\right]$$

$$\leq 2\|f''\|_\infty^2 \cdot E\left[\sum_{k=1}^m (M_{t_k} - M_{t_{k-1}})^4 + \sum_{k=1}^m (\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})^2\right]$$

$$\leq 2\|f''\|_\infty^2 \cdot E\left[V_t^{(4)}(\Pi) + \langle M\rangle_t \cdot \max_{1\leq k\leq m} (\langle M\rangle_{t_k} - \langle M\rangle_{t_{k-1}})\right].$$

From Lemma 1.5.10 and the bounded convergence theorem, we conclude that the last term above goes to zero as $\|\Pi\| \to 0$. Since convergence in $L^2(\Omega,\mathfrak{F},P)$ implies convergence in $L^1(\Omega,\mathfrak{F},P)$, we conclude that

$$J_3(\Pi) \xrightarrow[\|\Pi\| \to 0]{} \int_0^t f''(X_s) d\langle M \rangle_s$$

in $L^1(\Omega, \mathcal{F}, P)$.

<u>Step 4</u>: If $\{\Pi^{(n)}\}_{n=1}^{\infty}$ is a sequence of partitions of $[0,t]$ with $\|\Pi^{(n)}\| \xrightarrow[n \to \infty]{} 0$, then for some subsequence $\{\Pi^{(n_k)}\}_{k=1}^{\infty}$ we have, P - a.s.,

$$\lim_{k \to \infty} J_1\left(\Pi^{(n_k)}\right) = \int_0^t f'(X_s) dB_s,$$

$$\lim_{k \to \infty} J_2\left(\Pi^{(n_k)}\right) = \int_0^t Y_s \, dM_s,$$

$$\lim_{k \to \infty} J_3\left(\Pi^{(n_k)}\right) = \int_0^t f''(X_s) d\langle M \rangle_s.$$

Thus, passing to the limit in (3.4), we see that (3.3) holds P - a.s. for each $0 \leq t < \infty$. In other words, the processes on the two sides of equality (3.3) are modifications of one another. Since both of them are continuous, they are indistinguishable (Problem 1.1.5). □

We have the following, multi-dimensional version of Itô's rule.

<u>3.6. Theorem</u>: Let $\{M_t \triangleq (M_t^{(1)}, \ldots, M_t^{(d)}), \mathcal{F}_t; 0 \leq t < \infty\}$ be a vector of local martingales in $\mathbb{M}^{c,loc}$, $\{B_t \triangleq (B_t^{(1)}, \ldots, B_t^{(d)}), \mathcal{F}_t; 0 \leq t < \infty\}$ a vector of

adapted processes of bounded variation with $B_0 = 0$,

and set $X_t = X_0 + M_t + B_t;$ $0 \leq t < \infty$, where X_0 is an

$\mathcal{F}_0$-measurable random vector in $\mathbb{R}^d$.

Let $f(t,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ be of class $C^{1,2}$

(continuous, with continuous partial derivatives

$\frac{\partial}{\partial t} f, \frac{\partial}{\partial x_i} f, \frac{\partial^2}{\partial x_i \partial x_j} f;$ $1 \leq i, j \leq d)$. Then, a.s. P,

$$(3.5) \quad f(t,X_t) = f(0,X_0) + \int_0^t \frac{\partial}{\partial t} f(s,X_s) ds + \sum_{i=1}^{d} \int_0^t \frac{\partial}{\partial x_i} f(s,X_s) dB_s^{(i)}$$

$$+ \sum_{i=1}^{d} \int_0^t \frac{\partial}{\partial x_i} f(s,X_s) dM_s^{(i)}$$

$$+ \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(s,X_s) d\langle M^{(i)}, M^{(j)} \rangle_s, \quad 0 \leq t < \infty. \quad \square$$

3.7. Problem: Prove Theorem 3.6.

3.8. Example: With $M = W = $ Brownian motion, $X_0 = 0$, $B_t \equiv 0$

and $f(x) = x^2$, we deduce from (3.3):

$$W_t^2 = 2\int_0^t W_s \, dW_s + t.$$

Compare this with Problem 2.26.

3.9. <u>Example</u>: Again with $M = W =$ Brownian motion, let us consider $X \in \mathcal{P}$ and recall the exponential supermartingale of Problem 2.25:

$$Z_t = \exp(\zeta_t); \qquad 0 \le t < \infty$$

where

$$\zeta_t = \int_0^t X_s dW_s - \frac{1}{2}\int_0^t X_s^2 ds; \qquad 0 \le t < \infty.$$

We now check by application of Itô's rule that this process satisfies the <u>stochastic integral equation</u>

(3.6) $$Z_t = 1 + \int_0^t Z_s X_s dW_s; \qquad 0 \le t < \infty.$$

Indeed, $\{\zeta_t; \mathscr{F}_t; 0 \le t < \infty\}$ is a semimartingale, with local martingale part $M_t \triangleq \int_0^t X_s dW_s$ and bounded variation part $B_t \triangleq -\frac{1}{2}\int_0^t X_s^2 ds$. With $f(x) = e^x$, we have

$$Z_t = f(\zeta_t) = f(\zeta_0) + \int_0^t f'(\zeta_s) dM_s$$

$$+ \int_0^t f'(\zeta_s) dB_s + \frac{1}{2}\int_0^t f''(\zeta_s) d\langle M\rangle_t$$

$$= 1 + \int_0^t Z_s X_s dW_s - \frac{1}{2}\int_0^t Z_s X_s^2 ds$$

$$+ \frac{1}{2}\int_0^t Z_s X_s^2 ds.$$

$$= 1 + \int_0^t Z_s X_s dW_s.$$

The replacement of dM_s by $X_s dW_s$ in this equation is justified by Corollary 2.18 (actually, the extension of Corollary 2.18 to $X \in P$). It is usually more convenient to perform computations like this using <u>differential notation</u>. We write

$$d\zeta_t = X_t dW_t - \frac{1}{2}X_t^2 dt,$$

and, to reflect the fact that the martingale part of ζ has quadratic variation with differential $X_t^2 dt$, we let $(d\zeta_t)^2 = X_t^2 dt$. One may obtain this from the formal compuption

$$(d\zeta_t)^2 = (X_t dW_t - \frac{1}{2} X_t^2 dt)^2$$

$$= X_t^2 (dW_t)^2 - X_t^3 dW_t dt - dt + \frac{1}{4} X_t^4 (dt)^2$$

$$= X_t^2 dt,$$

using the conventional "multiplication table"

(3.7)

	dt	dW_t	$d\widetilde{W}_t$
dt	0	0	0
dW_t	0	dt	0
$d\widetilde{W}_t$	0	0	dt

where $W, \widetilde{W}$ are independent Brownian motions. With these formalisms, Itô's rule can be written as

$$df(\zeta_t) = f'(\zeta_t)d\zeta_t + \frac{1}{2}f''(\zeta_t)(d\zeta_t)^2,$$

and with $f(x) = e^x$, we obtain

$$dZ_t = Z_t X_t dW_t - \frac{1}{2}Z_t X_t^2 dt + \frac{1}{2}Z_t X_t^2 dt$$

$$= Z_t X_t dW_t.$$

Taking into account the initial condition $Z_0 = 1$, we can then recover (3.6).

3.10. Problem: With $\{Z_t; \ 0 \leq t < \infty\}$ as in Example 3.9, set $Y_t = \frac{1}{Z_t}; \ 0 \leq t < \infty$, which is well-defined because

$$P[\inf_{0 \leq t \leq T} Z_t > 0] = P[\inf_{0 \leq t \leq T} \zeta_t > -\infty] = 1. \quad \text{Show that } Y$$

satisfies the stochastic differential equation

$$dY_t = Y_t X_t^2 dt - Y_t X_t dW_t, \quad Y_0 = 1.$$

3.11. Problem: Suppose we have two continuous semimartingales

$$X_t = X_0 + M_t + B_t, \quad Y_t = Y_0 + N_t + C_t; \quad 0 \leq t < \infty,$$

where M and N are in $\mathfrak{m}^{c,loc}$ and B and C are adapted, continuous processes of bounded variation with $B_0 = C_0 = 0$ a.s. Prove the integration by parts formula

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \langle M, N \rangle_t.$$

3.12. Example: If $a(t)$, $b(t)$ are nonrandom functions satisfying:

$$\int_0^T |a(t)|\,dt + \int_0^T b^2(t)\,dt < \infty, \qquad 0 < T < \infty,$$

and if W is a Brownian motion, then the process

$$\xi_t = \exp[\int_0^t a(s)\,ds] \cdot \{\xi_0 + \int_0^t b(s)\exp[-\int_0^s a(u)\,du]\,dW_s\}; \qquad 0 \le t < \infty,$$

is well-defined, because $0 < T < \infty$:

$$\int_0^T b^2(s)\exp[-2\int_0^s a(u)\,du]\,ds$$

$$\le \exp[2\int_0^T |a(u)|\,du]\int_0^T b^2(s)\,ds < \infty.$$

According to Itô's rule (Theorem 3.6) with $f(x_1,x_2) = x_1 x_2$, $X_t^{(1)} = \exp[\int_0^t a(s)\,ds]$ and $X_t^{(2)} = \xi_0 + \int_0^t b(s)\exp[-\int_0^s a(u)\,du]\,dW_s$, we have

$$\xi_t = \xi_0 + \int_0^t a(s)\xi_s\,ds + \int_0^t b(s)\,dW_s. \qquad \square$$

In the hands of Kunita and Watanabe [1967], the change-of-variable formula (3.5) was shown to be the right tool for providing a simple proof of P. Lévy's celebrated <u>martingale characterization of Brownian motion</u> in $\mathbb{R}^d$. Let us recall here that if $\{B_t = (B_t^{(1)},\ldots,B_t^{(d)}), \mathcal{F}_t; \ 0 \le t < \infty\}$ is a d-dimensional Brownian motion on $(\Omega,\mathcal{F},P)$ with $P[B_0=0] = 1$, then $\langle B^{(i)},B^{(j)}\rangle_t = \delta_{ij}t; \ 1 \le i,j \le d, \ 0 \le t < \infty$ (Remark 1.5.6). It turns out that this property characterizes Brownian motion among <u>continuous</u> local martingales. The compensated Poisson process with intensity $\lambda = 1$ provides an example of a <u>discontinuous</u>,

square-integrable martingal with quadratic variati n $\langle M \rangle_t = t$ (c.f. Example 1.5.4), so the assumption of continuity in the following theorem is essential.

3.13. Theorem: P. Lévy (19)

Let $X = \{X_t = (X_t^{(1)}, \ldots, X_t^{(d)}, \mathcal{F}_t, \quad 0 \leq t < \infty\}$ be

a continuous, adapted proc s in $\mathbb{R}^d$ such th , for

every component $1 \leq i \leq d$, the process

$$M_t^{(i)} \triangleq X_t^{(i)} - X_0^{(i)}; \quad 0 \leq t < \infty,$$

is a continuous local martingale relative to $\{\mathcal{F}_t\}$, and the cross-variations are given by

(3.7)' $$\langle M^{(i)}, M^{(j)} \rangle_t = \delta_{ij} t; \quad 1 \leq i, \ j \leq d.$$

Then $\{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a d-dimensional Brownian motion.

Proof:

We must show that for $0 \leq s < t$, the random vector $X_t - X_s$ is independent of $\mathcal{F}_s$ and has the d-variate normal distribution with mean zero and covariance matrix equal to the $d \times d$ identity. In light of Lemma 2.6.12, it suffices to prove that for each $u \in \mathbb{R}^d$,

(3.8) $$E[e^{i(u, X_t - X_s)} | \mathcal{F}_s] = e^{-\frac{1}{2} \|u\|^2 (t-s)}, \quad \text{a.s. } P.$$

For fixed $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$, the function $f(x) = e^{i(u, x)}$ satisfies $\frac{\partial}{\partial x_j} f(x) = i u_j f(x)$, $\frac{\partial^2}{\partial x_j \partial x_k} f(x) = -u_j u_k f(x)$. Applying Theorem 3.6 to the real and imaginary parts of f, we obtain

(3.9) $\qquad e^{i(u,X_t)} = e^{i(u,X_s)} + i \sum_{j=1}^{d} u_j \int_s^t e^{i(u,X_v)} dM_v^{(j)}$

$$- \frac{1}{2} \sum_{j=1}^{d} u_j^2 \int_s^t e^{i(u,X_v)} dv.$$

Now $|f(x)| \leq 1$ for all $x \in \mathbb{R}^d$ and, because $\langle M^{(j)} \rangle_t = t$, we have $M^{(j)} \in \mathfrak{m}_2^c$. Thus, the real and imaginary parts of $\{\int_0^t e^{i(u,X_v)} dM_v^{(j)}, \mathfrak{F}_t; 0 \leq t < \infty\}$ are not only in $\mathfrak{m}^{c,loc}$, but also in $\mathfrak{m}_2^c$. Consequently,

$$E[\int_s^t e^{i(u,X_v)} dM_v^{(j)} | \mathfrak{F}_s] = 0, \quad P - a.s.$$

For $A \in \mathfrak{F}_s$, we may multiply (3.9) by $e^{-i\langle u, X_s \rangle} 1_A$ and take expectations to obtain

$$E[e^{i(u,X_t - X_s)} 1_A] = P(A) - \frac{1}{2} \|u\|^2 \int_s^t E[e^{i(u,X_v - X_s)} 1_A] dv.$$

This integral equation for the deterministic function $t \mapsto E[e^{i(u,X_t - X_s)} 1_A]$ is readily solved:

$$E[e^{i(u,X_t - X_s)} 1_A] = P(A) e^{-\frac{1}{2} \|u\|^2 (t-s)}, \quad \forall A \in \mathfrak{F}_s,$$

and (3.8) follows. $\qquad \square$

3.13'. Problem: Let $W_t = (W_t^{(1)}, W_t^{(2)}, W_t^{(3)})$ be a three-dimensional Brownian motion starting at the orgin, and define
$X = \prod_{i=1}^{3} \text{sgn}(W_1^{(i)})$. Define $M_t^{(1)} = W_t^{(1)}$, $M_t^{(2)} = W_t^{(2)}$,
$M_t^{(3)} = X W_t^{(3)}$. Show that each of the pairs $(M^{(1)}, M^{(2)})$,

$(M^{(1)}, M^{(3)})$ and $(M^{(2)}, M^{(3)})$ is a two-dimensional Brownian motion, but $(M^{(1)}, M^{(2)}, M^{(3)})$ is not a three-dimensional Brownian motion. Why doesn't this provide a counterexample to Theorem 3.13, i.e., a three-dimensional process which is not a Brownian motion, but each component process is in $\mathbb{m}^{c, loc}$ and (3.7)' is satisfied.

3.14. Problem: Let $W = \{W_t = (W_t^{(1)}, \ldots, W_t^{(d)}), \mathfrak{F}_t; \quad 0 \leq t < \infty\}$ be a d-dimensional Brownian motion starting at the origin, and let Q be a $d \times d$ orthogonal matrix ($Q^T = Q^{-1}$). Show that $\widetilde{W}_t \triangleq QW_t$ is also a d-dimensional Brownian motion. We express this property by saying that "d-dimensional Brownian motion starting at the origin is <u>rotationally invariant</u>".

Another use of the P. Levy Theorem 3.13 is to obtain an integral representation for the so-called <u>Bessel Process</u>. For an integer $d \geq 2$, let $W = \{W_t = (W_t^{(1)}, \ldots, W_t^{(d)}), \mathfrak{F}_t; \quad 0 \leq t < \infty\}$, $\{P^x\}_{x \in \mathbb{R}^d}$ be a d-dimensional Brownian family on some measurable space $(\Omega, \mathfrak{F})$. Define

$$(3.10) \quad R_t \triangleq \|W_t\| = \sqrt{(W_t^{(1)})^2 + \ldots + (W_t^{(d)})^2}; \quad 0 \leq t < \infty,$$

so $P^x[R_0 = \|x\|] = 1$. If $x, y \in \mathbb{R}^d$ and $\|x\| = \|y\|$, then there is an orthogonal matrix Q such that $y = Qx$. Under P^x, $\widetilde{W} = \{\widetilde{W}_t \triangleq QW_t, \mathfrak{F}_t; \quad 0 \leq t \leq \infty\}$ is a d-dimensional Brownian motion starting at y, but $\|\widetilde{W}_t\| = \|W_t\|$, so for any $F \in \mathcal{B}(C[0, \infty))$, we have

$$(3.11) \quad P^x[R. \in F] = P^x[\|\widetilde{W}.\| \in F] = P^y[R. \in F].$$

In other words, the distribution of the process R under P^x depends on x only through $\|x\|$.

3.15. Definition: Fix an integer $d \geq 2$, and let $W = \{W_t, \mathcal{F}_t; \ 0 \leq t < \infty\}$, $\{P^x\}_{x \in \mathbb{R}^d}$ be a d-dimensional Brownian family on $(\Omega, \mathcal{F})$. The process $R = \{R_t = \|W_t\|, \mathcal{F}_t; \ 0 \leq t < \infty\}$ together with the family of measures $\{P^{(r,0,\ldots,0)}\}_{r \geq 0}$ on $(\Omega, \mathcal{F})$ is called a <u>Bessel family with index d</u>. For fixed $r \geq 0$, we say that R on $(\Omega, \mathcal{F}, P^{(r,0,\ldots,0)})$ is a <u>Bessel process with index d starting at r</u>.

3.16. Problem: Show that for each $d \geq 2$, the Bessel family with index d is a strong Markov family (where we modify Definition 2.6.3 to account for the state space $\mathbb{R}^+ \triangleq [0, \infty))$.

3.17. Proposition: Let $d \geq 2$ be an integer and choose $r \geq 0$. The Bessel process R with index d starting at r satisfies the integral equation

$$(3.12) \qquad R_t = r + \int_0^t \frac{d-1}{2R_s} ds + B_t; \qquad 0 \leq t < \infty,$$

where $B = \{B_t, \mathcal{F}_t; \ 0 \leq t < \infty\}$ is a standard, one-dimensional, Brownian motion.

<u>Proof:</u>

We use the notation of Definition 3.15, except we write P in place of $P^{(r,0,\ldots,0)}$. Note first of all that R_t can be

zero only when $W_t^{(1)}$ is zero, an. so the Lebesgue measure of the set $\{0 \leq s \leq t; R_s = 0\}$ is zero, P-a.s. (Theorem 2. .). Consequently, the integrand $\frac{d-1}{2R_s}$ in (3.12) is defined for Lebesgue almost every s, P-a.s.

The process B is given by $B_t \triangleq \sum_{i=1}^{d} B_t^{(i)}$, where $B_t^{(i)} \triangleq \int_0^t \frac{1}{R_s} W_s^{(i)} dW_s^{(i)}$. Note that

$$E\int_0^t (\frac{1}{R_s} W_s^{(i)})^2 ds \leq t; \qquad 0 \leq t < \infty,$$

so each $B^{(i)} \in \mathfrak{m}_2^c$. For $t > 0$, we have

$$\langle B^{(i)}, B^{(j)} \rangle_t = \int_0^t \frac{1}{R_s^2} W_s^{(i)} W_s^{(j)} d \langle W^{(i)}, W^{(j)} \rangle_s$$

$$= \delta_{ij} \int_0^t \frac{1}{R_s^2} W_s^{(i)} W_s^{(j)} ds,$$

and so

$$\langle B \rangle_t = \sum_{i=1}^{d} \langle B^{(i)} \rangle_t = t.$$

We conclude from Theorem 3.13 that B is a standard, one-dimensional Brownian motion.

It remains to prove (3.12). A heuristic derivation is to apply Itô's rule (Theorem 3.6) to the function $f(x) \triangleq \|x\| = \sqrt{x_1^2 + \ldots + x_d^2} : \mathbb{R}^d \to [0, \infty)$, for which

$$\frac{\partial}{\partial x_i} f(x) = \frac{x_i}{\|x\|}, \quad \frac{\partial^2}{\partial x_i \partial x_j} f(x) = \frac{\delta_{ij}}{\|x\|} - \frac{x_i x_j}{\|x\|^3}; \qquad 1 \leq i, j \leq d,$$

hold on $\mathbb{R}^d \backslash \{0\}$. Then $R_t = f(W_t)$ and (3.12) follows from (3.5).

The difficulty here is that f is not differentiable at the origin, and so Theorem 3.6 cannot be applied directly to f. This problem is related to our uneasiness about whether the integral in (3.12) is finite. Here is a resolution of this problem. Define:

$$Y_t \triangleq \|W_t\|^2 = R_t^2,$$

and use Itô's rule to show that

$$Y_t = r^2 + 2 \sum_{i=1}^{d} \int_0^t W_s^{(i)} \, dW_s^{(i)} + td.$$

Let $g(y) = \sqrt{y}$, and for $\epsilon > 0$, define

$$g_\epsilon(y) = \begin{cases} \dfrac{3}{8}\sqrt{\epsilon} + \dfrac{3}{4\sqrt{\epsilon}} y - \dfrac{1}{8\epsilon\sqrt{\epsilon}} y^2; & y < \epsilon, \\[4mm] \sqrt{y}; & y \geq \epsilon, \end{cases}$$

so g_ϵ is of class C^2 and $\displaystyle \lim_{\epsilon \downarrow 0} g_\epsilon(y) = g(y)$ for all $y \geq 0$.

Now apply Itô's rule to obtain

$$(3.13) \qquad g_\epsilon(Y_t) = g_\epsilon(r^2) + \sum_{i=1}^{d} I_t^{(i)}(\epsilon) + J_t(\epsilon) + K_t(\epsilon),$$

where

$$I_t^{(i)}(\epsilon) \triangleq \int_0^t \left[1_{\{Y_s \geq \epsilon\}} \frac{1}{R_s} + 1_{\{Y_s < \epsilon\}} \frac{1}{2\sqrt{\epsilon}} \left(3 - \frac{Y_s}{\epsilon}\right) \right] W_s^{(i)} \, dW_s^{(i)},$$

$$J_t(\epsilon) \triangleq \int_0^t 1_{\{Y_s \geq \epsilon\}} \frac{d-1}{2R_s} \, ds,$$

$$K_t(\epsilon) \triangleq \int_0^t 1_{\{Y_s < \epsilon\}} \frac{1}{4\sqrt{\epsilon}} [3d - (d+2)\frac{Y_s}{\epsilon}] ds.$$

We now show that, as $\epsilon \downarrow 0$, (3.13) yields (3.12). From the monotone convergence theorem, we see that

$$\lim_{\epsilon \downarrow 0} J_t(\epsilon) = \int_0^t 1_{\{Y_s > 0\}} \frac{d-1}{2R_s} ds = \int_0^t \frac{d-1}{2R_s} ds, \quad \text{a.s.}$$

We also have $0 \leq EK_t(\epsilon) \leq \frac{3d}{4\sqrt{\epsilon}} \int_0^t P[Y_s < \epsilon] ds$. The probability in the integrand is bounded above by

$$P[(W_s^{(1)})^2 + (W_s^{(2)})^2 < \epsilon] = \int_0^{2\pi} \int_0^{\sqrt{\epsilon}} \frac{1}{2\pi s} e^{-\frac{\rho^2}{2s}} \rho \, d\rho \, d\theta,$$

and so the integral becomes, upon using Fubini's theorem and the change of variable $\xi = \frac{\rho}{\sqrt{s}}$:

$$\int_0^t P[Y_s < \epsilon] ds \leq \int_0^{\sqrt{\epsilon}} \rho \left(\int_0^t \frac{1}{s} e^{-\rho^2/2s} ds \right) d\rho$$

$$= 2\int_0^{\sqrt{\epsilon}} \rho \left(\int_{\frac{\rho}{\sqrt{t}}}^{\infty} \frac{1}{\xi} e^{-\xi^2/2} d\xi \right) d\rho.$$

But now it is easy to see that this expression is $o(\sqrt{\epsilon})$ as $\epsilon \downarrow 0$, using the rule of de l'Hôpital. Therefore, $\lim_{\epsilon \downarrow 0} EK_t(\epsilon) = 0$. Finally

$$E[B_t^{(i)} - I_t^{(i)}(\epsilon)]^2 = E\int_0^t 1_{\{Y_s < \epsilon\}} [\frac{1}{R_s} - \frac{1}{2\sqrt{\epsilon}} (3 - \frac{Y_s}{\epsilon})]^2 (W_s^{(i)})^2 ds$$

$$= E\int_0^t 1_{\{Y_s < \epsilon\}} [1 - \frac{1}{2}\sqrt{\frac{Y_s}{\epsilon}} (3 - \frac{Y_s}{\epsilon})]^2 (\frac{W_s^{(i)}}{R_s})^2 ds$$

$$\leq E \int_0^t P[Y_s < \epsilon] ds,$$

and we have already shown that this expression converges to zero as $\epsilon \downarrow 0$. This establishes (3.12). ☐

Let $\{R_t, \mathcal{F}_t; \ 0 \leq t < \infty\}$ be a Bessel process with index $d \geq 2$ starting at $r \geq 0$. Then, for each fixed $t > 0$, it is clear from (3.10) that $P[R_t > 0] = 1$. A more interesting question is whether the origin is nonattainable:

$$P[R_t > 0, \forall 0 < t < \infty] = 1.$$

The next proposition shows that this is indeed the case. Of course the situation is drastically different for the Bessel process with index 1, since $P[|W_t^{(1)}| > 0, \forall 0 < t < \infty] = 0$ (Remark 2.8.3).

3.18. Proposition: Nonattainability of the origin by the Brownian path in dimension $d \geq 2$.

Let $d \geq 2$ be an integer and $r \geq 0$. The Bessel process R with index d starting at r satisfies

$$P[R_t > 0, \forall 0 < t < \infty] = 1.$$

Proof:

It is sufficient to treat the case $d = 2$, since, for larger d, $\sqrt{(W_t^{(1)})^2 + \ldots + (W_t^{(d)})^2}$ can reach zero only if $\sqrt{(W_t^{(1)})^2 + (W_t^{(2)})^2}$ reaches zero.

We consider first the case $r > 0$. For each positive integer k satisfying $(\frac{1}{k})^k < r < k$, define stopping times

$$T_k = \begin{cases} \inf \{t \geq 0; R_t = (\frac{1}{k})^k\}; & \text{if } \{...\} \neq \emptyset, \\ \\ \infty & ; \quad \text{otherwise} \end{cases}$$

$$S_k = \begin{cases} \inf \{t \geq 0; R_t = k\}; & \text{if } \{...\} \neq \emptyset, \\ \\ \infty & ; \quad \text{otherwise} \end{cases}$$

$$\tau_k = T_k \wedge S_k \wedge n.$$

Because P-almost every Brownian path is unbounded (Theorem 2.9.21), we have

(3.14) $$P[\bigcap_{k=1}^{\infty} \{S_k < \infty\} \cap \{\lim_{k \to \infty} S_k = \infty\}] = 1.$$

Using (3.12), apply Itô's rule to $\ell n(R_t)$ to obtain

$$\ell n R_{\tau_k} = \ell n \, r + \int_0^{\tau_k} \frac{1}{R_s} dB_s.$$

This step is permissible because ℓn is of class C^2 on an open interval containing $[(\frac{1}{k})^k, k]$ and so can be modified outside this interval to obtain a C^2 function on $\mathbb{R}$. For $0 \leq s \leq \tau_k$, $|\frac{1}{R_s}|$ is bounded, and since τ_k is also bounded, we have $E \int_0^{\tau_k} \frac{1}{R_s} dB_s = 0$. Therefore

(3.15)
$$\ell n \, r = E[\log R_{T_k}]$$

$$= -k(\log k) P[T_k \leq S_k \wedge n]$$

$$+ (\log k) P[S_k \leq T_k \wedge n]$$

$$+ E[(\log R_n) 1_{\{n < S_k \wedge T_k\}}].$$

For every $n \geq 1$, $\ell n \, R_n$ on $\{n < S_k \wedge T_k\}$ is bounded between $-k(\log k)$ and $\log k$. According to (3.14), as $n \to \infty$, we have $P[n < S_k \wedge T_k] \to 0$. Thus, letting $n \to \infty$ in (3.15), we obtain

$$\ell n \, r = -k(\log k) P[T_k \leq S_k]$$

$$+ (\log k) P[S_k \leq T_k].$$

If we divide by $k(\log k)$ and let $k \to \infty$, we see that

(3.15)
$$\lim_{k \to \infty} P[T_k \leq S_k] = 0.$$

Now set

$$T = \begin{cases} \inf \{t > 0; \ R_t = 0\}; & \text{if } \{\ldots\} \neq \emptyset, \\ \infty; & \text{otherwise,} \end{cases}$$

so that $T_k \leq T$ for every $k \geq 1$. From (3.14) and (3.15), we have

$$P[T < \infty] = \lim_{k \to \infty} P[T \leq S_k]$$

$$\leq \lim_{k \to \infty} P[T_k \leq S_k] = 0.$$

It follows that $P[R_t > 0, \forall 0 < t < \infty] = 1$.

Finally, we consider the case $r = 0$. Recalling the indexing of probability measures in Definition 3.15, we have from Problem 3.16

$$P^{(0,0,\ldots,0)}[R_t>0, \epsilon<t<\infty]$$

$$= E^{(0,0,\ldots,0)}\{P^{(R_\epsilon,0,\ldots0)}[R_t>0, 0<t<\infty]\}$$

$$= 1$$

for any $\epsilon > 0$, by what was just proved and the fact that $\underline{P}^{(0,0,\ldots,0)}[R_\epsilon>0] = 1$. Letting $\epsilon \downarrow 0$, we obtain the desired result. $\qquad\square$

3.19. Problem:

Let $R = \{R_t, \mathscr{F}_t; 0 \le t < \infty\}$ be a Bessel process with index $d \ge 2$ starting at $r > 0$, and define

$$m = \inf_{0 \le t<\infty} R_t.$$

(i) Show that if $d = 2$, then $m = 0$ a.s.P.

(ii) Show that if $d \ge 3$, then m has the beta distribution

$$P[m \le c] = (\frac{c}{r})^{d-2}; \qquad 0 \le c \le r.$$

(<u>Hint</u>: Adapt the proof of Proposition 3.18. For (ii), an appropriate substitute for the function $f(r) = \ell n\, r$ must be used.)

Proposition 3.18 says that, with probability one, a two-dimensional Brownian motion never reaches the origin. Problem 3.19(i) shows, however, that it comes arbitrarily close. By translation, we can conclude that for any fixed point $z \in \mathbb{R}^2$,

a two-dimensional Brownian path, regardless of its starting position, never reaches the point z, but does reach every disc of positive radius centered at z. In the parlance of Markov chains, one says that "every singleton is nonrecurrent", but that "every disc of positive radius is recurrent." For a Brownian motion of dimension 3 or greater, Problem 3.19(ii) shows that, once it gets away from the origin, almost every path of the process remains bounded away from the origin; this lower bound depends, of course, on the particular path. Thus, d-dimensional spheres are nonrecurrent for d-dimensional Brownian motion when $d \geq 3$.

3.20. **Problem**: Let R be a Bessel process with index $d \geq 3$ starting at $r \geq 0$. Show that

$$P[\lim_{t \to \infty} R_t = \infty] = 1.$$

As a final application in this section of Itô's rule, we derive some useful bounds on the moments of stochastic integrals. The following problem illustrates the technique.

3.21. **Problem**: With $W = \{W_t, \mathcal{F}_t; \ 0 \leq t < \infty\}$ a standard, one-dimensional, Brownian motion and X a measurable, adapted process satisfying

(3.17) $$E\int_0^T |X_t|^{2m} dt < \infty$$

for some real numbers $T > 0$ and $m \geq 1$, show that

(3.18) $\qquad E|\int_0^T X_t dW_t|^{2m} \leq (m(2m-1))^m T^{m-1} E\int_0^T |X_t|^{2m} dt.$

(__Hint__: Consider the martingale $\{M_t = \int_0^t X_s dW_s, \mathcal{F}_t;$
$0 \leq t \leq T\}$, and apply Itô's rule to the submartingale
$|M_t|^{2m}$.)

Actually, with a bit of extra effort, we can obtain much stronger results.

__3.22. Proposition__: __Martingale moment inequalities (Millar__
   __(1968), Novikov (1971))__

Suppose $M \in \mathfrak{m}^{c,loc}$ and

(3.19) $\qquad E\langle M\rangle_T^m < \infty$

for some real numbers $T > 0$ and $m > 0$. Then

(3.20) $\qquad E|M_T|^{2m} \leq C_m E\langle M\rangle_T^m,$

where C_m is a universal constant depending only on m.
Furthermore, if $m > \frac{1}{2}$, there exists another constant
$B_m > 0$, depending only on m, such that

(3.21) $\qquad B_m E\langle M\rangle_T^m \leq E|M_T|^{2m}.$

__Remark__: If, in the notation of Problem 3.21, we take
$M_t = \int_0^t X_s dW_s$, then the Hölder inequality implies that

for $m \geq 1$,

$$\langle M \rangle_T^m = (\int_0^T X_t^2 dt)^m \leq T^{m-1} \int_0^T |X_t|^{2m} dt.$$

Thus, condition (3.19) is weaker than (3.17).

Proof: We assume for the moment that both M and $\langle M \rangle$ are bounded on $[0,T]$:

(3.22) $|M_t(\omega)| \leq N, \quad \langle M \rangle_t(\omega) \leq N; \quad 0 \leq t \leq T, \quad \omega \in \Omega,$

for some positive integer N. We consider the process

$$Y_t \triangleq \delta + \varepsilon \langle M \rangle_t + M_t^2 = \delta + (1+\varepsilon) \langle M \rangle_t + 2 \int_0^t M_s dM_s, \quad 0 \leq t \leq T,$$

where $\delta > 0$ and $\varepsilon \geq 0$ are constants to be chosen later. Applying the change-of-variable formula to $f(x) = x^m$, we obtain

(3.23) $Y_t^m = \delta^m + m(1+\varepsilon) \int_0^t Y_s^{m-1} d\langle M \rangle_s + 2m(m-1) \int_0^t Y_s^{m-2} M_s^2 d\langle M \rangle_s$

$$+ 2m \int_0^t Y_s^{m-1} M_s dM_s, \quad 0 \leq t \leq T.$$

Because M, Y are bounded and Y is bounded away from zero, the integrand in the last integral is bounded, so this martingale integral has expectation zero. Taking expectations in (3.23), we obtain our basic identity

$$(3.24) \quad EY_t^m = \delta^m + m(1+\epsilon)E\int_0^t Y_s^{m-1}d\langle M\rangle_s + 2m(m-1)E\int_0^t Y_s^{m-2}M_s^2 d\langle M\rangle_s,$$

$$0 \leq t \leq T.$$

Case 1: $0 < m \leq 1$, upper bound. The last term on the right-hand side of (3.24) is nonpositive; so, letting $\delta \downarrow 0$, we obtain

$$(3.25) \quad E[\epsilon\langle M\rangle_t + M_t^2]^m \leq m(1+\epsilon)E\int_0^t (\epsilon\langle M\rangle_s + M_s^2)^{m-1}d\langle M\rangle_s$$

$$\leq m(1+\epsilon)\epsilon^{m-1}E\int_0^t \langle M\rangle_s^{m-1}d\langle M\rangle_s$$

$$= (1+\epsilon)\epsilon^{m-1}E\langle M\rangle_t^m.$$

The second inequality uses the fact $0 < m \leq 1$. But for such m, the function $f(x) = x^m$; $x \geq 0$ is concave so

$$(3.26) \quad 2^{m-1}(x^m + y^m) \leq (x+y)^m; \quad x \geq 0, \quad y \geq 0,$$

and (3.25) yields: $\epsilon^m E\langle M\rangle_t^m + E|M_t|^{2m} \leq (1+\epsilon)(\frac{\epsilon}{2})^{m-1}E\langle M\rangle_t^m$,

whence

$$(3.27) \quad E|M_t|^{2m} \leq [(1+\epsilon)(\frac{2}{\epsilon})^{1-m} - \epsilon^m]E\langle M\rangle_t^m, \quad 0 \leq t \leq T.$$

Case 2: $m > 1$, lower bound. Now the last term in (3.24) is nonegative, and the direction of all inequalities (3.25) - (3.27) is reversed:

$$E|M_t|^{2m} \geq [(1+\epsilon)(\tfrac{\epsilon}{2})^{m-1} - \epsilon^m] \cdot E\langle M\rangle_t^m, \qquad 0 \leq t \leq T.$$

Here, ϵ has to be chosen in $(0, (2^{m-1}-1)^{-1})$.

Case 3: $\frac{1}{2} < m \leq 1$, lower bound. Let us evaluate (3.24) with $\epsilon = 0$ and then let $\delta \downarrow 0$. We obtain

$$(3.28) \qquad E|M_t|^{2m} = 2m(m-\tfrac{1}{2}) E\int_0^t |M_s|^{2(m-1)} d\langle M\rangle_s.$$

On the other hand, we have from (3.26), (3.24):

$$2^{m-1}[\epsilon^m E\langle M\rangle_t^m + E(\delta+M_t^2)^m] \leq E[\epsilon\langle M\rangle_t + (\delta+M_t^2)]^m$$

$$\leq \delta^m + m(1+\epsilon) E\int_0^t (\delta+M_s^2)^{m-1} d\langle M\rangle_s.$$

Letting $\delta \downarrow 0$, we see that

$$(3.29) \qquad 2^{m-1}[\epsilon^m E\langle M\rangle_t^m + E|M_t|^{2m}] \leq m(1+\epsilon) E\int_0^t |M_s|^{2(m-1)} d\langle M\rangle_s.$$

Relations (3.28) and (3.29) provide us with the lower bound

$$E|M_t|^{2m} \geq \epsilon^m \left(\frac{(1+\epsilon)2^{1-m}}{2m-1} - 1\right)^{-1} \cdot E\langle M\rangle_t^m; \qquad 0 \leq t \leq T,$$

valid for all $\epsilon > 0$.

Case 4: $m > 1$, upper bound. In this case, the inequality (3.29) is reversed, and we obtain

$$E|M_t|^{2m} \leqq \epsilon^m (\frac{(1+\epsilon) 2^{1-m}}{2m-1} - 1)^{-1} \cdot E\langle M\rangle_t^m, \qquad 0 \leq t \leq T,$$

where now ϵ has to satisfy $\epsilon > (2m-1) 2^{m-1} - 1$.

This analysis establishes (3.20) and (3.21) under the condition (3.22). For an arbitrary $M \in m^{c,loc}$, we consider the sequence of stopping times

$$\tau_N = \begin{cases} \inf\{0 \leq t \leq T; \quad |M_t| \geq N \text{ or } \langle M\rangle_t \geq N\} & \text{if } \{...\} \neq \emptyset, \\ \\ T, & \text{otherwise,} \end{cases}$$

which is increasing and converges almost surely to T. With $M_t^{(N)} \stackrel{\Delta}{=} M_{t\wedge\tau_N}$, the sequences

$$\{|M_T^{(N)}| = |M_{T\wedge\tau_N}|\}_{N=1}^{\infty} \quad \text{and} \quad \{\langle M^{(N)}\rangle_T = \langle M\rangle_{T\wedge\tau_N}\}_{N=1}^{\infty}$$

are also increasing and converge almost surely to $|M_T|$ and $\langle M\rangle_T$, respectively. We have proved (3.20) and (3.21) for each $M^{(N)}$, and letting $N \to \infty$, we obtain these relations for M from the monotone convergence theorem. $\square$

3.23. Problem: Prove the following d-dimensional version of Proposition 3.22. Suppose

$$M = \{M_t = (M_t^{(1)},...,M_t^{(2)}), \mathcal{F}_t; \quad 0 \leq t < \infty\} \text{ is an}$$

adapted, d-dimensional process with $M^{(i)} \in m^{c,loc}$; $1 \leq i \leq d$. Let $A_t = \sum_{i=1}^{d} \langle M^{(i)}\rangle_t$, and assume

$$EA_T^m < \infty$$

for some real numbers $T > 0$ and $m > 0$. Then

$$E\|M_T\|^{2m} \leq C_m' EA_T^m,$$

where C_m' is a constant depending only on d and m.
Furthermore, if $m > \frac{1}{2}$, there exists another such constant
$B_m' > 0$ such that

$$B_m' EA_T^m \leq E\|M_T\|^{2m}.$$

3.24. Problem: Prove the following vector stochastic integral
version of Proposition 3.22. Suppose
$W = \{W_t = (W_t^{(1)}, \ldots, W_t^{(r)}), \mathcal{F}_t; \quad 0 \leq t < \infty\}$ is an
r-dimensional Brownian motion starting at the origin,
and suppose $X = \{X_t = (X_t^{(i,j)}); 1 \leq i \leq d, 1 \leq j \leq r,$
$0 \leq t < \infty\}$ is a matrix of processes adapted to $\{\mathcal{F}_t\}$.
Let $\|X_t\|^2 = \sum\limits_{i=1}^{d} \sum\limits_{j=1}^{r} (X_t^{(i,j)})^2$ and assume

$$E[\int_0^T \|X_t\|^2 dt]^m < \infty$$

for some real numbers $T > 0$ and $m > 0$. Define
$M_t = (M_t^{(1)}, \ldots, M_t^{(d)})$ by

$$M_t^{(i)} = \sum\limits_{j=1}^{r} \int_0^t X_s^{(i,j)} dW_s^{(j)}.$$

Then

$$E\|M_T\|^{2m} \le C_m' E[\int_0^T \|X_t\|^2 dt]^m,$$

where C_m' is a constant depending only on d and m. Furthermore, if $m > \frac{1}{2}$, there exists another such constant $B_m' > 0$ such that

$$B_m' E[\int_0^T \|X_t\|^2 dt]^m \le E\|M_T\|^{2m}.$$

3.25. Problem: Prove the following bound on the maximum of a stochastic integral. Suppose $W = \{W_t, \mathcal{F}_t; \ 0 \le t \le T\}$ is a standard Brownian motion. If X is adapted to $\{\mathcal{F}_t\}$ and satisfies

$$E[\int_0^T X_t^2 dt]^m < \infty$$

for some real numbers $T > 0$ and $m > \frac{1}{2}$, then there is a constant C_m depending only on m such that

$$E[\max_{0 \le t \le T} |\int_0^t X_s dW_s|^{2m}] \le C_m E[\int_0^T X_t^2 dt]^m.$$

3.4: REPRESENTATIONS OF CONTINUOUS

MARTINGALES IN TERMS OF BROWNIAN MOTION

In this section we expound on the theme that Brownian motion is the fundamental continuous martingale, by showing how to represent other continuous martingales in terms of it. We give conditions under which a vector of d continuous, local martingales can be represented as stochastic integrals with respect an r-dimensional Brownian motion on a possibly extended probability space. Here we have $r \leq d$. We also discuss how a continuous, local martingale can be transformed into a Brownian motion by a random time change. In contrast to these representation results, in which one begins with a continuous local martingale, we will also prove a result in which one begins with a Brownian motion $W = \{W_t, \mathscr{F}_t; 0 \leq t < \infty\}$ and shows that every continuous local martingale with respect to the Brownian filtration $\{\mathscr{F}_t\}$ is a stochastic integral with respect to W. A related result is that for fixed $0 \leq T \leq \infty$, every $\mathscr{F}_T$-measurable random variable can be represented as a stochastic integral with respect to T.

We recall our standing assumption that every filtration satisfies the usual conditions, i.e., is right-continuous and contains all null sets

4.1 Remark: Our first representation theorem involves the notion of the extension of a probability space. Let $X = \{X_t, \mathscr{F}_t; 0 \leq t < \infty\}$ be an adapted process on some $(\Omega, \mathscr{F}, P)$. We may need an r-dimensional Brownian motion independent of X, but because $(\Omega, \mathscr{F}, P)$ may not be rich

enough to support this Brownian motion, we must extend the probability space to construct this. Let $(\hat{\Omega}, \hat{\mathfrak{F}}, \hat{P})$ be another probability space, on which we consider an r-dimensional Brownian motion $\hat{B} = \{B_t, \hat{\mathfrak{F}}_t; 0 \le t < \infty\}$, set $\tilde{\Omega} \triangleq \Omega \times \hat{\Omega}$, $\tilde{\mathcal{G}} \triangleq \mathfrak{F} \otimes \hat{\mathfrak{F}}$, $\tilde{P} \triangleq P \times \hat{P}$, and define a new filtration by $\tilde{\mathcal{G}}_t \triangleq \mathfrak{F}_t \otimes \hat{\mathfrak{F}}_t$. The latter may not satisfy the usual conditions, so we augment it and make it right-continuous by defining

$$\tilde{\mathfrak{F}}_t \triangleq \bigcap_{s>t} \sigma(\tilde{\mathcal{G}}_s \cup \mathcal{N}),$$

where $\mathcal{N}$ is the collection of $\tilde{P}$-null sets in $\tilde{\mathcal{G}}$. We also complete $\tilde{\mathcal{G}}$ be defining $\tilde{\mathfrak{F}} = \sigma(\tilde{\mathcal{G}} \cup \mathcal{N})$. We may extend X and B to $\{\tilde{\mathfrak{F}}_t\}$-adapted process on $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{P})$ be defining for $(\omega, \hat{\omega}) \in \tilde{\Omega}$,

$$\tilde{X}_t(\omega, \hat{\omega}) = X_t(\omega)$$

$$\tilde{B}_t(\omega, \hat{\omega}) = B_t(\hat{\omega}).$$

Then $\tilde{B} = \{\tilde{B}_t, \tilde{\mathfrak{F}}_t; 0 \le t < \infty\}$ is an r-dimensional Brownian motion, independent of $\tilde{X}_t = \{\tilde{X}_t, \tilde{\mathfrak{F}}_t; 0 < t < \infty\}$. Indeed, $\tilde{B}$ is independent of the extension to $\tilde{\Omega}$ of any $\mathfrak{F}$-measurable random variable on Ω. To simplify notation, we henceforth write X and B instead of $\tilde{X}$ and $\tilde{B}$ in the context of extensions. $\square$

Let us recall (Definition 2.21 and the discussion preceding it) that if $W = \{W_t, \mathfrak{F}_t; 0 \le t < \infty\}$ is a standard Brownian motion and X is a measurable, adapted process with $P[\int_0^t X_s^2 ds < \infty] = 1$ for every $0 \le t < \infty$, then the stochastic integral $I_t(X) = \int_0^t X_s dW_s$ is a continuous, local martingale with quadratic variation process $\langle I(X) \rangle_t = \int_0^t X_s^2 ds$ which is an absolutely continuous function of t, P-a.s. Our first representation result provides the converse to this statement; its one-dimensional version is due to Doob [1953].

4.2 Theorem: Suppose $M = \{M_t = (M_t^{(1)}, \ldots, M_t^{(d)}), \mathfrak{F}_t; 0 \le t < \infty\}$ is defined on $(\Omega, \mathfrak{F}, P)$ and each $M^{(i)} \in \mathfrak{m}^{c,loc}$, $1 \le i \le d$. Suppose also that for $1 \le i, j \le d$, the cross-variation $\langle M^{(i)}, M^{(j)} \rangle_t(\omega)$ is an absolutely continuous function of t for P-almost every ω. Then there is an extension $(\widetilde{\Omega}, \widetilde{\mathfrak{F}}, \widetilde{P})$ of $(\Omega, \mathfrak{F}, P)$ which is rich enough to support a d-dimensional Brownian motion
$W = \{W_t = (W_t^{(1)}, \ldots, W_t^{(d)}), \widetilde{\mathfrak{F}}_t; 0 \le t < \infty\}$, a matrix
$X = \{(X_t^{(i,k)})_{i,k=1}^d, \widetilde{\mathfrak{F}}_t; 0 \le t < \infty\}$ of measurable, adapted processes with

$$(4.1) \qquad P[\int_0^t (X_s^{(i,k)})^2 ds < \infty] = 1; \quad 1 \le i, k \le d; \ 0 < t < \infty,$$

such that we have, $\widetilde{P}$-a.s., the representations

$$(4.2) \qquad M_t^{(i)} = \sum_{k=1}^d \int_0^t X_s^{(i,k)} dW_s^{(k)}; \quad 1 \le i \le d; \ 0 \le t < \infty,$$

$$(4.3) \quad \langle M^{(i)}, M^{(j)} \rangle_t = \sum_{k=1}^{d} \int_0^t X_s^{(i,k)} X_s^{(j,k)} ds; \quad 1 \leq i, j \leq d; \quad 0 \leq t < \infty,$$

Proof:

We prove this theorem by a random, time-dependent, rotation of coordinates which reduces it to d separate, one-dimensional cases. We begin by defining

$$(4.4) \quad z_t^{i,j} = z_t^{j,i} = \frac{d}{dt} \langle M^{(i)}, M^{(j)} \rangle_t$$

$$= \lim_{n \to \infty} n[\langle M^{(i)}, M^{(j)} \rangle_t - \langle M^{(i)}, M^{(j)} \rangle_{(t - \frac{1}{n})^+}],$$

so that the matrix-valued process $Z = \{Z_t = (z_t^{i,j})_{i,j=1}^{d}, \mathcal{F}_t; 0 \leq t < \infty\}$ is symmetric and progressively measurable. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$, we have

$$\sum_{i=1}^{d} \sum_{j=1}^{d} \alpha_i z_t^{i,j} \alpha_j = \frac{d}{dt} \langle \sum_{i=1}^{d} \alpha_i M^{(i)} \rangle_t \geq 0,$$

so Z_t is positive-semidefinite for Lebesgue-almost every t, P-a.s.

Any symmetric, positive-semidefinite matrix Z can be diagonalized by an orthogonal matrix Q, i.e., $Q^{-1} = Q^{transpose}$, $Q^{-1}ZQ = \Lambda$, and Λ is diagonal with the (nonnegative) eigenvalues of Z as its diagonal elements. There are several algorithms which compute such a Q and Λ from Z, and one can easily verify that these algorithms typically obtain Q and Λ as Borel-measurable functions of Z. In our case, we have a progressively measurable, symmetric, positive-semidefinite matrix process Z,

and so there exist progressively measurable, matrix-valued processes $\{Q_t(\omega) = (q_t^{i,j}(\omega))_{i,j=1}^d; \mathcal{F}_t; 0 \le t < \infty\}$ and $\{\Lambda_t(\omega) = (\delta_{ij}\lambda_t^i(\omega))_{i,j=1}^d, \mathcal{F}_t; 0 \le t < \infty\}$ such that for Lebesgue-almost every t, we have

$$(4.5) \quad \sum_{k=1}^d q_t^{k,i} q_t^{k,j} = \sum_{k=1}^d q_t^{i,k} q_t^{j,k} = \delta_{ij}; \quad 1 \le i, j \le d,$$

$$(4.6) \quad \sum_{k=1}^d \sum_{\ell=1}^d q_t^{k,i} z_t^{k,\ell} q_t^{\ell,j} = \delta_{i,j}\lambda_t^i \ge 0; \quad 1 \le i, j \le d,$$

a.s. P. From (4.5) with $i = j$, we see that $(q_t^{k,i})^2 \le 1$, so

$$\int_0^t (q_s^{k,i})^2 d\langle M^{(k)} \rangle_s \le \langle M^{(k)} \rangle_t < \infty,$$

and we can define continuous, local martingales by the prescription

$$(4.7) \quad N_t^{(i)} \triangleq \sum_{k=1}^d \int_0^t q_s^{k,i} dM_s^{(k)}; \quad 1 \le i \le d; \quad 0 \le t < \infty.$$

From (4.4) and (4.6), we have, a.s. P,

$$(4.8) \quad \langle N^{(i)}, N^{(j)} \rangle_t = \sum_{k=1}^d \sum_{\ell=1}^d \int_0^t q_s^{k,i} q_s^{\ell,j} d\langle M^{(k)}, M^{(\ell)} \rangle_s$$

$$= \sum_{k=1}^d \sum_{\ell=1}^d \int_0^t q_s^{k,i} z_s^{k,\ell} q_s^{\ell,j} ds$$

$$= \delta_{ij} \int_0^t \lambda_s^i ds.$$

We see, in particular, that

$$(4.9) \qquad \int_0^t \lambda_s^i \, ds = \langle N^{(i)} \rangle_t < \infty; \quad 1 \le i \le d; \quad 0 \le t < \infty.$$

We now represent the vector of local martingales
$N = \{(N_t^{(1)}, \ldots, N_t^{(d)}), \mathfrak{F}_t; \ 0 \le t < \infty\}$ as a vector of stochastic
integrals on an extended probability space $(\widetilde{\Omega}, \widetilde{\mathfrak{F}}, \widetilde{P})$, which supports
a d-dimensional Brownian motion $B = \{B_t = (B_t^{(1)}, \ldots, B_t^{(d)}), \widetilde{\mathfrak{F}}_t; \ 0 \le t < \infty$
independent of N (c.f. Remark 4.1.). Since

$$\int_0^t 1_{\{\lambda_s^i > 0\}} \frac{1}{\lambda_s^i} \, d\langle N^{(i)} \rangle_s = \int_0^t 1_{\{\lambda_s^i > 0\}} ds \le t,$$

we can define continuous, local martingales

$$(4.10) \qquad W_t^{(i)} \overset{\Delta}{=} \int_0^t 1_{\{\lambda_s^i > 0\}} \frac{1}{\sqrt{\lambda_s^i}} \, dN_s^{(i)} + \int_0^t 1_{\{\lambda_s^i = 0\}} dB_s^{(i)}; \quad 1 \le i \le d.$$

From (4.8) we have

$$\langle W^{(i)}, W^{(j)} \rangle_t = \delta_{ij} t; \quad 1 \le i, j \le d; \quad 0 \le t < \infty,$$

so, according to Theorem 3.13, $W = \{W_t = (W_t^{(1)}, \ldots, W_t^{(d)}), \mathfrak{F}_t; \ 0 \le t < \infty\}$
is a d-dimensional Brownian motion. Moreover,

$$(4.11) \qquad \int_0^t \lambda_s^i \, dW_s^{(i)} = \int_0^t 1_{\{\lambda_s^i > 0\}} dN_s^{(i)} = N_t^{(i)}; \quad 1 \le i \le d; \quad 0 \le t < \infty,$$

because the martingale $\int_0^t 1_{\{\lambda_s^i = 0\}} dN_s^{(i)}$, having quadratic variation

$$\int_0^t 1_{\{\lambda_s^i=0\}} d\langle N^{(i)}\rangle_s = \int_0^t 1_{\{\lambda_s^{(i)}=0\}} \lambda_s^i ds = 0,$$

is itself identically zero.

Having thus obtained the stochastic integral representation (4.11) for N in terms of the d-dimensional Brownian motion W, we invert the rotation of coordinates (4.7) to obtain a representation for M. Let us first observe that for $1 \leq i, k \leq d$,

$$\int_0^t (q_s^{i,k})^2 \lambda_s^k ds \leq \int_0^t \lambda_s^k ds < \infty; \quad 0 \leq t < \infty,$$

by (4.9), so with $x_t^{(i,k)} \triangleq q_t^{i,k}\sqrt{\lambda_t^k}$, condition (4.1) holds. Furthermore, (4.11), (4.7), and (4.5) imply

$$(4.12) \quad \sum_{k=1}^d \int_0^t x_s^{(i,k)} dW_s^{(k)} = \sum_{k=1}^d \int_0^t q_s^{i,k} dN_s^{(k)}$$

$$= \sum_{j=1}^d \sum_{k=1}^d \int_0^t q_s^{i,k} q_s^{j,k} dM_s^{(j)}$$

$$= \sum_{j=1}^d \delta_{i,j} \int_0^t dM_s^{(j)} = M_t^{(i)},$$

which establishes (4.2). Equation (4.3) is an immediate consequence of (4.2). $\square$

4.3 Remark: If the matrix-values process $z_t(\omega) = (z_t^{i,j}(\omega))_{i,j=1}^d$ has constant rank r, $1 \leq r \leq d$, for Lebesgue-almost every t, a.s. P, then the Brownian motion W used in the representation (4.2) can be chosen to be r-dimensional, and there is no need

to introduce the extended probability space $(\tilde{\Omega},\tilde{\mathfrak{F}},\tilde{P})$. Indeed, we may take $\lambda_t^1,\ldots,\lambda_t^r$ to be the r strictly positive eigenvalues of Z_t, and replace (4.10) by

$$(4.10)' \qquad W_t^{(i)} = \int_0^t \frac{1}{\sqrt{\lambda_s^i}}\, dN_s^{(i)}; \quad 1 \le i \le r.$$

Since $N_t^{(i)} = 0$; $r+1 \le i \le d$; $0 \le t < \infty$ (witness (4.9)), (4.12) becomes

$$(4.12)' \qquad \sum_{k=1}^r \int_0^t X_s^{(i,k)}\, dW_s^{(k)} = \sum_{k=1}^d \int_0^t q_s^{i,k}\, dN_s^{(k)} = M_t^{(i)}, \quad 1 \le i \le d.$$

Because $(4.10)'$ defines $W^{(1)},\ldots,W^{(r)}$ without reference to the Brownian motion B, there is no need to extend the original probability space.

4.4 Problem: This problem shows that any vector of continuous, local martingales can be transformed by a random time-change into a vector of continuous, local martingales satisfying the hypotheses of Theorem 4.2. Let $M = (M_t^{(1)},\ldots,M_t^{(d)})$, $\mathfrak{F}_t$; $0 \le t < \infty\}$ be a vector of continuous, local martingales on some $(\Omega,\mathfrak{F},P)$, and define

$$A^{(i,j)} = \langle M^{(i)},M^{(j)}\rangle, \quad A_t(\omega) \overset{\Delta}{=} \sum_{i=1}^d \sum_{j=1}^d \overset{\vee}{A}_t^{(i,j)}(\omega),$$

where $\overset{\vee}{B}_t$ denotes total variation of B on $[0,t]$. Let $T_s(\omega)$ be the inverse of the function $A_t(\omega) + t$, i.e.,

$$A_{T_s(\omega)}(\omega) + T_s(\omega) = s; \quad 0 \leq s < \infty.$$

(i) Show that for each s, T_s is a stopping time of $\{\mathcal{F}_t\}$.

(ii) Define $\mathcal{G}_s \triangleq \mathcal{F}_{T_s}$; $0 \leq s < \infty$. Show that if $\{\mathcal{F}_t\}$ satisfies the usual conditions, then $\{\mathcal{G}_s\}$ does also.

(iii) Define

$$N_s^{(i)} \triangleq M_{T_s}^{(i)}; \quad 1 \leq i \leq d; \quad 0 \leq s < \infty.$$

Show that for each $1 \leq i \leq d$: $N^{(i)} \in \mathcal{M}^{c,loc}$, and the cross variation $\langle N^{(i)}, N^{(j)} \rangle_s$ is an absolutely continuous function of s, a.s. P.

The time change in Problem 4.4 is straightforward because the function $A_t + t$ is strictly increasing and continuous in t, and so has a strictly increasing, continuous inverse T_s. Our next representation result requires us to consider the inverse of the quadratic variation of a continuous, local martingale, and because such a quadratic variation may not be strictly increasing, we begin with a problem describing this situation in some detail.

<u>4.5 Problem</u>: Let $A = \{A(t); 0 \leq t < \infty\}$ be a continuous, nondecreasing function with $A(0) = 0$, $A(\infty) = \infty$, and define for $0 \leq s < \infty$:

$$T(s) = \inf\{t \geq 0; A(t) > s\}.$$

The function $T = \{T(s); \ 0 \leq s < \infty\}$ has the following properties:

(i) T is nondecreasing and right-continuous;

(ii) $A(T(s)) = s; \quad 0 \leq s < \infty$.

(iii) For $0 \leq t, s < \infty$; $s < A(t) \iff T(s) < t$ and $T(s) \leq t \Rightarrow s \leq A(t)$.

(iv) If G is a bounded, measurable, real-valued function defined on $[a,b] \subset [0,\infty)$, then

(4.13)
$$\int_a^b G(t) \, dA(t) = \int_{A(a)}^{A(b)} G(T(s)) \, ds.$$

4.6 Theorem: Time-change for martingales

Let $M = \{M_t, \mathcal{F}_t; \ 0 \leq t < \infty\} \in \overline{\mathbb{m}}^{c,loc}$ satisfy $\lim_{t \to \infty} \langle M \rangle_t = \infty$, a.s. P. Define, for each $0 \leq s < \infty$, the optional time

(4.14)
$$T(s) = \inf\{t \geq 0; \ \langle M \rangle_t > s\}.$$

Then the "time-changed process"

$$B = \{B_s \overset{\Delta}{=} M_{T(s)}, \ \mathcal{G}_s \overset{\Delta}{=} \mathcal{F}_{T(s)}; \ 0 \leq s < \infty\}$$

is a standard one-dimensional Brownian motion. In particular, the filtration $\{\mathcal{G}_s\}$ satisfies the usual conditions and we have, a.s. P:

(4.15)
$$M_t = B_{\langle M \rangle_t}; \quad 0 \leq t < \infty.$$

<u>Proof:</u>

Each T(s) is optional because, by Problem 4.5(iii),

$\{T(s) < t\} = \{\langle M \rangle_t > s\} \in \mathfrak{F}_t$. Just as in Problem 4.4(ii), the standing assumption that $\{\mathfrak{F}_t\}$ satisfies the usual conditions implies that $\{\mathcal{G}_s\}$ does also. Furthermore, for each t, $\langle M \rangle_t$ is a stopping time for the filtration $\{\mathcal{G}_s\}$ because, again by Problem 4.5(iii),

$$\{\langle M \rangle_t \leq s\} = \{T(s) \geq t\} \in \mathfrak{F}_{T(s)} = \mathcal{G}_s; \quad 0 \leq s < \infty.$$

Let us choose $0 \leq s_1 < s_2$ and consider the martingale $\{\widetilde{M}_t = M_{t \wedge T(s_2)}, \mathfrak{F}_t; 0 \leq t < \infty\}$ for which we have

$$\langle \widetilde{M} \rangle_t = \langle M \rangle_{t \wedge T(s_2)} \leq \langle M \rangle_{T(s_2)} = s_2; \quad 0 \leq t < \infty$$

by Problem 4.5(ii). It follows from Problem 1.5.22 that both $\widetilde{M}$ and $\widetilde{M}^2 - \langle \widetilde{M} \rangle$ are uniformly integrable. The Optional Sampling Theorem 1.3.20 implies

$$E[B_{s_2} - B_{s_1} | \mathcal{G}_{s_1}] = E[\widetilde{M}_{T(s_2)} - \widetilde{M}_{T(s_1)} | \mathfrak{F}_{T(s_1)}] = 0; \quad \text{a.s. } P,$$

$$E[(B_{s_2} - B_{s_1})^2 | \mathcal{G}_{s_1}] = E[(\widetilde{M}_{T(s_2)} - \widetilde{M}_{T(s_1)})^2 | \mathfrak{F}_{T(s_1)}]$$

$$= E[\langle \widetilde{M} \rangle_{T(s_2)} - \langle \widetilde{M} \rangle_{T(s_1)} | \mathfrak{F}_{T(s_1)}]$$

$$= s_2 - s_1; \qquad \text{a.s. } P.$$

Consequently, $B = \{B_s, \mathcal{G}_s; 0 \leq s < \infty\}$ is a square-integrable martingale with quadratic variation $\langle B \rangle_s = s$. We shall know that

B is a standard Brownian motion as soon as we establish its continuity (Theorem 3.13).

For fixed $\omega \in \Omega$, $s \mapsto B_s(\omega)$ is the composition of the right-continuous function $s \mapsto T_s(\omega)$ and the continuous function $t \mapsto M_t(\omega)$. The jumps in $T_s(\omega)$ correspond to flat stretches in $\langle M \rangle_t(\omega)$, i.e., $t_1 \stackrel{\Delta}{=} T_{s-}(\omega) < T_s(\omega) \stackrel{\Delta}{=} t_2$ if and only if $\langle M \rangle_{t_1}(\omega) = \langle M \rangle_{t_2}(\omega)$. We must show that for all ω in some $\Omega^* \subseteq \Omega$ with $P(\Omega^*) = 1$, we have:

(4.16) $\langle M \rangle_{t_1}(\omega) = \langle M \rangle_t(\omega)$ for some $0 \leq t_1 < t \Rightarrow M_{t_1}(\omega) = M_t(\omega)$.

If implication (4.16) is valid under the additional assumption that t_1 is rational, then, because of the continuity of $\langle M \rangle$ and M, it is valid even without this assumption. For $t_1 \geq 0$, t_1 rational, define

$$\sigma = \inf\{t > t_1 : \langle M \rangle_t > \langle M \rangle_{t_1}\},$$

$$N_s = M_{(t_1+s) \wedge \sigma} - M_{t_1}, \quad 0 \leq s < \infty,$$

so $\{N_s, \mathcal{F}_{t_1+s}; 0 \leq s < \infty\}$ is in $\mathbb{m}^{c,loc}$ and

$$\langle N \rangle_s = \langle M \rangle_{(t_1+s) \wedge \sigma} - \langle M \rangle_{t_1} = 0, \quad \text{a.s. } P.$$

It follows that there is a set $\Omega(t_1) \subseteq \Omega$ with $P(\Omega(t_1)) = 1$ such that for all $\omega \in \Omega(t_1)$,

$$\langle M\rangle_{t_1}(\omega) = \langle M\rangle_t(\omega), \text{ for some } t > t_1 \Rightarrow M_{t_1}(\omega) = M_t(\omega).$$

The union of all such sets $\Omega(t_1)$ as t_1 ranges over the nonnegative rationals will serve as Ω^*, so that implication (4.16) is valid for each $\omega \in \Omega^*$.

It remains to prove (4.15) for all $0 \leq t < \infty$. If, for $\omega \in \Omega^*$, we have t in the range of $T.(\omega)$, then there is some $s \geq 0$ for which $t = T_s(\omega)$ and (4.15) is a consequence of the definition $B_s = M_{T(s)}$ and Problem 4.5(ii). Now $\lim\limits_{s \to \infty} T_s(\omega) = \infty$, so if t is not in the range of $T.(\omega)$, then there must be some $s \geq 0$ such that $t_1 \triangleq T_{s-}(\omega) \leq t < T_s(\omega) \triangleq t_2$ where we define $T_{0-}(\omega) = 0$. This means that $s = \langle M\rangle_t(\omega) = \langle M\rangle_{t_2}(\omega)$, and implication (4.16) yields

$$M_t(\omega) = M_{t_2}(\omega) = M_{T_s(\omega)}(\omega) = B_s(\omega) = B_{\langle M\rangle_t(\omega)}(\omega).$$

$\square$

<u>4.7 Problem</u>: We cannot expect to be able to define the stochastic integral $\int_0^1 X_s dW_s$ with respect to Brownian motion W for measurable adapted processes X which do not satisfy $\int_0^1 X_s^2 ds < \infty$ a.s. Indeed, show that if

$$\int_0^t X_s^2 ds < \infty \text{ a.s.}, \quad 0 \leq t < 1,$$

but

$$\int_0^1 X_s^2 ds = \infty \quad \text{a.s.,}$$

then

$$P[\overline{\lim_{t \uparrow 1}} \int_0^t X_s dW_s = - \varliminf_{t \uparrow 1} \int_0^t X_s dW_s = +\infty] = 1.$$

Let us state and discuss the multivariate extension of Theorem 4.6. The proof will be given later in this section.

4.8 Theorem: F. Knight (1971)

Let $M = \{M_t = (M_t^{(1)}, \ldots, M_t^{(d)}), \mathcal{F}_t; 0 \le t < \infty\}$ be a continuous, adapted process with $M^{(i)} \in m^{c,loc}$,

$$\lim_{t \to \infty} \langle M^{(i)} \rangle_t = \infty; \quad \text{a.s. } P, \text{ and}$$

(4.17) $\langle M^{(i)}, M^{(j)} \rangle_t = 0; \quad 1 \le i \ne j \le d, \quad 0 \le t < \infty.$

Define

$$T_i(s) = \inf\{t \ge 0; \langle M^{(i)} \rangle_t > s\}; \quad 0 \le s < \infty, \quad 1 \le i \le d,$$

so that for each i and s, the random time $T_i(s)$ is optional for $\{\mathcal{F}_t\}$. Then the processes

$$B_s^{(i)} \triangleq M_{T_i(s)}^{(i)}; \quad 0 \le s < \infty, \quad 1 \le i \le d,$$

are __independent__, standard, one-dimensional Brownian motions.

<u>Discussion of Theorem 4.8</u>: The only assertion in Theorem 4.8 which is not already contained in Theorem 4.6 is the independence of the Brownian motions $B^{(i)}$; $1 \leq i \leq d$. Theorem 4.6 states, in fact, that $B^{(i)}$ is a Brownian motion relative to the filtration $\{\mathcal{G}_s^{(i)} \triangleq \mathcal{F}_{T_i(s)}\}_{s \geq 0}$, but, of course, these filtrations are not independent for different values of i because $\mathcal{G}_\infty^{(i)} = \mathcal{F}_\infty$; $1 \leq i \leq d$. The independence claim is that the σ-fields $\mathcal{F}_\infty^{B^{(1)}}, \mathcal{F}_\infty^{B^{(2)}}, \dots, \mathcal{F}_\infty^{B^{(d)}}$ are independent, where $\{\mathcal{F}_s^{B^{(i)}}\}$ is the filtration generated by $B^{(i)}$. This claim would follow easily if assumption (4.17) were sufficient to guarantee the independence of $M^{(i)}, M^{(j)}$ for $i \neq j$; in general, however, this is not the case. Indeed, if $W = \{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ is a standard Brownian motion, then, with

$$M_t^{(1)} \triangleq \int_0^t 1_{\{W_s \geq 0\}} dW_s, \quad M_t^{(2)} = \int_0^t 1_{\{W_s < 0\}} dW_s; \quad 0 \leq t < \infty,$$

we have $M^{(1)}, M^{(2)} \in \mathfrak{m}^{c,loc}$ and

$$\langle M^{(1)}, M^{(2)} \rangle_t = \int_0^t 1_{\{W_s \leq 0\}} 1_{\{W_s < 0\}} ds = 0; \quad 0 \leq t < \infty.$$

But $M^{(1)}$ and $M^{(2)}$ are not independent, for if they were, $\langle M^{(1)} \rangle$ and $\langle M^{(2)} \rangle$ would also be independent. On the contrary, we have

$$\langle M^{(1)} \rangle_t + \langle M^{(2)} \rangle_t = \int_0^t 1_{\{W_s \geq 0\}} ds + \int_0^t 1_{\{W_s < 0\}} ds = t; \quad 0 \leq t < \infty.$$

F. Knight's remarkable theorem states that when we apply the proper time-changes to these two intricately connected margingales, and then _forget the time-changes_, independent martingales are obtained. Forgetting the time changes is accomplished by passing from the filtrations $\{\mathcal{G}_s^{(i)}\}$ to the less informative filtrations $\{\mathcal{F}_s^{B^{(i)}}\}$.

We shall use this example in Section 5. to prove the independence of the positive and negative excursion processes associated with a one-dimensional Brownian motion. □

In preparation for the proof of Theorem 4.8, we consider a different class of representation results, those for which we begin with a Brownian motion rather than constructing it. We take as given a standard, one-dimensional Brownian motion $W = \{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$ on a probability space $(\Omega, \mathcal{F}, P)$, and we assume $\{\mathcal{F}_t\}$ satisfies the usual conditions. For $0 < T < \infty$, we recall from Lemma 2.1' that $\mathcal{L}_T^*$ is a closed subspace of the Hilbert space $\mathcal{H}_T$. The mapping

$$\mathcal{H}_T \supseteq \mathcal{L}_T^* \ni X \mapsto I_T(X) \in \mathcal{L}_2(\Omega, \mathcal{F}, P)$$

preserves inner products (see (2.20)):

$$E \int_0^T X_t Y_t \, dt = E[I_T(X) I_T(Y)].$$

Since any convergent sequence in

(4.18)
$$\mathbb{R}_T \triangleq \{I_T(X); \quad X \in \mathcal{L}_T^*\}$$

is also Cauchy, its preimage sequence in $\mathcal{L}_T^*$ must have a limit in $\mathcal{L}_T^*$. It follows that $\mathbb{R}_T$ is closed in $\mathcal{L}_2(\Omega, \mathfrak{F}_T, P)$, a fact we shall need shortly.

Let us denote by $\mathbb{m}_2^*$ the subset of $\mathbb{m}_2^c$ which consists of stochastic integrals

$$I_t(X) = \int_0^t X_s dW_s; \quad 0 \le t < \infty,$$

of processes $X \in \mathcal{L}^*$:

(4.19)
$$\mathbb{m}_2^* \triangleq \{I(X); \ X \in \mathcal{L}^*\} \subseteq \mathbb{m}_2^c \subseteq \mathbb{m}_2.$$

Recall from Definition 1.5.5 the concept of orthogonality in $\mathbb{m}_2$. We have the following fundamental decomposition result.

<u>4.9 Proposition</u>: For every $M \in \mathbb{m}_2$, we have the decomposition $M = N + Z$, where $N \in \mathbb{m}_2^*$, $Z \in \mathbb{m}_2$, and Z is orthogonal to every element of $\mathbb{m}_2^*$.

<u>Proof</u>:

We have to show the existence of a process $Y \in \mathcal{L}^*$ such that $M = I(Y) + Z$, where $Z \in \mathbb{m}_2$ has the property

(4.20)
$$\langle Z, I(X) \rangle = 0; \quad \forall X \in \mathcal{L}^*.$$

Such a decomposition is unique (up to indistinguishability); indeed, if we have $M = I(Y') + Z' = I(Y'') + Z''$ with $Y', Y'' \in \mathcal{L}$

and both Z and Z'' satisfy (4.20), then

$$Z \stackrel{\Delta}{=} Z'' - Z' = I(Y'-Y'')$$

is in $\mathbb{m}_2^c$ and $\langle Z \rangle = \langle Z, I(Y'-Y'') \rangle = 0$. It follows that $P[Z_t = 0 \text{ for every } 0 \leq t < \infty] = 1$.

It suffices, therefore, to establish the decomposition for every finite time interval $[0,T]$; by uniqueness, we can then extend it to the entire half-line $[0,\infty)$. Let us fix $T > 0$, let $\mathbb{R}_T$ be the closed subspace of $\mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ defined by (4.18), and let $\mathbb{R}_T^\perp$ denote its orthogonal complement. The random variable M_T is in $\mathcal{L}_2(\Omega, \mathcal{F}_T, P)$, so it admits the decomposition

$$(4.21) \qquad M_T = I_T(Y) + Z_T,$$

where $Y \in \mathcal{L}_T^*$ and $Z_T \in \mathcal{L}_2(\Omega, \mathcal{F}_T, P)$ satisfies

$$(4.22) \qquad E[Z_T I_T(X)] = 0; \qquad \forall X \in \mathcal{L}_T^*.$$

Let us denote by $Z = \{Z_t, \mathcal{F}_t; 0 \leq t < \infty\}$ a right-continuous version of the martingale $E(Z_T | \mathcal{F}_t)$ (Theorem 1.3.11). Note that $Z_t = Z_T$ for $t \geq T$. Obviously $Z \in \mathbb{m}_2$ and, conditioning (4.21) on $\mathcal{F}_t$, we obtain

$$(4.23) \qquad M_t = I_t(Y) + Z_t; \quad 0 \leq t \leq T, \text{ a.s. } P.$$

It remains to show that Z is orthogonal to every square-integrable martingale of the form $I(X)$; $X \in \mathcal{L}^*$, or equivalently, that $\{Z_t I_t(X), \mathcal{F}_t; 0 \leq t < \infty\}$ is a martingale. But we know

from Problem 1.3.24 that this amounts to having $E[Z_S I_S(X)] = 0$ for every bounded stopping time S of the filtration $\{\mathfrak{F}_t\}$. For such an S, we have

$$Z_S I_S(X) = Z_{S \wedge T} I_{S \wedge T}(X), \quad \text{a.s. } P$$

because $Z_t = Z_T$, $X_t = 0$ for $t > T$. Thus, we need only consider $S \leq T$. From (2.21) we have $I_S(X) = I_T(\tilde{X})$, where $\tilde{X}_t(\omega) = X_t(\omega) 1_{\{t \leq S(\omega)\}}$ is a process in $\mathcal{L}_T^*$. Therefore,

$$E[Z_S I_S(X)] = E[E(Z_T | \mathfrak{F}_S) I_S(X)]$$

$$= E[Z_T I_T(\tilde{X})] = 0$$

by virtue of (4.22). □

It is useful to have sufficient conditions under which the classes $\mathfrak{m}_2^c$ and $\mathfrak{m}_2^*$ actually coincide; in other words, the component Z in the decomposition of Proposition 4.9 is actually the trivial martingale $Z \equiv 0$. One such condition is that the filtration $\{\mathfrak{F}_t\}$ is the augmentation under P of the filtration $\{\mathfrak{F}_t^W\}$ generated by the Brownian motion W. We recall from Problem 2.7.6 and Proposition 2.7.7 that this augmented filtration is continuous. We state and prove this result in several dimensions.

4.10 Theorem: <u>Representation of Brownian, square-integrable</u>
<u>martingales as stochastic integrals</u>

Let $W = \{W_t = (W_t^{(1)}, \ldots, W_t^{(d)}), \mathfrak{F}_t; 0 \leq t < \infty\}$ be a d-dimensional Brownian motion on $(\Omega, \mathfrak{F}, P)$, and let $\{\mathfrak{F}_t\}$ be

the augmentation under P of the filtration $\{\mathfrak{F}_t^W\}$ generated by W. Then, for any right-continuous, square-integrable martingale $M = \{M_t, \mathfrak{F}_t; 0 \leq t < \infty\}$ <u>relative to the Brownian filtration</u> $\{\mathfrak{F}_t\}$ with $M_0 = 0$ a.s., there exist progressively measurable processes $Y^{(j)} = \{Y_t^{(j)}, \mathfrak{F}_t; 0 \leq t < \infty\}$ such that

(4.23) $\qquad E \int_0^T (Y_t^j)^2 dt < \infty; \quad 1 \leq j \leq d, \quad 0 < T < \infty,$

and

(4.24) $\qquad M_t = \sum_{j=1}^d \int_0^t Y_s^{(j)} dW_s^{(j)}; \quad 0 \leq t < \infty.$

In particular, M is a.s. continuous.

<u>Proof</u>: We shall say that a progressively measurable process X satisfying $E \int_0^T X_t^2 dt < \infty; \quad 0 < T < \infty$, is in $\mathcal{L}^*$. We first prove by induction on d that there are processes $Y^{(1)}, \ldots, Y^{(d)}$ in $\mathcal{L}^*$ such that

(4.25) $\qquad Z_t \triangleq M_t - \sum_{j=1}^d \int_0^t Y_s^{(j)} dW_s^{(j)}; \quad 0 \leq t < \infty$

is orthogonal to every martingale of the form $\sum_{j=1}^d \int_0^t X_s^{(j)} dW_s^{(j)}$, where $X^{(j)} \in \mathcal{L}^*; \quad 1 \leq j \leq d$. If $d = 1$, this is a direct consequence of Proposition 4.9. Suppose such processes exist for $d - 1$, i.e.,

$$\tilde{Z}_t \triangleq M_t - \sum_{j=1}^{d-1} \int_0^t Y_s^{(j)} dW_s^{(j)}; \quad 0 \le t < \infty,$$

is orthogonal to $\sum_{j=1}^{d-1} \int_0^t X_s^{(j)} dW_s^{(j)}$ for all $X^{(j)} \in \mathcal{L}^*$; $1 \le j \le d$. Apply Proposition 4.9 to write

$$\tilde{Z}_t = \int_0^t Y_s^{(d)} dW_s^{(d)} + Z_t; \quad 0 \le t < \infty,$$

for some $Y^{(d)} \in \mathcal{L}^*$, where Z is orthogonal to $\int_0^t X_s^{(d)} dW_s^{(d)}$ for all $X^{(d)} \in \mathcal{L}^*$. For $1 \le j \le d - 1$ and $X^{(j)} \in \mathcal{L}^*$, we have

$$\langle Z, I^{W^{(j)}}(X^{(j)}) \rangle = \langle \tilde{Z}, I^{W^{(j)}}(X^{(j)}) \rangle - \langle I^{W^{(d)}}(Y^{(d)}), I^{W^{(j)}}(X^{(j)}) \rangle = 0.$$

Thus, we have the decomposition (4.25) for M. In particular

$$(4.26) \qquad \langle M, W^{(j)} \rangle_t = \int_0^t Y_s^{(j)} ds; \quad 0 \le t < \infty, \quad 1 \le j \le d.$$

Following Liptser and Shiryayev [1977, pp. 162-163], we now show that, P-a.s.,

$$Z_t = 0; \quad 0 \le t < \infty.$$

First, we show by induction on n that if $0 = s_0 \le s_1 \le \ldots \le s_n \le t$, and if the functions $f_k : \mathbb{R}^d \to \mathbb{C}$, $0 \le k \le n$ are bounded and measurable, then

$$(4.27) \qquad E[Z_t \cdot \prod_{k=0}^n f_k(W_{s_k})] = 0.$$

When $n = 0$, (4.27) can be verified by conditioning on $\mathfrak{F}_0$ and using the fact $Z_0 = 0$ a.s. Suppose now that (4.27) holds for some n and choose $s_n < t$. For $\theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$ fixed and $s_n \leq s \leq t$, define

$$\varphi(s) = E[Z_t \cdot \prod_{k=0}^{n} f_k(W_{s_k}) e^{i(\theta, W_s)}]$$

$$= E[Z_s \cdot \prod_{k=0}^{n} f_k(W_{s_k}) e^{i(\theta, W_s)}].$$

Using Itô's rule to justify the identity

$$e^{i(\theta, W_s)} = e^{i(\theta, W_{s_n})} + \sum_{j=1}^{d} i\theta_j \int_{s_n}^{s} e^{i(\theta, W_u)} dW_u^{(j)}$$

$$- \frac{\|\theta\|^2}{2} \int_{s_n}^{s} e^{i(\theta, W_u)} du,$$

we may write

$$(4.28) \quad E[Z_s e^{i(\theta, W_s)} | \mathfrak{F}_{s_n}] = Z_{s_n} e^{i(\theta, W_{s_n})} + \sum_{j=1}^{d} i\theta_j E[Z_s \int_{s_n}^{s} e^{i(\theta, W_u)} dW_u^{(j)} | \mathfrak{F}_{s_n}]$$

$$- \frac{\|\theta\|^2}{2} E[Z_s \int_{s_n}^{s} e^{i(\theta, W_u)} du | \mathfrak{F}_{s_n}].$$

But (4.25) and (4.26) imply

$$E[Z_s \int_{s_n}^{s} e^{i(\theta, W_u)} dW_u^{(j)} | \mathfrak{F}_{s_n}] = E[(Z_s - Z_{s_n}) \int_{s_n}^{s} e^{i(\theta, W_u)} dW_u^{(j)} | \mathfrak{F}_{s_n}] = 0.$$

Multiplying (4.28) by $\prod_{k=0}^{n} f_k(W_{s_k})$ and taking expectations, we obtain

$$(4.29) \quad \varphi(s) = \varphi(s_n) - \frac{\|\theta\|^2}{2} \int_{s_n}^{s} E[Z_s \cdot \prod_{k=1}^{n} f_k(W_{s_k}) e^{i(\theta, W_u)}] du$$

$$= \varphi(s_n) - \frac{\|\theta\|^2}{2} \int_{s_n}^{s} \varphi(u) du; \quad s_n \leq s \leq t.$$

By our induction hypothesis, $\varphi(s_n) = 0$, and the only solution to the integral equation (4.29) satisfying this initial condition is $\varphi(s) = 0$; $s_n \leq s \leq t$. Thus

$$(4.30) \quad E[Z_t \cdot \prod_{k=0}^{n} f_k(W_{s_k}) e^{i(\theta, W_s)}] = 0; \quad \theta \in \mathbb{R}^d.$$

With $D^{\pm} \triangleq \max\{\pm Z_t \cdot \prod_{k=0}^{n} f_k(W_{s_k}), 0\}$, we define two measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ by

$$\mu^{\pm}(\Gamma) = E[D^{\pm} 1_{\Gamma}(W_s)]; \quad \Gamma \in \mathcal{B}(\mathbb{R}^d).$$

Equation (4.30) implies

$$\int_{\mathbb{R}^d} e^{i(\theta, x)} \mu^{+}(dx) = \int_{\mathbb{R}^d} e^{i(\theta, x)} \mu^{-}(dx); \quad \theta \in \mathbb{R}^d,$$

and by the uniqueness theorem for Fourier transforms, we see that $\mu^{+} = \mu^{-}$. Thus

$$E[D^{+} f(W_s)] = E[D^{-} f(W_s)]; \quad s_n \leq s \leq t,$$

for any bounded, measurable $f : \mathbb{R}^d \to \mathbb{C}$. This proves (4.27) for $n + 1$ and completes the induction step.

A standard argument using the Dynkin System Theorem 2.5.1'
now shows that

$$(4.31) \qquad E[Z_t \xi] = 0$$

for every $\mathfrak{F}_t^W$-measurable indicator ξ, and thus, for every
$\mathfrak{F}_t^W$-measurable, bounded ξ. Since $\mathfrak{F}_t$ differs from $\mathfrak{F}_t^W$ only
by P-null sets, (4.31) also holds for every $\mathfrak{F}_t$-measurable,
bounded ξ. Setting $\xi = \text{sgn}(Z_t)$, we conclude that $Z_t = 0$ a.s. P.
Indistinguishability of Z from the process which is identically
zero follows now by right-continuity of its paths (Problem 1.1.5). □

4.11 **Problem**: Let $W = \{W_t = (W_t^{(1)}, \ldots, W_t^{(d)}), \mathfrak{F}_t; 0 \le t < \infty\}$ be a
d-dimensional Brownian motion as in Theorem 4.10. Let
$M = \{M_t, \mathfrak{F}_t; 0 \le t < \infty\}$ be a right-continuous local
martingale such that $M_0 = 0$ a.s. and $P[\lim_{s \uparrow t} M_s$ exists
and is finite for $0 < t < \infty] = 1$. Then there exists
progressively measurable processes $Y^{(j)} = \{Y_t^{(j)}, \mathfrak{F}_t;$
$0 \le t < \infty\}$ such that

$$\int_0^T (Y_t^{(j)})^2 dt < \infty; \quad 1 \le j \le d, \ 0 \le T < \infty,$$

and

$$M_t = \sum_{j=1}^d \int_0^t Y_s^{(j)} dW_s^{(j)}; \quad 0 \le t < \infty.$$

In particular, M is a.s. continuous.

4.12 Problem: Under the hypotheses of Theorem 4.10 and with

$0 < T < \infty$, let ξ be an $\mathfrak{F}_T$-measurable random variable

with $E\xi^2 < \infty$. Prove that there are progressively measurable

processes $Y^{(1)}, \ldots, Y^{(d)}$ satisfying

$$E \int_0^T (Y_t^{(j)})^2 dt < \infty; \quad 1 \leq j \leq d,$$

such that

(4.32) $\xi = E(\xi) + \sum_{j=1}^{d} \int_0^T Y_s^{(j)} dW_s^{(j)}$;   a.s. P.   □

We extend Problem 4.12 to include the case $T = \infty$. Recall

that for $M \in \mathfrak{m}_2^c$, we denote by $\mathcal{L}_\infty^*(M)$ the set of processes X

which are progressively measurable with respect to the filtration

of M and which satisfy $E \int_0^\infty X_t^2 d\langle M \rangle_t < \infty$. According to

Problem 1.5.22, when $X \in \mathcal{L}_\infty^*(M)$, we have $\int_0^\infty X_t dM_t$ defined a.s.P. If W is a d-dimensional Brownian motion, we denote by $\mathcal{L}_\infty^*(W)$ the set of processes X which are progressively measurable with respect to the filtration of W and which satisfy $E \int_0^\infty X_t^2 dt < \infty$.

4.13 <u>Corollary</u>: Under the hypotheses of Theorem 4.10, assume that ξ is an $\mathcal{F}_\infty$-measurable random variable with $E\xi^2 < \infty$. Then there are processes $Y^{(1)}, \ldots, Y^{(d)}$ in $\mathcal{L}_\infty^*(W)$ such that

$$\xi = E(\xi) + \sum_{j=1}^d \int_0^\infty Y_s^{(j)} dW_s^{(j)}; \quad \text{a.s.P.}$$

<u>Proof</u>:

Assume without loss of generality that $E(\xi) = 0$, and let M_t be a right-continuous modification of $E(\xi | \mathcal{F}_t)$. According to Theorem 4.10, there exist progressively measurable $Y^{(1)}, \ldots, Y^{(d)}$ satisfying (4.23) and (4.24). Jensen's inequality implies $M_t^2 \leq E(\xi^2 | \mathcal{F}_t)$, so

$$\sum_{j=1}^d E \int_0^t (Y_s^{(j)})^2 ds = E\langle M \rangle_t = E(M_t^2) \leq E(\xi^2) < \infty; \quad 0 \leq t < \infty.$$

Hence, $Y^{(j)} \in \mathcal{L}_\infty^*(W)$ and $M_\infty \overset{\Delta}{=} \int_0^\infty Y_s^{(j)} dW_s^{(j)}$ is defined for $1 \leq j \leq d$. Problem 1.3.18 shows that $M_\infty = E(\xi | \mathcal{F}_\infty) = \xi$. $\qquad \square$

In one dimension, there is a representation result similar to that of Corollary 4.13 in which Brownian motion is replaced by a continuous, local martingale M. This result is instrumental in our proof of Theorem 4.8.

4.14 Proposition: Let $M = \{M_t, \mathfrak{F}_t; 0 \leq t < \infty\}$ be in $\mathfrak{m}^{c,loc}$ and assume that $\lim_{t \to \infty} \langle M \rangle_t = \infty$, a.s.P. Define $T(s)$ by (4.14) and let B be the one-dimensional Brownian motion

$$B = \{B_s \overset{\Delta}{=} M_{T(s)}; \quad 0 \leq s < \infty\}$$

as in Theorem 4.6, except now we take the filtration $\{\mathcal{E}_s\}$ for B to be the augmentation with respect to P of the filtration $\{\mathfrak{F}_s^B\}$ generated by B. Then, for every $\mathcal{E}_\infty$ -measurable random variable ξ satisfying $E\xi^2 < \infty$, there is a process $X \in \mathcal{L}_\infty^*(M)$ for which

$$(4.33) \qquad \xi = E(\xi) + \int_0^\infty X_t dM_t; \quad \text{a.s.P.}$$

Proof:

Let $Y = \{Y_s, \mathcal{E}_s; 0 \leq s < \infty\}$ be the progressively measurable process of Corollary 4.13 for which we have

$$(4.34) \qquad E \int_0^\infty Y_s^2 ds < \infty,$$

$$(4.35) \qquad \xi = E(\xi) + \int_0^\infty Y_s dB_s.$$

Define $\tilde{X}_t = Y_{\langle M \rangle_t}; \quad 0 \leq t < \infty.$

We show how to modify $\tilde{X}$ so as to obtain a progressively measurable process X. Note that because $\{\mathcal{G}_s \overset{\Delta}{=} \mathcal{F}_{T(s)}\}$ contains $\{\mathcal{F}_s^B\}$ and satisfies the usual conditions (Theorem 4.6), we have $\mathcal{E}_s \subseteq \mathcal{G}_s$; $0 \leq s \leq \infty$. Consequently, Y is progressively measurable relative to $\{\mathcal{G}_s\}$. If Y is a simple process, it is left-continuous (c.f. Definition 2.2), and it is straightforward to show using Problem 4.5 that $\{Y_{\langle M \rangle_t}; 0 \leq t < \infty\}$ is a left-continuous process adapted to $\{\mathcal{F}_t\}$, and hence progressively measurable (Proposition 1.1.13). In the general case, let $\{Y^{(n)}\}_{n=1}^{\infty}$ be a sequence of progressively measurable (relative to $\{\mathcal{E}_s\}$) simple processes for which

$$\lim_{n \to \infty} E \int_0^{\infty} |Y_s^{(n)} - Y_s|^2 ds = 0.$$

(Use Proposition 2.7 and (4.34)). A change of variables (Problem 4.5(iv)) yields

(4.36) $$\lim_{n \to \infty} E \int_0^{\infty} |X_t^{(n)} - \tilde{X}_t|^2 d\langle M \rangle_t = 0,$$

where $X_t^{(n)} \overset{\Delta}{=} Y_{\langle M \rangle_t}^{(n)}$. In particular, the sequence $\{X_t^{(n)}\}_{n=1}^{\infty}$ is Cauchy in $\mathcal{L}_{\infty}^*(M)$, and so, by Lemma 2.1', converges to a limit $X \in \mathcal{L}_{\infty}^*(M)$. From (4.36), we must have

(4.37) $$E \int_0^{\infty} |\tilde{X}_t - X_t|^2 d\langle M \rangle_t = 0.$$

It remains to prove (4.33), which, in light of (4.35), will follow from

$$(4.38) \qquad \int_0^\infty Y_s dB_s = \int_0^\infty X_t dM_t; \quad \text{a.s.P.}$$

We leave the proof of this equality as a problem. □

4.15 <u>Problem</u>: Prove (4.38). (<u>Hint</u>: Consider first the case where Y is simple).

<u>Proof of F. Knight's Theorem 4.8</u>:

Our proof is based on that of Meyer [1971]. Under the hypotheses of Theorem 4.8, let $\{\mathcal{E}_s^{(i)}\}$ be the augmentation of the filtration $\{\mathcal{F}_s^{B^{(i)}}\}$ generated by $B^{(i)}$; $1 \leq i \leq d$. All we need to show is that $\mathcal{E}_\infty^{(1)}, \ldots, \mathcal{E}_\infty^{(d)}$ are independent.

For each i, let $\xi^{(i)}$ be a bounded, $\mathcal{E}_\infty^{(i)}$-measurable random variable. According to Proposition 4.14, there is, for each i, a progressively measurable process $X^{(i)} = \{X_t^{(i)}, \mathcal{F}_t; 0 \leq t < \infty\}$ which satisfies

$$E \int_0^\infty (X_t^{(i)})^2 d\langle M^{(i)}\rangle_t < \infty, \quad 1 \leq i \leq d,$$

and for which

$$\xi^{(i)} = E(\xi^{(i)}) + \int_0^\infty X_t^{(i)} dM_t^{(i)}; \quad 1 \leq i \leq d.$$

Let us assume for the moment that

$$(4.39) \qquad E(\xi^{(i)}) = 0; \quad 1 \leq i \leq d,$$

and define $\{\mathfrak{F}_t\}$-martingales

$$\xi_t^{(i)} \triangleq \int_0^t X_s^{(i)} dM_s^{(i)}; \quad 0 \leq t < \infty, \quad 1 \leq i \leq d.$$

Ito's rule and (4.17) imply that

$$(4.40) \quad \prod_{i=1}^d \xi_t^{(i)} = \sum_{i=1}^d \int_0^t \prod_{j \neq i} \xi_s^{(j)} X_s^{(i)} dM_s^{(i)}; \quad 0 \leq t < \infty.$$

In order to let $t \to \infty$ in (4.40), we must show that

$$(4.41) \quad E \int_0^\infty (\prod_{j \neq i} \xi_s^{(j)} X_s^{(i)})^2 d\langle M^{(i)} \rangle_s < \infty; \quad 1 \leq i \leq d.$$

Repeated application of Holder's inequality yields

$$E \int_0^t (\prod_{j \neq i} \xi_s^{(j)} X_s^{(i)})^2 d\langle M^{(i)} \rangle_s$$

$$\leq E\{ \prod_{j \neq i} [\sup_{0 \leq s \leq t} (\xi_s^{(j)})^2] \cdot \langle \xi^{(i)} \rangle_t \}$$

$$\leq [E \sup_{0 \leq s \leq t} (\xi_s^{(1)})^4]^{1/2} \cdot [E \sup_{0 \leq s \leq t} (\xi_s^{(2)})^8]^{1/4} \cdot \dots$$

$$\cdot \left[E \sup_{0 \leq s \leq t} (\xi_s^{(d)})^{2^{d+1}} \right]^{2^{-d}} \cdot \left[E\langle \xi^{(i)} \rangle_t^{2^d} \right]^{2^{-d}}; \quad 0 \leq t < \infty.$$

For $m \geq 1$, Doob's maximal inequality (Theorem 1.3.6(iv)) gives

$$E \sup_{0 \leq s \leq t} (\xi_s^{(j)})^{2m} \leq \left(\frac{2m}{2m-1} \right)^{2m} E(\xi_t^{(j)})^{2m}$$

$$\leq \left(\frac{2m}{2m-1} \right)^{2m} E(\xi^{(j)})^{2m} < \infty.$$

We have from Proposition 3.22:

$$B \cdot E \langle \xi^{(i)} \rangle_t^{2^d} \leq E(\xi_t^{(i)})^{2^{d+1}} \leq E(\xi^{(i)})^{2^{d+1}} < \infty,$$

for some positive constant B which does not depend on t. Thus, (4.41) holds, and letting $t \to \infty$ in (4.40), we obtain the representation

$$\prod_{i=1}^{d} \xi^{(i)} = \sum_{i=1}^{d} \int_0^\infty \prod_{j \neq i} \xi_s^{(j)} X_s^{(i)} dM_s^{(i)}.$$

The right-hand side, being a sum of martingale last elements (Problem 2.16'), has expectation zero. Thus, under assumption (4.39), we have $E \prod_{i=1}^{d} \xi^{(i)} = 0$. Equivalently, we have shown that for any set of bounded random variables $\xi^{(1)}, \ldots, \xi^{(d)}$, where each $\xi^{(i)}$ is $\mathcal{E}_\infty^{(i)}$-measurable, the equality

$$(4.42) \qquad E \prod_{i=1}^{d} [\xi^{(i)} - E(\xi^{(i)})] = 0$$

holds. Using (4.42), one can show by a simple argument of induction on d that

$$E \prod_{i=1}^{d} \xi^{(i)} = \prod_{i=1}^{d} E\xi^{(i)}.$$

Taking $\xi^{(i)} = 1_{A_i}$; $A_i \in \mathcal{E}_\infty^{(i)}$, $1 \leq i \leq d$, we conclude that the σ-fields $\mathcal{E}_\infty^{(1)}, \ldots, \mathcal{E}_\infty^{(d)}$ are independent. $\qquad \square$

What happens if the random variable ξ in Problem 4.12 is not square-integrable, but merely a.s. finite? It is reasonable to guess that there is still a representation of the form (4.32), where now the integrands $Y^{(1)}, \ldots, Y^{(d)}$ can only be expected to satisfy

$$(4.43) \qquad \int_0^T (Y_t^{(j)})^2 dt < \infty; \quad \text{a.s. } P.$$

In fact, an even stronger result is true. For any fixed j, there is a progressively measurable process $Y^{(j)}$ such that (4.43) holds and

$$\xi = \int_0^T Y_s^{(j)} dW_s^{(j)}; \quad \text{a.s. } P.$$

Thus, we only need the one-dimensional Brownian motion $\{W_t^{(j)}, \mathcal{F}_t; 0 \le t < \infty\}$ for the representation, even though $\{\mathcal{F}_t\}$ is the augmentation of the filtration generated by the d-dimensional Brownian motion $\{(W_t^{(1)}, \ldots, W_t^{(d)}); 0 \le t < \infty\}$. This is a special case of the following theorem. The left-continuity of $\{\mathcal{F}_t\}$ follows from Problem 2.7.6.

4.16 **Theorem: Dudley (1977).**

Let $W = \{W_t, \mathcal{F}_t; 0 \le t < \infty\}$ be a standard, one-dimensional Brownian motion, where, in addition to satisfying the usual conditions, $\{\mathcal{F}_t\}$ is left-continuous. If $0 < T < \infty$ and ξ is an $\mathcal{F}_T$-measurable, a.s. finite random variable, then there exists a progressively measurable process

$$Y = \{Y_t, \mathcal{F}_t; 0 \le t \le T\} \text{ satisfying}$$

(4.44)
$$\int_0^T Y_t^2 dt < \infty; \quad P\text{-a.s.,}$$

such that

(4.45)
$$\xi = \int_0^T Y_t dW_t; \quad P\text{-a.s.}$$

$\square$

We present the beautiful proof of this result provided by Dudley [1977], a proof which uses the representation of stochastic integrals as time-changed Brownian motion.

4.17   Lemma:

Consider numbers $0 \le a < b < \infty$ and a measurable, nonrandom, function $\varphi : [a,b) \to \mathbb{R}$ for which

$A(t) \triangleq \int_a^t \varphi^2(s)\,ds$ is finite and positive on (a,b), with $\lim_{t \uparrow b} A(t) = \infty$. Let $W = \{W_t, \mathcal{F}_t; 0 \le t < \infty\}$ be a standard, one-dimensional, Brownian motion and X an $\mathcal{F}_a$-measurable, a.s. finite random variable. We set

$M_t = \int_a^t \varphi(s)\,dW_s; \quad a \le t < b$, and

$$\tau(\omega) \triangleq \begin{cases} \inf\{t \in [a,b); M_t(\omega) = X(\omega)\}; & \text{if } \{\ldots\} \ne \emptyset, \\ b & ; \text{ otherwise} \end{cases}$$

Then τ is a stopping time of $\{\mathcal{F}_t\}$ with $P[a \le \tau < b] = 1$. Furthermore, the random variable $G(\omega) \triangleq A(\tau(\omega))$ obeys

$$P[G > u | \mathcal{F}_a] \le \frac{|X|}{\sqrt{u}} \wedge 1; \quad \text{a.s. } P, \ u > 0.$$

Proof: The change of clock

$$T(s) \triangleq \inf\{t \in [a,b); A(t) > s\}; \quad 0 \leq s < \infty,$$

is deterministic, and $T(\infty) \triangleq \lim_{s \to \infty} T(s) = b$.   The continuous

local martingale $B = \{B_s \triangleq M_{T(s)}) \; \mathcal{G}_s \triangleq \mathcal{F}_{T(s)}; \quad 0 \leq s < \infty\}$ has

quadratic variation $\langle B \rangle_s = A(T(s)) = s$ (c.f. Problem 4.5(ii)),

and so B is a Brownian motion.

Now X and $\{B_s; \; 0 \leq s < \infty\}$ are independent, so

$B - X = \{B_s - X, \; \mathcal{G}_s; \; 0 \leq s < \infty\}$ is a Brownian motion with

initial distribution $\mu(dx) = P[-X \in dx]$.   Define the passage

time

$$\sigma = \begin{cases} \inf\{0 \leq s < \infty; \quad B_s - X = 0\}; & \text{if } \{...\} \neq \emptyset, \\ \infty, & \text{otherwise.} \end{cases}$$

We have from (2.6.3) and the Markov property:

$$P[\sigma \in ds \mid \mathcal{G}_0] = \frac{|x|}{\sqrt{2\pi s^3}} e^{-\frac{x^2}{2s}} ds; \quad 0 \leq s < \infty.$$

In particular, $P[0 \leq \sigma < \infty] = 1$. Now $\tau = T(\sigma)$, so $P[a \leq \tau < b] = 1$.

Furthermore, $G = A(T(\sigma)) = \sigma$ and $\mathcal{F}_a = \mathcal{G}_0$,   so

$$P[G > u \mid \mathcal{F}_a] \leq 1 \wedge \int_u^\infty \frac{|x|}{\sqrt{2\pi s^3}} ds \leq \frac{|x|}{\sqrt{u}} \wedge 1; \quad \text{a.s.P, } u \geq 0.$$

$\square$

Proof of Theorem 4.16:

Let $\eta = \arctan \xi$, so that $|\eta(\omega)| \leq \frac{\pi}{2}$ for all $\omega \in \Omega$.   For

any sequence of positive numbers $\{a_n\}_{n=1}^\infty$ strictly increasing to T,

the discrete-time martingale $\{\eta_n \triangleq E[\eta|\mathfrak{I}_{a_n}], \mathfrak{I}_{a_n}; n \geq 1\}$ converges

a.s. to $E[\eta|\sigma(\bigcup_{n=1}^{\infty} \mathfrak{I}_{a_n})]$ (Chung [1974], Theorem 9.4.6] or Ash

[1972, Theorem 7.6.2]), and this limit is actually $E[\eta|\mathfrak{I}_T] = \eta$

because of the left-continuity of $\{\mathfrak{I}_t\}$. Consequently, $\xi_n \triangleq \tan \eta_n$

converges a.s. to ξ, and we can extract a subsequence, which we

also denote $\{\xi_n\}_{n=1}^{\infty}$, for which

$$(4.46) \qquad P[|\xi_n - \xi| > \frac{1}{n^3}] \leq \frac{1}{n^2}; \quad n \geq 1.$$

Because $\frac{1}{(n-1)^3} + \frac{1}{n^3} \leq \frac{4}{n(n-1)^2}$ for $n \geq 2$, we have from (4.46):

$$(4.47) \quad P[|\xi_n - \xi_{n-1}| > \frac{4}{n(n-1)^2}] \leq P[|\xi_{n-1} - \xi| > \frac{1}{(n-1)^3}] + P[|\xi_n - \xi| > \frac{1}{n^3}]$$

$$\leq \frac{1}{(n-1)^2} + \frac{1}{n^2} \leq \frac{2}{(n-1)^2}; \quad n \geq 2.$$

Now we construct the progressively measurable process Y

satisfying (4.44) and (4.45). For $n \geq 1$, we let $\varphi_n(t) = \frac{1}{a_{n+1} - t}$;

$a_n \leq t < a_{n+1}$ and observe that $A_n(t) \triangleq \int_{a_n}^{t} \varphi_n^2(s) ds$ is positive

and finite on (a_n, a_{n+1}), and increases to infinity as $t \uparrow a_{n+1}$.
According to Lemma 4.16, the stopping time

$$\tau_n \triangleq \begin{cases} \inf\{t \in [a_n, a_{n+1}); \int_{a_n}^{t} \varphi(s) dW_s = \xi_n - \xi_{n-1}\}; \\ \qquad\qquad\qquad \text{if } \{\cdots\} \neq \emptyset, \\ a_{n+1}; \quad \text{otherwise} \end{cases}$$

satisfies $P[a_n \leq \tau_n < a_{n+1}] = 1$ for every $n \geq 1$, where we take $\xi_0 \triangleq 0$. It follows that for P-a.e. $\omega \in \Omega$, the sequence $\{\tau_n(\omega)\}_{n=1}^{\infty}$ is strictly increasing and converges to T as $n \to \infty$. We define

$$Y_t(\omega) \triangleq \sum_{n=1}^{\infty} \varphi_n(t) 1_{[a_n, \tau_n(\omega))}(t); \quad 0 \leq t \leq T.$$

This defines an adapted, right-continuous (and hence, progressively measurable) process such that for every $n \geq 1$,

$$(4.48) \qquad \int_0^{a_{n+1}} Y_t dW_t = \sum_{j=1}^{n} \int_{a_j}^{\tau_j} Y_t dW_t = \sum_{j=1}^{n} (\xi_j - \xi_{j-1}) = \xi_n,$$

and

$$(4.49) \qquad \int_0^T Y_t^2 dt = \lim_{n \to \infty} \sum_{j=1}^{n} \int_{a_j}^{\tau_j} \varphi_j^2(t) dt = \sum_{j=1}^{\infty} G_j$$

where $G_j(\omega) = A_j(\tau_j(\omega))$. Lemma 4.17 gives

$$P[G_n > \frac{1}{n^2} | \mathcal{F}_{a_n}] \leq n|\xi_n - \xi_{n-1}| \wedge 1.$$

But, from (4.47),

$$E[n|\xi_n - \xi_{n-1}| \wedge 1] \leq \int_{\{n|\xi_n - \xi_{n-1}| \geq \frac{4}{(n-1)^2}\}} (n|\xi_n - \xi_{n-1}| \wedge 1) dP + \frac{4}{(n-1)^2}$$

$$\leq \frac{6}{(n-1)^2}; \quad n \geq 2,$$

and so

$$P[G_n > \frac{1}{n^2}] \leqq \frac{6}{(n-1)^2}; \qquad n \geq 2.$$

By the Borel-Cantelli lemma, there exists an event Ω^* of probability one, such that for every $\omega \in \Omega^*$, there exists an integer $n_0(\omega) \geq 2$ with:

$$G_n(\omega) \leq \frac{1}{n^2}; \qquad n \geq n_0(\omega).$$

We conclude that $\sum\limits_{n=1}^{\infty} G_n(\omega)$ is a convergent series on Ω^*, and (4.49) gives us (4.44).

Because of (4.44), the stochastic integral $\{I_t(Y) = \int_0^t Y_s dW_s;$ $0 \leq t \leq T\}$ is defined and almost surely continuous. Letting $n \to \infty$ in (4.48), we obtain (4.45). $\qquad\qquad \square$

4.18 Problem: Extend Theorem 4.16 to the case $T = \infty$.

4.19 Remark: It is instructive to compare the representations (4.32) and (4.45) in the case where $\{\mathcal{F}_t\}$ is the augmentation of the filtration $\{\mathcal{F}_t^W\}$ generated by the one-dimensional Brownian motion W. The expectation $E\xi$ does not appear in (4.45) (Theorem 4.15 does not assume the integrability of ξ). We do not know if the proof of Theorem 4.16 can be modified in the case $E\xi^2 < \infty$, $E\xi = 0$ to give the result of Problem 4.12, i.e., the representation (4.45) with $E \int_0^T Y_t^2 dt < \infty$.

3.5 THE GIRSANOV THEOREM

In order to motivate the results of this section, let us consider independent normal random variables $Z_1, \ldots, Z_d$ on $(\Omega, \mathfrak{F}, P)$ with $EZ_i = 0$, $EZ_i^2 = 1$. Given a vector $(\mu_1, \ldots, \mu_d) \in \mathbb{R}^d$, we consider the new probability measure $\widetilde{P}$ on $(\Omega, \mathfrak{F})$ given by

$$\widetilde{P}(d\omega) = \exp[\sum_{i=1}^{d} \mu_i Z_i(\omega) - \frac{1}{2} \sum_{i=1}^{d} \mu_i^2] \cdot P(d\omega).$$

Then $\widetilde{P}[Z_1 \in dz_1, \ldots, Z_d \in dz_d]$ is given by

$$\exp[\sum_{i=1}^{d} \mu_i z_i - \frac{1}{2} \sum_{i=1}^{d} \mu_i^2] \cdot P[Z_1 \in dz_1, \ldots, Z_d \in dz_d]$$

$$= (2\pi)^{-d/2} \exp[-\frac{1}{2} \sum_{i=1}^{d} (z_i - \mu_i)^2] dz_1 \ldots dz_d.$$

Therefore, under $\widetilde{P}$ the random variables $Z_1, \ldots, Z_d$ are independent and normal with $\widetilde{E}Z_i = \mu_i$ and $\widetilde{E}Z_i^2 = 1$. In other words, $\{\widetilde{Z}_i = Z_i - \mu_i; \ 1 \leq i \leq d\}$ are independent, standard normal random variables on $(\Omega, \mathfrak{F}, \widetilde{P})$. The Girsanov Theorem 5.1 below extends this idea of "invariance of Gaussian finite-dimensional distributions", under appropriate translations and changes of the underlying probability measure, from the static to the dynamic setting. Rather than beginning with a d-dimensional vector $(Z_1, \ldots, Z_d)$ of independent, standard normal random variables, we begin with a d-dimensional Brownian motion under P, and then compute a new measure $\widetilde{P}$ under which a "translated" process is a d-dimensional Brownian motion.

Throughout this section, we shall have a probability space $(\Omega, \mathfrak{F}, P)$, and a d-dimensional Brownian motion $W = \{W_t = (W_t^{(1)}, \ldots, W_t^{(d)}), \mathfrak{F}_t; \ 0 \leq t < \infty\}$ defined on it, with $P[W_0 = 0] = 1$. We assume that the filtration $\{\mathfrak{F}_t\}$ satisfies the usual conditions. Let $X = \{X_t = (X_t^{(1)}, \ldots, X_t^{(d)}), \mathfrak{F}_t; \ 0 \leq t < \infty\}$ be a vector of measurable, adapted processes satisfying

$$(5.1) \qquad P[\int_0^T (X_t^{(i)})^2 dt < \infty] = 1; \quad 1 \leq i \leq d, \quad 0 \leq T < \infty.$$

Then, for each i, the stochastic integral $I^{W^{(i)}}(X^{(i)})$ is defined and is a member of $\mathfrak{m}^{c,loc}$. We set

$$(5.2) \qquad Z_t(X) \triangleq \exp[\sum_{i=1}^{d} \int_0^t X_s^{(i)} dW_s^{(i)} - \frac{1}{2} \int_0^t \|X_s\|^2 ds].$$

Just as in Example 3.9, we have

$$(5.3) \qquad Z_t(X) = 1 + \sum_{i=1}^{d} \int_0^t Z_s(X) X_s^{(i)} dW_s^{(i)},$$

which shows that $Z(X)$ is a continuous, local martingale with $Z_0(X) = 1$.

Under certain conditions on X, to be discussed later, $Z(X)$ will in fact be a martingale, and so $EZ_t(X) = 1$; $0 \leq t < \infty$. In this case we can define, for each $0 \leq T < \infty$, a probability measure $\tilde{P}_T$ on $\mathfrak{F}_T$ by

$$(5.4) \qquad \tilde{P}_T(A) \triangleq E[1_A Z_T(X)]; \quad A \in \mathfrak{F}_T.$$

The martingale property shows that the family of probability measures $\{\tilde{P}_T; \ 0 \leq T < \infty\}$ satisfies the consistency condition

$$(5.5) \qquad \tilde{P}_T(A) = \tilde{P}_t(A); \qquad A \in \mathfrak{F}_t, \quad 0 \leq t \leq T.$$

5.1 Theorem: Girsanov (1960), Cameron & Martin (1944)

Assume that $Z(X)$ defined by (5.2) is a martingale. Define a process $\tilde{W} = \{\tilde{W}_t = (\tilde{W}_t^{(1)}, \ldots, \tilde{W}_t^{(d)}), \ \mathfrak{F}_t; \ 0 \leq t < \infty\}$ by

$$(5.6) \qquad \tilde{W}_t^{(i)} \stackrel{\Delta}{=} W_t^{(i)} - \int_0^t X_s^{(i)} ds; \quad 1 \leq i \leq d, \quad 0 \leq t < \infty,$$

For each fixed $T \in [0, \infty)$, the process $\{\tilde{W}_t, \ \mathfrak{F}_t; \ 0 \leq t \leq T\}$ is a d-dimensional Brownian motion on $(\Omega, \mathfrak{F}_T, \tilde{P})$. $\qquad \square$

The preparation for the proof of this result starts with Lemma 5.3 below; the reader may proceed there directly, skipping the ensuing discussion on first reading.

Discussion: Occasionally, one wants to consider $\tilde{W}$ as a process defined for all $t \in [0, \infty)$, and for this purpose the measures $\{\tilde{P}_T; \ 0 \leq T < \infty\}$ are inadeqate. We would like to have a single measure $\tilde{P}$ defined on $\mathfrak{F}_\infty$, so that $\tilde{P}$ restricted to any $\mathfrak{F}_T$ agrees with $\tilde{P}_T$; however, such a measure does not exist in general. We thus content ourselves with a measure $\tilde{P}$ defined only on $\mathfrak{F}_\infty^W$, the σ-field generated by W,

such that $\widetilde{P}$ restricted to any $\mathcal{F}_T^W$ agrees with $\widetilde{P}_T$, i.e.,

(5.7) $\widetilde{P}(A) = E[1_A Z_T(X)]$; $A \in \mathcal{F}_T^W$, $0 \leq T < \infty$.

If such a $\widetilde{P}$ exists, it is clearly unique. The existence of $\widetilde{P}$ follows from the Daniell-Kolmogorov Extension Theorem 2.2.2. We show this when $d = 1$; only notational changes are required for the multidimensional case.

Let $\underset{\sim}{t} = (t_1, \ldots, t_n)$ be a finite sequence of distinct, nonnegative numbers, as in Definition 2.2.1, and let $t = \max\{t_1, \ldots, t_n\}$. Define

$$Q_{\underset{\sim}{t}}(A) = \widetilde{P}_t[\omega \in \Omega; \ (W_{t_1}(\omega), \ldots, W_{t_n}(\omega)) \in A]; \ A \in \mathcal{B}(\mathbb{R}^n).$$

Then $\{Q_{\underset{\sim}{t}}\}$ is a consistent family of finite-dimensional distributions, so there is a probability measure Q on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$ such that

$$Q_{\underset{\sim}{t}}(A) = Q[w \in \mathbb{R}^{[0,\infty)}; \ (w(t_1), \ldots, w(t_n)) \in A]; \ A \in \mathcal{B}(\mathbb{R}^n).$$

But the typical set in $\mathcal{F}_\infty^W$ has the form $\{\omega \in \Omega; \ W_.(\omega) \in B\}$, where $B \in \mathcal{B}(\mathbb{R}^{[0,\infty)})$. Consequently, Q induces a probability measure $\widetilde{P}$ on $\mathcal{F}_\infty^W$ defined by

$$\widetilde{P}[\omega \in \Omega; \ W_.(\omega) \in B] \triangleq Q(B); \ B \in \mathcal{B}(\mathbb{R}^{[0,\infty)}),$$

and this measure satisfies (5.7).

The process $\widetilde{W}$ in Theorem 5.1 is adapted to the filtration $\{\mathfrak{F}_t\}$, and so is the process $\{\int_0^t X_s^{(i)} ds; \ 0 \leq t < \infty\}$; this can be seen as in part (c) of the proof of Lemma 2.3, which uses the completeness of $\mathfrak{F}_t$. However, when working with the measure $\widetilde{P}$ which is defined only on $\mathfrak{F}_\infty^W$, we wish $\widetilde{W}$ to be adapted to $\{\mathfrak{F}_t^W\}$. This filtration does not satisfy the usual conditions, and so we must impose the stronger condition of progressive measurability on X. We have the following corollary to Theorem 5.1.

5.2 Corollary: Let $W = \{W_t, \mathfrak{F}_t; \ 0 \leq t < \infty\}$ be a d-dimensional Brownian motion on $(\Omega, \mathfrak{F}, P)$ with $P[W_0 = 0] = 1$. Assume that the filtration $\{\mathfrak{F}_t\}$ satisfies the usual conditions. Let $X = \{X_t, \mathfrak{F}_t^W; \ 0 \leq t < \infty\}$ be a d-dimensional, progressively measurable process satisfying (5.1). If $Z(X)$ defined by (5.2) is a martingale, then $\widetilde{W} = \{\widetilde{W}_t, \mathfrak{F}_t^W; \ 0 \leq t < \infty\}$ defined by (5.6) is a d-dimensional Brownian motion on $(\Omega, \mathfrak{F}_\infty^W, \widetilde{P})$.

Proof:

For $0 \leq t_1 < \ldots < t_n \leq t$, we have

$$\widetilde{P}[(\widetilde{W}_{t_1}, \ldots, \widetilde{W}_{t_n}) \in A] = \widetilde{P}_t[(\widetilde{W}_{t_1}, \ldots, \widetilde{W}_{t_n}) \in A]; \quad A \in \mathcal{B}(\mathbb{R}^{dn})$$

The result now follows from Theorem 5.1. $\qquad \square$

3.5.6

Under the assumptions of Corollary 5.2, the probability measure P and $\tilde{P}$ are mutually absolutely continuous <u>when restricted to</u> $\mathcal{F}_T^W$; $0 \le T < \infty$. However, considered as probability measures on $\mathcal{F}_\infty^W$, P and $\tilde{P}$ may not be mutually absolutely continuous. For example, when $d = 1$ and $X_t = \mu$, a nonzero constant, then

$$Z_t(X) = \exp[\mu W_t - \frac{1}{2}\mu^2 t]; \quad 0 \le t < \infty$$

is easily seen to be a martingale. Corollary 5.2 and the law of large numbers imply

$$\tilde{P}[\lim_{t\to\infty} \frac{1}{t} W_t = \mu] = \tilde{P}[\lim_{t\to\infty} \frac{1}{t} \tilde{W}_t = 0] = 1,$$

$$P[\lim_{t\to\infty} \frac{1}{t} W_t = \mu] = 0.$$

In particular, the P-null set $\{\lim_{t\to\infty} \frac{1}{t} W_t = \mu\}$ is in $\mathcal{F}_T$ for every $0 \le T < \infty$, so $\tilde{P}$ and $\tilde{P}_T$ cannot agree on $\mathcal{F}_T$. This is the reason we require (5.7) to hold only for $A \in \mathcal{F}_T^W$. $\quad\square$

We now proceed with the proof of Theorem 5.1. We denote by $\tilde{E}_T$ $(\tilde{E})$ the expectation operator with respect to $\tilde{P}_T$ $(\tilde{P})$.

5.3 Lemma: Fix $0 \le T < \infty$ and assume that $Z(X)$ is a martingale. If $0 \le s \le t \le T$ and Y is an $\mathcal{F}_t$-measurable random variable satisfying $\tilde{E}_T|Y| < \infty$, then we have the <u>Bayes' Rule</u>:

$$\tilde{E}_T[Y|\mathcal{F}_s] = \frac{1}{Z_s(X)} E[YZ_t(X)|\mathcal{F}_s], \text{ a.s. } P \text{ and } \tilde{P}_T.$$

Proof:

Using the definition of $\widetilde{E}_T$, the definition of conditional expectation, and the martingale property, we have for any $A \in \mathfrak{F}_s$:

$$\widetilde{E}_T\{1_A \frac{1}{Z_s(X)} E[YZ_t(X)|\mathfrak{F}_s]\} = E\{1_A E[YZ_t|\mathfrak{F}_s]\}$$

$$= E[1_A YZ_t] = \widetilde{E}_T[1_A Y]. \qquad \square$$

We denote by $\mathbb{m}_T^{c,loc}$ the set of continuous, local martingales $M = \{M_t, \mathfrak{F}_t; 0 \leq t \leq T\}$ on $(\Omega, \mathfrak{F}_T, P)$ satisfying $P[M_0 = 0]$. We define $\widetilde{\mathbb{m}}_T^{c,loc}$ similarly, with P replaced by $\widetilde{P}_T$.

5.4 Proposition: Fix $0 \leq T < \infty$ and assume that $Z(X)$ is a martingale. If $M \in \mathbb{m}_T^{c,loc}$, then the process

$$(5.8) \quad \widetilde{M} = \{\widetilde{M}_t \triangleq M_t - \sum_{i=1}^d \int_0^t X_s^{(i)} d\langle M, W^{(i)}\rangle_s, \mathfrak{F}_t; \ 0 \leq t \leq T\}$$

is in $\widetilde{\mathbb{m}}_T^{c,loc}$. If $N \in \mathbb{m}_T^{c,loc}$ and

$$\widetilde{N}_t \triangleq N_t - \sum_{i=1}^d \int_0^t X_s^{(i)} d\langle N, W^{(i)}\rangle_s; \ 0 \leq t \leq T,$$

then

$$\langle \widetilde{M}, \widetilde{N}\rangle_t = \langle M, N\rangle_t; \ 0 \leq t \leq T, \quad \text{a.s.} \quad P \quad \text{and} \quad \widetilde{P}_T,$$

where the cross variations are computed under the appropriate measures.

<u>Proof</u>: We consider only the case where M and N are bounded martingales with bounded quadratic variations, and assume also that $Z_t(X)$ and $\sum_{j=1}^{d} \int_0^t (X_s^{(j)})^2 ds$ are bounded in t and ω; the general case can be reduced to this one by localization. Since (Problem 3.2.16)

$$|\int_0^t X_s^{(i)} d\langle M, W^{(i)} \rangle_s|^2 \leq \langle M \rangle_t \cdot \int_0^t (X_s^{(i)})^2 ds,$$

we see that $\widetilde{M}$ is also bounded. The integration-by-parts formula (Problem 3.11) gives

$$Z_t(X)\widetilde{M}_t = \int_0^t Z_u(X) dM_u + \sum_{i=1}^{d} \int_0^t \widetilde{M}_u X_u^{(i)} Z_u(X) dW_u^{(j)},$$

which is a martingale under P. Therefore, for $0 \leq s \leq t \leq T$, we have from Lemma 5.3:

$$\widetilde{E}_T[\widetilde{M}_t | \mathcal{F}_s] = \frac{1}{Z_s(X)} E[Z_t(X)\widetilde{M}_t | \mathcal{F}_s] = \widetilde{M}_s, \quad \text{a.s.} \quad P \text{ and } \widetilde{P}_T.$$

It follows that $\widetilde{M} \in \widetilde{\mathcal{M}}^{c,loc}$.

The change-of-variable formula also implies:

$$\widetilde{M}_t \widetilde{N}_t - \langle M, N \rangle_t = \int_0^t \widetilde{M}_u dN_u + \int_0^t \widetilde{N}_u dM_u$$

$$- \sum_{i=1}^{d} [\int_0^t \widetilde{M}_u X_u^{(i)} d\langle N, W^{(i)} \rangle_u + \int_0^t \widetilde{N}_u X_u^{(i)} d\langle M, W^{(i)} \rangle_u],$$

as well as

$$Z_t(X)(\widetilde{M}_t\widetilde{N}_t - \langle M,N\rangle_t) = \int_0^t Z_u(X)\widetilde{M}_u dN_u + \int_0^t Z_u(X)\widetilde{N}_u dM_u$$

$$+ \sum_{i=1}^d \int_0^t (\widetilde{M}_u\widetilde{N}_u - \langle M,N\rangle_u) X_u^{(i)} Z_u(X) dW_u^{(i)}.$$

This last process is consequently a martingale under P, and so Lemma 5.3 implies that for $0 \le s \le t \le T$

$$\widetilde{E}[\widetilde{M}_t\widetilde{N}_t - \langle M,N\rangle_t | \mathfrak{F}_s] = \widetilde{M}_s\widetilde{N}_s - \langle M,N\rangle_s; \quad \text{a.s.} \quad P \quad \text{and} \quad \widetilde{P}_T.$$

This proves that $\langle \widetilde{M},\widetilde{N}\rangle_t = \langle M,N\rangle_t;$ $0 \le t \le T$, a.s. $\widetilde{P}_T$ and P.

__Proof of Theorem 5.1__: We show that the continuous process $\widetilde{W}$ on $(\Omega,\mathfrak{F}_T,\widetilde{P}_T)$ satisfies the hypotheses of P. Lévy's Theorem 3.13. Setting $M = W^{(j)}$ in Proposition 5.4 we obtain $\widetilde{M} = \widetilde{W}^{(j)}$ from (5.8), so $\widetilde{W}^{(j)} \in \mathfrak{M}_T^{c,loc}$. Setting $N = W^{(k)}$, we obtain

$$\langle \widetilde{W}^{(j)},\widetilde{W}^{(k)}\rangle_t = \langle W^{(j)},W^{(k)}\rangle_t = \delta_{j,k}t; \quad 0 \le t \le T, \quad \text{a.s.} \quad \widetilde{P}_T \quad \text{and} \quad P.$$

$\square$

Let $\{M_t, \mathfrak{F}_t; 0 \le t \le T\}$ be a continuous, local martingale under P $(M \in \mathfrak{M}_T^{c,loc})$. With the hypotheses of Theorem 5.1, Proposition 5.4 shows that M is a continuous semimartingale under $\widetilde{P}_T$ (Definition 3.1). The converse is also true; if $\{\widetilde{M}_t, \mathfrak{F}_t; 0 \le t \le T\}$ is a continuous martingale under $\widetilde{P}_T$, then Lemma 5.3 implies that for $0 \le s \le t \le T$:

$$E[Z_t(X)\widetilde{M}_t | \mathcal{F}_s] = Z_s(X)\widetilde{E}_T[\widetilde{M}_t | \mathcal{F}_s] = Z_s(X)\widetilde{M}_s; \quad \text{a.s.} \quad P \quad \text{and} \quad \widetilde{P}_T,$$

so $Z(X)\widetilde{M}$ is a martingale under P. If $\widetilde{M} \in \widetilde{\mathbb{m}}_T^{c,loc}$, a localization argument shows that $Z(X)\widetilde{M} \in \mathbb{m}_T^{c,loc}$. But $Z(X) \in \mathbb{m}^c$, and so Itô's rule implies that $\widetilde{M} = \frac{1}{Z(X)}[Z(X)\widetilde{M}]$ is a continuous semimartingale under P (cf. Remark 3.4). Thus, given $\widetilde{M} \in \widetilde{\mathbb{m}}_T^{c,loc}$, we have a decomposition

$$\widetilde{M}_t = M_t + B_t; \quad 0 \le t < T,$$

where $M \in \mathbb{m}_T^{c,loc}$ and B is the difference of two continuous, nondecreasing, adapted processes with $B_0 = 0$, P-a.s. According to Proposition 5.4, the process

$$\widetilde{M}_t - (M_t - \sum_{i=1}^{d} \int_0^t X_s^{(i)} d\langle M, W^{(i)}\rangle_s)$$

$$= B_t + \sum_{i=1}^{d} \int_0^t X_s^{(i)} d\langle M, W^{(i)}\rangle_s; \quad 0 \le t \le T,$$

is in $\widetilde{\mathbb{m}}_T^{c,loc}$, and being of bounded variation this process must be indistinguishable from the identically zero process (Problem 3.2). We have proved the following result.

5.5 Proposition: Assume the hypotheses of Theorem 5.1. Then every $\widetilde{M} \in \widetilde{\mathbb{m}}_T^{c,loc}$ has the representation (5.8) for some $M \in \mathbb{m}_T^{c,loc}$. □

We note now that integrals with respect to $d\widetilde{W}_t^{(i)}$ have two possible interpretations. On the one hand, we may interpret them by replacing $d\widetilde{W}_t^{(i)}$ by $dW_t^{(i)} - X_t^{(i)} dt$ so as to obtain the sum of an Itô integral (under P) and a Lebesgue-Stieltjes integral. On the other hand, $\widetilde{W}^{(i)}$ is a Brownian motion under $\widetilde{P}_T$, so we may regard integrals with respect to $d\widetilde{W}_t^{(i)}$ as Itô integrals under $\widetilde{P}_T$. Fortunately, these two interpretations coincide, as the next problem shows.

5.6 Problem: Assume the hypotheses of Theorem 5.1 and suppose $Y = \{Y_t, \mathfrak{F}_t; 0 \le t < \infty\}$ is a measurable adapted process satisfying $P[\int_0^t Y_t^2 dt < \infty] = 1;\ 0 \le T < \infty$. Under P we may define the Itô integral $\int_0^t Y_s dW_s^{(i)}$, whereas, under $\widetilde{P}_T$, we may define the Itô integral $\int_0^t Y_s d\widetilde{W}_s^{(i)},\ 0 \le t \le T$. Show that for $1 \le i \le d$, we have

$$\int_0^t Y_s d\widetilde{W}_s^{(i)} = \int_0^t Y_s dW_s^{(i)} - \int_0^t Y_s X_s^{(i)} ds;\quad 0 \le t \le T,$$

$$\text{a.s. } P \text{ and } \widetilde{P}_T.$$

(Hint: Use Proposition 2.22.)

We now discuss a rather simple, but interesting, application of the Girsanov theorem: the distribution of passage times for Brownian motion with drift. Let us consider a Brownian motion $W = \{W_t, \mathfrak{F}_t; 0 \le t < \infty\}$ and recall from Remark 2.8.3.,

that the passage time to level $b \neq 0$,

$$T_b \triangleq \begin{cases} \inf\{t \geq 0; \ W_t = b\}; & \text{if } \{\ldots\} \neq \emptyset, \\ + \infty; & \text{if } \{\ldots\} = \emptyset, \end{cases}$$

has density and moment generating function, respectively:

$$(5.9) \qquad P[T_b \in dt] = \frac{|b|}{\sqrt{2\pi t^3}} \exp[-\frac{b^2}{2t}]dt; \quad t > 0,$$

$$(5.10) \qquad E e^{-\alpha T_b} = e^{-|b|\sqrt{2\alpha}}; \quad \alpha > 0.$$

For any real number $\mu \neq 0$, the process $\tilde{W} = \{\tilde{W}_t \triangleq W_t - \mu t, \ \mathfrak{F}_t^W; \ 0 \leq t < \infty\}$ is a Brownian motion under the unique measure $P^{(\mu)}$ which satisfies

$$P^{(\mu)}(A) = E[1_A \exp\{\mu W_t - \frac{1}{2}\mu^2 t\}]; \quad A \in \mathfrak{F}_t^W$$

(Corollary 5.2). We say that, <u>under $P^{(\mu)}$</u>, $W_t = \mu t + \tilde{W}_t$ <u>is a Brownian motion with drift</u> μ. On the set $\{T_b \leq t\} \in \mathfrak{F}_t^W \cap \mathfrak{F}_{T_b}^W = \mathfrak{F}_{t \wedge T_b}^W$, we have $Z_{t \wedge T_b} = Z_{T_b}$, so the Optional Sampling Theorem 1.3.20 and Problem 1.3.21(i) imply

$$(5.11) \qquad P^{(\mu)}[T_b \leq t] = E[1_{\{T_b \leq t\}} Z_t] = E[1_{\{T_b \leq t\}} E\{Z_t | \mathfrak{F}_{t \wedge T_b}^W\}]$$

$$= E[1_{\{T_b \leq t\}} Z_{t \wedge T_b}] = E[1_{\{T_b \leq t\}} Z_{T_b}]$$

$$= E[1_{\{T_b \leq t\}} e^{\mu b - \frac{1}{2}\mu^2 T_b}]$$

$$= \int_0^t \exp\{\mu b - \frac{1}{2}\mu^2 s\} P[T_b \in ds].$$

Relation (5.11) has several consequences. First, together with (5.9) it yields the density of T_b under $P^{(\mu)}$:

$$(5.12) \qquad P^{(\mu)}[T_b \in dt] = \frac{|b|}{\sqrt{2\pi t^3}} \exp[-\frac{(b-\mu t)^2}{2t}]dt, \quad t > 0.$$

Secondly, letting $t \to \infty$ in (5.11), we see that

$$P^{(\mu)}[T_b < \infty] = e^{\mu b}E[\exp(-\frac{1}{2}\mu^2 T_b)],$$

and so we obtain from (5.10):

$$(5.13) \qquad P^{(\mu)}[T_b < \infty] = \exp[\mu b - |\mu b|].$$

In particular, <u>a Brownian motion with drift $\mu \neq 0$ reaches the</u> <u>level $b \neq 0$ with probability one if and only if μ and b have</u> <u>the same sign</u>. If μ and b have opposite signs, the density in (5.12) is "defective", in the sense that $P^{(\mu)}[T_b < \infty] < 1$.

<u>5.7 Problem</u>: Let T be a stopping time of the filtration $\{\mathcal{F}_t^W\}$ with $P[T < \infty] = 1$. A necessary and sufficient condition for the validity of the <u>Wald identity</u>:

$$(5.14) \qquad E[\exp\{\mu W_T - \frac{1}{2}\mu^2 T\}] = 1,$$

where μ is a given real number, is that

$$(5.15) \qquad P^{(\mu)}[T < \infty] = 1.$$

In particular, if $\mu \in \mathbb{R}$ and $\mu b < 0$, then this condition holds for the stopping time

(5.16)
$$S_b \triangleq \begin{cases} \inf\{t \geq 0; \; W_t - \mu t = b\}; & \text{if} \;\; \{\ldots\} \neq \emptyset, \\ \infty; & \text{if} \;\; \{\ldots\} = \emptyset. \end{cases}$$

5.8 Problem: Denote by

$$h(t;b,\mu) \triangleq \frac{|b|}{\sqrt{2\pi t^3}} \exp\left[-\frac{(b-\mu t)^2}{2t}\right]; \quad t > 0, \quad b \neq 0, \quad \mu \in \mathbb{R},$$

the (possibly defective) density on the right-hand side of (5.12). Show that

$$h(\cdot;b_1+b_2,\mu) = h(\cdot;b_1,\mu) \ast h(\cdot;b_2,\mu); \quad b_1 b_2 > 0, \;\; \mu \in \mathbb{R},$$

where $\ast$ denotes convolution.

5.9 Problem: With $\mu > 0$ and $W_\ast \triangleq \inf\limits_{t>0} W_t$, under $P^{(\mu)}$ the random variable $-W_\ast$ is exponentially distributed with parameter 2μ, i.e.,

$$P^{(\mu)}[-W_\ast \in db] = 2\mu e^{-2\mu b} db, \quad b > 0.$$

5.10 Problem: Show that

$$E^{(\mu)} e^{-\alpha T_b} = \exp(\mu b - |b|\sqrt{\mu^2+2\alpha}), \quad \alpha > 0.$$

5.11 Exercise: (Robbins & Siegmund [1973]) Consider, for $\mu > 0$ and $b > 1$, the stopping time of $\{\mathfrak{F}_t^W\}$:

$$R_b = \begin{cases} \inf\{t \geq 0; \ \exp(\mu W_t - \frac{1}{2}\mu^2 t) = b\}; & \text{if } \{\ldots\} \neq \emptyset, \\ \infty; & \text{if } \{\ldots\} = \emptyset. \end{cases}$$

Show that

$$P[R_b < \infty] = \frac{1}{b}, \qquad E^{(\mu)} R_b = \frac{2 \ln b}{\mu^2}.$$

In order to use Girsanov's theorem effectively, we need some fairly general conditions under which the process $Z(X)$ defined by (5.2) is a martingale. This process is a local martingale because of (5.3). Indeed, with

$$T_n \triangleq \begin{cases} \inf\{t \geq 0; \ \max_{0 \leq i \leq d} \int_0^t (Z_s(X) X_s^{(i)})^2 ds = n\}; & \text{if } \{\ldots\} \neq \emptyset, \\ \infty; & \text{if } \{\ldots\} = \emptyset, \end{cases}$$

the "stopped" processes $Z^{(n)} \triangleq \{Z_t^{(n)} \triangleq Z_{t \wedge T_n}(X), \ \mathfrak{F}_t; \ 0 \leq t < \infty\}$ are martingales. Consequently, we have

$$E[Z_{t \wedge T_n} | \mathfrak{F}_s] = Z_{s \wedge T_n}; \quad 0 \leq s \leq t, \quad n \geq 1,$$

and using Fatou's lemma as $n \to \infty$, we obtain

$$E[Z_t(X) | \mathfrak{F}_s] \leq Z_s; \quad 0 \leq s \leq t.$$

In other words, $Z(X)$ is always a supermartingale, and is a martingale if and only if

(5.17) $EZ_t(X) = 1;$ $0 \leq t < \infty$

(Problem 1.3.23). We provide now sufficient conditions for (5.17).

<u>5.12 Proposition</u>: Let $M = \{M_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be in $\mathcal{M}^{c,loc}$
and define

$$Z_t = \exp[M_t - \frac{1}{2} \langle M \rangle_t]; \quad 0 \leq t < \infty.$$

If

(5.18) $E[\exp\{\frac{1}{2} \langle M \rangle_t\}] < \infty;$ $0 \leq t < \infty,$

then $EZ_t = 1; \quad 0 \leq t < \infty.$

<u>Proof</u>:

We must show that for an arbitrary, positive t we have $EZ_t = 1$. Once t is fixed, we may alter M_u for $u \geq t$ if necessary to assure that

(5.19) $P[\lim_{u \to \infty} \langle M \rangle_u = \infty] = 1.$

We assume henceforth that (5.19) holds.

Let $T(s) = \inf\{t \geq 0; \langle M \rangle_t > s\}$, so the time-changed process

$$B = \{B_s \triangleq M_{T(s)}, \mathcal{G}_s \triangleq \mathcal{F}_{T(s)}; 0 \leq s < \infty\}$$

is a Brownian motion (Theorem 4.6). For $b < 0$, we define the stopping time for $\{\mathcal{G}_s\}$ as in (5.16):

$$S_b = \begin{cases} \inf\{s \geq 0; \ B_s - s = b\}; & \text{if } \{\ldots\} \neq \emptyset. \\ \infty; & \text{if } \{\ldots\} = \emptyset. \end{cases}$$

Problem 5.7 yields the Wald identity $E[\exp\{B_{S_b} - \frac{1}{2} S_b\}] = 1$, whence $E[\exp\{\frac{1}{2} S_b\}] = e^{-b}$. Consider the exponential martingale $\{Z_s \triangleq \exp(B_s - \frac{s}{2}), \ \mathcal{G}_s; \ 0 \leq s < \infty\}$ and define $\{N_s \triangleq Z_{s \wedge S_b}, \ \mathcal{G}_s; \ 0 \leq s < \infty\}$. According to Problem 1.3.22(i), N is a martingale, and because $P[S_b < \infty] = 1$, we have

$$N_\infty = \lim_{s \to \infty} N_s = \exp(B_{S_b} - \frac{1}{2} S_b).$$

Fatou's lemma implies

$$N_r = \lim_{s \to \infty} E[N_s | \mathcal{F}_r] \geq E[N_\infty | \mathcal{F}_r], \quad 0 \leq r < \infty,$$

so $N = \{N_s, \ \mathcal{G}_s; \ 0 \leq s \leq \infty\}$ is a supermartingale with a last element. However, $E N_\infty = 1 = E N_0$, so $N = \{N_s, \ \mathcal{G}_s; \ 0 \leq s \leq \infty\}$ has constant expectation; thus N is actually a martingale with a last element (Problem 1.3.23). This allows us to use the Optional Sampling Theorem 1.3.20 to conclude that for any stopping time R of the filtration $\{\mathcal{G}_s\}$:

$$E[\exp\{B_{R \wedge S_b} - \frac{1}{2}(R \wedge S_b)\}] = 1.$$

But $\{<M>_t > s\} = \{T(s) < t\} \in \mathcal{F}_{T(s)} = \mathcal{G}_s$ (Problem 4.5(iii)),

so $<M>_t$ is a stopping time of $\{\mathcal{G}_s\}$. Therefore,

(5.20) $\quad E[1_{\{S_b \leq <M>_t\}} \exp(b + \frac{1}{2} S_b)]$

$$+ E[1_{\{<M>_t < S_b\}} \exp\{M_t - \frac{1}{2} <M>_t\}] = 1;$$

$$0 \leq t < \infty, \quad b < 0.$$

The first expectation in (5.20) is bounded above by

$e^b E[\exp\{\frac{1}{2} <M>_t\}]$, which converges to zero as $b \downarrow -\infty$, thanks

to assumption (5.18). As $b \downarrow -\infty$, the second expectation in

(5.20) converges to EZ_t because of the monotone convergece

theorem. Therefore, $EZ_t = 1$; $0 \leq t < \infty$. $\qquad \square$

5.13 Corollary: Novikov (1972)

Let $W = \{W_t = (W_t^{(1)}, \ldots, W_t^{(d)}), \mathcal{F}_t; 0 \leq t < \infty\}$ be a

d-dimensional Brownian motion, and let $X = \{X_t = (X_t^{(1)}, \ldots, X_t^{(d)}),$

$\mathcal{F}_t; 0 \leq t < \infty\}$ be a vector of measurable, adapted processes

satisfying (5.1). If

(5.21) $\quad E[\exp\{\frac{1}{2} \int_0^t \|X_s\|^2 ds\}] < \infty; \quad 0 \leq t < \infty,$

then $Z(X)$ defined by (5.2) is a martingale.

Proof: Let $M_t = \sum\limits_{i=1}^{d} \int_0^t X_s^{(i)} dW_s^{(i)}$ in Proposition 5.12

and recall the discussion preceding (5.17). □

5.14 Corollary: Corollary 5.13 still holds if (5.21) is replaced by
the following assumption: there exists an increasing sequence
$\{t_n\}_{n=0}^{\infty}$ of real numbers with $0 = t_0 < t_1 < \ldots$ and
$\lim\limits_{n\to\infty} t_n = \infty$, such that

$$(5.22) \quad E[\exp\{\tfrac{1}{2} \int_{t_{n-1}}^{t_n} \|X_s\|^2 ds\}] < \infty; \quad n \geq 1.$$

Proof:

Let $X_t(n) = (X_t^{(1)} 1_{[t_{n-1} \leq t < t_n)}, \ldots, X_t^{(d)} 1_{[t_{n-1} \leq t < t_n)})$, so
$Z(X(n))$ is a martingale by Corollary 5.13. In particular,

$$E[Z_{t_n}(X(n)) | \mathfrak{F}_{t_{n-1}}] = Z_{t_{n-1}}(X(n)) = 1; \quad n \geq 1.$$

But then,

$$E[Z_{t_n}(X)] = E[Z_{t_{n-1}}(X) E\{Z_{t_n}(X(n)) | \mathfrak{F}_{t_{n-1}}\}] = E[Z_{t_{n-1}}(X)],$$

and by induction on n we can show that $E[Z_{t_n}(X)] = 1$ holds for

all $n \geq 1$. Since $E[Z_t(X)]$ is nondecreasing in t and
$\lim\limits_{n\to\infty} t_n = \infty$, we obtain (5.17). □

5.15 Definition: Let $C^d[0,\infty)$ be the space of continuous functions $x : [0,\infty) \to \mathbb{R}^d$. For $0 \leq t < \infty$, define $\mathcal{G}_t \triangleq \sigma\{x(s); 0 \leq s \leq t\}$, and set $\mathcal{G} = \mathcal{G}_\infty$ (cf. Problems 2.4.1 and 2.4.2). A _progressively_ _measurable_ functional on $C^d[0,\infty)$ is a mapping $\mu : [0,\infty) \times C^d[0,\infty) \to \mathbb{R}$ which has the property: for each fixed $0 \leq t < \infty$, μ restricted to $[0,t] \times C^d[0,\infty)$ is $\mathcal{B}[0,t] \otimes \mathcal{G}_t/\mathcal{B}(\mathbb{R})$ -measurable.

If $\mu = (\mu^{(1)},\ldots,\mu^{(d)})$ is a vector of progressively measurable functionals on $C^d[0,\infty)$ and $W = \{W_t = (W_t^{(1)},\ldots,W_t^{(d)}), \mathcal{F}_t; 0 \leq t < \infty\}$ is a d-dimensional Brownian motion on some $(\Omega,\mathcal{F},P)$, then the processes

$$(5.23) \quad X_t^{(i)}(\omega) \triangleq \mu^{(i)}(t, W_t(\omega)); \; 0 \leq t < \infty, \; 1 \leq i \leq d,$$

are progressively measurable relative to $\{\mathcal{F}_t\}$.

5.16 Corollary: Beneš (1971)

If the vector $\mu = (\mu^{(1)},\ldots,\mu^{(d)})$ of progressively measurable functionals on $C^d[0,\infty)$ satisfies, for each $0 \leq T < \infty$ and some $K_T > 0$ depending on T, the condition

$$(5.24) \quad \|\mu(t,x)\| \leq K_T(1 + x^*(t)); \; 0 \leq t \leq T,$$

where $x^*(t) \triangleq \max_{0 \leq s \leq t} \|x(s)\|$, then with $X_t = (X_t^{(1)},\ldots,X_t^{(d)})$ defined by (5.23), $Z(X)$ is a martingale.

Proof: If, for arbitrary $T > 0$, we can find $\{t_0, \ldots, t_{n(T)}\}$ such that $0 = t_0 < t_1 < \ldots < t_{n(T)} = T$ and (5.22) holds for $1 \leq n \leq n(T)$, then we can construct a sequence $\{t_n\}_{n=0}^{\infty}$ satisfying the hypotheses of Corollary 5.14. Thus, fix $T > 0$. We have from (5.23),(5.24) that whenever $0 \leq t_{n-1} < t_n \leq T$, then

$$\int_{t_{n-1}}^{t_n} \|X_s\|^2 ds \leq (t_n - t_{n-1}) K_T^2 (1 + W_T^*)^2,$$

where $W_T^* \triangleq \max_{0 \leq t \leq T} \|W_t\|$. According to (2.   ),

$$P[W_T^* \in dm] = \frac{2}{\sqrt{2\pi T}} e^{-\frac{m^2}{2T}} dm; \quad m \geq 0,$$

so (5.22) holds provided $t_n - t_{n-1} < \dfrac{1}{TK_T^2}$. This allows us to construct $\{t_0, \ldots, t_{n(T)}\}$ as described above. $\qquad \square$

5.17 Remark: Lipster and Shiryaev (1977), p. 222, show that when $d = 1$ and if $0 < \varepsilon < \frac{1}{2}$, then there is a process X satisfying the hypotheses of Corollary 5.13 but with (5.21) replaced by the weaker condition

$$E[\exp\{(\tfrac{1}{2} - \varepsilon) \int_0^T X_t^2 dt\}] < \infty; \quad 0 \leq T < \infty,$$

such that $Z(X)$ is $\underline{not}$ a martingale.

The next exercise, taken from Lipster and Shiryaev (1977),
p. 224, provides a simple example in which $Z(X)$ is not a
martingale. In particular, it shows that a local martingale
(cf. (5.3)) need not be a martingale.

5.18 **Exercise:** With $W = \{W_t, \mathcal{F}_t; 0 \leq t \leq 1\}$ a Brownian motion,
we define

$$T = \inf\{0 \leq t \leq 1; \ t + W_t^2 = 1\},$$

$$X_t = \begin{cases} -\dfrac{2}{(1-t)^2} W_t \mathbf{1}_{\{t \leq T\}}; & 0 \leq t < 1, \\[2mm] 0; & t = 1. \end{cases}$$

(i) Prove that $P[T<1] = 1$, and therefore $\displaystyle\int_0^1 X_t^2 dt < \infty$ a.s.

(ii) Apply Itô's rule to the process $\left\{\left(\dfrac{W_t}{1-t}\right)^2; 0 \leq t < 1\right\}$
to conclude that

$$\int_0^1 X_t dW_t - \frac{1}{2}\int_0^1 X_t^2 dt = -1 - 2\int_0^T \left[\frac{1}{(1-t)^4} - \frac{1}{(1-t)^3}\right] W_t^2 dt \leq -1.$$

(iii) The exponential supermartingale $\{Z_t(X), \mathcal{F}_t; 0 \leq t \leq 1\}$
is not a martingale; however, for each $n \geq 1$ and
$\sigma_n = 1 - \dfrac{1}{\sqrt{n}}$, $\{Z_{t \wedge \sigma_n}(X), \mathcal{F}_t; 0 \leq t \leq 1\}$ is a
martingale.

5.19 Exercise: Let $W = \{W_t, \mathcal{F}_t; \ 0 \le t < \infty\}$ be a Brownian motion

on $(\Omega, \mathcal{F}, P)$ with $P[W_0 = 0] = 1$, and assume $\{\mathcal{F}_t\}$ satisfies

the usual conditions. Suppose that, for each $0 \le T < \infty$,

there is a probability measure $\tilde{P}_T$ on $\mathcal{F}_T$ which is mutually

absolutely continuous with respect to P , and that the family

of probability measures $\{\tilde{P}_T; \ 0 \le T < \infty\}$ satisfies the

consistency condition (5.5). Show that there exists a

measurable, adapted process $X = \{X_t, \mathcal{F}_t; \ 0 \le t < \infty\}$

satisfying (5.1), such that $Z(X)$ defined by (5.2) is a

martingale and (5.4) holds for $0 \le T < \infty$.

3.6 LOCAL TIME AND A GENERALIZED ITÔ RULE

In this section we devise a method for measuring the amount of time spent by the Brownian path in the vicinity of a point $x \in \mathbb{R}$. We saw in Section 2.9 that the Lebesgue measure of the level set $Z_\omega(x) = \{0 \le t < \infty;\ W_t(\omega) = x\}$ turns out to be zero, i.e.,

(6.1)   meas $Z_\omega(x) = 0$,   for   P-a.e. $\omega \in \Omega$,

yielding no information whatsoever about the amount of time spent in the vicinity of the point x (Theorem 2.9.6 and Remark 2.9.7). In search of a nontrivial measure for this amount of time, P. Lévy introduced the two-parameter random field

(6.2) $\ell_t(x) = \lim\limits_{\varepsilon \downarrow 0} \dfrac{1}{2\varepsilon}$ meas$\{0 \le s \le t;\ |W_s - x| \le \varepsilon\}$; $t \in [0, \infty)$, $x \in \mathbb{R}$

and showed that this limit exists and is finite, as well as positive (e.g. for $x = 0$, $t > 0$). We shall show how $\ell_t(x)$ can be chosen to be jointly continuous in (t, x) and, for fixed x, nondecreasing in t and constant on each interval in the complement of the closed set $Z_\omega(x)$. Therefore, $\dfrac{d}{dt} \ell_t(x)$ exists and is zero for Lebesgue almost every t; i.e., the function $t \to \ell_t(x)$ is singularly continuous. P. Lévy called $\ell_t(x)$ the mesure du voisinage, or "measure of the time spent by the Brownian path in the vicinity of the point x." We shall refer to $\ell_t(x)$ as local time.

This new concept provides a very powerful tool for the study of Brownian sample paths. In this section, we show how it allows us to generalize Itô's change-of-variable rule to convex but not necessarily differentiable functions, and we use it to study certain

additive functionals of the Brownian path. These functionals will be employed in Chapter 5 to provide solutions of stochastic differential equations by the method of random time change. Local time will be further developed in Chapter 6, where we shall use it to prove that the Brownian path has no point of increase (Theorem 2.9.13). In this section, the reader can appreciate the application of local time to the study of sample paths by providing a simple proof of the nondifferentiability of Brownian paths (Problem 6.6). This problem shows that jointly continuous local time cannot exist for processes whose sample paths are of bounded variation on bounded intervals.

Throughout this section, $\{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$, $(\Omega, \mathcal{F})$, $\{P^z\}_{z \in \mathbb{R}}$ denotes the one-dimensional Brownian family on the canonical space $\Omega = C[0, \infty)$. This assumption entails no loss of generality, because every standard Brownian motion induces Wiener measure on $C[0, \infty)$ (Remark 2.4.19), and results proved for the latter can be carried back to the original probability space. We take the filtration $\{\mathcal{F}_t\}$ to be $\{\widetilde{\mathcal{F}}_t\}$ defined by (2.7.3), and we set $\mathcal{F} = \widetilde{\mathcal{F}}_\infty$. This filtration satisfies the usual conditions, and for each $z \in \mathbb{R}$ and $F \in \mathcal{F}$ there is a set $G^z \in \mathcal{B}(C[0, \infty))$ such that $P^z(F \triangle G^z) = 0$. In this situation, P^z is just a translate of P^0, i.e.,

(6.3) $\qquad P^z(F) = P^0(F-z); \qquad F \in \mathcal{F},$

(cf. (2.5.1)). We also have at our disposal the shift operators $\{\theta_s\}_{s \geq 0}$ defined by (2.5.11).

6.1 <u>Definition</u>: A measurable, adapted, real-valued process

$A = \{A_t, \mathcal{F}_t; 0 \le t < \infty\}$ is called an <u>additive functional</u>

if, for every $z \in \mathbb{R}$ and P^z-a.e. $\omega \in \Omega$, we have

(6.4) $A_{t+s}(\omega) = A_s(\omega) + A_t(\theta_s\omega); \quad 0 \le s, t < \infty.$

6.2 <u>Example</u>: For every fixed Borel set $B \in \mathcal{B}(\mathbb{R})$, we define the

<u>occupation time</u> of B by the Brownian path up to time t as

(6.5) $\Gamma_t(B) \triangleq \int_0^t 1_B(W_s)\,ds = \text{meas}\{0 \le s \le t; W_s \in B\}; \quad 0 \le t < \infty,$

where "meas" denotes Lebesgue measure. The resulting process

$\Gamma(B) = \{\Gamma_t(B), \mathcal{F}_t; 0 \le t < \infty\}$ is adapted and continuous,

thus progressively measurable (Problem 1.2.18), and is easily

seen to be an additive functional.

Equation (6.2) indicates that local time $\ell_t(x)$ should

serve as a density with respect to Lebesgue measure for

occupation time. In other words, we should have

(6.6) $\Gamma_t(B,\omega) = \int_B \ell_t(x,\omega)\,dx; \quad 0 \le t < \infty, \quad B \in \mathcal{B})\mathbb{R}).$

We take this property as part of the definition of local time.

6.3 <u>Definition</u>: Let

$\ell = \{\ell_t(x,\omega); (t,x) \in [0,\infty) \times \mathbb{R}, \quad \omega \in \Omega\}$

be a random field with values in $[0,\infty)$, such that for each

fixed value of the parameter pair (t,x) the random variable

$\ell_t(x)$ is $\mathfrak{F}_t$-measurable. Suppose that there is a set $\Omega^* \in \mathfrak{F}$ with $P^z(\Omega^*) = 1$ for every $z \in \mathbb{R}$ and such that, for each $\omega \in \Omega^*$, the function $(t,x) \mapsto \ell_t(x,\omega)$ is continuous and (6.6) holds. Then we call ℓ <u>Brownian local time</u>.

6.4 <u>Remark</u>: There is no universal agreement as to whether ℓ in Definition 6.3 or $\frac{1}{2}\ell$ is to be called local time. We shall sometimes use the symbol $L = \frac{1}{2}\ell$, and somewhat loosely refer to both ℓ and L as local time.

6.5 <u>Remark</u>: With ℓ as in Definition 6.3 and $\omega \in \Omega^*$, one can immediately derive (6.2) from (6.6) and the continuity of $x \mapsto \ell_t(x,\omega)$. Further, $\ell(a) = \{\ell_t(a), \mathfrak{F}_t; 0 \leq t < \infty\}$ is easily seen to inherit the additive functional property (6.4) from its progenitor, the occupation time Γ (Example 6.2).

6.6 <u>Exercise</u>: Assume that Brownian local time exists and show that for each $\omega \in \Omega^*$ of Definition 6.3, the sample path $t \mapsto B_t(\omega)$ can have no point of differentiability. (<u>Hint</u>: If $t \mapsto B_t(\omega)$ is differentiable at t, then for some sufficiently large C and sufficiently small $\delta > 0$ we must have $|B_{t+h}(\omega) - B_t(\omega)| \leq Ch$; $0 \leq h \leq \delta$).

6.7 Problem:
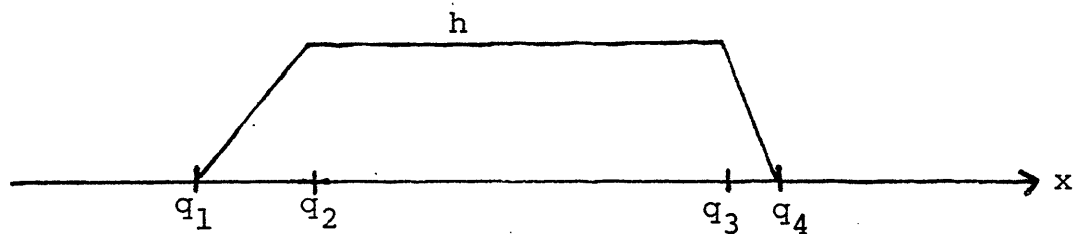
(i) Show that the validity of (6.6) is equivalent to

(6.7)
$$\int_0^t f(W_s(\omega))\,ds = \int_{-\infty}^\infty f(x)\,\ell_t(x,\omega)\,dx; \quad 0 \le t < \infty,$$

for every Borel function $f : \mathbb{R} \to [0,\infty)$.

(ii) Let $\mathcal{H}$ be the set of continuous functions $h : \mathbb{R} \to [0,1]$
of the form

$$h(x) = \begin{cases} 0; & x \le q_1, \\[1mm] \dfrac{x-q_1}{q_2-q_1}; & q_1 \le x \le q_2, \\[1mm] 1; & q_2 \le x \le q_3, \\[1mm] \dfrac{q_4-x}{q_4-q_3}; & q_3 \le x \le q_4, \\[1mm] 0; & x \ge q_4, \end{cases}$$

where $q_1 < q_2 < q_3 < q_4$ are rational numbers.

Show that if (6.7) holds for all $h \in \mathcal{H}$, then it holds
for every Borel function $f : \mathbb{R} \to [0,\infty)$. ◻

We have not yet established the existence of Brownian local time. One could take the representation (6.2) as a starting point for the existence analysis, but it turns out that there is a more convenient representation for this purpose, the Tanaka formula, which we now develop. Let us fix a number $a \in \mathbb{R}$, and take $f(x)$ in (6.7) to be the Dirac delta evaluated at $x-a$, thus deriving formally the representation

$$(6.9) \qquad \ell_t(a,\omega) = \int_0^t \delta(W_s-a)\,ds.$$

But the integral on the right-hand side is only formal, so to give it meaning we consider the nondecreasing, convex function $u(x) = (x-a)^+$, which is continuously differentiable on $\mathbb{R} \setminus \{a\}$ and whose second derivative in the distributional sense is $u''(x) = \delta(x-a)$. Bravely assuming that Itô's rule can be applied in this highly irregular situation we write

$$(6.10) \qquad (W_t-a)^+ - (z-a)^+ = \int_0^t 1_{[a,\infty)}(W_s)\,dW_s + \frac{1}{2}\int_0^t \delta(W_s-a)\,ds;$$
$$0 \le t < \infty,$$

and in conjunction with (6.9) and Remark (6.4) we have

$$(6.11) \qquad L_t(a) = (W_t-a)^+ - (z-a)^+ - \int_0^t 1_{[a,\infty)}(W_s)\,dW_s; \quad 0 \le t < \infty$$

P^z-a.s. for every $z \in \mathbb{R}$. Despite the heuristic nature of both (6.9) and (6.10), the representation (6.11) for local time is valid and will be established rigorously.

6.8 <u>Proposition</u>: Let us assume that Brownian local time exists, and fix a number $a \in \mathbb{R}$. Then the process $\ell(a) = \{\ell_t(a), \mathcal{F}_t, 0 \leq t < \infty\}$ is a nonnegative, continuous additive functional which satisfies P^z-a.s. for every $z \in \mathbb{R}$, the formula (6.11) and the companion representations

$$(6.12) \qquad L_t(a) = (W_t - a)^- - (z-a)^- + \int_0^t 1_{(-\infty, a)} W_s \, dW_s; \quad 0 \leq t < \infty,$$

$$(6.13) \qquad \ell_t(a) = |W_t - a| - |z - a| - \int_0^t \mathrm{sgn}(W_s - a) \, dW_s; \qquad 0 \leq t < \infty.$$

6.9 <u>Remark</u>: Any of the formulas (6.11), (6.12) or (6.13) is referred to as the <u>Tanaka formula for Brownian local time</u>. We need establish only (6.11); then (6.12) follows by symmetry and (6.13) by addition, since

$$P^z[\int_0^t 1_{\{a\}}(W_s) \, dW_s = 0; \quad 0 \leq t < \infty] = 1; \quad z \in \mathbb{R}.$$

In particular, it does not matter how we define $\mathrm{sgn}(0)$ in (6.13); we shall define sgn so as to make it right-continuous, i.e.,

$$(6.14) \qquad \mathrm{sgn}(x) = \begin{cases} 1; & x \geq 0 \\ -1; & x < 0. \end{cases}$$

6.10 <u>Remark</u>: The process $\{(W_t-a)^+, \mathfrak{F}_t; 0 \le t < \infty\}$ is a continuous, nonnegative submartingale (Proposition 1.3.5); it admits, therefore, a unique Doob-Mayer decomposition (under P^z, for any $z \in \mathbb{R}$):

(6.15) $(W_t-a)^+ = (z-a)^+ + M_t(a) + A_t(a); \quad 0 \le t < \infty,$

where $A(a)$ is a continuous, increasing process (Definition 1.4.4) and $M(a)$ is a martingale (Problem 1.4.9(a), Theorem 1.4.10, Remark 1.4.13, and Theorem 1.4.14). The Tanaka formula (6.11) identifies both parts of this decomposition, as $A_t(a) = L_t(a)$ and

(6.16) $M_t(a) = \int_0^t 1_{[a,\infty)}(W_s)\,dW_s; \quad 0 \le t < \infty.$

Similar remarks apply to the representations (6.12) and (6.13).

<u>Proof of Proposition 6.8</u>:

If local time exists, then it satisfies (6.2) as well as the additive functional property (Remark 6.5). In order to make rigorous the heuristic discussion which led to (6.11), we must approximate the Dirac delta $\delta(x)$ by a sequence of probability densities with increasing concentration at the origin. More specifically, let us start with the C^∞ function

(6.17) $\rho(x) \triangleq \begin{cases} c \exp[\dfrac{1}{(x-1)^2-1}]; & 0 < x < 2, \\ \\ 0; & \text{otherwise} \end{cases}$

which satisfies $\int_{-\infty}^{\infty} \rho(x)\,dx = 1$ by appropriate choice of the constant c, and use it to define the probability density functions (called _mollifiers_)

$$(6.18) \qquad \rho_n(x) \triangleq n\rho(nx)$$

as well as

$$u_n(x) \triangleq \int_{-\infty}^{x} \int_{-\infty}^{y} \rho_n(z-a)\,dz\,dy; \quad x \in \mathbb{R}, \quad n \geq 1.$$

We observe that $u_n'(x) = \int_{-\infty}^{x} \rho_n(z-a)\,dz$, and so we have the limiting relations

$$\lim_{n\to\infty} u_n'(x) = 1_{(a,\infty)}(x), \quad \lim_{n\to\infty} u_n(x) = (x-a)^+, \quad x \in \mathbb{R}.$$

We now choose an arbitrary $z \in \mathbb{R}$. According to Itô's rule,

$$(6.19) \qquad u_n(W_t) - u_n(z) = \int_0^t u_n'(W_s)\,dW_s + \frac{1}{2}\int_0^t \rho_n(W_s-a)\,ds; \quad 0 \leq t < \infty,$$
$$\text{a.s. } P^z.$$

But now from (6.7) and the continuity of local time,

$$\int_0^t \rho_n(W_s-a)\,ds = \int_{-\infty}^{\infty} \rho_n(x-a)\ell_t(x)\,dx \xrightarrow[n\to\infty]{} \ell_t(a); \quad \text{a.s. } P^z.$$

On the other hand,

$$E^Z | \int_0^t u_n'(W_s) dW_s - \int_0^t 1_{(a,\infty)}(W_s) dW_s |^2$$

$$= E^Z \int_0^t |u_n'(W_s) - 1_{(a,\infty)}(W_s)|^2 ds$$

$$\leq \int_0^t P^Z [|W_s - a| \leq \frac{2}{n}] ds,$$

which converges to zero as $n \to \infty$. Therefore, for each fixed t, the stochastic integral in (6.19) converges in quadratic mean to the one in (6.16), and (6.11) for each fixed t follows by letting $n \to \infty$ in (6.19). Because of the continuity of the processes in (6.11), we obtain that, except on a P^Z-null event, (6.11) holds for $0 \leq t < \infty$. $\square$

We are now ready to use the Tanaka representation (6.11) to settle the question of existence of Brownian local time.

6.11 Theorem: Trotter (1958)

Brownian local time exists.

Proof:

We start by showing that the two-parameter random field, obtained by setting $z = 0$ on the right-hand side of (6.11), admits a continuous modification under P^0. The term $(W_t - a)^+ - (-a)^+$ is obviously jointly continuous in the pair (t,a). For the random field $\{M_t(a); 0 \leq t < \infty, a \in \mathbb{R}\}$ in (6.16) we have, with $a < b$, $0 \leq s < t \leq T$ and any even integer $n \geq 2$:

$$E^0|M_t(a) - M_s(b)|^{2n} \leq 4^n \{ E^0 | \int_s^t 1_{[a,\infty)}(W_u)\,dW_u|^{2n}$$

$$+ E^0|\int_0^s 1_{[a,b)}(W_u)\,dW_u|^{2n} \}$$

$$\leq 4^n C_n [E^0(\int_s^t 1_{[a,\infty)}(W_u)\,du)^n + E^0(\int_0^s 1_{[a,b)}(W_u)\,du)^n],$$

thanks to (3.3.20). The first expectation is bounded above by $(t-s)^n$, whereas the second is dominated by

$$E^0[\int_0^T 1_{[a,b)}(W_t)\,dt]^n = \int_0^T \cdots \int_0^T E^0[1_{[a,b)}(W_{t_1})\cdots 1_{[a,b)}(W_{t_n})]\,dt_n\cdots dt_1$$

$$= n! \int_0^T \int_{t_1}^T \cdots \int_{t_{n-1}}^T E^0[1_{[a,b)}(W_{t_1})1_{[a,b)}(W_{t_2})\cdots 1_{[a,b)}(W_{t_n})]\,dt_n\cdots dt_2 dt_1.$$

With $0 = t_0 < t_1 < t_2 < \ldots < t_n < T$, we have for every $y \in [a,b]$:

$$P^0[a \leq W_{t_j} < b | W_{t_{j-1}} = y] \leq P^0[a \leq W_{t_j} < b | W_t = \tfrac{a+b}{2}]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\frac{b-a}{2\sqrt{t_j - t_{j-1}}}} e^{-\frac{1}{2}z^2}\,dz \leq \frac{b-a}{2\sqrt{t_j - t_{j-1}}}; \quad 1 \leq j \leq n,$$

and so

$$E^0\left[\int_0^T 1_{[a,b)}(W_t)\,dt\right]^n$$

$$\leq n!(\tfrac{b-a}{2})^n \int_0^T \int_{t_1}^T \cdots \int_{t_{n-1}}^T [t_1(t_2-t_1)\cdots(t_n-t_{n-1})]^{-\frac{1}{2}}\,dt_n\cdots dt_2 dt_1$$

$$\leq \hat{C}_{n,T}(b-a)^n,$$

where $\hat{C}_{n,T}$ is a constant depending on n and T but not on a and b. Therefore, with $a < b$ and $0 \leq s < t \leq T$, we have

(6.20) $E^0 |M_t(a) - M_s(b)|^{2n} \leq C_{n,T}[(t-s)^n + (b-a)^n]$

$\leq C_{n,T} \|(t,a) - (s,b)\|^n$

for some constant $C_{n,T}$. By the version of the Kolmogorov-Čentsov theorem for random fields (Problem 2.2.9), there exists a two parameter random field $\{I_t(a); (t,a) \in [0,\infty) \times \mathbb{R}\}$ such that the mapping $(t,a) \mapsto I_t(a,\omega)$ is locally Hölder continuous with any exponent $\gamma \in (0,\frac{1}{2})$, for P^0-a.e. $\omega \in \Omega$, and for each fixed pair (t,a) we have

(6.21) $P^0[I_t(a) = M_t(a)] = 1.$

Now we define

$$L_t(a) \overset{\Delta}{=} (W_t - a) - (-a)^+ - I_t(a); \quad 0 \leq t < \infty, \quad a \in \mathbb{R}.$$

For fixed (t,a), $L_t(a)$ is an $\mathcal{F}_t$-measurable random variable, and the random field L is P^0-a.s. continuous in the pair (t,a). Indeed, because W_t and $I_t(a)$ are both locally Hölder continuous with any exponent $\gamma \in (0,\frac{1}{2})$, the local time L also has this property: for every $\gamma \in (0,\frac{1}{2})$ and positive T,K, there exists a P^0-a.s. positive random variable $h(\omega)$ and a constant $\delta > 0$ such that

$$(6.22) \quad P^0 \left[\begin{array}{c} \omega \in \Omega; \quad \sup_{\substack{0 < \|(t,a)-(s,b)\| < h(\omega) \\ 0 \leq s,t \leq T \\ -K \leq a,b \leq K}} \frac{|L_t(a,\omega) - L_s(b,\omega)|}{\|(t,a)-(s,b)\|^{\gamma}} \leq \delta \end{array} \right] = 1.$$

Our next task is to show that the random field $\ell_t(a) \overset{\Delta}{=} 2L_t(a)$ satisfies the identity (6.6), or equivalently (6.7), for every function h in the collection $\mathcal{H}$ defined in Problem 6.7. For $h \in \mathcal{H}$, define

$$H(x) \overset{\Delta}{=} \int_{-\infty}^{\infty} h(u)(x-u)^+ du = \int_{-\infty}^{x} \int_{-\infty}^{y} h(u) du\, dy; \quad x \in \mathbb{R},$$

and observe the identities

$$H'(x) = \int_{-\infty}^{\infty} h(u) 1_{[u,\infty)}(x) du = \int_{-\infty}^{x} h(u) du, \quad H''(x) = h(x).$$

By virtue of Itô's rule and Problem 6.12 below, we have P^0-a.s. for fixed $t \geq 0$:

$$\frac{1}{2} \int_0^t h(W_s) ds = H(W_t) - H(0) - \int_0^t H'(W_s) dW_s$$

$$= \int_{-\infty}^{\infty} h(a) \{ (W_t-a)^+ - (-a)^+ \} da - \int_0^t (\int_{-\infty}^{\infty} h(a) 1_{[a,\infty)}(W_s) du) dW_s$$

$$= \int_{-\infty}^{\infty} h(a) \{ (W_t-a)^+ - (-a)^+ - \int_0^t 1_{[a,\infty)}(W_s) dW_s \} da$$

$$= \int_{-\infty}^{\infty} h(a) L_t(a) da + \int_{-\infty}^{\infty} h(a) \{ I_t(a) - M_t(a) \} da.$$

But $E^0 \int_{-\infty}^{\infty} (I_t(a)-M_t(a))^2 da = \int_{-\infty}^{\infty} E^0 (I_t(a)-M_t(a))^2 da = 0$ by

(6.21). Thus, for each fixed $t \geq 0$, we have for P^0-a.e. ω

$$(6.23) \qquad \int_0^t h(W_s(\omega)) ds = \int_{-\infty}^{\infty} h(x) \ell_t(x,\omega) dx.$$

Since both sides of (6.23) are continuous in t and $\mathcal{H}$ is countable, it is possible to find a set $\Omega_0^* \in \mathcal{F}$ with $P^0(\Omega_0^*) = 1$ such that for every $\omega \in \Omega_0^*$, (6.23) holds for every $h \in \mathcal{H}$ and every $t \geq 0$. Problem 6.7 now implies that for every $\omega \in \Omega_0^*$, (6.7) holds for every Borel function $f : \mathbb{R} \to [0,\infty)$.

Recall finally that $\Omega = C[0,\infty)$ and that P^z assigns probability one to the set $\Omega_z \triangleq \{\omega \in \Omega; \omega(0) = z\}$. We may assume that $\Omega_0^* \subseteq \Omega_0$, and we may redefine $\ell_t(x,\omega)$ for $\omega \notin \Omega_0$ by setting

$$\ell_t(x,\omega) \triangleq \ell_t(x-\omega(0), \omega-\omega(0)).$$

We set $\Omega^* = \{\omega \in \Omega; \omega-\omega(0) \in \Omega_0^*\}$, so that $P^z(\Omega^*) = 1$ for every $z \in \mathbb{R}$ (cf. (6.3)). It is easily verified that ℓ and Ω^* have all the properties set forth in Definition 6.3.

6.12 Problem: For a continuous function $h : \mathbb{R} \to [0,\infty)$ with compact support, the following interchange of Lebesgue and Itô integrals is permissible:

$$(6.24) \qquad \int_{-\infty}^{\infty} h(a) (\int_0^t 1_{[a,\infty)} (W_s) dW_s) da = \int_0^t (\int_{-\infty}^{\infty} h(a) 1_{[a,\infty)} (W_s) da) dW_s,$$

a.s. P^0. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

6.13 **Problem:** We may cast (6.13) in the form

$$(6.25) \qquad |W_t - a| = |z - a| - B_t(a) + \ell_t(a); \quad 0 \le t < \infty,$$

where $B_t(a) \triangleq -\int_0^t \operatorname{sgn}(W_s - a) \, dW_s$, for fixed $a \in \mathbb{R}$.

  (i)   Show that for any $z \in \mathbb{R}$, the process $B(a) = \{B_t(a), \mathcal{F}_t; 0 \le t < \infty\}$ is a Brownian motion under P^z, with $P^z[B_0(a) = 0] = 1$.

  (ii)   Using (6.25) and the representation (6.2), show that $\ell(a) = \{\ell_t(a), \mathcal{F}_t; 0 \le t < \infty\}$ is a continuous, increasing process (Definition 1.4.4) which satisfies

$$(6.26) \qquad \int_0^\infty 1_{\mathbb{R} \setminus \{a\}}(W_t) \, d\ell_t(a) = 0; \quad \text{a.s. } P^z.$$

In other words, the path $t \mapsto \ell_t(a, \omega)$ is "flat" off the level set $Z_\omega(a) = \{0 \le t < \infty; W_t(\omega) = a\}$ of the Brownian path.

 (iii) Show that for P^0-a.e. ω, we have $\ell_t(0, \omega) > 0$ for all $t > 0$.

  (iv) Show that for every $z \in \mathbb{R}$ and P^z-a.e. ω, every point of $Z_\omega(a)$ is a point of increase of $t \mapsto \ell_t(a, \omega)$.

□

Our next goal in this section is to provide a new proof of the celebrated result of P. Lévy (1948) already discussed in Problem 2.8.7, according to which <u>the processes</u>

(6.27) $\{M_t^W - W_t \triangleq \max_{0 \leq s \leq t} W_s - W_t; \ 0 \leq t < \infty\}$ and $\{|W_t|; \ 0 \leq t < \infty\}$

<u>have the same finite-dimensional distributions under</u> P^0. In particular, we shall present the ingenious method of A. V. Skorohod (1961), which provides as a by-product the fact <u>that the processes</u>

(6.28) $\{M_t^W \triangleq \max_{0 \leq s \leq t} W_s; \ 0 \leq t \leq \infty\}$ and $\{\ell_t(0); \ 0 \leq t < \infty\}$

<u>also have the same finite-dimensional distributions under</u> P^0.

<u>6.14 Lemma: The Skorohod (1961) equation.</u>

Let $z \geq 0$ be a given number and $y(\cdot) = \{y(t); \ 0 \leq t < \infty\}$ a continuous function with $y(0) = 0$. There exists a unique continuous function $k(\cdot) = \{k(t); \ 0 \leq t < \infty\}$, such that

(i) $x(t) \triangleq z + y(t) + k(t) \geq 0; \quad 0 \leq t < \infty,$

(ii) $k(0) = 0, \ k(\cdot)$ is nondecreasing, and

(iii) $k(\cdot)$ is flat off $\{t \geq 0; \ x(t) = 0\}$, i.e., $\int_0^\infty 1_{\{x(s)>0\}} dk(s) = 0.$

This function is given by

(6.29) $k(t) = \max[0, \ \max_{0 \leq s \leq t} \{-(z+y(s))\}], \quad 0 \leq t < \infty.$

Proof:

To prove uniqueness, let $k(\cdot)$ and $\tilde{k}(\cdot)$ be continuous functions with properties (i) and (ii), where $x(\cdot)$ and $\tilde{x}(\cdot)$ correspond to $k(\cdot)$ and $\tilde{k}(\cdot)$, respectively. Suppose there exists a number $T > 0$ with $x(T) > \tilde{x}(T)$, and let $\tau \triangleq \max\{0 \leq t < T; \; x(t) - \tilde{x}(t) = 0\}$ so that $x(t) > \tilde{x}(t) \geq 0$, $\forall\, t \in (\tau, T]$. But $k(\cdot)$ is flat on $\{u \geq 0; \; x(u) > 0\}$, so $k(\tau) = k(T)$. Therefore,

$$0 < x(T) - \tilde{x}(T) = k(T) - \tilde{k}(T) \leq k(\tau) - \tilde{k}(\tau) = x(\tau) - \tilde{x}(\tau),$$

a contradiction. It follows that $x(T) \leq \tilde{x}(T)$ for all $T \geq 0$, so $k \leq \tilde{k}$. Similarly, $k \geq \tilde{k}$.

We now take $k(\cdot)$ to be defined by (6.29). Conditions (i) and (ii) are obviously satisfied. In order to verify (iii), it suffices to show that $\int_0^\infty 1_{\{x(s) > \varepsilon\}} dk(s) = 0$ for every $\varepsilon > 0$. Let (t_1, t_2) be a component of the open set $\{s : x(s) > \varepsilon\}$ and note that

$$-(z + y(s)) = k(s) - x(s) \leq k(t_2) - \varepsilon; \qquad t_1 \leq s \leq t_2.$$

But then

$$k(t_2) = \max[k(t_1), \max_{t_1 \leq s \leq t_2} \{-(z + y(s))\}] \leq \max[k(t_1), k(t_2) - \varepsilon],$$

which shows that $k(t_2) = k(t_1)$ and thus $\int_{t_1}^{t_2} dk(s) = 0$.

6.15 Remark: For every $z \geq 0$ and $y(\cdot) \in C[0, \infty)$ with $y(0) = 0$, we denote by $\mathcal{K}$ the class of functions $k \in C[0, \infty)$ which satisfy conditions (i) and (ii) of Lemma 6.14, and introduce the mappings

(6.30) $T_t(z;y) \triangleq \max[0, \max_{0 \leq s \leq t} \{-(z+y(s))\}]; \qquad 0 \leq t < \infty$

(6.31) $R_t(z;y) \triangleq z + y(t) + T_t(z;y); \qquad 0 \leq t < \infty.$

In terms of these, the solution to the Skorohod equation is given by

(6.32) $k(t) = T_t(z;y), \qquad x(t) = R_t(z;y)$

and $T(z;y)$ is the minimal element of $\mathcal{K}$, as can be seen in the first part of the proof of Lemma 6.14.   □

<u>6.16 Proposition</u>: Let $z \geq 0$ be a given number, and $B = \{B_t, \mathcal{G}_t; 0 \leq t < \infty\}$ a Brownian motion on some probability space $(\Theta, \mathcal{G}, Q)$ with $Q[B_0 = 0] = 1$. We suppose there exists a continuous process $k = \{k_t, \mathcal{G}_t; 0 \leq t < \infty\}$ such that, for Q-a.e. $\theta \in \Theta$, we have

(i) $X_t(\theta) \triangleq z - B_t(\theta) + k_t(\theta) \geq 0; \qquad 0 \leq t < \infty,$

(ii) $k_0(\theta) = 0, \quad t \mapsto k_t(\theta)$ is nondecreasing, and

(iii) $\int_0^\infty 1_{(0,\infty)}(X_s(\theta)) dk_s(\theta) = 0.$

Then $X = \{X_t; 0 \leq t < \infty\}$ under Q has the same finite-dimensional distributions as $|W| = \{|W_t|; 0 \leq t < \infty\}$ under P^z.

Proof: The finite-dimensional distributions of the pair (k, X) are uniquely determined, since by Lemma 6.14 $k_t(\theta) = T_t(z; -B(\theta))$, $X_t(\theta) = R_t(z; -B(\theta))$; $0 \leq t < \infty$, for $\underset{\sim}{Q}$-a.e. $\theta \in \Theta$. It suffices, therefore, on our given measurable space $(\Omega, \mathcal{F})$ equipped with the Brownian family $\{W_t, \mathcal{F}_t; 0 \leq t < \infty\}$, $\{P^x\}_{x \in \mathbb{R}}$, to exhibit a standard Brownian motion $B = \{B_t, \mathcal{F}_t, 0 \leq t < \infty\}$ and a continuous nondecreasing process $k = \{k_t, \mathcal{F}_t; 0 \leq t < \infty\}$ such that, for P^z-a.e. $\omega \in \Omega$:

$$|W_t(\omega)| = z - B_t(\omega) + k_t(\omega); \quad 0 \leq t < \infty,$$

(6.33) $\qquad k_0(\omega) = 0, \quad t \to k_t(\omega)$ is nondecreasing, and

$$\int_0^\infty 1_{\mathbb{R} \setminus \{0\}}(W_s(\omega)) \, dk_s(\omega) = 0.$$

But this has already been accomplished in Problem 6.13 (relations (6.25), (6.26) with $a = 0$), if we make the identifications

$$B_t \equiv -\int_0^t \text{sgn } W_s \, dW_s, \quad k_t \equiv \ell_t(0).$$

6.17 Theorem: P. Lévy (1948)

The pairs of processes $\{(M_t^W - W_t, M_t^W), \mathcal{F}_t; 0 \leq t < \infty\}$ and $\{(|W_t|, \ell_t(0)), \mathcal{F}_t; 0 \leq t < \infty\}$ as in (6.27), (6.28) have the same finite-dimensional distributions under P^0.

Proof:

Because of uniqueness in the Skorohod equation, we have from (6.33)

(6.34) $\ell_t(0,\omega) \equiv M_t^B(\omega)$, $|W_t(\omega)| \equiv M_t^B(\omega) - B_t(\omega)$; $0 \leq t < \infty$

for P^0-a.e. $\omega \in \Omega$, upon observing that

(6.35) $M_t^B(\omega) = \max\limits_{0 \leq s \leq t} B_s(\omega) = T_t(0;-B(\omega))$

(Remark 6.15). The assertion follows, since both W and B are Brownian motions starting at the origin under P^0. We also notice the useful identity, valid for every fixed $t \in [0,\infty)$:

(6.36) $M_t^B = \lim \frac{1}{2\varepsilon} \text{meas}\{0 \leq s \leq t;\ M_s^B - B_s \leq \varepsilon\}$, a.s.$P^0$. $\square$

6.18 Problem: Show that for every real numbers a,z we have

$$P^z[\omega \in \Omega;\ \lim\limits_{t \to \infty} \ell_t(a,\omega) = \infty] = 1.$$ $\square$

The function $f_1(x) = (x-a)^+$, $f_2(x) = (x-a)^-$ and $f_3(x) = |x-a|$ in the Tanaka formulas (6.11)-(6.13) share an important property, namely <u>convexity</u>:

(6.37) $f(\lambda x + (1-\lambda)z) \leq \lambda f(x) + (1-\lambda)f(z)$; $x < z,\ 0 < \lambda < 1$,

which can be put in the equivalent form

(6.38) $f(y) \leq \frac{z-y}{z-x} f(x) + \frac{y-x}{z-x} f(z)$; $x < y < z$,

upon substituting $y = \lambda x + (1-\lambda)z$. Our success in representing $f(W_t)$ explicitly as a semimartingale, for the particular choices

$f(x) = (x-a)^{\pm}$ and $f(x) = |x-a|$, makes us wonder whether it might be possible to obtain a <u>generalized Itô formula</u> for convex functions which are not necessarily twice differentiable. This possibility was explored by Meyer (1976) and Wang (1977). We derive the pertinent Itô formula in Theorem 6.22, after a brief digression on the fundamental properties of convex functions.

6.19 Problem: Every convex function $f : \mathbb{R} \to \mathbb{R}$ is continuous.

For fixed $x \in \mathbb{R}$, the difference quotient

$$(6.39) \qquad \Delta f(x;h) \triangleq \frac{f(x+h) - f(x)}{h} ; \quad h \neq 0$$

is a nondecreasing function of $h \in \mathbb{R} \setminus \{0\}$, and therefore the right- and left-derivatives

$$(6.40) \qquad D^{\pm} f(x) \triangleq \lim_{h \to 0\pm} \frac{1}{h}[f(x+h) - f(x)]$$

exist and are finite for every $x \in \mathbb{R}$. Furthermore,

$$(6.41) \qquad D^{+} f(x) \leq D^{-} f(y) \leq D^{+} f(y); \quad x < y,$$

and $D^{+}f(\cdot)$ (respectively, $D^{-}f(\cdot)$) is right- (respectively, left-) continuous and nondecreasing on $\mathbb{R}$.

Finally, there exist sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ of real numbers, such that

(6.42) $f(x) = \sup_{n \geq 1}(\alpha_n x + \beta_n); \quad x \in \mathbb{R}.$

(<u>Hint</u>: Use (6.38) extensively).   □

<u>6.20 Problem</u>: Let the function $\varphi : \mathbb{R} \to \mathbb{R}$ be nondecreasing, and define

$$\varphi_{\pm}(x) = \lim_{y \to x\pm} \varphi(y), \qquad \Phi(y) = \int_0^x \varphi(u)\,du.$$

(i) The functions φ_+ and φ_- are right- and left-continuous, respectively, with

(6.43) $\varphi_-(x) \leq \varphi(x) \leq \varphi_+(x); \quad x \in \mathbb{R}.$

(ii) The functions $\varphi_{\pm}$ have the same set of continuity points, and equality holds in (6.43) on this set; in particular, except for x in a countable set N, we have

$\varphi_{\pm}(x) = \varphi(x).$

(iii) The function Φ is convex, with

$$D^-\Phi(x) = \varphi_-(x) \leq \varphi(x) \leq \varphi_+(x) = D^+\Phi(x); \quad x \in \mathbb{R}.$$

(iv) If $f : \mathbb{R} \to \mathbb{R}$ is any other convex function for which

(6.44) $D^-f(x) \leq \varphi(x) \leq D^+f(x); \quad x \in \mathbb{R},$

then we have

$f(x) = f(0) + \Phi(x); \quad x \in \mathbb{R}.$

6.21 Problem: For any convex function $f : \mathbb{R} \to \mathbb{R}$, there is a countable set $N \subseteq \mathbb{R}$ such that f is differentiable on $\mathbb{R} \setminus N$, and

$$(6.45) \qquad f'(x) = D^+ f(x) = D^- f(x); \quad x \in \mathbb{R} \setminus N.$$

Moreover

$$(6.46) \qquad f(x) - f(0) = \int_0^x f'(u)\,du = \int_0^x D^+ f(u)\,du = \int_0^x D^- f(u)\,du; \quad x \in \mathbb{R}. \qquad \square$$

The preceding problems show that convex functions are "essentially" differentiable, but Itô's rule requires the existence of a second derivative. For a convex function f, we use in place of the second derivative the measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by

$$(6.47) \qquad \mu((a,b]) = D^+ f(b) - D^+ f(a); \quad -\infty < a < b < \infty.$$

Of course, if f'' exists, then $\mu(dx) = f''(x)\,dx$. Even without the existence of f'', we may integrate Riemann-Stieltjes integrals by parts to obtain the formula

$$(6.48) \qquad \int_{-\infty}^{\infty} g(x)\,\mu(dx) = -\int_{-\infty}^{\infty} g'(x) D^+ f(x)\,dx$$

for every function $g : \mathbb{R} \to \mathbb{R}$ which is of class C^1 and has compact support.

6.22 Theorem: A generalized Itô rule for convex functions

Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function and μ the "second derivative measure" introduced in (6.47). Then, for every $z \in \mathbb{R}$, we have

$$(6.49) \qquad f(W_t) = f(z) + \int_0^t D^+ f(W_s)\, dW_s + \frac{1}{2} \int_{-\infty}^\infty \ell_t(x)\, \mu(dx); \quad 0 \leq t < \infty,$$
$$\text{a.s. } P^z.$$

Proof:

It suffices to prove (6.49) with t replaced by $t \wedge T_{-n} \wedge T_n$, and by such a localization we may assume without loss of generality that $D^+ f$ is uniformly bounded on $\mathbb{R}$. We employ the mollifiers ρ_n; $n \geq 1$, of (6.18) to obtain convex, infinitely differentiable approximation to f by convolution:

$$(6.50) \qquad f_n(x) \triangleq \int_{-\infty}^\infty \rho_n(y-x) f(y)\, dy; \quad n \geq 1.$$

It is not hard to verify that $f_n(x) = \int_{-\infty}^\infty \rho(z) f(x + \frac{z}{n})\, dz$ and

$$(6.51) \qquad \lim_{n \to \infty} f_n(x) = f(x), \qquad \lim_{n \to \infty} f_n'(x) = D^+ f(x)$$

hold for every $x \in \mathbb{R}$. In particular, the nondecreasing functions $D^+ f$ and $\{f_n'\}_{n=1}^\infty$ are uniformly bounded on compact subsets of $\mathbb{R}$. If $g : \mathbb{R} \to \mathbb{R}$ is of class C^1 and has compact support, then because of (6.48),

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} g(x) f_n''(x) \, dx = -\lim_{n \to \infty} \int_{-\infty}^{\infty} g'(x) f_n'(x) \, dx$$

$$= -\int_{-\infty}^{\infty} g'(x) D^+ f(x) \, dx = \int_{-\infty}^{\infty} g(x) \mu(dx).$$

The general continuous g with compact support can be uniformly approximated by functions of class C^1, so for such a g we have

$$(6.52) \qquad \lim_{n \to \infty} \int_{-\infty}^{\infty} g(x) f_n''(x) \, dx = \int_{-\infty}^{\infty} g(x) \mu(dx).$$

In other words, the measures $f_n''(x) \, dx$ converge weakly to the measure $\mu(dx)$.

We can now apply the change-of-variable formula (Theorem 3.3) to $f_n(W_s)$, and obtain, for fixed $t \in (0, \infty)$:

$$f_n(W_t) - f_n(z) = \int_0^t f_n'(W_s) \, dW_s + \frac{1}{2} \int_0^t f_n''(W_s) \, ds, \quad \text{a.s. } P^z.$$

When $n \to \infty$, the left-hand side converges almost surely to $f(W_t) - f(z)$, and the stochastic integral converges in L^2 to $\int_0^t D^+ f(W_s) \, dW_s$ because of (6.51) and the uniform boundedness of the functions involved. We also have from (6.7) and (6.52):

$$\lim_{n \to \infty} \int_0^t f_n''(W_s) \, ds = \lim_{n \to \infty} \int_{-\infty}^{\infty} f_n''(x) \ell_t(x) \, dx = \int_{-\infty}^{\infty} \ell_t(x) \mu(dx), \quad \text{a.s. } P^z.$$

because, for P^z-a.e. $\omega \in \Omega$, the continuous function $x \mapsto \ell_t(x, \omega)$ has support on the compact set $[\min_{0 \le s \le t} W_s(\omega), \max_{0 \le s \le t} W_s(\omega)]$. This

proves (6.49) for each fixed t, and because of continuity it is also seen to hold simultaneously for all $t \in [0,\infty)$, a.s. P^z.

6.23 Corollary: If $f : \mathbb{R} \to \mathbb{R}$ is the difference of convex functions, then (6.49) holds again for every $z \in \mathbb{R}$; now, μ defined by (6.47) is a signed measure, finite on each bounded subinterval of $\mathbb{R}$.

6.24 Problem: Let $a_1 < a_2 < \ldots < a_n$ be real numbers, and denote $D = \{a_1, \ldots, a_n\}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and f' and f'' exist and are continuous on $\mathbb{R} \setminus D$. Suppose further that the limits

$$f'(a_k \pm) \triangleq \lim_{x \to a_k \pm} f'(x), \quad f''(a_k \pm) = \lim_{x \to a_k \pm} f''(x)$$

exist and are finite. Show that f is the difference of convex functions and, for every $z \in \mathbb{R}$,

$$(6.53) \qquad f(W_t) = f(z) + \int_0^t f'(W_s)\, dW_s + \frac{1}{2} \int_0^t f''(W_s)\, ds$$

$$+ \frac{1}{2} \sum_{k=1}^{n} \ell_t(a_k)[f'(a_k+) - f'(a_k-)]; \quad 0 \leq t < \infty,$$
$$\text{a.s. } P^z.$$

6.25 Exercise: Obtain the Tanaka formulas (6.11)-(6.13) as corollaries of the generalized Itô rule (6.49). $\qquad \square$

Our next application of local time concerns the study of the continuous, nondecreasing, additive functional

$$A_t(\omega) = \int_0^t f(W_s(\omega)) \, ds; \quad 0 \le t < \infty,$$

where $f : \mathbb{R} \to [0, \infty)$ is a given Borel measurable function. We shall be interested in questions of finiteness and asymptotics, but first we need an auxiliary result.

6.26 Lemma: Let $f : \mathbb{R} \to [0, \infty)$ be Borel measurable; fix $x \in \mathbb{R}$, and suppose there exists a random time T with

$$P^0[0 < T < \infty] = 1, \quad P^0[\int_0^T f(x+W_s) \, ds < \infty] > 0.$$

Then, for some $\varepsilon > 0$, we have

(6.54) $\int_{-\varepsilon}^{\varepsilon} f(x+y) \, dy < \infty.$

Proof:

From (6.7) and Problem 6.13(iii), we know there exists an event Ω^* with $P^0(\Omega^*) = 1$, such that for every $\omega \in \Omega^*$:

$$\int_0^{T(\omega)} f(x+W_s(\omega)) \, ds = \int_{-\infty}^{\infty} f(x+y) \ell_{T(\omega)}(y, \omega) \, dy$$

and $\ell_{T(\omega)}(0, \omega) > 0$. By assumption, we may choose $\omega \in \Omega^*$ such that $\int_0^{T(\omega)} f(x+W_s(\omega)) \, ds < \infty$ as well. With this choice of ω,

we may appeal to the continuity of $\ell_{T(\omega)}(\cdot,\omega)$ to choose positive numbers ε and c such that $\ell_{T(\omega)}(y,\omega) \geq c$ whenever $|y| \leq \varepsilon$. Therefore,

$$c \int_{-\varepsilon}^{\varepsilon} f(x+y)\,dy \leq \int_{0}^{T(\omega)} f(x+W_s(\omega))\,ds < \infty,$$

which yields (6.54). □

6.27 Proposition: Engelbert-Schmidt (1981) Zero-One Law

Let $f : \mathbb{R} \to [0,\infty)$ be Borel measurable. The following three assertions are equivalent.

(i) $P^0[\int_0^t f(W_s)\,ds < \infty; \ 0 \leq t < \infty] > 0,$

(ii) $P^0[\int_0^t f(W_s)\,ds < \infty; \ 0 \leq t < \infty] = 1,$

(iii) f is locally integrable, i.e., for every compact set $K \subseteq \mathbb{R}$, we have $\int_K f(y)\,dy < \infty$.

Proof:

For the implication (i) ⇒ (iii) we fix $b \in \mathbb{R}$ and consider the first passage time T_b. Because $P^0[T_b < \infty] = 1$, (i) gives $P^0[\int_0^{t+T_b} f(W_s)\,ds < \infty; \ 0 \leq t < \infty] > 0$. But then

$$\int_0^{t+T_b(\omega)} f(W_s(\omega))\,ds \geq \int_{T_b(\omega)}^{t+T_b(\omega)} f(W_s(\omega))\,ds = \int_0^t f(b+B_s(\omega))\,ds,$$

where $B_s(\omega) \triangleq W_{s+T_b(\omega)}(\omega) - b$; $0 \le s < \infty$ is a new Brownian motion under P^0. It follows that for each $t > 0$, $P^0[\int_0^t f(b+B_s(\omega))ds < \infty] > 0$, and Lemma 6.26 guarantees the existence of an open neighborhood $U(b)$ of b such that $\int_{U(b)} f(y)dy < \infty$. If $K \subseteq \mathbb{R}$ is compact, the family $\{U(b)\}_{b \in K}$, being an open cover of K, has a finite subcover. It follows that $\int_K f(y)dy < \infty$.

For the implication (iii) $\Rightarrow$ (ii) we have again from (6.7):

$$\int_0^t f(W_s(\omega))ds = \int_{-\infty}^{\infty} f(y)\ell_t(y,\omega)dy = \int_{m_t(\omega)}^{M_t(\omega)} f(y)\ell_t(y,\omega)dy$$

$$\le [\max_{m_t(\omega) \le y \le M_t(\omega)} \ell_t(y,\omega)] \cdot \int_{m_t(\omega)}^{M_t(\omega)} f(y)dy; \quad 0 \le t < \infty,$$

where $m_t(\omega) = \min_{0 \le s \le t} W_s(\omega)$, $M_t(\omega) = \max_{0 \le s \le t} W_s(\omega)$. The last integral is P^0-a.s. finite by assumption, because the set $K = [m_t(\omega), M_t(\omega)]$ is P^0-a.s. compact. $\square$

6.28 **Corollary:** For $0 < \alpha < \infty$, we have the following dichotomy:

$$P^0[\int_0^t \frac{ds}{|W_s|^\alpha} < \infty; \; 0 \le t < \infty] = \begin{cases} 1; & \text{if } 0 < \alpha < 1 \\ 0; & \text{if } \alpha \ge 1 \end{cases}.$$

6.29 **Problem:** The conditions of Proposition 6.27 are also equivalent to the following assertions:

(iv) $P^0[\int_0^t f(W_s)\,ds < \infty] = 1$, for some $0 < t < \infty$;

(v) $P^x[\int_0^t f(W_s)\,ds < \infty; \; 0 \le t < \infty] = 1$ for every $x \in \mathbb{R}$;

(vi) for every $x \in \mathbb{R}$, there exists a Brownian motion $\{B_t, \mathcal{G}_t; \; 0 \le t < \infty\}$ and a random time S on a suitable probability space $(\Theta, \mathcal{G}, Q)$, such that $Q[B_0 = 0, \; 0 < S < \infty] = 1$ and

$$Q[\int_0^S f(x+B_s)\,ds < \infty] > 0.$$

(Hint: It suffices to justify the implications (ii) $\Rightarrow$ (iv) $\Rightarrow$ (vi) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (vi), the first and last of which are obvious).

6.30 Problem: Suppose that the Borel measurable function

$f : \mathbb{R} \to [0, \infty)$ satisfies $\text{meas}\{y \in \mathbb{R}; \; f(y) > 0\} > 0$. Show that

(6.55) $P^x[\omega \in \Omega; \; \int_0^\infty f(W_s(\omega))\,ds = \infty] = 1$

holds for every $x \in \mathbb{R}$. Assume further that f has compact support and consider the sequence of continuous processes

$X^{(n)} \triangleq \{\frac{1}{\sqrt{n}} \int_0^{nt} f(W_s)\,ds; \; 0 \le t < \infty\}$, $n \ge 1$; establish then,

under P^0, the convergence

(6.56) $\qquad x^{(n)} \xrightarrow[n \to \infty]{\mathscr{L}} x \triangleq \{\overline{f}\ell_t(0); \ 0 \leq t < \infty\}$

in the sense of Definition 2.4.4, where $\overline{f} = \int_{-\infty}^{\infty} f(y)\,dy > 0.$

3.S: SOLUTIONS TO PROBLEMS

2.4 Solution:

(a) Since $t-s - \frac{1}{2^n} \leq \varphi_n(t-s) < t-s$, we have $t - \frac{1}{2^n} \leq \varphi_n(t-s)+s < t$.
Consequently, $X_t^{(n,s)}$ is $\mathfrak{F}_t$-measurable, and since φ_n takes
only discrete values, $X^{(n,s)}$ is simple.

(b) The procedure (2.4) results in measurable (but perhaps not
adapted) processes $\{\widetilde{X}^{(m)}\}_{n=1}^\infty$, such that

$$\lim_{m \to \infty} E\int_{-\infty}^{T} |X_t^\epsilon - X_t|^2 \, dt \leq \epsilon^2.$$

By the Minkowski inequality we have

$$\left(E\int_0^T |X_t - X_{t-h}|^2 \, dt\right)^{\frac{1}{2}}$$

$$\leq \left(E\int_0^T |X_t - X_t^c|^2 \, dt\right)^{\frac{1}{2}} + \left(E\int_0^T |X_t^\epsilon - X_{t-h}^\epsilon|^2 \, dt\right)^{\frac{1}{2}}$$

$$+ \left(E\int_0^T |X_{t-h}^\epsilon - X_{t-h}|^2 \, dt\right)^{\frac{1}{2}}$$

$$\leq 2\epsilon + \left(E\int_0^T |X_t^\epsilon - X_{t-h}^\epsilon|^2 \, dt\right)^{\frac{1}{2}}.$$

We can now let $h\downarrow 0$ and conclude, from the continuity of
X^ϵ and the bounded convergence theorem, that

$$\overline{\lim_{h\downarrow 0}} \, E\int_0^T |X_t - X_{t-h}|^2 \, dt \leq 4\epsilon^2.$$

(c) Let i be any nonnegative integer. As s ranges over $[\frac{i}{2^n}, \frac{i+1}{2^n})$, $\varphi_n(t-s)+s$ ranges over $[t - \frac{1}{2^n}, t)$. Therefore,

$$E\int_0^T \int_0^1 |X_t^{(n,s)} - X_t|^2 ds\, dt = 2^n E\int_0^T \int_0^{2^{-n}} |X_t - X_{t-h}|^2\, dh\, dt$$

$$= 2^n \int_0^{2^{-n}} [E\int_0^T |X_t - X_{t-h}|^2 dt]dh \leq \max_{0 \leq h \leq 2^{-n}} E\int_0^T |X_t - X_{t-h}|^2 dt,$$

which converges to zero as $n \to \infty$ because of (b).

(d) From (c) we have that there is a sequence $\{n_k\}_{k=1}^\infty$ of integers, increasing to infinity as $k \to \infty$, such that for $\lambda \times \lambda \times P$ - a.e. triple (s,t,ω) in $[0,1] \times [0,T] \times \Omega$, we have

(S.2) $$\lim_{k \to \infty} |X_t^{(n_k, s)}(\omega) - X_t(\omega)|^2 = 0.$$

Therefore, we can select $s \in [0,1]$ such that for $\lambda \times P$ -a.e. pair (t,ω) in $[0,T] \times \Omega$, we have (S.2). Setting $X^{(k)} \triangleq X^{(n_k, s)}$, we obtain (2.5) from the bounded convergence theorem.

2.11 Solution:

We may write $W_{t \wedge T} = I_t^W(X)$, where $X_t(\omega) = 1_{\{t \leq T(\omega)\}}$; $0 \leq t < \infty$. Because

$$\langle I^W(X) \rangle_t = \int_0^t X_s^2 ds = t \wedge T,$$

we have $\langle I^W(X) \rangle_\infty = T$ and $E\langle I^W(X) \rangle_\infty < \infty$. It follows from Problem 1.5.19 that both $\{W_{t \wedge T}; \ 0 \leq t < \infty\}$ and $\{W^2_{t \wedge T}; \ 0 \leq t < \infty\}$ are uniformly integrable, so (2.22) is justified.

2.12 Solution:

If ET_b were finite, then we would have $W_{T_b} = b$, a.s.P, as well as $E(W_{T_b}) = 0$ from Problem 2.11. But this is absurd.

2.16 Solution:

$$\left| \int_0^t X_u Y_u \ d\langle M, N \rangle_u \right|^2 = \left| \langle I^M(X), \ I^N(Y) \rangle_t \right|^2$$

$$\leq \langle I^M(X) \rangle_t \ \langle I^N(Y) \rangle_t = \int_0^t X_u^2 \ d\langle M \rangle_u \cdot \int_0^t Y_u^2 \ d\langle N \rangle_u.$$

2.16′ Solution: By assumption, we have

$$E\langle I^M(X) \rangle_\infty = E\int_0^\infty X_s^2 \ d\langle M \rangle_s < \infty.$$

Uniform integrability and the existence of a last element for $I^M(X)$ follow from Problem 1.5.22, as does uniform integrability of $(I^M(X))^2$. The same is true for $I^N(Y)$.

Applying Problem 2.16 with X_u, Y_u replaced by $X_u 1_{[u \geq T]}$, $Y_u 1_{[u \geq T]}$ respectively, we obtain

$$\left| \int_T^{T+t} X_s Y_s \ d\langle M, N \rangle_s \right| \leq \left(\int_T^{T+t} X_s^2 \ d\langle M \rangle_s \cdot \int_T^{T+t} Y_s^2 \ d\langle N \rangle_s \right)^{1/2},$$

whence

$$\left| \int_T^\infty X_s Y_s \ d\langle M, N \rangle_s \right| \leq \left(\int_T^\infty X_s^2 \ d\langle M \rangle_s \cdot \int_T^\infty Y_s^2 \ d\langle N \rangle_s \right)^{1/2}$$

a.s. P. As $T \to \infty$, the right-hand side of this inequality converges to zero; therefore,

$$\langle I^M(X), I^N(Y) \rangle_t = \int_0^t X_s Y_s \, d\langle M, N \rangle_s$$

converges as $t \to \infty$, and is bounded by the integrable random variable

$$\left(\int_0^\infty X_s^2 \, d\langle M \rangle_s \right)^{1/2} \cdot \left(\int_0^\infty Y_s^2 \, d\langle N \rangle_s \right)^{1/2},$$

a.s. P. The dominated convergence theorem gives then

$$\lim_{t \to \infty} E[I_t^M(X) I_t^N(Y)] = \lim_{t \to \infty} E\langle I^M(X), I^N(Y) \rangle_t =$$

$$E\langle I^M(X), I^N(Y) \rangle_\infty = E \int_0^\infty X_s Y_s \, d\langle M, N \rangle_s .$$

We also have

$$E[I_\infty^M(X) \, I_\infty^N(Y) = E[(I_\infty^M(X) - I_t^M(X))(I_\infty^N(Y) - I_t^N(Y))]$$

$$+ E[I_t^M(X)(I_\infty^N(Y) - I_t^N(Y))]$$

$$+ E[I_t^N(Y)(I_\infty^M(X) - I_t^M(X))]$$

$$+ E[I_t^M(X) \, I_t^N(Y)].$$

We have just shown that the fourth term on the right-hand side converges to $E\langle I^M(X), I^N(Y) \rangle_\infty$ as $t \to \infty$. The other three terms converge to zero because of Hölder's inequality and the uniform integrability of $(I^M(X))^2$ and $(I^N(Y))^2$.

2.23 Solution: For any $\tilde{N} \in \mathcal{M}_2^c$,

$$\langle \alpha I^M(X) + \beta I^N(X), \tilde{N} \rangle_t = \alpha \langle I^M(X), \tilde{N} \rangle_t + \beta \langle I^N(X), \tilde{N} \rangle_t$$

$$= \alpha \int_0^t X_s \, d\langle M, \tilde{N} \rangle_s + \beta \int_0^t X_s \, d\langle N, \tilde{N} \rangle_s$$

$$= \int_0^t X_s \, d\langle \alpha M + \beta N, \tilde{N} \rangle_s,$$

and the result follows from Proposition 2.21.

2.24 Solution:

With X a measurable, adapted process satisfying $P[\int_0^T X_t^2(\omega) dt < \infty] = 1$ for every $0 < T < \infty$, we construct the sequences of stopping times

$$S_N(\omega) = \begin{cases} \inf\{0 \leqslant t \leqslant N; \int_0^t X_s^2(\omega) ds \geqslant N, \\ \\ N, \quad \text{if } \{\ldots\} = \emptyset, \end{cases}$$

and processes $X_t^{(N)}(\omega) = X_t(\omega) 1_{\{S_N(\omega) \geqslant t\}}; \; 0 \leqslant t < \infty$, indexed by $N \geqslant 1$. We have $E \int_0^\infty (X_t^{(N)}(\omega))^2 dt \leqslant N < \infty$, so for each $N \geqslant 1$ the process $X^{(N)}$ is in $\mathcal{L}$, and therefore can be approximated by a sequence of simple processes $\{X^{(n,N)}\}_{n=1}^\infty \subseteq \mathcal{L}_0$ in the sense

$$\lim_{n \to \infty} E \int_0^T |X_t^{(n,N)} - X_t^{(N)}|^2 \, dt = 0, \qquad \forall \, T < \infty$$

(Proposition 2.5). Let us fix a positive number $T<\infty$, and consider $N>T$; we have $P[\int_0^T X_t^2 \, dt > 0] = P[\int_0^T X_t^2 \, 1_{\{S_N < t\}} \, dt > 0] \leq P[S_N < T] = P[\int_0^T X_t^2 \, dt > N]$, and the last quantity converges to zero as $N \to \infty$, by assumption. Now, given any $\epsilon > 0$, we have

$$P[\int_0^T |X_t^{(n,N)} - X_t|^2 \, dt > \epsilon] \leq P[\int_0^T |X_t^{(n,N)} - X_t^{(N)}|^2 \, dt > \frac{\epsilon}{2}] +$$

$$P[\int_0^T |X_t^{(N)} - X_t|^2 \, dt > 0] \leq \frac{2}{\epsilon} E \int_0^T |X_t^{(n,N)} - X_t^{(N)}|^2 \, dt + P[S_N < T]$$

by the Čebyšev inequality, as well as

$$P[\sup_{0 \leq t \leq T} |I_t(X^{(n,N)}) - I_t(X)| \geq \epsilon] \leq$$

$$P[\{\sup_{0 \leq t \leq T} |I_t(X^{(n,N)}) - I_t(X^{(N)})| \geq \epsilon\} \cap \{S_N \geq T\}] + P[S_N < T] \leq$$

$$\frac{1}{\epsilon^2} E|I_T(X^{(n,N)}) - I_T(X^{(N)})|^2 + P[S_N < T] =$$

$$\frac{1}{\epsilon^2} E \int_0^T |X_t^{(n,N)} - X_t^{(N)}|^2 + P[S_N < T].$$

We have employed Corollary 2.19, the first submartingale inequality (Theorem 1.3.6), and (2.11). For any given $\delta > 0$ we can select $N_\delta > T$ so that $P[S_N < T] < \frac{\delta}{4}$ for every $N \geq N_\delta$, and for each such value of N we can find an integer $n_N \geq 1$ so that

$$E \int_0^T |X_t^{(n,N)} - X_t^{(N)}|^2 \, dt < \frac{\delta}{2} \left(\frac{2}{\epsilon} + \frac{1}{\epsilon^2} \right)^{-1}$$

holds for every $n \geq n_N$. It follows that for every $\epsilon > 0$, $\delta > 0$ there exists an integer N_δ such that, for every $N \geq N_\delta$, we have with $Y^{(N)} \underset{=}{\triangle} X^{(n_N, N)}$:

$$P[\int_0^T |Y_t^{(N)} - X_t|^2 \, dt > \epsilon] + P[\sup_{0 \leq t \leq T} |I_t(Y^{(N)}) - I_t(X)| > \epsilon] < \delta.$$

In other words, we can construct a sequence of simple processes $\{Y^{(N)}\}_{N=1}^\infty$ such that both sequences of random variables

$$\int_0^T |Y_t^{(N)} - X_t|^2 \, dt, \quad \sup_{0 \leq t \leq T} |I_t(Y^{(N)} - I_t(X)|$$

converge to zero in probability, as $N \to \infty$. There exists then a subsequence for which the convergence takes place almost surely. Having done this construction for T fixed, we now appeal to the first paragraph of the proof of Lemma 2.3 to obtain a sequence which works for all T.

2.25 Solution:

Consider first a simple process X. Using the notation of Definition 2.2, we have

$$\zeta_t^0(X) = \sum_{i=0}^\infty [\xi_i(W_{t \wedge t_{i+1}} - W_{t \wedge t_i}) - \frac{1}{2} \xi_i^2(t \wedge t_{i+1} - t \wedge t_i)],$$

where the sum is really a finite one, and in this case the martingale property: $E[\exp \zeta_t^S(X) | \mathcal{F}_s] = 1$ a.s. P, for $0 \leq s < t < \infty$, amounts to showing

$$E[\exp\{\xi_i(W_{t\wedge t_{i+1}}-W_{t\wedge t_i}) - \tfrac{1}{2}\xi_i^2(t\wedge t_{i+1}-t\wedge t_i)\}|\mathfrak{F}_s] =$$

$$= \exp\{\xi_i(W_{s\wedge t_{i+1}}-W_{s\wedge t_i}) - \tfrac{1}{2}\xi_i^2(s\wedge t_{i+1}-s\wedge t_i)\}, \quad \text{a.s. } P$$

for any $i\geq 0$. The reader will have no difficulty verifying this (reminder: ξ_i is a bounded, $\mathfrak{F}_{t_i}$-measurable random variable).

For general $X\in P$, there exists a sequence $\{X^{(n)}\}_{n=1}^{\infty} \subseteq \mathcal{L}_0$ such that, for P - a.e. $\omega\in\Omega$, we have

$$\lim_{n\to\infty}\int_0^T |X_t^{(n)}(\omega) - X_t(\omega)|^2 dt = 0, \quad \lim_{n\to\infty}\sup_{0\leq t\leq T}|\int_0^t X_s^{(n)}(\omega)dW_s - \int_0^t X_s(\omega)dW_s|=0$$

for every $T<\infty$; by Theorem 4.5.1 in Chung [ ], we also have

$$\lim_{n\to\infty}\int_0^T (X_t^{(n)}(\omega))^2 dt = \int_0^T X_t^2(\omega)dt, \quad \text{a.s. } P.$$

Therefore, with $T\geq t\geq s\geq 0$ one sees that $\lim_{n\to\infty}\exp\zeta_t^s(X^{(n)}) = \exp\zeta_t^s(X)$, a.s. and by Fatou's lemma:

$$E[\exp\zeta_t^s(X)|\mathfrak{F}_s] \leq \lim_{n\to\infty} E[\exp\zeta_t^s(X^{(n)})|\mathfrak{F}_s] = 1, \quad \text{a.s. } P.$$

2.26 Solution:

Let us take a partition $\Pi = \{t_0, t_1,\ldots,t_m\}$ of $[0,t]$ with $0 = t_0<t_1<\ldots<t_m = t$ and consider the corresponding simple process

$$X_s^{\Pi}(\omega) = \sum_{i=0}^{m-1} W_{t_i}(\omega)\, 1_{(t_i, t_{i+1}]}(s); \quad 0\leq s<t.$$

Now

$$E \int_0^t |X_s^{\Pi} - W_s|^2 ds = \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} E\, |W_{t_i} - W_t|^2\, dt$$

$$\le \sum_{i=0}^{m-1} (t_{i+1} - t_i) \cdot E(\sup_{t_i \le t \le t_{i+1}} |W_{t_i} - W_t|^2).$$

But $\{|W_{t_i} - W_t|, \mathfrak{F}_t^W;\ t_i \le t < \infty\}$ is a submartingale, and so, by Doob's maximal inequality (Theorem 1.3.6 (iv)),

$$E(\sup_{t_i \le t \le t_{i+1}} |W_{t_i} - W_t|^2) \le 4E\, |W_{t_i} - W_{t_{i+1}}|^2$$

$$= 4(t_{i+1} - t_i) \le 4\|\Pi\|,$$

where $\|\Pi\| \triangleq \max_{0 \le i \le m-1} (t_{i+1} - t_i)$. It follows that

$$\lim_{\|\Pi\| \to 0} E \int_0^t |X_s^{\Pi} - W_s|^s ds \le \lim_{\|\Pi\| \to 0} 4t\|\Pi\| = 0.$$

By definition, $I_t^W(W)$ is the L^2-limit of

$$I_t^W(X^{\Pi}) = \sum_{i=0}^{m-1} W_{t_i} (W_{t_{i+1}} - W_{t_i})$$

$$= \frac{1}{2} \sum_{i=0}^{m-1} \{(W_{t_{i+1}}^2 - W_{t_i}^2) - (W_{t_{i+1}} - W_{t_i})^2\}$$

$$= \frac{1}{2} W_t^2 - \frac{1}{2} \sum_{i=0}^{m-1} (W_{t_{i+1}} - W_{t_i})^2,$$

which converges in L^2 to $\frac{1}{2} W_t^2 - \frac{1}{2} t$ (Problem 2.9.8).

In fact, if we have a sequence of partitions $\{\Pi_n\}_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} \|\Pi_n\| < \infty$, then this last convergence takes place almost surely as well; c.f. Problem 2.9.8 .

This example provides a nice illustration of the sensitivity of the stochastic integral to the selection of the point where the integrand is evaluated. On the interval $(t_i, t_{i+1}]$, the process X^{Π} defined above takes the value of W at the left end-point, and is thereby adapted to the filtration of W. If, in place of $I_t^W(X^{\Pi})$, we were to consider for $\theta_i \triangleq t_i + \epsilon(t_{i+1} - t_i)$, $0 \le \epsilon \le 1$, the approximating sum

$$R(\Pi) \triangleq \sum_{i=0}^{m-1} W_{\theta_i}(W_{t_{i+1}} - W_{t_i})$$

$$= -\sum_{i=0}^{m-1} (W_{t_i} - W_{\theta_i})(W_{t_{i+1}} - W_{\theta_i}) + \sum_{i=0}^{m-1} (W_{\theta_i} - W_{t_i})^2$$

$$+ \sum_{i=0}^{m-1} W_{t_i}(W_{t_{i+1}} - W_{t_i}),$$

we would get a substantially different answer. Indeed, as $\|\Pi\| \to 0$, we have

$$R(\Pi) \to \sum_{i=0}^{m-1} (\theta_i - t_i) + \frac{1}{2}(W_t^2 - t) = \frac{1}{2} W_t^2 + (\epsilon - \frac{1}{2})t$$

in L^2. (Work this out carefully; Problem 1.2.10 is helpful here.) Different choices of ϵ thus lead to different values of the integral; the choice $\epsilon = 0$ is the only one which preserves the

martingale property, and this is the Itô definition with the increments of the integrating martingale "sticking out into the future". With $\epsilon = \frac{1}{2}$, we obtain the <u>Fisk-Stratonovich integral</u> which obeys the rules of standard calculus such as $\int_0^t W_s \, dW_s = \frac{1}{2} W_t^2$. This integral exists only under assumptions more restrictive than those necessary for the construction of the Itô integral, and, when it exists, it is related to the corresponding Itô integral by a simple "correction formula". Choosing $\epsilon = 1$ leads us to the so-called <u>backward Itô integral</u>.

3.2 Solution: We have

$$X_t = X_0 + M_t + B_t = X_0 + \widetilde{M}_t + \widetilde{B}_t,$$

so

$$M_t - \widetilde{M}_t = \widetilde{B}_t - B_t; \quad 0 \leq t < \infty.$$

We may localize by setting

$$T_n = \inf\{0 \leq t < \infty; \; |M_t - \widetilde{M}_t| \geq n\},$$

so that

$$N_t^{(n)} \triangleq M_{t \wedge T_n} - \widetilde{M}_{t \wedge T_n}$$

is a continuous martingale of bounded variation. It follows that $N_t^{(n)} = 0$; $0 \leq t < \infty$, and since $T_n \uparrow \infty$ a.s. as $n \rightarrow \infty$, we have

$$M_t = \widetilde{M}_t, \quad B_t = \widetilde{B}_t; \quad 0 \leq t < \infty.$$

3.7 **Solution:** The proof is much like that of Theorem 3.3.

The Taylor expansion in Step 2 of that proof is replaced by:

$$f(t_k, X_{t_k}) - f(t_{k-1}, X_{t_{k-1}}) =$$

$$= [f(t_k, X_{t_k}) - f(t_{k-1}, X_{t_k})] + [f(t_{k-1}, X_{t_k}) - f(t_{k-1}, X_{t_{k-1}})]$$

$$= \frac{\partial}{\partial t} f(\tau_k, X_{t_k})(t_k - t_{k-1}) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} f(t_{k-1}, X_{t_{k-1}})\left(x_{t_k}^{(i)} - x_{t_{k-}}^{(i)}\right)$$

$$+ \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} f(t_{k-1}, \eta_k)\left(x_{t_k}^{(i)} - x_{t_{k-1}}^{(i)}\right)\left(x_{t_k}^{(j)} - x_{t_{k-1}}^{(j)}\right),$$

where $t_{k-1} \le \tau_k \le t_k$ and η_k is as before.

3.10 Solution: Let $f(z) = \frac{1}{z}$, so $f'(z) = -\frac{1}{z^2}$ and $f''(z) = \frac{2}{z^3}$.
We have

$$dY_t = f'(Z_t)dZ_t + \frac{1}{2}f''(Z_t)(dZ_t)^2$$

$$= -\frac{X_t}{Z_t}dW_t + \frac{X_t^2}{Z_t}dt$$

$$= -Y_tX_t\,dW_t + Y_tX_t^2\,dt.$$

3.11 Solution: Let $f(x,y) = xy$ and apply Theorem 3.6 to compute

$$f(X_t,Y_t) = f(X_0,Y_0) + \int_0^t \frac{\partial}{\partial x}f(X_s,Y_s)dX_s + \int_0^t \frac{\partial}{\partial y}f(X_s,Y_s)dY_s$$

$$+ \frac{1}{2}\int_0^t\left[\frac{\partial^2}{\partial x\partial y}f(X_s,Y_s) + \frac{\partial^2}{\partial y\partial x}f(X_s,Y_s)\right]d\langle M,N\rangle_s.$$

3.13' Solution:

Note that X is independent of each pair $(W^{(1)},W^{(2)})$, $(W^{(1)},W^{(3)})$ and $(W^{(2)},W^{(3)})$. It is clear that $(M^{(1)},M^{(2)})$ is a two-dimensional Brownian motion. For $\Gamma \in \mathcal{B}(C[0,\infty)^2)$, we have

$$P[(M^{(1)},M^{(3)}) \in \Gamma]$$

$$= P[(W^{(1)},W^{(3)}) \in \Gamma | X = 1]P[X=1]$$

$$+ P[(W^{(1)},-W^{(3)}) \in \Gamma | X = -1]P[X=-1]$$

$$= \frac{1}{2}P[(W^{(1)},W^{(3)}) \in \Gamma] + \frac{1}{2}P[(W^{(1)},-W^{(3)} \in \Gamma]$$

$$= P[(W^{(1)},W^{(3)}) \in \Gamma].$$

The same argument also applies to $(M^{(2)}, M^{(3)})$. The triple $(M^{(1)}, M^{(2)}, M^{(3)})$ is not a three-dimensional Brownian motion because

$$P[M_1^{(1)} M_1^{(2)} M_1^{(3)} \geq 0] = 1.$$

3.14 Solution: Let $Q = \{q_{ik}\}_{1 \leq i,k \leq d}$, so $\widetilde{W}_t^{(i)} = \sum\limits_{k=1}^{d} q_{ik} W_t^{(k)}$ is in $\mathfrak{m}_2^c$ and

$$\langle \widetilde{W}^{(i)}, \widetilde{W}^{(j)} \rangle_t = \sum\limits_{k=1}^{d} q_{ik} q_{jk} \langle W^{(k)}, W^{(k)} \rangle_t$$

$$= \delta_{ij} t,$$

by the orthogonality of Q. Now appeal to Theorem 3.13.

3.16 Solution: We check condition (e) of Proposition 2.6.7. For $r \in \mathbb{R}^+$, $t \geq 0$, $\Gamma \in \mathcal{B}(\mathbb{R}^+)$ and any optional time S of $\{\mathcal{F}_t\}$, we have from the strong Markov property for W and equation (3.11):

$$P^{(r,0,\ldots,0)}[R_{S+t} \in \Gamma | \mathcal{F}_{S+}] = P^{W_{S+}}[R_t \in \Gamma]$$

$$= P^{(R_{S+},0,\ldots,0)}[R_t \in \Gamma], \quad P^{(r,0,\ldots,0)} - a.s.$$

$$\text{on } \{S < \infty\}.$$

3.19 Solution: Let $f_d(x) = \ln x$ if $d = 2$ and $f_d(x) = \dfrac{1}{x^{d-2}}$ if $d \geq 3$. For $0 < c < r$, let

$$T_c = \begin{cases} \inf\{t \geq 0; R_t = c\}; & \text{if } \{\cdots\} \neq \emptyset, \\ \\ \infty; & \text{otherwise.} \end{cases}$$

For $k > r$, k an integer, let

$$S_k = \begin{cases} \inf\{t \geq 0; \ R_t = k\}; & \text{if } \{\cdots\} \neq \emptyset, \\ \\ \infty; & \text{otherwise.} \end{cases}$$

Set $\tau_k = T_c \wedge S_k \wedge n$. Applying Itô's rule, we have

$$f_d(R_{\tau_k}) - f_d(r) = \begin{cases} \displaystyle\int_0^{\tau_k} \frac{1}{R_s} \, dB_s; & d = 2, \\ \\ \displaystyle -\int_0^{\tau_k} \frac{d-2}{R_s^{d-1}} \, dB_s; & d \geq 3, \end{cases}$$

so

$$f_d(r) = E f_d(R_{\tau_k})$$

$$= f_d(c) P[T_c \leq S_k \wedge n] + f_d(k) P[S_k \leq T_c \wedge n]$$

$$+ E[f_d(R_n) 1_{\{n < S_k \wedge T_c\}}].$$

Let $n \to \infty$ to obtain

(S.1) $\qquad f_d(r) = f_d(c) P[T_c \leq S_k] + f_d(k) P[S_k \leq T_c].$

If $d = 2$, we divide (S.1) by $f_d(k) = \ln k$ and let $k \to \infty$ to obtain $\lim_{k\to\infty} P[S_k \leq T_c] = 0$, which means $T_c < \infty$, P-a.s. Thus, $m \leq c$ a.s. for every $0 < c < r$, so $m = 0$ a.s. If $d \geq 3$, let $k \to \infty$ in (S.1) to obtain

$$\frac{1}{r^{d-2}} = \frac{1}{c^{d-2}} P[T_c < \infty].$$

But $\{T_c < \infty\} = \{m \leq c\}$.

3.20 <u>Solution</u>:   We denote the starting point of the Bessel process

by a superscript on the probability. Let $0 \leq r < a < b < \infty$

be given and define sequences of stopping times by $T_0 = 0$,

and for $k = 0, 1, \ldots$,

$$S_{k+1} = \begin{cases} \inf\{t \geq T_k;\ R_t = b\}; & \text{if } \{\ \} \neq \emptyset, \\ \\ \infty; & \text{otherwise,} \end{cases}$$

$$T_{k+1} = \begin{cases} \inf\{t \geq S_{k+1};\ R_t = a\}; & \text{if } \{\ \} \neq \emptyset, \\ \\ \infty; & \text{otherwise.} \end{cases}$$

It is clear from Theorem 2.?? that $P^r[\overline{\lim_{t \to \infty}} R_t = \infty] = 1$,

so on the event $\{T_k < \infty\}$, we have $S_{k+1} < \infty$ a.s. On the

event $\{S_{k+1} < \infty\}$, the strong Markov property asserts

$$P^r[T_{k+1} < \infty \mid \mathcal{F}_{S_{k+1}}] = P^b[\min_{0 \leq t < \infty} R_t \leq a]$$

$$= \left(\frac{a}{b}\right)^{d-2},$$

and since T_k is $\mathcal{F}_{S_{k+1}}$-measurable, we have

$$P^r[T_{k+1} < \infty] = P^r[T_k < \infty,\ T_{k+1} < \infty]$$

$$= E^r\{1_{\{T_k < \infty\}} P^r[T_{k+1} < \infty \mid \mathcal{F}_{S_{k+1}}]\}$$

$$= \left(\frac{a}{b}\right)^{d-2} P^r[T_k < \infty].$$

Induction on k yields $P^r[T_k < \infty] = \left(\frac{a}{b}\right)^{k(d-2)}$, $k = 0, 1, \ldots$,

and so $P^r[T_k < \infty \ \forall\ k \geq 0] = 0$. This shows that

$P[\varliminf_{t\to\infty} R_t \geq a] = 1$, and since a can be as large as we please,

we must have $P[\lim_{t\to\infty} R_t = \infty] = 1$.

3.21 Solution: For $m \geq 1$, $f(x) \triangleq |x|^{2m}$ is of class C^2. Itô's

rule implies

(S.2) $|M_t|^{2m} = 2m \int_0^t |M_s|^{2m-1} (\text{sgn } M_s) X_s dW_s + m(2m-1) \int_0^t |M_s|^{2m-2} X_s^2 ds$; $0 \leq t \leq T$.

For $N > 0$, let

$$\tau_N = \begin{cases} \inf\{0 \leq t \leq T: |M_t| = N\}; & \text{if } \{\cdots\} \neq \emptyset, \\ \\ T & ; \quad \text{otherwise} \end{cases}$$

so $E \int_0^{\tau_N} (|M_s|^{2m-1} X_s)^2 ds < \infty$. We may replace t by τ_N

in (S.2) and take expectations to obtain

$$E|M_{\tau_N}|^{2m} = m(2m-1) E \int_0^{\tau_N} |M_s|^{2m-2} X_s^2 ds$$

$$\leq m(2m-1) E \int_0^T |M_s|^{2m-2} X_s^2 ds.$$

Letting $N \to \infty$ and using Hölder's inequality and the

submartingale inequality $E|X_s|^{2m} \leq E|X_T|^{2m}$; $0 \leq s \leq T$,

we may write

$$E|M_T|^{2m} \leq m(2m-1) E \int_0^T |M_s|^{2m-2} X_s^2 ds$$

$$\leq m(2m-1) \left(E \int_0^T |M_s|^{2m} ds\right)^{\frac{m-1}{m}} \left(E \int_0^T |X_s|^{2m} ds\right)^{\frac{1}{m}}$$

$$\leq m(2m-1) T^{\frac{m-1}{m}} \left(E|M_T|^{2m}\right)^{\frac{m-1}{m}} \left(E \int_0^T |X_s|^{2m} ds\right)^{\frac{1}{m}}.$$

Raising both sides to the mth power and then dividing by $\left(E|M_t|^{2m}\right)^{m-1}$, we obtain the desired result.

3.23 Solution: For $x_i \geq 0$, $i = 1, \ldots, d$, we have

$$x_1^m + \ldots + x_d^m \leq d(x_1 + \ldots + x_d)^m \leq d^{m+1}(x_1^m + \ldots + x_d^m).$$

Therefore

(S.3) $$\|M_T\|^{2m} = \left[\sum_{i=1}^{d} \left(M_T^{(i)}\right)^2\right]^m \leq d^m \sum_{i=1}^{d} \left|M_T^{(i)}\right|^{2m}$$

and

(S.4) $$\sum_{i=1}^{d} \langle M^{(i)}\rangle_T^m \leq d \cdot \left(\sum_{i=1}^{d} \langle M^{(i)}\rangle_T\right)^m = d \cdot A_T^m.$$

Taking expectations in (S.3), (S.4) and applying (3.20) to each $M^{(i)}$, we obtain

$$E \|M_T\|^{2m} \leq d^{m+1} C_m E A_T^m.$$

A similar proof can be given for the lower bound on $E \|M_T\|^{2m}$.

3.24 Solution: We have

$$\sum_{i=1}^{d} \langle M^{(i)}\rangle_T = \sum_{i=1}^{d} \sum_{j=1}^{r} \int_0^T \left(X_s^{(i,j)}\right)^2 dt = \int_0^T \|X_t\|^2 dt.$$

Now apply Problem 3.23.

<u>3.25 Solution</u>: If $M_t = \int_0^t X_s \, dW_s$ is a martingale (rather than merely a local martingale), the desired inequality follows from Doob's maximal inequality (Theorem 1.3.6(iv)) applied to $|M_t|$ and relation (3.20). When $M \in \mathfrak{m}^{c,loc}$, a localization argument now gives the result.

4.4 Solution:

(i) We have $\{T_s \le t\} = \{A_t + t \ge s\} \in \mathcal{F}_t$, so T_s is a stopping time of $\{\mathcal{F}_t\}$.

(ii) Since $\mathcal{F}_t$ contains every P-null set in $\mathcal{F}$, it is clear from the definition of $\mathcal{F}_{T_s}$ that $\mathcal{G}_s$ also contains every P-null set in $\mathcal{F}$. We now prove right-continuity of $\{\mathcal{G}_s\}$. Let $\{s_n\}_{n=1}^{\infty}$ be a sequence with $s_n \downarrow s$, so $\{T_{s_n}\}_{n=1}^{\infty}$ is a sequence of optional times with $T_{s_n} \downarrow T_s$. According to Problem 1.2.22,

$$\mathcal{G}_{s+} \overset{\Delta}{=} \bigcap_{n=1}^{\infty} \mathcal{G}_{s_n} = \bigcap_{n=1}^{\infty} \mathcal{F}_{T_{s_n}} = \mathcal{F}_{T_s+},$$

where $\mathcal{F}_{T_s+}$ agrees with $\mathcal{F}_{T_s} = \mathcal{G}_s$ under the assumption of right-continuity of $\{\mathcal{F}_t\}$ (Definition 1.2.19).

(iii) Because T_s is a continuous function of s, $N_s^{(i)}$ is continuous. For fixed s, T_s is a bounded stopping time, so the optional sampling theorem can be used to prove that $N_s^{(i)}$ is a local martingale. Furthermore, the same theorem shows that for $0 \le s_1 \le s_2$,

$$|\langle N^{(i)}, N^{(j)} \rangle_{s_2} - \langle N^{(i)}, N^{(j)} \rangle_{s_1}|$$

$$= |\langle M^{(i)}, M^{(j)} \rangle_{T_{s_2}} - \langle M^{(i)}, M^{(j)} \rangle_{T_{s_1}}|$$

$$\le A_{T_{s_2}} - A_{T_{s_1}}$$

$$= s_2 - s_1 - (T_{s_2} - T_{s_1})$$

$$\le s_2 - s_1,$$

so $\langle N^{(i)}, N^{(j)} \rangle_s$ is an absolutely continuous function of s.

4.5 Solution:

(i) The nondecreasing character of T is obvious. Thus, for right-continuity, we need only show that $\lim_{\theta \downarrow s} T(\theta) \leq T(s)$.

Set $t = T(s)$. The definition of $T(s)$ implies that for each $\varepsilon > 0$, we have $A(t+\varepsilon) > s$, and for $s < \theta < A(t+\varepsilon)$, we have $T(\theta) \leq t + \varepsilon$. Therefore, $\lim_{\theta \downarrow s} T(\theta) \geq t$.

(ii) Set $t = T(s)$ and choose $\varepsilon > 0$. We have $A(t+\varepsilon) > s$, and letting $\varepsilon \downarrow 0$, we see from the continuity of A that $A(T(s)) \geq s$. If $t = T(s) = 0$, we are done. If $t > 0$, then for $0 < \varepsilon < t$, the definition of $T(s)$ implies $A(t-\varepsilon) \leq s$. Letting $\varepsilon \downarrow 0$, we obtain $A(T(s)) \leq s$.

(iii) This is a direct consequence of the definition of T and the continuity of A.

(iv) For $a \leq t_1 < t_2 \leq b$, let $G(t) = 1_{[t_1, t_2)}(t)$. According to (iii), $t_1 \leq T(s) < t_2$ if and only if $A(t_1) \leq s < A(t_2)$, so

$$\int_a^b G(t)\,dA(t) = A(t_2) - A(t_1) = \int_{A(a)}^{A(b)} G(T(s))\,ds.$$

Linearity of the integral and the monotone convergence theorem imply that the collection of sets $C \in \mathcal{B}[a,b]$ for which

(S.4)
$$\int_a^b 1_C(t)\,dA(t) = \int_{A(a)}^{A(b)} 1_C(T(s))\,ds$$

forms a Dynkin System. Since it contains all intervals of the form $[t_1, t_2) \subseteq [a,b]$, and these are closed under finite intersection and generate $\mathcal{B}[a,b]$, we have (S.4) for every $C \in \mathcal{B}[a,b]$ (Dynkin System Theorem 2.5.1'). The proof of (iv) is now straightforward.

4.7 Solution: Let φ be a deterministic, strictly increasing function mapping $[0, \infty)$ onto $[0,1)$, and define $M \in \mathcal{M}^{c, loc}$ by

$$M_t = \int_0^{\varphi(t)} X_s dW_s; \quad 0 \le t < \infty,$$

so

$$\langle M \rangle_t = \int_0^{\varphi(t)} X_s^2 ds; \quad 0 \le t < \infty,$$

and $\langle M \rangle_t \to \infty$ a.s. as $t \to \infty$. According to Theorem 4.6, there is a Brownian motion B such that

$$\int_0^t X_s dW_s = B_{\langle M \rangle(\varphi^{-1}(t))}.$$

As $t \uparrow 1$, $\langle M \rangle(\varphi^{-1}(t)) \to \infty$, so, by the law of the iterated logarithm for Brownian motion,

$$P[\overline{\lim_{t \uparrow 1}} B_{\langle M \rangle(\varphi^{-1}(t))} = -\underline{\lim_{t \uparrow 1}} B_{\langle M \rangle(\varphi^{-1}(t))} = +\infty] = 1.$$

4.11 Solution: For simplicity of notation, we take d to be 1.
For each positive integer n, define

$$T_n = \begin{cases} \inf\{0 < t < \infty; \ |M_t| > n\}; & \text{if } \{\ldots\} \neq \emptyset, \\ \infty; & \text{if } \{\ldots\} = \emptyset, \end{cases}$$

and set $T_0 = 0$. Because M is right-continuous with left
limits, we have $T_n \uparrow \infty$ a.s. According to Problem 1.2.5
and Proposition 1.2.3, each T_n is a stopping time for $\{\mathcal{F}_t\}$.
The martingale (Problem 1.3.22)

$$M^{(n)} \triangleq \{M^{(n)} = M_{t \wedge T_n}, \ \mathcal{F}_t; \ 0 \leq t < \infty\}$$

is bounded, as is $M^{(n)} - M^{(n-1)}$, and so Theorem 4.10 guarantees
the existence of a progressively measurable $Y^{(n)} = \{Y_t^{(n)}, \mathcal{F}_t;$
$0 \leq t < \infty\}$ satisfying $E \int_0^T (Y_t^{(n)})^2 dt < \infty; \ 0 \leq t < \infty$, and

$$M_t^{(n)} - M_t^{(n-1)} = \int_0^t Y_s^{(n)} dW_s; \ 0 \leq t < \infty, \ n \geq 1.$$

Because

$$\langle M \rangle_{t \wedge T_n} - \langle M \rangle_{t \wedge T_{n-1}} = \langle M^{(n)} - M^{(n-1)} \rangle_t = \int_0^t (Y_s^{(n)})^2 ds,$$

we must have $Y_s^{(n)}(\omega) = Y_s^{(n)}(\omega) 1_{[T_{n-1}(\omega), T_n(\omega))}(s)$ for
Lebesgue a.e. s, P a.e. ω. Setting $Y_t = \sum_{n=1}^{\infty} Y_t^{(n)}$, we
have the desired representation:

$$M_t = \int_0^t Y_s dW_s; \ 0 \leq t < \infty.$$

4.12 <u>Solution</u>: Let M_t in Theorem 4.10 be a right-continuous modification of the martingale $E(\xi|\mathfrak{F}_t) - E(\xi)$.

4.15 <u>Solution</u>: Suppose

(S.5)
$$Y_s = \xi_0 1_{\{0\}}(s) + \sum_{i=0}^{m} \xi_i 1_{(s_i, s_{i+1}]}(s),$$

where each ξ_i is $\mathcal{E}_{s_i}$-measurable. Then (4.38) reduces to

$$\xi_i (B_{s_{i+1}} - B_{s_i}) = \int_0^\infty \xi_i 1_{(s_i, s_{i+1}]}(\langle M\rangle_t) dM_t; \quad 0 \le i \le m,$$

which, because of the definition of B and Problem 4.5(iii), is equivalent to

$$\xi_i (M_{T(s_{i+1})} - M_{T(s_i)}) = \int_0^\infty \xi_i 1_{(T(s_i), T(s_{i+1})]}(t) dM_t; \quad 0 \le i \le m.$$

We show that whenever $T_1 \le T_2$ are stopping times for $\{\mathfrak{F}_t\}$ such that $E\langle M\rangle_{T_2} < \infty$ and ξ is an $\mathfrak{F}_{T_1}$-measurable, bounded, random variable, then

(S.6)
$$\xi (M_{T_2} - M_{T_1}) = \int_0^\infty \xi 1_{(T_1, T_2]}(t) dM_t, \quad \text{a.s.} P.$$

Replacing M_t by $M_{T_2 \wedge t}$, we may assume that $E\langle M\rangle_\infty < \infty$ and $T_2 = \infty$ (Corollary 2.19 and Definition 2.21). Now let

$$T^{(n)} = \sum_{k=0}^{\infty} \frac{k+1}{2^n} \mathbf{1}_{[k2^{-n}, (k+1)2^{-n})}(T_1),$$

so that $\xi_t^{(n)} \triangleq \xi \mathbf{1}_{\{t \geq T^{(n)}\}}$ is a simple process. For this process, we have

$$\xi(M_\infty - M_{T^{(n)}}) = \int_0^\infty \xi_t^{(n)} dM_t,$$

and letting $n \to \infty$, we obtain (S.6).

If Y is not simple, there is a sequence $\{Y^{(n)}\}_{n=1}^\infty$ of simple processes of the form (S.5) such that

$$\lim_{n \to \infty} E \int_0^\infty (Y_s - Y_s^{(n)})^2 ds = 0. \quad \text{With } X_t^{(n)} \triangleq Y_{\langle M \rangle_t}^{(n)}, \text{ we have}$$

from Problem 2.16':

$$\int_0^\infty Y_s dB_s = \lim_{n \to \infty} \int_0^\infty Y_s^{(n)} dB_s = \lim_{n \to \infty} \int_0^\infty X_t^{(n)} dM_t = \int_0^\infty X_t dM_t.$$

4.18 <u>Solution</u>: The proof already given for Theorem 4.16 applies to the case $T = \infty$ once we show that $\int_0^\infty Y_s^2 ds < \infty$, a.s.P, implies the existence of $\lim_{t \to \infty} \int_0^t Y_s dW_s$. Let

$T_n = \inf\{t \geq 0; \int_0^t Y_s^2 ds = n\}$, so $Y^{(n)} \triangleq \{Y_{t \wedge T_n}, \mathfrak{F}_t;$

$0 \leq t < \infty\}$ is in $\mathcal{L}_\infty^*$ and $\lim_{t \to \infty} \int_0^t Y_s^{(n)} dW_s$ exists.

On the set $\{T_n = \infty\}$, we have $\int_0^t Y_s^{(n)} dW_s = \int_0^t Y_s dW_s$ for

all $0 \leq t < \infty$. But $\lim_{n \to \infty} P[T_n = \infty] = 1$, so we have the desired result.

5.6 <u>Solution</u>: We define $M_t = \int_0^t Y_s dW_s^{(i)}$, so $\tilde{M}$ given by (5.8)

is

$$\tilde{M}_t = \int_0^t Y_s dW_s^{(i)} - \int_0^t Y_s X_s^{(i)} ds; \quad 0 \le t \le T.$$

We shall identify $\tilde{M}_t$ as the Itô integral (under $\tilde{P}_T$)
$\int_0^t Y_s d\tilde{W}_s^{(i)}$, by appealing to Proposition 2.22. Now, according
to Proposition 5.5, every element $\tilde{N}$ of $\tilde{\mathbb{m}}_T^{c,loc}$ admits a
representation of the form

$$\tilde{N}_t = N_t - \sum_{j=1}^d \int_0^t X_s^{(j)} d\langle N, W^{(j)}\rangle_s; \quad 0 \le t \le T,$$

for some $N \in \mathbb{m}_T^{c,loc}$. Proposition 5.4 implies

$$\langle \tilde{M}, \tilde{N}\rangle_t = \langle M, N\rangle_t = \int_0^t Y_s d\langle N, W^{(i)}\rangle_s$$

$$= \int_0^t Y_s d\langle \tilde{N}, \tilde{W}^{(i)}\rangle_s; \quad 0 \le t \le T,$$

a.s. P and $\tilde{P}_T$.

5.7 <u>Solution</u>: As in (5.11), we have with $Z_t \triangleq \exp\{\mu W_t - \frac{1}{2}\mu^2 t\}$:

$$P^{(\mu)}[T \le t] = E[1_{\{T \le t\}} Z_T].$$

Let $t \to \infty$ and use the monotone convergence theorem to conclude

$$P^{(\mu)}[T < \infty] = E Z_T.$$

Under $P^{(\mu)}$, the process $\{W_t - \mu t, \mathcal{F}_t; \ 0 \le t < \infty\}$ is a standard

Brownian motion, so $P^{(\mu)}[S_b < \infty] = 1$. We also have

$P[S_b < \infty] = 1$, since $\mu b < 0$.

5.8 <u>Solution</u>: We have $P[T_{b_1} < \infty] = 1$ and

$$
T_{b_1+b_2} - T_{b_1} =
\begin{cases}
\inf\{t \ge 0; \ W_{T_{b_1}+t} - W_{T_{b_1}} = b_2\}; & \text{if } \{\ldots\} \ne \emptyset, \\[2ex]
\infty; & \text{if } \{\ldots\} = \emptyset.
\end{cases}
$$

Theorem 2.6.15 states that, under P, the process

$\{W_{T_{b_1}+t} - W_{T_{b_1}}, \ 0 \le t < \infty\}$ is a standard Brownian motion

independent of $\mathcal{F}_{T_{b_1}}$. It follows that, under P, $T_{b_1+b_2} - T_{b_1}$

is independent of T_{b_1} and $P[T_{b_1+b_2} - T_{b_1} \in dt] = h(t; b_2, 0)dt$.

Using the same justifications as in (5.11), we have

$$
P^{(\mu)}[T_{b_1} \le s, \ T_{b_1+b_2} - T_{b_1} \le t]
$$

$$
= E[1_{\{T_{b_1} \le s, \ T_{b_1+b_2} - T_{b_1} \le t\}} Z_{s+t}]
$$

$$
= E[1_{\{T_{b_1} \le s, \ T_{b_1+b_2} - T_{b_1} \le t\}} Z_{T_{b_1+b_2}}]
$$

$$
= \int_0^t \int_0^s \exp\{\mu(b_1+b_2) - \tfrac{1}{2}\mu^2(\tau+\sigma)\} h(\sigma; b_1, 0)
$$

$$
\cdot h(\tau; b_2, 0) d\sigma \, d\tau
$$

$$
= \int_0^t \int_0^s h(\sigma; b_1, \mu) h(\tau; b_2, \mu) d\sigma \, d\tau.
$$

Therefore

$$h(t; b_1+b_2, \mu)\,dt = P^{(\mu)}[T_{b_1} \in dt]$$

$$= \int_{s=0}^{t} P^{(\mu)}[T_{b_1} \in ds, \; T_{b_1+b_2} - T_{b_1} \in dt - s]$$

$$= \int_{s=0}^{t} h(s; b_1, \mu)\,h(t-s; b_2, \mu)\,ds \; dt$$

$$= [h(\cdot; b_1, \mu) * h(\cdot; b_1, \mu](t)\,dt.$$

$\underline{5.11\ \text{Solution}}$: The process $\widetilde{W}_t \triangleq W_t - \mu t$ is a Brownian motion under $P^{(\mu)}$, so the law of large numbers implies

$$P^{(\mu)}[\lim_{t\to\infty} \frac{W_t - \mu t}{t} = 0] = 1.$$

Therefore, $\lim_{t\to\infty} \frac{1}{t}(\mu W_t - \frac{1}{2}\mu^2 t) = \frac{1}{2}\mu^2$, $P^{(\mu)}$ - a.s., so

with $Z_t \triangleq \exp(\mu W_t - \frac{1}{2}\mu^2 t)$, we have $\lim_{t\to\infty} Z_t = \infty$,

$P^{(\mu)}$ - a.s., and so $P^{(\mu)}[R_b < \infty] = 1$. Now

$$\frac{1}{Z_t} = \exp(-\mu V_t - \frac{1}{2}\mu^2 t),$$

so $\{\frac{1}{Z_t}, \mathcal{F}_t; 0 \le t < \infty\}$ is a martingale under $P^{(\mu)}$.

Using the same justifications as in (5.11), we have

$$P[R_b \le t] = E^{(\mu)}[1_{\{R_b \le t\}} \frac{1}{Z_t}] = E^{(\mu)}[1_{\{R_b \le t\}} \frac{1}{Z_{T_b}}]$$

$$= \frac{1}{b} P^{(\mu)}[R_b \le t].$$

Letting $t \to \infty$, we obtain the desired result $P[R_b < \infty] = \frac{1}{b}$.

For the second claim, note that for every finite $t > 0$, we have

$$0 = E^{(\mu)}V_{t \wedge R_b} = E^{(\mu)}W_{t \wedge R_b} - \mu E^{(\mu)}(t \wedge R_b).$$

But

$$W_{t \wedge R_b} - \frac{1}{2}\mu (t \wedge R_b) = \frac{1}{\mu} \log Z_{t \wedge R_b} \le \frac{\log b}{\mu},$$

so

$$E^{(\mu)}(t \wedge R_b) = 2E[W_{t \wedge R_b} - \frac{1}{2}\mu(t \wedge R_b)] \le \frac{2 \log b}{\mu} < \infty.$$

Letting $t \to \infty$, we obtain

$$E^{(\mu)}R_b \le \frac{2 \log b}{\mu} < \infty.$$

From Problem 2.11, we have

$$0 = E^{(\mu)}V_{R_b} = E^{(\mu)}W_{R_b} - \mu E^{(\mu)}R_b.$$

We also have

$$E^{(\mu)}W_{R_b} - \frac{1}{2}\mu E^{(\mu)}R_b = \frac{1}{\mu}E^{(\mu)}\log Z_{R_b} = \frac{\log b}{\mu}.$$

Solving these two equations for $E^{(\mu)}R_b$ yields the desired

result $E^{(\mu)}R_b = \frac{2\log b}{\mu^2}$.

5.19 Solution: Let $Z_T = \frac{d\widetilde{P}_T}{dP}$, i.e.,

$$\widetilde{P}_T(A) = E[1_A Z_T]; \quad A \in \mathcal{F}_T, \quad 0 \le T < \infty.$$

The consistency condition (5.5) implies that $Z = \{Z_t, \mathcal{F}_t;$ $0 \le t \le \infty\}$ is a martingale, and because of Theorem 1.3.11, we may assume that for P-a.e. ω, $t \mapsto Z_t(\omega)$ is a right-continuous function with left limits. By the construction of $\mathcal{F}_0$, every set in this σ-field has P-probability zero or one, so $Z_0 = 1$, a.s. P. Problem 4.11 implies that Z has the representation

$$Z_t = 1 + \sum_{i=1}^{d} \int_0^t Y_s^{(i)} dW_s^{(i)}; \quad 0 \le t < \infty,$$

where each $Y^{(i)}$ is progressively measurable and

$$P[\int_0^t (Y_s^{(i)})^2 ds < \infty] = 1, \quad \text{for every } 0 \le t < \infty, \quad 1 \le i \le d.$$

Let

$$S = \begin{cases} \inf\{0 \le t < \infty; \quad Z_t = 0\}; & \text{if } \{\ldots\} \neq \emptyset \\ \infty; & \text{if } \{\ldots\} = \emptyset. \end{cases}$$

For each $0 \le T < \infty$, the Optional Sampling Theorem implies (cf. (5.11))

$$\widetilde{P}_T[S \le T] = E[1_{\{S \le T\}} Z_S] = 0,$$

and by the absolute continuity of P with respect to $\widetilde{P}_T$ on $\mathfrak{F}_T$ we conclude:

$$P[S \le T] = 0; \quad 0 \le T < \infty.$$

It follows that $P[S < \infty] = 0$, so $\log(Z_t)$ is defined for $0 \le t < \infty$, a.s. P, and $X = \{X_t = (X_t^{(1)}, \ldots, X_t^{(d)}), \mathfrak{F}_t; 0 \le t < \infty\}$ defined by

$$X_t^{(i)} = \frac{1}{Z_t} Y_t^{(i)}; \quad 1 \le i \le d, \quad 0 \le t < \infty,$$

satisfies (5.1). Ito's rule gives

$$\log(Z_t) = \sum_{i=1}^{d} \int_0^t X_s^{(i)} dW_s^{(i)} - \frac{1}{2} \int_0^t \|X_s\|^2 ds,$$

so $Z(X) = Z$ is a martingale and (5.4) holds.

6.6 Solution: If $|B_{t+h}(\omega) - B_t(\omega)| \leq Ch;\ 0 \leq h \leq \delta$, then by (6.6) and the additive functional property of ℓ we have

$$1 = \frac{1}{\delta} \int_0^\delta 1\{|B_{t+h}(\omega) - B_t(\omega)| \leq C\delta\} dh$$

$$= \frac{1}{\delta} \int_{B_t(\omega)-C\delta}^{B_t(\omega)+C\delta} [\ell_{t+h}(x,\omega) - \ell_t(x,\omega)] dx$$

$$= 2C \max_{\substack{|x-B_t(\omega)| \leq C\delta \\ 0 \leq h \leq \delta}} [\ell_{t+h}(x,\omega) - \ell_t(x,\omega)].$$

The last term converges to zero as $\delta \downarrow 0$, because of the joint continuity of $\ell_t(x,\omega)$ in (t,x). This contradiction establishes the nondifferentiability of $B_\cdot(\omega)$.

6.7 Solution: (i) It is clear that (6.7) implies (6.6). If (6.6) holds, then (6.7) holds for every linear combination of Borel measurable indicator functions, and it is possible to find a sequence of these which converges everywhere from below to a given Borel measurable $f : \mathbb{R} \to [0,\infty)$. Equation (6.7) follows then from the monotone convergence theorem.

(ii) For any $a < b$, the indicator $1_{(a,b]}$ can be written as the limit (everywhere) of a sequence of functions in $\mathcal{H}$. By the bounded convergence theorem, (6.6) holds for every B of the form $(a,b]$. The collection of all Borel sets B for which (6.6) holds forms a Dynkin system and so, by the Dynkin System Theorem 2.5.1', (6.6) holds for every $B \in \mathcal{B}(\mathbb{R})$.

We saw in part (i) that this implies (6.7) for every Borel function $f : \mathbb{R} \to [0, \infty)$.

6.12 **Solution:** Let h have support in $[0,b]$, consider the sequence of partitions

$$D_n = \{b_k^{(n)} = \frac{k}{2^n} b; \quad k = 0,1,\ldots,2^n\}; \quad n \geq 1$$

of this interval, and set $D = \bigcup_{n=1}^{\infty} D_n$. The Lebesgue integral on the left-hand side of (6.24) is approximated, as $n \to \infty$, by the sum

$$\sum_{k=0}^{2^n-1} \frac{b}{2^n} h(b_k^{(n)}) (\int_0^t 1_{[b_k^{(n)},\infty)} (W_s) dW_s) = \int_0^t F_n(W_s) dW_s,$$

where the function

$$F_n(x) \triangleq \sum_{k=1}^{2^n-1} \frac{b}{2^n} h(b_k^{(n)}) 1_{[b_k^{(n)},\infty)} (x)$$

converges uniformly, as $n \to \infty$, to the function

$$F(x) \triangleq \int_{-\infty}^{\infty} h(a) 1_{[a,\infty)} (x) da.$$

Therefore, the sequence of stochastic integrals $\{\int_0^t F_n(W_s) dW_s\}_{n=1}^{\infty}$ converges in L^2 to the stochastic integral $\int_0^t F(W_s) dW_s$, which is the right-hand side of (6.24).

6.13 Solution: (i) Under any P^z, B(a) is a continuous, square-integrable martingale with quadratic variation process

$$\langle B(a) \rangle_t = \int_0^t [sgn(W_s - a)]^2 ds = t; \quad 0 \leq t < \infty, \text{ a.s. } P^z.$$

According to Theorem 3.3.13, B(a) is a Brownian motion.

(ii) For ω in the set Ω^* of Definition 6.3, we have (6.2) (Remark 6.5), and from this we see immediately that $\ell_0(a, \omega) = 0$ and $\ell_t(a, \omega)$ is nondecreasing in t. For each $z \in \mathbb{R}$, there is a set $\tilde{\Omega} \in \mathcal{F}$ with $P^z(\tilde{\Omega}) = 1$ such that $Z_\omega(a)$ is closed for all $\omega \in \tilde{\Omega}$. For $\omega \in \tilde{\Omega} \cap \Omega^*$, the complement of $Z_\omega(a)$ is the countable union of open intervals $\bigcup_{\alpha \in \mathbb{N}} I_\alpha$. To prove (6.26), it suffices to show that

$\int_{I_\alpha} d\ell_t(\omega) = 0$ for each $\alpha \in \mathbb{N}$. Fix an index α and let $I_\alpha = (u, v)$. Since $W.(\omega) - a$ has no zero in (a,b), we know that $|W.(\omega) - a|$ is bounded away from the origin on $[u + \frac{1}{n}, v - \frac{1}{n}]$, where $n > \frac{2}{v-u}$. Thus, for all sufficiently small $\varepsilon > 0$

$$meas\{0 \leq s \leq u + \frac{1}{n}; |W_s - a| \leq \varepsilon\}$$

$$= meas\{0 \leq s \leq v - \frac{1}{n}; |W_s - a| \leq \varepsilon\},$$

and thus $\ell_{u+\frac{1}{n}}(a, \omega) = \ell_{v-\frac{1}{n}}(a, \omega)$. It follows that

$\int_{[u + \frac{1}{n}, v - \frac{1}{n}]} d\ell_t(a, \omega) = 0$, and letting $n \to \infty$ we obtain the desired result.

(iii) Set $z = a = 0$ in (6.25) to obtain

$$|W_t| = -B_t(0) + \ell_t(0); \quad 0 \leq t < \infty, \quad P^0\text{-a.s.}$$

The left-hand side of this relation is nonnegative, while $B_t(0)$ changes sign infinitely often in any interval $[0, \varepsilon]$, $\varepsilon > 0$ (Problem 2.7.17). It follows that $\ell_t(0)$ cannot remain zero in any such interval.

(iv) It suffices to show that for any two rational numbers $0 \leq q < r < \infty$, if $W_t(\omega) = a$ for some $t \in (q, r)$ then $\ell_q(a, \omega) < \ell_r(a, \omega)$, P^z-a.e. ω. Let $T(\omega) \triangleq \inf\{t \geq q; \ W_t(\omega) = a\}$. Applying (iii) to the Brownian motion $\{W_{s+T} - a; \ 0 \leq s < \infty\}$ we conclude that

$$\ell_{T(\omega)}(a, \omega) < \ell_{T(\omega)+s}(a, \omega) \quad \text{for all} \quad s > 0, \quad P^z\text{-a.e. } \omega,$$

by the additive functional property of local time (Definition 6.1 and Remark 6.5). For every $\omega \in \{T < r\}$ we may take $s = r - T(\omega)$ above, and this yields $\ell_q(a, \omega) < \ell_r(a, \omega)$.

6.18 Solution: It is certainly sufficient to take $z = 0$. From (6.34) and the fact that B is Brownian motion under P^0, we have

$$\lim_{t \to \infty} \ell_t(0, \omega) = \lim_{t \to \infty} (\max_{0 \leq s \leq t} B_s(\omega)) \equiv \infty$$

for P^0-a.e. $\omega \in \Omega$. By the additive functional property of local time, we have for every $a \neq 0$:

$$\ell_{t+T_a(\omega)}(a,\omega) = \ell_t(a,\theta_{T_a}\omega); \quad 0 \leq t < \infty, \quad \text{for } P^0\text{-a.e. } \omega \in \Omega,$$

and by the strong Markov property of Brownian motion:

$$P^0[\omega \in \Omega; \lim_{t\to\infty} \ell_t(a,\omega) = \infty] = P^0[\omega \in \Omega; \lim_{t\to\infty} \ell_t(a,\theta_{T_a}\omega) = \infty]$$

$$= P^a[\omega \in \Omega; \lim_{t\to\infty} \ell_t(a,\omega) = \infty] = P^0[\omega \in \Omega; \lim_{t\to\infty} \ell_t(0,\infty) = \infty] = 1.$$

<u>6.19 Solution</u>: From (6.38) we obtain $\overline{\lim\limits_{y\downarrow x}} f(y) \leq f(x)$,

$\overline{\lim\limits_{y\uparrow z}} f(y) \leq f(z)$ and $f(y) \leq \underline{\lim\limits_{x\uparrow y}} f(x)$, $f(y) \leq \underline{\lim\limits_{z\downarrow y}} f(z)$.

This establishes the continuity of f on $\mathbb{R}$.

For $\xi \in \mathbb{R}$ fixed and $0 < h_1 < h_2$, we have from (6.38), with $x = \xi$, $y = \xi + h_1$, $z = \xi + h_2$:

(S.7) $\qquad \Delta f(\xi; h_1) \leq \Delta f(\xi; h_2).$

On the other hand, applying (6.38) with $x = \xi - h_2$, $y = \xi - h_1$ and $z = \xi$ yields

(S.8) $\qquad \Delta f(\xi; -h_2) \leq \Delta f(\xi; -h_1).$

Finally, with $x = \xi - \varepsilon$, $y = \xi$, $z = \xi + \delta$, we have

(S.9) $\qquad \Delta f(\xi; -\varepsilon) \leq \Delta f(\xi; \delta); \quad \varepsilon, \delta > 0.$

Relations (S.7)-(S.9) establish the requisite monotonicity in h of the difference quotient (6.39), and hence the existence and finiteness of the limits in (6.40).

In particular, (S.9) gives $D^-f(x) \leq D^+f(x)$ upon letting
$\varepsilon \downarrow 0$, $\delta \downarrow 0$, which establishes the second inequality in (6.41).
On the other hand, we obtain easily from (S.7) and (S.8) the
bounds

(S.10) $(y-x)D^+f(x) \leq f(y) - f(x) \leq (y-x)D^-f(y)$; $x < y$,

which establish (6.41).

For the right-continuity of the function $D^+f(\cdot)$, we
begin by observing the inequality

$$D^+f(x) \leq \lim_{y \downarrow x} D^+f(y); \quad x \in \mathbb{R},$$

which is a consequence of (6.41). In the opposite direction,
we employ the continuity of f, as well as (S.10), to obtain
for $x < z$:

$$\frac{f(z) - f(x)}{z - x} = \lim_{y \downarrow x} \frac{f(z) - f(y)}{z - y} \geq \lim_{y \downarrow x} D^+f(y).$$

Upon letting $z \downarrow x$, we obtain $D^+f(x) \geq \lim_{y \downarrow x} D^+f(y)$. Left-
continuity of $D^-f(\cdot)$ is proved similarly.

From (S.10) we observe that, for any function $\varphi : \mathbb{R} \to \mathbb{R}$
satisfying

(S.11) $D^-f(x) \leq \varphi(x) \leq D^+f(x)$; $x \in \mathbb{R}$,

we have for fixed $y \in \mathbb{R}$,

(S.12) $\qquad f(x) \geq G_y(x) \overset{\Delta}{=} f(y) + (x-y)\varphi(y); \qquad x \in \mathbb{R}.$

The function $G_y(\cdot)$ is called a <u>line of support</u> for the convex function $f(\cdot)$. It is immediate from (S.12) that $f(x) = \sup\limits_{y \in \mathbb{R}} G_y(x)$, but the point is that $f(\cdot)$ can be expressed as the supremum of <u>countably many</u> lines of support. Indeed, let E be a countable, dense subset of $\mathbb{R}$. For any $x \in \mathbb{R}$, take a sequence $\{y_n\}_{n=1}^{\infty}$ of numbers in E, converging to x. Because this sequence is bounded, so are the sequences $\{D^{\pm}f(y_n)\}_{n=1}^{\infty}$ (by monotonicity and finiteness of the functions $D^{\pm}f(\cdot)$) and $\{\varphi(y_n)\}_{n=1}^{\infty}$ (by (S.11)). Therefore, $\lim\limits_{n \to \infty} G_{y_n}(x) = f(x)$, which implies that $f(x) = \sup\limits_{y \in E} G_y(x)$.

6.20 <u>Solution</u>: (iii) For any $x < y < z$, we have

(S.13) $\qquad \varphi(x) \leq \dfrac{\Phi(y) - \Phi(x)}{y - x} = \dfrac{1}{y - x} \int_x^y \varphi(u)\, du \leq \varphi(y)$

$\qquad\qquad\qquad \leq \dfrac{1}{z - y} \int_y^z \varphi(u)\, du = \dfrac{\Phi(z) - \Phi(y)}{z - y} \leq \varphi(z).$

This gives

$\qquad\qquad \Phi(y) \leq \dfrac{z - y}{z - x} \Phi(x) + \dfrac{y - x}{z - x} \Phi(z),$

which verifies convexity in the form (6.38). Now let $x \uparrow y$, $z \downarrow y$ in (S.13), to obtain

(S.14) $\varphi_-(y) \leq D^-\hat{\Phi}(y) \leq \varphi(y) \leq D^+\hat{\Phi}(y) \leq \varphi_+(y);$ $y \in \mathbb{R}.$

At every continuity point x of φ, we have $\varphi_\pm(x) = \varphi(x) = D^\pm\hat{\Phi}(x)$. The left- (respectively, right-) continuity of φ_- and $D^-\varphi$ (respectively, φ_+ and $D^+\hat{\Phi}$) implies $\varphi_-(y) = D^-\hat{\Phi}(y)$ (respectively, $\varphi_+(y) = D^+\hat{\Phi}(y)$) for all $y \in \mathbb{R}$.

(iv) Letting $x \downarrow y$ (respectively, $x \uparrow y$) in (6.44), we obtain

$$D^-f(y) \leq \varphi_-(y) \leq \varphi(y) \leq \varphi_+(y) \leq D^+f(y);$$ $y \in \mathbb{R}.$

But now from (6.41) one gets

$$\varphi_+(x) \leq D^+f(x) \leq D^-f(y) \leq \varphi_-(y) \leq \varphi(y);$$ $x < y,$

and letting $y \downarrow x$ we conclude: $\varphi_+(x) = D^+f(x);$ $x \in \mathbb{R}.$ Similarly, we conclude $\varphi_-(x) = D^-f(x);$ $x \in \mathbb{R}.$ Now consider the function $G \triangleq f - \hat{\Phi}$, and simply notice the consequences

$$D^\pm G(x) = D^\pm f(x) - D^\pm\hat{\Phi}(x) = 0;$$ $x \in \mathbb{R},$

of the above discussion; in other words, G is differentiable on $\mathbb{R}$ with derivative which is identically zero. It follows that G is identically constant.

6.21 <u>Solution</u>: Just take $\varphi = D^+f$ or $\varphi = D^-f$ in the preceding problem.

6.24 <u>Solution:</u> Let $d_k = f'(a_k+) - f'(a_k-)$, and for $x \in \mathbb{R}$, set $g_1(x) = f''(x+) \vee 0$, $g_2(x) = (-f''(x+)) \vee 0$. Choose $a_0 < a_1$. We have then

$$f'(y+) = f'(a_0) + \sum_{k=1}^{n} d_k 1_{[a_k, \infty)}(x) + \int_{a_0}^{y} (g_1(z) - g_2(z))dz;$$
$$y \in \mathbb{R},$$

and upon further integration, with $q_{\pm} = (\pm f'(a_0)) \vee 0$:

$$f(x) = [f(a_0) + q_+(x-a_0) + \sum_{k=1}^{n} d_k^+ (x-a_k)^+ + \int_{a_0}^{x} \int_{a_0}^{y} g_1(z)dz]$$

$$- [q_-(x-a_0) + \sum_{k=1}^{n} d_k^- (x-a_k)^+ + \int_{a_0}^{x} \int_{a_0}^{y} g_2(z)dz]; \quad x \in \mathbb{R}.$$

This provides us with the desired decomposition of f into the difference of convex functions. Equation (6.49) takes the form (6.53) in this special case of Corollary 6.23.

6.29 <u>Solution:</u> <u>(iv) ⇒ (vi):</u> Let $t \in (0, \infty)$ be such that

$P^0[\int_0^t f(W_s)ds < \infty] = 1$. For $x = 0$, just take $S = t$. For $x \neq 0$, consider the first passage time T_x and notice that $P^0[0 < T_x < \infty] = 1$, $P^0[2T_x \leq t] > 0$, and that $\{B_s \overset{\Delta}{=} W_{s+T_x} - x, \ 0 \leq s < \infty\}$ is a Brownian motion under P^0. Now, for every $\omega \in \{2T_x \leq t\}$:

$$\int_0^{T_x(\omega)} f(x + B_s(\omega))ds = \int_{T_x(\omega)}^{2T_x(\omega)} f(W_u(\omega))du \leq \int_0^t f(W_u(\omega))du < \infty,$$

whence $\{2T_x \leq t\} \subseteq \{\int_0^{T_x} f(x+B_s)\,ds < \infty\}$, a.s. P^0. We conclude

that this latter event has positive probability under P^0,

and (vi) follows upon taking $S = T_x$.

(vi) $\Rightarrow$ (iii): Lemma 6.26 gives, for each $x \in K$, the

existence of an open neighbourhood $U(x)$ of x with

$\int_{U(x)} f(y)\,dy < \infty$. Now (iii) follows from the compactness

of K.

(iii) $\Rightarrow$ (v): For fixed $x \in \mathbb{R}$, define $g_x(y) = f(x+y)$

and apply the known implication (iii) $\Rightarrow$ (ii) to the function g_x.

6.30 Solution: Relation (6.55) follows from (6.7) and Problem (6.18):

$$\int_0^\infty f(W_s)\,ds = \lim_{t \to \infty} \int_0^t f(W_s)\,ds = \int_{-\infty}^\infty f(y) \cdot \lim_{t \to \infty} \ell_t(y)\,dy = \infty;$$
$$\text{a.s. } P^x,$$

by the monotone convergence theorem.

For (6.56), we observe first that $X_t^{(n)}$ can be written as

$\dfrac{1}{\sqrt{n}} \int_{-\infty}^\infty f(y) \ell_{nt}(y)\,dy$, thanks to (6.7). Now the crucial observation

is that, by the scaling property of Brownian motion (Lemma 2.9.4(i))

and the definition of local time, the random fields

$\{\dfrac{1}{\sqrt{n}} \ell_{nt}(y); \quad 0 \leq t < \infty, \quad y \in \mathbb{R}\}$ and $\{\ell_t(\dfrac{y}{\sqrt{n}}); \quad 0 \leq t < \infty, \quad y \in \mathbb{R}\}$

induce the same distribution on $C([0,\infty) \times \mathbb{R})$ for each

$n \geq 1$. Thus, the processes $X^{(n)}$ and $Z^{(n)} \triangleq \{\int_{-\infty}^\infty f(y) \ell_t(\dfrac{y}{\sqrt{n}})\,dy;$

$0 \leq t < \infty\}$ have the same finite-dimensional distributions.

Now it is easily seen that

$$\max_{0 \le t \le T} |z_t^{(n)} - X_t| \xrightarrow[n \to \infty]{} 0, \quad \text{a.s. } P^0$$

holds for every finite $T > 0$, and (6.56) follows.

3.8: Notes

Section 3.2: The concept of the stochastic integral with respect to Brownian motion was introduced by K. Itô (1942,1944) in order to achieve a rigorous treatment of the stochastic differential equation which governs the diffusion processes of A. N. Kolmogorov (1931). Doob (1953) was the first to study the stochastic integral as a martingale, and to suggest a unified treatment of stochastic integration as a chapter of martingale theory. This task was accomplished by Courrège (1962/63), Fisk (1963), Kunita & Watanabe (1967), Meyer (1967), Millar (1968), Doléans-Dade & Meyer (1970). Much of this theory has become standard, and has received monograph treatment; we mention in this respect the books by McKean (1969), Gihman & Skorohod (1972), Arnold (1973), Friedman (1975), Lipster & Shiryaev (1977), Stroock & Varadhan (1979), Ikeda & Watanabe (1981), Elliott (1982), Kopp (1984), the monographs by Skorohod (1965), Kussmaul (1977) and Chung & Williams (1983), and the detailed accounts of the contributions of the "French school" in Meyer (1976), Dellacherie & Meyer (1975/1980). Our presentation draws on most of these sources, but is closer in spirit to Ikeda & Watanabe (1981) and Lipster & Shiryaev (1977). The approach suggested by Lemma 2.3 and Problem 2.4 is due to Doob (1953).

Section 3.3: Theorem 3.13 was discovered by P. Lévy (1948; p. 78); a different proof appears on p. 384 of Doob (1953).

Section 3.4: The idea of extending the probability space in order
to accommodate the Brownian motion W in the representation
Theorem 4.2 is due to Doob (1953; pp. 449-451) for the case d = 1.
Problem 4.7 is essentially from McKean (1969; p. 31). Chapters II
of Ikeda & Watanabe (1981) and 12 of Elliott (1982) are good sources
for further reading on the subject matter of sections 3.3 and 3.4.

Section 3.5: The celebrated Theorem 5.1 was proved by Cameron &
Martin (1944) for nonrandom integrands X, and by Girsanov (1960)
in the present generality. Our treatment of it was inspired by
the lecture notes of S. Orey (1974).

Section 3.6: Brownian local time is the creation of P. Lévy (1948),
although the first rigorous proof of its existence was given by
Trotter (1958). Our approach to Theorem 6.11 follows that of
Ikeda & Watanabe (1981), McKean (1969). One can study the local
time of a nonrandom function divorced from probability theory,
and the general pattern that develops is that regular local times
correspond to irregular functions; for instance, for the highly
irregular Brownian paths we obtained Hölder continuous local times
(relation (6.22)). See Geman & Horowitz (1980) for more information
on this topic. Local time for semimartingales is discussed in the
volume edited by Azéma & Yor (1978); see in particular the articles
by Azéma & Yor (pp. 3-16) and Yor (pp. 23-35). Local time for Markov
processes is treated by Blumenthal & Getoor (1968).

The generalized Itô rule (Theorem 6.22) is due to Meyer (1976) and Wang (1977). There is a converse to Corollary 6.22: if $f(W_t)$ is a continuous semimartingale, then f is the difference of convex functions (Wang (1977), Cinlar, Jacod, Protter & Sharpe (1980)). A multidimensional version of Theorem 6.22, in which convex functions are replaced by potentials, has been proved by Brosamler (1970).

Tanaka's formula (6.11) provides a representation of the form $f(W_t) - f(W_0) + \int_0^t g(W_s) dW_s$ for the continuous additive functional $\ell_t(a)$, with $a \in \mathbb{R}$ fixed. In fact, any continuous additive functional has such a representation, where f may be chosen to be continuous; see Ventcel (1962), Tanaka (1963).

We follow Ikeda & Watanabe (1981) in our exposition of Theorem 6.17 and in the proof of (6.56), Problem 6.30. For more information on the subject matter of this problem, the reader is referred to Papanicolaou, Stroock & Varadhan (1977).

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