Problems for 18.311, MIT.

Rodolfo R. Rosales*

May 01, 2003

Contents

*MIT, Department of Mathematics, room 2-337, Cambridge, MA 02139.

1 GaDy (Isentropic Gas Dynamics Problems).

1.1 Introduction to Isentropic Gas Dynamics.

Equations that govern the behavior of a Gas can be derived using the same *Conservation Equation* techniques that were used in the lectures, the notes, etc., to derive equations for the examples of Traffic Flow, River Flows, Shallow Water Waves, Modulations of Dispersive Waves, etc. The conserved quantities in this case are the mass, the momentum and the energy, with the resulting equations known as the **Euler equations of Gas Dynamics.**

Under certain conditions, one can assume that the Entropy is a constant throughout the flow, in which case the equations can be simplified a bit, with the elimination of the equation for the conservation of energy. The **one dimensional isentropic (constant entropy) Euler equations of Gas Dynamics** are

$$\left.\begin{aligned} \rho_t + (\rho u)_x &= 0\,, \\ (\rho u)_t + (\rho u^2 + p)_x &= 0\,, \end{aligned}\right\} \tag{1.1.1}$$

where $\rho = \rho(x,t)$, $u = u(x,t)$, and $p = p(x,t)$ are the gas mass density, flow velocity, and pressure, respectively. The first equation here implements the conservation of mass, and the second the conservation of momentum. In the usual way, we also need some equations relating the fluxes of the conserved quantities with the conserved densities — the analogue of the equation $q = Q(\rho)$ in Traffic Flow. In this case this is provided by an **equation of state**, relating the pressure to the density. This takes the form

$$p = P(\rho)\,, \tag{1.1.2}$$

where P is a function satisfying $\boxed{\dfrac{dP}{d\rho} > 0.}$ For example, for an ideal gas $P = \kappa\rho^\gamma$, where $\kappa > 0$ and $1 < \gamma < 2$ are constants.

To summarize, the system of equations given by (1.1.1) is known as the

Conservation Form for the Isentropic Equations of Gas Dynamics, in Eulerian Coordinates.

Remark 1.1.1 *Normally, the pressure is a function of both the density and some other thermodynamic variable, such as the temperature. But the isentropic assumption allows us to write the pressure as a function of the gas mass density only.*

Introduce now the function $c = c(x,t) > 0$ (**c is the sound speed**) by

$$c = C(\rho)\,, \quad \text{where } C(\rho) = \sqrt{\frac{dP}{d\rho}(\rho)}\,. \tag{1.1.3}$$

An alternative form of the equations (1.1.1) is then given by

$$\left.\begin{aligned} \rho_t + u\rho_x + \rho u_x &= 0\,, \\ u_t + \frac{c^2}{\rho}\,\rho_x + u u_x &= 0\,. \end{aligned}\right\} \tag{1.1.4}$$

As mentioned earlier, in the form given by (1.1.1), the equations are said to be written in **Eulerian (or laboratory) Coordinates.** The equations can also be written in **Lagrangian Coordinates**, where the "space" coordinate is a label for each mass particle, rather than a position in space.

To transform the equations in (1.1.1) to Lagrangian coordinates, introduce the variable

$$\sigma = \sigma(x,t) = \int^x \rho(\zeta, t)\, d\zeta\,. \tag{1.1.5}$$

Then σ denotes the **mass to the left of a given point in space** x, so that σ is constant if an only if x changes to track a fixed mass point in the gas. That is: **σ is a Lagrangian Coordinate.** In fact σ is defined so that:

$$\frac{\partial\sigma}{\partial x} = \rho\,, \quad \text{and} \quad \frac{\partial\sigma}{\partial t} = -\rho\, u\,. \tag{1.1.6}$$

Then, if a change of independent variables (from (x,t) to (σ,t)) is made, you can check that the equations take the form:

$$\left.\begin{aligned} \mathrm{v}_t - u_\sigma &= 0\,, \\ u_t + p_\sigma &= 0\,, \end{aligned}\right\} \tag{1.1.7}$$

where $\mathrm{v} = 1/\rho$ is the specific volume, and $p = p(\mathrm{v})$ — note that $\boxed{\dfrac{dp}{d\mathrm{v}} = -\rho^2\, c^2,}$ where c is the sound speed (defined in (1.1.3)).

To summarize, the system of equations given by (1.1.7) is known as the

Conservation Form for the Isentropic Equations of Gas Dynamics, in Lagrangian Coordinates.

1.2 GaDy01 Problem. Riemann Invariants in Isentropic Gas Dynamics.

PART I. Derive the form of the equations given in (1.1.4) from (1.1.1).

PART II. Let $B = B(\rho)$ be given by $\boxed{\dfrac{dB}{d\rho} = \dfrac{C(\rho)}{\rho}.}$ Equivalently $\boxed{B(\rho) = \int^{\rho} C(z)\dfrac{dz}{z}.}$ Then define

$$s = s(x,t) = u - B(\rho) \quad \text{and} \quad r = r(x,t) = u + B(\rho)\,. \tag{1.2.1}$$

Show that there exist some functions $\boxed{R = R(\rho, u)}$ and $\boxed{S = S(\rho, u)}$ such that the Euler equations (1.1.4) can be written in the following **characteristic form:**

- Along the curves $\dfrac{dx}{dt} = R\,,$ we have $\dfrac{dr}{dt} = 0\,.$ (1.2.2)
- Along the curves $\dfrac{dx}{dt} = S\,,$ we have $\dfrac{ds}{dt} = 0\,.$ (1.2.3)

Note that you are requested to find EXPLICIT formulas for R and S.

Hint 1.2.1 *Note that, since along* $\dfrac{dx}{dt} = R$ *we have* $\dfrac{dr}{dt} = r_t + R r_x$*, it follows that it must be* $R = -\dfrac{r_t}{r_x}$*. A similar expression holds for* S*.*

WARNING: *Both* R *and* S *are functions of* ρ *and* u *only — no derivatives are involved. But here we see that it should also be* $R = -\dfrac{r_t}{r_x}$ *and* $S = -\dfrac{s_t}{s_x}$*. This means that (if you compute* r_t*,* r_x*, etc., using the equations, and substitute the answer into these quotients) the derivatives should cancel out!* **Do not** *give an answer for* R *and* S *where there are derivatives left;* R *and* S *are functions of* ρ *and* u **only!**

Remark 1.2.1 Read this remark AFTER you have done the problem. The points made here are not necessary for the solution, and may even confuse you if you read them before you solve the problem.

1. *The curves defined by the equations* $\dfrac{dx}{dt} = R$ *and* $\dfrac{dx}{dt} = S$ *are the* **characteristics** *for the Euler equations (1.1.1). Notice that we have* **two sets of characteristic curves,** *with* **two different characteristic speeds:** R **and** S**.** *This should be compared with the Traffic Flow or the River Flow examples, where there is a single set of characteristics with a single characteristic speed.* **The system (1.1.1) can support two types of waves.** *This is very*

similar to the situation that occurs for the (linear) string equation, except that here the waves are now nonlinear, and interact with each other.

2. *The quantities r and s are known by the name of* **Riemann Invariants***, in honor of Riemann, who first introduced them. Notice that these quantities are constants along their corresponding characteristics.*

3. *The Euler equations (1.1.1) model the dynamics of gas motion in one dimension, under certain assumptions. The physical interpretation of the variables was given earlier and it is as follows: ρ is the gas mass density, u is the gas flow velocity, p is the gas pressure and c is the sound speed. The equations implement the conservation of mass and momentum. The two types of waves the model supports are the sound waves moving to the right and to the left. The sound waves move at speed c, relative to the gas (which moves at speed u). Look at the expressions for R and S you just obtained;* **which one corresponds to the right (left) moving wave?**.

4. *The assumptions made to obtain (1.1.1) as a model for Gas Dynamics are: neglect viscosity and thermal conductivity, and assume a constant entropy. Then the first equation follows from the conservation of mass, and the second from the conservation of linear momentum.*

5. *For an ideal gas with constant specific heats (polytropic gas) one has $P = \kappa\rho^\gamma$, where $\kappa > 0$ and $1 < \gamma < 2$ are constants — γ is the ratio of the specific heats.*

6. *If one sets $p = \rho^2$ in (1.1.1), then the equations are equivalent to the equations for Shallow Water Waves.*

1.3 GaDy02 Problem. Isentropic Gas Dynamics Viscous Shock Profile.

1.3.1 Characteristic Form, and Riemann Invariants.

The equations in (1.1.7) can be rewritten in the form

$$\left.\begin{aligned} \mathrm{v}_t &= u_\sigma\,, \\ u_t &= \rho^2\, c^2\, \mathrm{v}_\sigma\,, \end{aligned}\right\} \tag{1.3.1}$$

using the fact that $\dfrac{dp}{d\text{v}} = -\rho^2\,c^2$, where c is the sound speed (defined in (1.1.3)). By doing linear combinations of these equations, **SHOW THAT** they can be written in the **characteristic form**

$$\left.\begin{aligned} \frac{du}{dt} + \rho\,c\,\frac{d\text{v}}{dt} = 0 \qquad \text{along} \qquad \frac{d\sigma}{dt} = -\rho\,c\,, \\ \frac{du}{dt} - \rho\,c\,\frac{d\text{v}}{dt} = 0 \qquad \text{along} \qquad \frac{d\sigma}{dt} = +\rho\,c\,, \end{aligned}\right\} \tag{1.3.2}$$

Assume $p = (1/2)\,\rho^2 = (1/2)\,\text{v}^{-2}$, and **SHOW THAT** you can define a (simple) function $g = g(\text{v})$ such that the equations above can be written in the form

$$\frac{dL}{dt} = 0 \quad \text{along} \quad \frac{d\sigma}{dt} = -\rho\,c\,, \qquad \text{and} \qquad \frac{dR}{dt} = 0 \quad \text{along} \quad \frac{d\sigma}{dt} = +\rho\,c\,, \tag{1.3.3}$$

where $L = u + g$ and $R = u - g$ — L and R are called the **Riemann Invariants**.

1.3.2 Viscosity, and Shocks.

The equations for Isentropic Gas Dynamics, in Lagrangian Coordinates, as displayed in the system (1.1.7), neglect the effects of viscous transport of momentum (i.e. transport of momentum due to the random motions of the atoms in the gas). As argued in the lectures, this effect can (easily) be incorporated into the equations. After an appropriate non-dimensionalization, the equations with viscous transport of momentum incorporated take the form:

$$\left.\begin{aligned} \text{v}_t &= +\,u_\sigma\,, \\ u_t &= -\,p_\sigma + \epsilon\,(\rho\,u_\sigma)_\sigma\,, \end{aligned}\right\} \tag{1.3.4}$$

where (generally) $\boxed{0 < \epsilon \ll 1.}$ Alternatively, we can also write

$$\left.\begin{aligned} \text{v}_t &= u_\sigma\,, \\ u_t &= \rho^2\,c^2\,\text{v}_\sigma + \epsilon\,(\rho\,u_\sigma)_\sigma\,, \end{aligned}\right\} \tag{1.3.5}$$

In the lectures we showed that these equations, for small deviations from equilibrium, accept smooth solutions that connect two different states on each side. These solutions, as $\epsilon \to 0$, become discontinuities (i.e. shocks) connecting the two states on each side. These solutions have the form of traveling waves, that is to say: functions of $\sigma - U\,t$.

The objective of this part of the problem is for you to develop the analog of the calculation done in class, for the case of the equations as written in (1.3.5) above. Proceed as follows:

A. Assume a simple form for the dependence of the pressure on the density. Specifically, assume that

$$p = \frac{1}{2}\rho^2 = \frac{1}{2}\mathrm{v}^{-2}\,.$$

B. Assume a solution of the equations of the form $\boxed{u = u(z)}$ and $\boxed{\mathrm{v} = \mathrm{v}(z),}$ where z is the traveling wave coordinate $\boxed{z = (\sigma - U\,t)/\epsilon,}$ and $\boxed{U}$ is the wave speed.

C. Substitute the forms in part **B** into the equations in (1.3.4 – 1.3.5). This should give ode's for $u = u(z)$ and $\mathrm{v} = \mathrm{v}(z)$. You should be able to integrate these ode's once (in terms of two constants of integration), and then you should also be able to eliminate u from the equations.

After this, you should end up with an equation of the form

$$\frac{d\mathrm{v}}{dz} = -\frac{U}{\mathrm{v}}\,Q(\mathrm{v})\,,$$

where $Q = Q(\mathrm{v})$ is a cubic polynomial, whose coefficients depend on U and the constants of integration.

D. Remember that we are only interested in solutions where $\mathrm{v} > 0$, and that (in particular) we are looking here for a solution of the equation in **C** such that

$$\lim_{z\to+\infty} \mathrm{v}(z) = \mathrm{v}_r \quad \text{and} \quad \lim_{z\to-\infty} \mathrm{v}(z) = \mathrm{v}_l\,,$$

where $\mathrm{v}_r > 0$ and $\mathrm{v}_l > 0$ are constants.

E. Using the facts in **D**, argue that Q in **C** must have the form

$$Q = (\mathrm{v} - \mathrm{v}_r)\,(\mathrm{v} - \mathrm{v}_l)\,(\mathrm{v} - \mathrm{v}_0)$$

where $\mathrm{v}_0 < 0$ is a constant. You should then be able to write v_0 in terms of v_r and v_l (using an obvious property of the polynomial Q), and then also write all the coefficients of Q in terms of v_l and v_r — in particular, you should get a formula expressing U^2 in terms of these constants.

F. Now you should be in a position to argue that the ode in **C** has solutions with the desired properties (see **D**), provided that:

For $U > 0$ **(wave moves to the right)** .... $\mathbf{v}_l < \mathbf{v}_r$, **i.e.** $\rho_l > \rho_r$.

For $U < 0$ **(wave moves to the left)** .... $\mathbf{v}_l > \mathbf{v}_r$, **i.e.** $\rho_l < \rho_r$.

That is to say, the **wave is compressive,** and raises the density of the fluid as it travels through it. This extends the results that we found in the lectures for the weak shock case

G. Show that:

For $U > 0$ **(wave moves to the right)** .... $+(\rho_l)^{1.5} > U > +(\rho_r)^{1.5}$.

For $U < 0$ **(wave moves to the left)** .... $-(\rho_l)^{1.5} > U > -(\rho_r)^{1.5}$.

Relate these result to the Generalized Entropy Condition (characteristics converge on shocks) that was introduced in the lectures — the characteristic form (1.3.2) will be useful for this.

This problem requires you to do all of the above, and to justify all your steps. Do not fall into the trap of arguing that something should happen just because the problem statement says so. Make sure that you fill in the proper and correct arguments!

THE END.