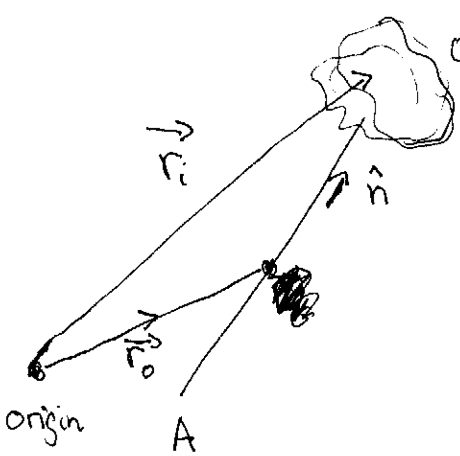
64] Note on Terminology (with some physics!)

We commonly speak of "angular momentum around an axis". The relation between this, and the fundamental definition of $\vec{L}$ is as follows.

angular momentum around axis A

= (Component of $\vec{L}$ in the direction of A, with the origin taken to be any point on A)



So, referring to the diagram,

$$L_{\substack{\text{around}\\ A}} = \left[\sum_i m_i (\vec{r}_i - \vec{r}_o) \times \vec{v}_i \right] \cdot \hat{n}$$

Example: $\vec{r}_o = 0$, $\hat{z}$ component

$$L_{\substack{\text{around}\\ z\text{-axis}\\ \text{through origin}}} = \sum_i m_i (x_i \dot{y}_i - y_i \dot{x}_i)$$

Similarly for moments of inertia around axes.

Note that it doesn't matter <u>which</u> point on the axis you choose, because taking $\vec{r}_0{}' = \vec{r}_0 + \lambda \hat{n}$ does not change $L_{\substack{\text{around} \\ A}}$. (Prove it!)

<u>65)</u> Translation + Rotation : Angular Momentum

<u>Theorem</u>: $\vec{L} = \vec{L}_{spin} + \vec{L}_{orbit}$, with

$\vec{L}_{spin}$ = angular momentum around CM

$\vec{L}_{orbit}$ = angular momentum of CM, with whole mass of body assigned there.

<u>Proof</u>: $\vec{L} = \sum m_i \vec{r}_i \times \vec{v}_i$

$$= \sum m_i \left(\vec{r} - \vec{r}_{CM} + \vec{r}_{CM} \right) \times \left(\vec{v}_i - \vec{v}_{CM} + \vec{v}_{CM} \right)$$

$$= \underbrace{\sum_i m_i (\vec{r} - \vec{r}_{CM}) \times (\vec{v} - \vec{v}_{CM})}_{\vec{L}_{spin} \ \left(= \sum m_i (\vec{r} - \vec{r}_{CM}) \times (\dot{\vec{r}} - \dot{\vec{r}}_{CM})\right)} + \underbrace{\sum_{m_i} m_i \vec{r}_{CM} \times \vec{v}_{CM}}_{\vec{L}_{orbit} \ \left(= M \sum \vec{r}_{CM} \times \dot{\vec{r}}_{CM}\right)}$$

+ crossterms

where

$$\text{cross-terms} = \sum m_i (\vec{r}_i - \vec{r}_{CM}) \times \vec{v}_{CM} + \sum_i m_i \vec{r}_{CM} \times (\vec{v} - \vec{v}_{CM})$$

$$= \Big(\sum m_i (\vec{r}_i - \vec{r}_{CM})\Big) \times \vec{v}_{CM} + \vec{r}_{CM} \times \Big(\sum m_i (\dot{\vec{r}} - \vec{r}_{CM})\Big)$$

$$\stackrel{\checkmark}{=} \vec{0} \text{, by df}^{\underline{n}} \text{ of CM!}$$

66] Translation + Rotation : Energy

<u>Theorem</u>: $\mathcal{E}_{kinetic} = \mathcal{E}_{rot.} + \mathcal{E}_{trans.}$, with

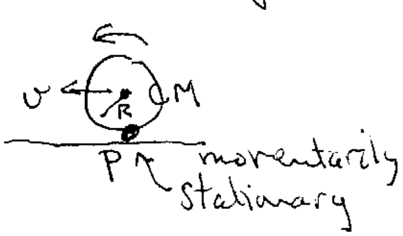
$\mathcal{E}_{rot.}$ = Energy of rotation around CM

$\mathcal{E}_{trans.}$ = Energy of CM translation (i.e. $\frac{1}{2} M v_{CM}^2$)

<u>Proof</u>: In $\mathcal{E}_{kinetic} = \frac{1}{2} \sum m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i$, write

$\vec{r}_i = (\vec{r}_i - \vec{r}_{CM}) + \vec{r}_{CM}$ and proceed as before.

<u>Example</u>: Rolling without slipping



Method 1: Pure rolling around P

$$\mathcal{E}_{kin.} = \frac{1}{2} I_P \omega^2 \left(= \frac{1}{2} I_P \left(\frac{v}{R}\right)^2\right)$$

Method 2 : Translation + Rotation around CM

$$\mathcal{E}_{kin.} = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M v^2 .$$

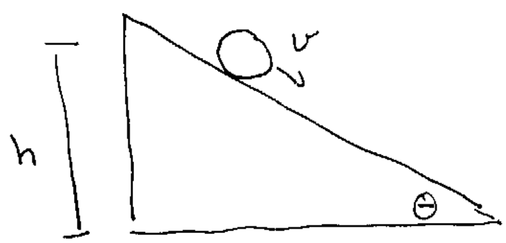
These are = by the ∥ axis theorem!

$$\frac{1}{2} I_P \left(\frac{v}{R}\right)^2 \stackrel{?}{=} \frac{1}{2} I_{CM} \left(\frac{v}{R}\right)^2 + \frac{1}{2} M v^2$$

$$\Updownarrow$$

$$I_P = I_{CM} + MR^2$$

67 Rolling Down Incline



$$\mathcal{E} = \frac{1}{2} M v^2 + \frac{1}{2} I_{cm} \left(\frac{v}{R}\right)^2 + Mgh$$

We start from rest at $h = H$, so $\mathcal{E} = MgH$, and

$$v^2 = \frac{2Mg(H-h)}{M + \frac{I}{R^2}}$$

Since $\dot{h} = -v \sin\theta$, we get

$$\dot{h}^2 = \frac{2Mg(H-h)}{M + I/R^2} \sin^2\theta$$

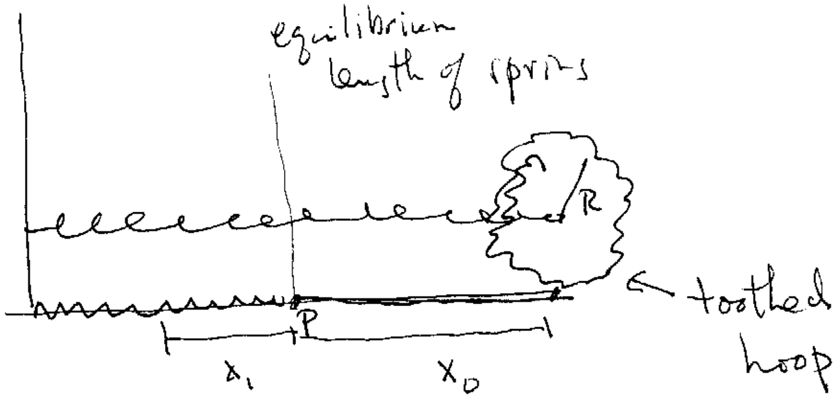
and by differentiating

$$\ddot{h} = -\frac{Mg}{M + \frac{I}{R^2}} \sin^2\theta = -a$$

Using familiar formulas for constant acceleration from rest, to fall to h=0 requires time

$$\frac{1}{2} a t^2 = H, \quad t^2 = \frac{2H}{a} = \frac{2H}{g \sin^2\theta}\left(1 + \underbrace{\frac{I}{MR^2}}_{\text{influence of rotation!}}\right)$$

68] Problem 6.40



Just before 1st contact:

$$\frac{1}{2} M v_B^2 = \frac{1}{2} k x_0^2 \qquad \left(\mathcal{E} = \underbrace{\frac{1}{2} M v^2}_{\text{no rolling, yet!}} + \frac{1}{2} k (x - x_{eq.})^2 = \frac{1}{2} k x_0^2\right)$$

L_P is conserved at contact, since torque = 0.

$$\underbrace{MRv_B}_{\text{orbit.}} = \underbrace{MRv_A}_{\text{orbit.}} + \underbrace{MR^2 \frac{v_A}{R}}_{\text{spin}} \quad (\omega = v/R)$$

$$\Rightarrow v_{After} = \frac{v_{Before}}{2}$$

Note that energy is not conserved in collision:

$$\mathcal{E}_B = \frac{1}{2} M v_B^2 \qquad \mathcal{E}_A = \frac{1}{2} M v_A^2 + \frac{1}{2} M R^2 \left(\frac{v_A}{R}\right)^2$$

$$= M v_A^2 = \frac{1}{4} M v_B^2$$

But it is subsequently, for rolling without slipping

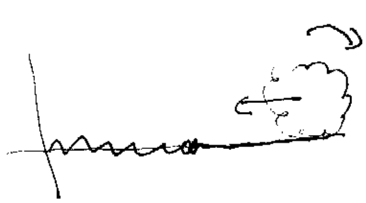
So at closest approach to wall (maximum compression)

$$\mathcal{E} = \frac{1}{2} k x_1^2 = \frac{1}{4} M v_B^2 = \frac{1}{2}\left(\frac{1}{2} M v_B^2\right) = \frac{1}{2} \cdot \frac{1}{2} k x_1^2,$$

and $x_1 = \frac{1}{\sqrt{2}} x_0$

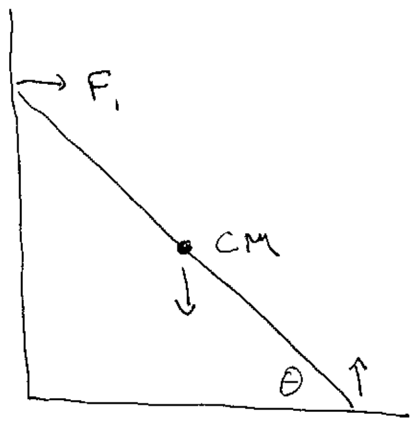
At second encounter L_{spin} and L_{orbit} are equal and opposite! $\Rightarrow$ It stops dead, since

$v_{\text{after 2nd}}$ & $L_{total} = 0$.



[Note: this is special for a hoop. You might amuse yourself by considering a toothed disc.]

69] Problem 6.41



$$CM = \frac{\ell}{2}(\cos\theta, \sin\theta)$$

$$F_1 = (\text{horizontal acceleration of CM}) \cdot M$$

$$= M\frac{\ell}{2}(\cos\theta)^{\cdot\cdot} = \frac{M\ell}{2}(-\dot\theta\sin\theta)^{\cdot} = \frac{M\ell}{2}(-\ddot\theta\sin\theta - \dot\theta^2\cos\theta)$$

$$\mathcal{E} = \underbrace{\frac{1}{2}M\left(\frac{\ell}{2}\right)^2\dot\theta^2}_{\text{CM, kinetic}} + \underbrace{\frac{1}{2}\,\frac{1}{12}M\ell^2\dot\theta^2}_{\text{rotation around CM}} + \underbrace{Mg\frac{\ell}{2}\sin\theta}_{\text{gravity potential}}$$

$$= \frac{1}{6}M\ell^2\dot\theta^2 + \frac{Mg\ell}{2}\sin\theta = \frac{Mg\ell}{2}\sin\theta_0$$

$$\Rightarrow \boxed{\dot\theta^2 = \frac{3g}{\ell}(\sin\theta_0 - \sin\theta)}$$

Also, differentiating,

$$\boxed{\ddot\theta = -\frac{3}{2}\frac{g}{\ell}\cos\theta}$$

So $F_1/(Ml/2) = -\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta$

$$= \frac{3}{2}\frac{g}{l}\cos\theta\sin\theta - \frac{3g}{l}(\sin\theta_0 - \sin\theta)\cos\theta$$

$$= \frac{3g}{2l}\cos\theta\,[3\sin\theta - 2\sin\theta_0]$$

It ceases to be > 0 when $\boxed{\sin\theta_c = \frac{2}{3}\sin\theta_0}$

– the ladder leaves the wall $\frac{1}{3}$ of the way down!

70) Comments

1) It's a jungle out there! Much of the art of physics is learning to focus on the few things ~~[illegible]~~ we can say something intelligent about.

2) You see how energy becomes the leading concept in hard mechanics problems. This is a hint of things to come (Lagrangians, Hamiltonians) in more advanced courses.