56) Angular Momentum Recollection + Application

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}$$

$$m\,\vec{r}\times\frac{d^2\vec{r}}{dt^2} = \vec{r}\times\vec{F} \equiv \text{torque} \equiv \vec{\tau}$$

$$\frac{d}{dt}\left(m\,\vec{r}\times\frac{d\vec{r}}{dt}\right) \equiv \frac{d}{dt}\vec{L} \qquad \Rightarrow \qquad \frac{d\vec{L}}{dt} = \vec{\tau}$$

$$\vec{L} = m\,\vec{r}\times\vec{v} = \vec{r}\times\vec{p} = \text{angular momentum}$$

Analogous to $\frac{d\vec{p}}{dt} = \vec{F}$ for linear momentum.

For central forces $\vec{\tau} = 0 \Rightarrow \vec{L}$ is conserved. We used this, ~~because~~ recently (Kepler's 2nd).

example: Gravitational capture

far-away body, $m \ll M$

choose x-axis along $\vec{v}$

ignore CM reduction

$$|L| = mbv$$

$$\mathcal{E} = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r} = \frac{1}{2}m\dot{r}^2 + \frac{mb^2v^2}{2r^2} - \frac{GMm}{r}$$

$$= \mathcal{E}_0 = \frac{1}{2}mv^2$$

~~At close~~ Is $r = R$ allowed?

Requires $(\dot{r}^2 > 0)$ $\frac{1}{2}mv^2 - \frac{mb^2v^2}{2R^2} + \frac{GMm}{R} \geq 0$

$$\boxed{v^2\left(\frac{b^2}{R^2} - 1\right) \leq \frac{2GM}{R}} \quad \text{or} \quad \boxed{\frac{b^2}{R^2} \leq 1 + \frac{2GM}{Rv^2}}$$

This is $\rightarrow b \leq R$ as $v \to \infty$, but is enhanced over "geometrical" capture for finite v.

57] Angular Momentum for Systems

$$m_i \frac{d^2\vec{r}_i}{dt^2} = \underbrace{\vec{F}_i}_{\text{external forces}} + \underbrace{\sum_j \vec{F}_{ij}}_{\text{internal forces}}$$

$$\sum_i m_i \vec{r}_i \times \frac{d^2\vec{r}_i}{dt^2} = \sum_i \vec{r}_i \times \vec{F}_i + \underbrace{\sum_i \sum_j \vec{r}_i \times \vec{F}_{ij}}_{\text{work on this}}$$

$$\sum_i \sum_j \vec{r}_i \times \vec{F}_{ij} = \frac{1}{2}\left(\sum_i\sum_j + \sum_j\sum_i\right)(\vec{r}_i \times \vec{F}_{ij})$$

$$= \frac{1}{2}\sum_i\sum_j\left(\vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \vec{F}_{ji}\right)$$

$$\underset{3^{rd}\text{ Law}}{=} \frac{1}{2}\sum_i\sum_j (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij}$$

Now if $\vec{F}_{ij} \propto \vec{r}_i - \vec{r}_j$ (central force), this $= 0$!

Then

$$\sum_i m_i \vec{r}_i \times \frac{d^2\vec{r}_i}{dt^2} = \sum_i \vec{r}_i \times \vec{F}_i$$

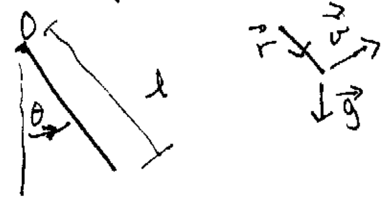
$$\parallel \qquad\qquad \parallel$$

$$\sum_i \frac{d}{dt}\vec{L}_i = \sum_i \vec{\tau}_i \, ;$$

or $\frac{d}{dt}\vec{L} = \vec{\tau}$ with $\vec{L} \equiv \sum_i \vec{L}_i$, $\vec{\tau} = \sum_i \vec{\tau}_i$

The result is more general than this derivation.

<u>example</u>: rod-pendulum



$\lambda =$ mass/length

$$\vec{L} = \int_0^\ell dm\ \vec{r}\times\vec{v} = \underbrace{\hat{z}}_{\text{into plane}} \int_0^\ell \lambda\, dr \cdot r \cdot r\dot{\theta} = \hat{z}\,\frac{\ell^3}{3}\lambda\dot{\theta} = \hat{z}\,\frac{M\ell^2}{3}\dot{\theta}$$

general theorem for near-Earth gravity

$$\vec{\tau} = \int \vec{r}\times d\vec{F}, \quad d\vec{F} = dm\,\vec{g}$$

$$\Downarrow$$

$$\vec{\tau} = \int dm\ \vec{r}\times\vec{g} = M\vec{r}_{CM}\times\vec{g} = \vec{r}_{CM}\times M\vec{g}$$

$$\left(\begin{aligned} &= \left(\int dm\,\vec{r}\right)\times\vec{g} \\ &= M\vec{r}_{CM}\times\vec{g} \end{aligned}\right)$$

$\Rightarrow$ The torque may be calculated as if the force is acting at the CM

<u>NB</u>. Near-Earth gravity only!!

In present example:

$$\vec{\tau} = Mg\frac{\ell}{2}\sin\theta\ \hat{z} \quad \text{(evaluating x-product)}$$

so $\dfrac{d\vec{L}}{dt} = \vec{\tau}$ reads $-\dfrac{M\ell^2}{3}\ddot{\theta} = \dfrac{Mg\ell}{2}\sin\theta$, or $\ddot{\theta} = -\dfrac{3g}{2\ell}\sin\theta$

small θ: $\ddot{\theta} = -\dfrac{3g}{2\ell}\theta$; $\omega^2 = \dfrac{3g}{2\ell}$ oscillations.

58] Statics

If a rigid body is not moving
(no internal motion!)

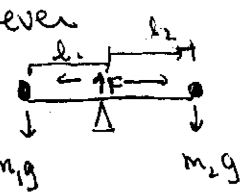
i) $\sum \vec{F}^{ext.} = 0$

ii) $\sum \vec{\tau}^{ext.} = 0$

since $\frac{d\vec{p}}{dt} = 0$, $\frac{d\vec{L}}{dt} = 0$. And if $\sum \vec{F}^{ext.} \neq 0$ or $\sum \vec{\tau}^{ext.} \neq 0$, for a rigid body, you will have visible motion. So
(bulk)

i) and ii) are the fundamental equations of statics. This is a vast subject in engineering (you want buildings, bridges, etc. to be static!)

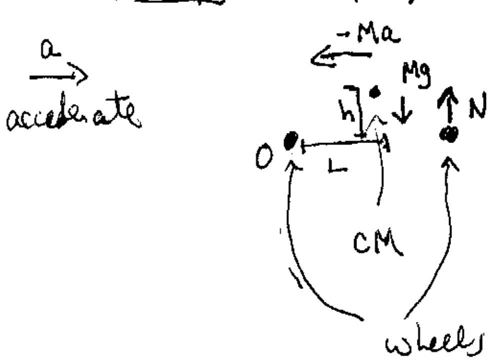
example: lever



$$F = m_1 g + m_2 g$$

around fulcrum $l_2 m_2 g - l_1 m_1 g = 0 \qquad \frac{l_1}{l_2} = \frac{m_2}{m_1}$

example: motorcycle lift-off



torques around O: $hMa - LMg + 2LN = 0$
with $N \geq 0$

possible only for $a < \frac{L}{h} g$

with bigger a, "lift off"

Note high CM and limited wheelbase are conducive to liftoff.

—·—

Converse? If $\Sigma F = 0$, $\Sigma \tau = 0$ we have $\vec{P} = \vec{P}_0$, $\vec{L} = \vec{L}_0$. The first is removed "trivially" by going to CM frame. The 2^{nd} ($\vec{L}_0 \neq 0$) is much more interesting. It does not correspond to a constant rotation, ~~will~~ in general. We'll treat it down the road.

—·—