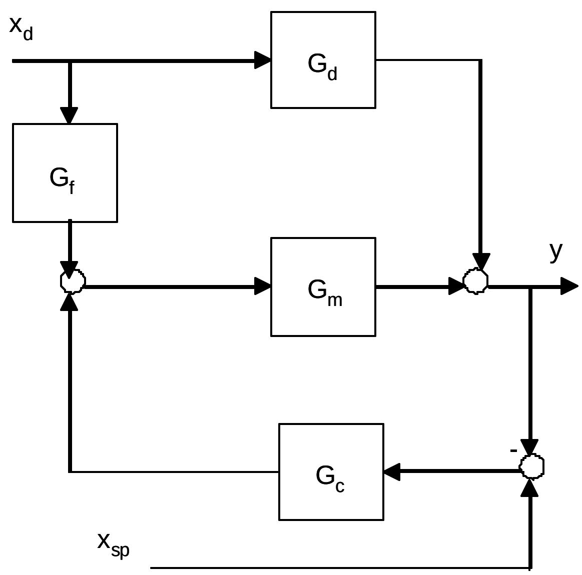
10.450 Process Dynamics, Operations, and Control

Spring 2003 **Exam 1**

from the 10.450 Policies: "The grade for either test may be raised by reworking the test out of class and turning it in the next class meeting. Final test grade will then be 2/3 in-class and 1/3 at-home. Up to 5 bonus points will be added for creativity in the presentation of the at-home portion."

(1) (25%) Represent this block diagram as an equation in the Laplace domain. The equation should be arranged so that a dependent variable is expressed as the sum of independent variables, each of these multiplied by some transfer function. (Note in the diagram that y is subtracted from x_{sp}.) If you could choose G_f to be anything you wanted, how could you use it to improve the response of y to the disturbance x_d?



Build the equations by starting at the output signal y and moving backwards through the diagram:

$$
\begin{aligned}
y &= G_d x_d + G_m x_m \\
&= G_d x_d + G_m \left(G_f x_d + G_c e\right) \\
&= G_d x_d + G_m \left(G_f x_d + G_c \left(x_{sp} - y\right)\right) \\
&= G_d x_d + G_m G_f x_d + G_m G_c x_{sp} - G_m G_c y
\end{aligned}
$$

$$
y = \frac{G_d + G_m G_f}{1 + G_m G_c} x_d + \frac{G_m G_c}{1 + G_m G_c} x_{sp}
$$

If we know the disturbance and manipulated variable transfer functions, G_d and G_m, perhaps we can create a device G_f to be $-G_m/G_d$. In this case, the denominator of the <u>closed loop</u> disturbance transfer function will be zero, making response variable y independent of disturbances!

(2) (20%) An old 18.03 exam says

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$$u(t-2) = \frac{3}{2}u(t-8) - \frac{1}{2}y - 2\frac{dy}{dt} \qquad \text{at } t=0,\ y=2$$

where u(t-a) is the unit step function (0 for t < a; 1 thereafter). Write the solution y(t), and sketch the plot. State the time at which the maximum absolute value of y occurs.

First rewrite in standard form to clarify the structure of the problem.

$$4\frac{dy}{dt} + y = -2u(t-2) + 3u(t-8) \qquad y(0) = 2$$

This is a first-order system that will react immediately to its initial condition and be subsequently disturbed by two step changes.

by-the-book Laplace transform solution

$$4\left(sy(s) - y(0)\right) + y(s) = \frac{-2}{s}e^{-2s} + \frac{3}{s}e^{-8s}$$

$$(4s+1)y(s) = 8 + \frac{-2}{s}e^{-2s} + \frac{3}{s}e^{-8s}$$

$$y(s) = \frac{8}{4s+1} + \frac{-2}{s(4s+1)}e^{-2s} + \frac{3}{s(4s+1)}e^{-8s}$$

The denominator product is easily resolved by partial fraction expansion

$$\frac{1}{s(4s+1)} = \frac{A}{s} + \frac{B}{4s+1} = \frac{1}{s} + \frac{-4}{4s+1}$$

Substituting

$$y(s) = \frac{8}{4s+1} + \left(\frac{-2}{s} + \frac{8}{4s+1}\right)e^{-2s} + \left(\frac{3}{s} + \frac{-12}{4s+1}\right)e^{-8s}$$

On being inverted, the terms in parentheses will be delayed in time by the exponential functions in s.

$$y(t) = \frac{8}{4}e^{-t/4} + \left(-2u(t) + \frac{8}{4}e^{-t/4}\right)\bigg|_{delayed 2} + \left(3u(t) + \frac{-12}{4}e^{-t/4}\right)\bigg|_{delayed 8}$$

$$= 2e^{-t/4} + 2\left(-u(t-2) + e^{-(t-2)/4}\right) + 3\left(u(t-8) - e^{-(t-8)/4}\right)$$

At t = 0, the solution satisfies the initial condition. At long time, the solution goes to 3 - 2 = 1.

by-the-book integrating factor solution

$$y(t) = 2e^{-t/4} + \frac{1}{4}e^{-t/4}\int_0^t e^{t/4}\left[-2u(t-2) + 3u(t-8)\right]dt$$

$$= 2e^{-t/4} + \frac{-2}{4}e^{-t/4}\int_0^t e^{t/4}u(t-2)dt + \frac{3}{4}e^{-t/4}\int_0^t e^{t/4}u(t-8)dt$$

$$= 2e^{-t/4} + \frac{-2}{4}e^{-t/4}\Big|_{t\ge 2}\int_2^t e^{t/4}dt + \frac{3}{4}e^{-t/4}\Big|_{t\ge 8}\int_8^t e^{t/4}dt$$

$$= 2e^{-t/4} + \frac{-2}{4}e^{-t/4}\Big|_{t\ge 2} 4\left[e^{t/4}\right]_2^t + \frac{3}{4}e^{-t/4}\Big|_{t\ge 8} 4\left[e^{t/4}\right]_8^t$$

$$= 2e^{-t/4} - 2e^{-t/4}\Big|_{t\ge 2}\left[e^{t/4} - e^{2/4}\right] + 3e^{-t/4}\Big|_{t\ge 8}\left[e^{t/4} - e^{8/4}\right]$$

$$= 2e^{-t/4} - 2\left[1 - e^{-(t-2)/4}\right]\Big|_{t\ge 2} + 3\left[1 - e^{-(t-8)/4}\right]\Big|_{t\ge 8}$$

It would be inappropriate to collect the constants to write the solution as

$$y(t) \ne 1 + 2e^{-t/4} + 2e^{-(t-2)/4} - 3e^{-(t-8)/4}$$

Although the long-term result is the same, this expression takes account of the step changes before they happen, and thus fails to satisfy the initial condition.

superimposition solution

We have experience with block diagrams showing a response variable as the sum of several disturbances, each processed through its own transfer function. We can regard this problem in the same way, treating each disturbance, and the initial condition, separately:

$$y = y_1 + y_2 + y_3$$

$$4\frac{dy_1}{dt} + y_1 = 0 \qquad y_1(0) = 2$$

$$4\frac{dy_2}{dt} + y_2 = -2u(t-2) \qquad y_2(0) = 0$$

$$4\frac{dy_3}{dt} + y_3 = 3u(t-8) \qquad y_3(0) = 0$$

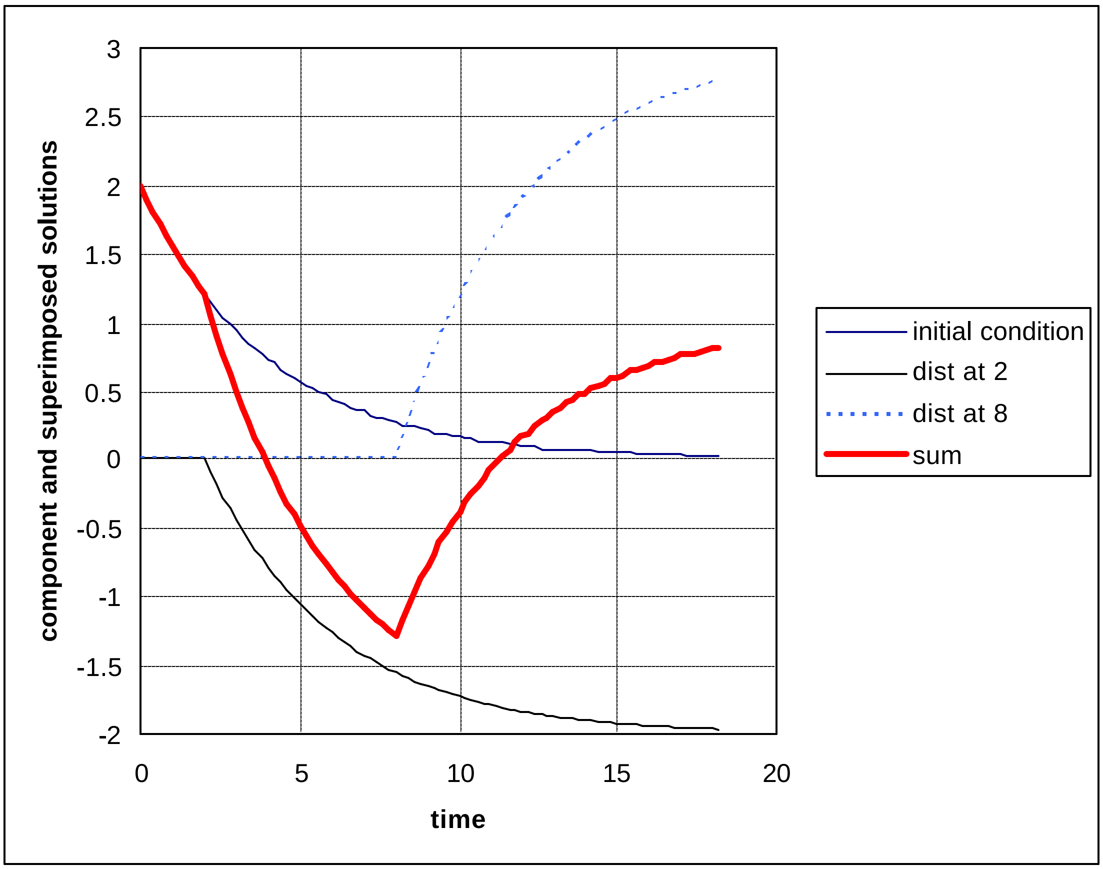
If the three differential equations are summed, we recover the original problem. Now we solve each equation:

$$y_1(t) = 2e^{-t/4}$$

$$y_2(t) = -2\left[1 - e^{-(t-2)/4}\right]_{t \geq 2}$$

$$y_3(t) = 3\left[1 - e^{-(t-8)/4}\right]_{t \geq 8}$$

and sum them to get the solution. By the way, the initial condition is as far from zero as it gets.



(3) (30%) After some experimentation, you have represented your process as

$$G_m(s) = \frac{y_c(s)}{x_m(s)} = \frac{K_m}{ts+1}e^{-qs}$$

where G_m relates the controlled and manipulated variables. K_m is 0.6 (dimensionless, because you have expressed both y_c and x_m as 0-100% scaled variables), t is 5 minutes and ? is 3 minutes.

(a) Use Ziegler-Nichols open-loop tuning to select PID parameters.

From Table 10.5 in Marlin,

$$K_c = \frac{1.2}{0.6}\frac{5}{3} = 3.33$$
$$T_i = 2(3) = 6\,\text{min}$$
$$T_d = 0.5(3) = 1.5\,\text{min}$$

(Using ZN closed-loop tuning would give the same result, but take longer to do.)

> Now, you recognize that your process has periods of reduced production, so that the dead time increases to 6 minutes. Examine the stability of the process when it is controlled using the controller parameters derived for the original production rate in (a).
>
> (b) Express your results in terms of the gain margin (GM) and phase margin (PM), where these are defined
>
> $$GM = \frac{1}{\text{amplitude ratio at the crossover frequency}}$$
> $$PM = 180^\circ + \text{phase angle at which amplitude ratio is 1}$$
>
> (these are covered by Marlin on p.339)

We examine the stability of a process under feedback control by determining the frequency response of the product of transfer functions around the loop, what has been called the "open-loop transfer function".

$$G_{OL} = G_m G_c = \frac{K_m}{ts+1}e^{-qs}K_c\left(1+\frac{1}{T_i s}+T_d s\right)$$

The amplitude and phase angle of G_{OL} are

$$|G_{OL}| = |G_m||G_c| = \frac{K_m}{\sqrt{1+(tw)^2}}K_c\sqrt{1+\left(T_d w-\frac{1}{T_i w}\right)^2}$$
$$\angle G_{OL} = \angle G_m + \angle G_c = \tan^{-1}(-tw) - qw + \tan^{-1}\left(T_d w - \frac{1}{T_i w}\right)$$

Notice how the deadtime ? contributes both negative phase angle (delay) and no reduction in amplitude ratio, both tending to destabilize the loop. At the crossover frequency $?_{co}$, the phase angle is -180º, or -p radians. Remembering to keep the units consistent (knowing how the calculator presents the result of the arctangent function), we find $?_{co}$ = 0.359 rad min^{-1}. By the Bode stability criterion, we check the amplitude ratio at this frequency and find $|G_{OL}| = 0.975$, very close to the stability limit! The gain margin is the factor by which the controller gain would have to be multiplied to reach the stability limit.

$$GM = \frac{1}{0.975} = 1.026$$

At a slightly lower frequency than crossover, the amplitude ratio will be 1. From the amplitude ratio expression, we find

$$w_1 = 0.346 \text{ rad min}^{-1}$$

For disturbances at frequency ? $_1$ the phase angle is computed to be -176.8º. The phase margin is the difference between ? $_1$ and crossover.

$$PM = -176.8 - (-180) = 3.2^{\circ}$$

In summary, the controller was tuned for ? = 3 min; when ? becomes 6 min, the process is near instability.

It may be useful to review the ZN closed-loop tuning method in this stability context. The process (at the new, larger deadtime of 6 min) is placed in closed loop with P-mode control. We examine the G_{OL} frequency response.

$$|G_{OL}| = |G_m||G_c| = \frac{K_m}{\sqrt{1+(tw)^2}} K_c$$
$$\angle G_{OL} = \angle G_m + \angle G_c = \tan^{-1}(-tw) - qw$$

Notice that the controller contributes no phase lag to the loop.

The ZN closed-loop tuning method seeks the controller gain at which the controlled variable oscillates with steady amplitude. This can be done empirically with real equipment (although it may be dangerous) or analytically with a process model, as we have above. Analytically, we find the crossover frequency by setting the phase angle to -180º, and then solve for the gain K_c that makes the amplitude ratio equal to 1.

This procedure is a method of tuning to accommodate the new dead time. However, it does not answer the question asked in this problem: how will the process operate at the new dead time with the old tuning parameters, in which integral and derivative modes influence phase lag!

(4) (25%) A colleague complains that the Ciancone tuning correlation you recommended to him gave bad results. You look carefully at his process model, and he does seem to have computed K_c, T_i, and T_d correctly from the correlation. However, in

examining his controller, you discover that it is not the ideal PID algorithm (for which Ciancone was formulated) but instead

$$G_c(s) = K_c'\left(1 + \frac{1}{T_i' s}\right)\left(1 + T_d' s\right)$$

You have 3 choices: get a new controller, make a new correlation for this particular controller, or figure out how to apply the Ciancone correlation. You opt for the third: given K_c, T_i, and T_d from Ciancone, how do you set K'_c, T'_i, and T'_d?

The trick is to put this controller in the form of a standard PID. If it can be put in that form, then it can show equivalent behavior, and perhaps the recommended PID parameters can be achieved by suitable settings of the actual instrument.

By algebra we find

$$G_c(s) = K_c'\left(1 + \frac{T_d'}{T_i'}\right)\left(1 + \frac{1}{\left(T_i' + T_d'\right)s} + \frac{T_i' T_d'}{\left(T_i' + T_d'\right)} s\right)$$

so that the entered settings interact to determine the actual gain, integral, and derivative performance of the controller. Inverting the process is not as easy, it turns out. Therefore, it is possible that no combination of K'_c, T'_i, and T'_d can be found to realize a particular K_c, T_i, and T_d.