Spring 2003 10.450 Process Dynamics, Operations, and Control

Problem Sets - 1

Solve these equations. Viewing them as dynamic systems, plot the disturbances and responses over appropriate time intervals.

1. $0.3y - 0.8x = -\frac{dy}{dt} \qquad y(0) = -1 \qquad x(t) = \begin{cases} 0 & t < 2 \\ -2 & t \geq 2 \end{cases}$

Write in standard form. This first order system is disturbed by a delayed step function. Ultimately it will come to the value of the step (-2) multiplied by the gain (8/3).

$$\frac{1}{0.3}\frac{dy}{dt} + y = \frac{8}{3}x(t)$$

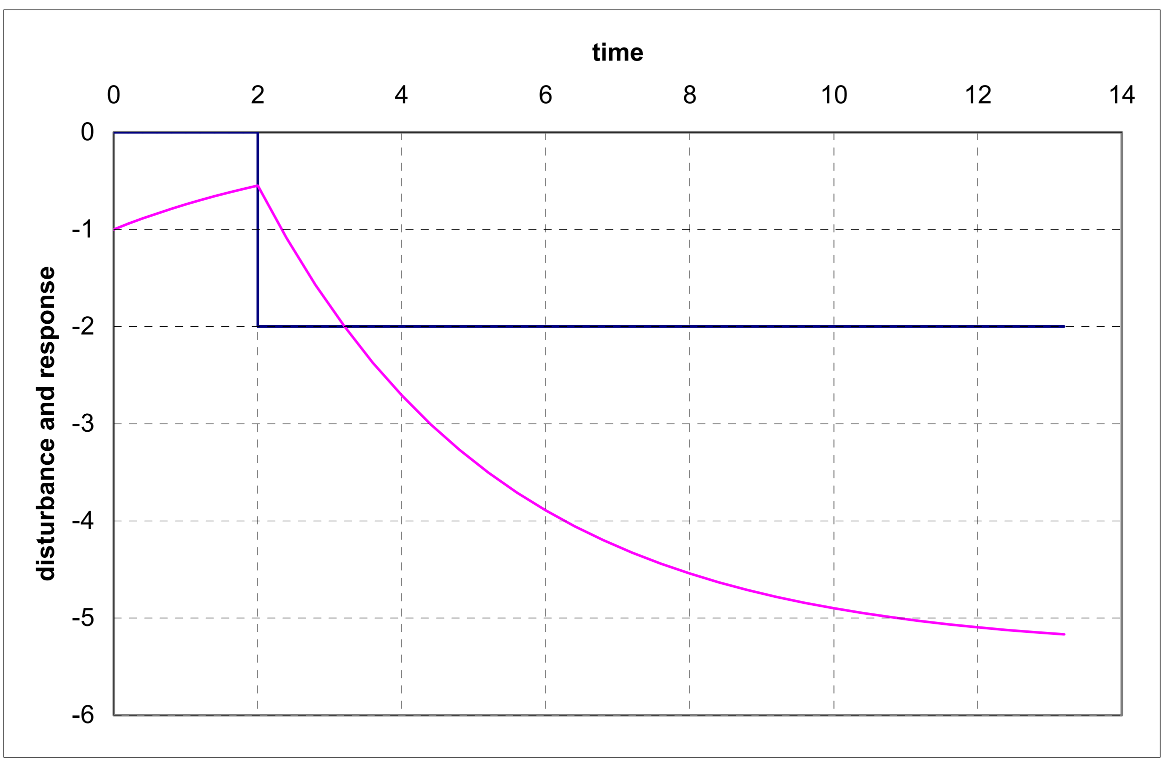
The solution of this equation (1st order, linear, constant coefficient) is

$$y(t) = \frac{e^{-t/\tau}}{\tau}\int_0^t e^{t/\tau} Kx(t)dt + y(0)e^{-t/\tau}$$

For the case of a delayed step disturbance x,

$$y(t) = Kx\left(1 - e^{-(t-t_d)/\tau}\right) + y(0)e^{-t/\tau}$$

The initial condition determines the response until the disturbance occurs at t_d. Substituting the given numbers leads to the plot:



2. $\frac{d^2y}{dt^2} = -0.6\frac{dy}{dt} - 1.2y \qquad y(0) = 1, \left.\frac{dy}{dt}\right|_0 = -0.2$

Write in standard form. This second-order system has no disturbances; it will only react to its initial conditions. If it is stable, it will ultimately tend to zero.

$$\frac{1}{1.2}\frac{d^2y}{dt^2} + 0.5\frac{dy}{dt} + y = 0$$

The roots of the characteristic equation are complex.

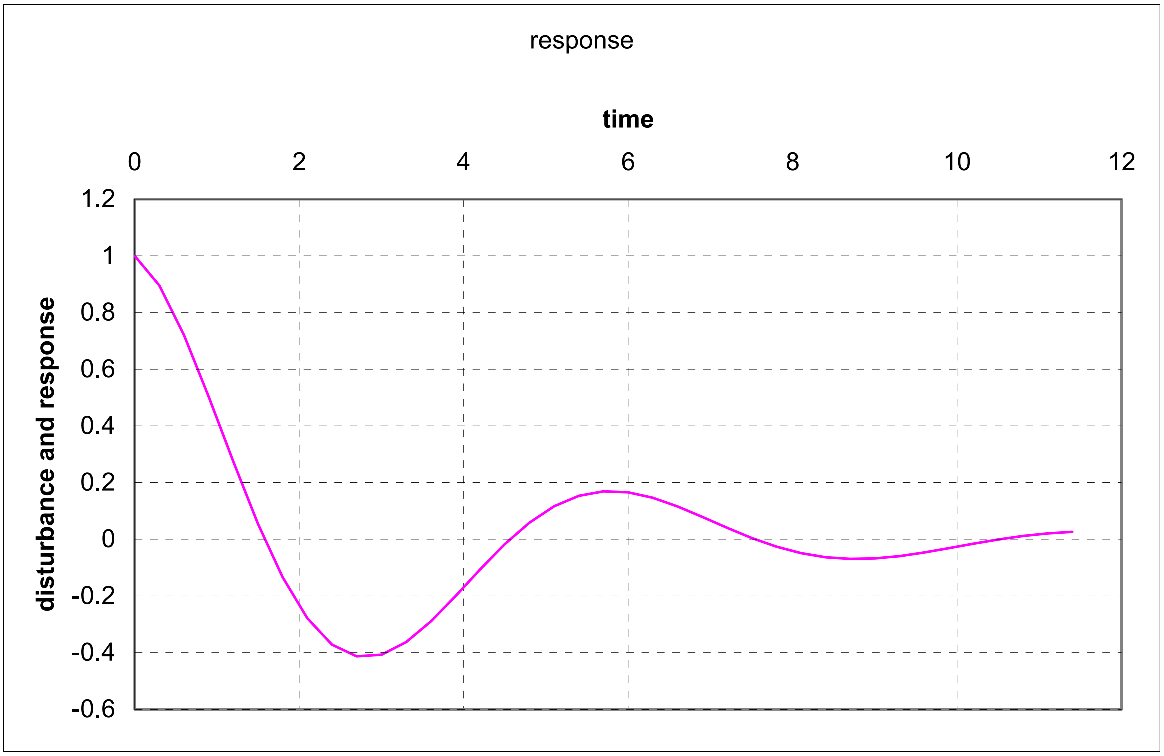
$$r_{1,2} = \frac{-0.5 \pm \sqrt{0.5^2 - 4/1.2}}{2/1.2} = a \pm jb$$

The solution is

$$y(t) = e^{at}\left(C_1 \cos bt + C_2 \sin bt\right)$$

Coefficients C_1 and C_2 are found by applying the initial conditions. The result is

$$y(t) = e^{-0.3t}\left(\cos 1.054t + 0.0949 \sin 1.054t\right)$$



3.
$$\frac{dy_1}{dt} + y_1 = 1 \qquad y_1(0) = 0$$
$$\frac{dy_2}{dt} + y_2 = y_1(t-2) \qquad y_2(0) = 0$$

These first-order systems are arranged in series, so that y_2 is disturbed by y_1, although the disturbance is delayed by two time units before affecting y_2. System y_1 is disturbed by a unit step at time zero. Because of the series arrangement, the equations can be solved sequentially.

Both equations have a solution of the form

$$y(t) = \frac{K}{\tau} e^{-t/\tau} \int_0^t e^{t/\tau} x(t)dt + y(0)e^{-t/\tau}$$

where for both systems, gain K and time constant t are unity, and the initial condition is zero. Substituting the step change for y_1,

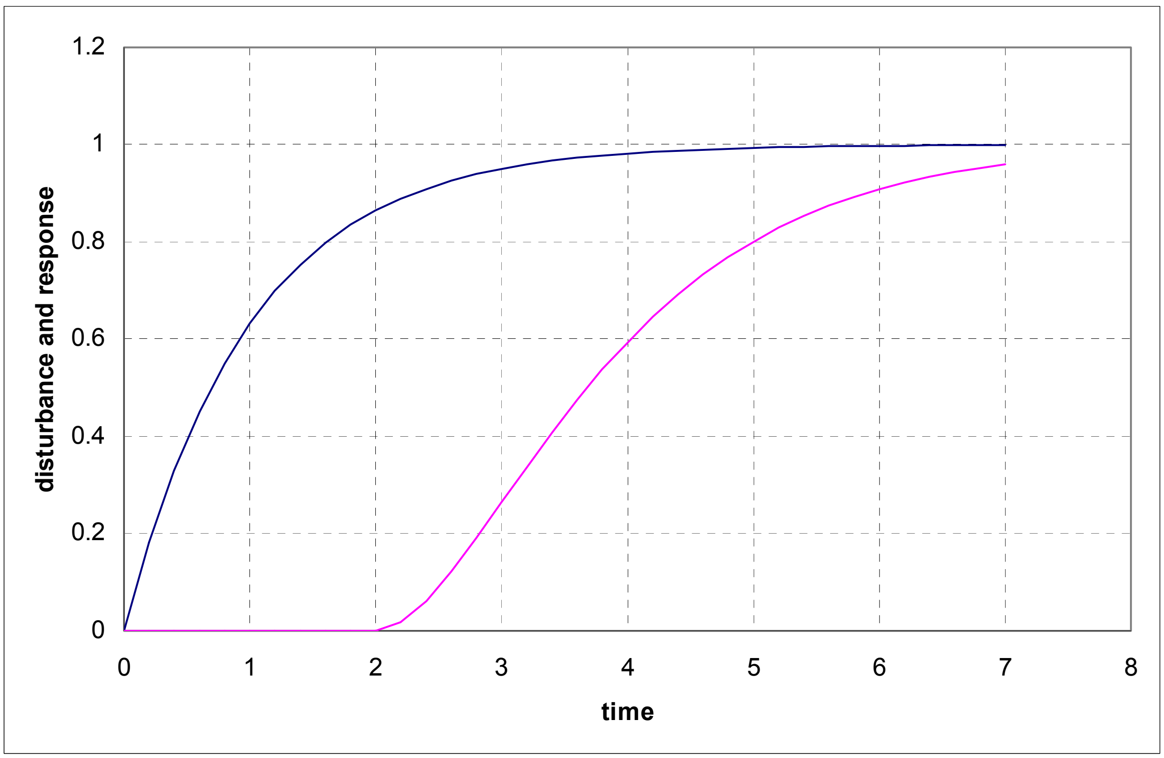
$$y_1(t) = 1 - e^{-t}$$

This function disturbs system 2. However, during the first 2 time units, there is no disturbance, due to the delay of y_1. Hence we may rewrite the second equation as

$$\frac{dy_2}{dt} + y_2 = \begin{cases} 0 & t < 2 \\ 1 - e^{-(t-2)} & t \geq 2 \end{cases}$$

Substituting into the general solution, we find

$$\begin{aligned} y_2(t) &= e^{-t}\int_2^t e^t\left(1 - e^{-(t-2)}\right)dt \\ &= 1 - e^{-(t-2)}\left(1 + (t-2)\right) \qquad t \geq 2 \end{aligned}$$



Because of the series arrangement of these two systems, we may view y_1 as an intermediate response to the step disturbance and y_2 as the output of the coupled system. The intermediate and output responses show different initial behaviors, but have the same ultimate fate.