Drift, Diffusion, and Dynamic Instability

In this problem set we explore the origin and characteristics of equations in the form

\[
\frac{\partial}{\partial t} p(x, t) = -\frac{\partial}{\partial x} (vp) + D \frac{\partial^2 p}{\partial x^2} - rxp .
\]

The first term on the right describes drift with velocity \( v \), the second is a consequence of diffusion with coefficient \( D \), while the final term arises from a dynamic instability that can occur at a rate \( r \) along any point along the interval \([0, x > 0]\).

1. Dispersion: Consider a slightly generalized model of microtubule growth and shrinkage [M. Dogterom and S. Leibler, Phys. Rev. Lett. 70, 1347 (1993).] described by the equations

\[
\begin{align*}
\partial_t p_+(x, t) &= -f_+ p_+ + f_- p_+ - \partial_z (v_+ p_+) + d \partial_x^2 p_+ \\
\partial_t p_-(x, t) &= +f_+ p_- - f_- p_- - \partial_z (v_- p_-) + d \partial_x^2 p_- .
\end{align*}
\]

(a) Such coupled linear equations are usually solved by first Fourier transforming to

\[
\tilde{p}(k, \omega) = \int dx dt e^{i(kx-\omega t)} p(x, t).
\]

Find the dispersion relations for allowed \( \omega(k) \).

(b) Expand the ‘slowly varying’ mode as \( \omega = vk - iDk^2 + \mathcal{O}(k^3) \), and hence obtain the dependence of the drift velocity and diffusion coefficient of the microtubule length on the parameters describing the growing and shrinking states.

(c) Typical values of parameters for microtubules growing in a tubulin solution of concentration \( c \approx 10 \mu M \) are \( v_+ \approx 2 \mu m/min, v_- \approx 20 \mu m/min, f_{+-} \approx 0.004 s^{-1}, f_{-+} \approx 0.05 s^{-1} \). Use these parameters (along with \( d = 0 \)) to estimate a time scale \( \tau \) beyond which diffusion effects are less important than the average drift. (Hence microtubules that have survived to a time \( \tau \) are unlikely to be completely eliminated by catastrophes.)

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2. Chemotaxis: The motion of \textit{E. Coli} in a solution of nutrients consists of an alternating sequence of runs and tumbles. During a run the bacterium proceeds along a straight line for a time \( t_r \) with a velocity \( v \). It then tumbles for a time \( t_t \), after which it randomly chooses a new direction \( \hat{n} \) to run along. Let us assume that the times \( t_r \) and \( t_t \) are independently selected from probability distributions

\[
p_r(t_r) = \frac{4t_r}{\tau^2_r} \exp \left(-\frac{2t_r}{\tau_r} \right) , \quad \text{and} \quad p_t(t_t) = \frac{4t_t}{\tau^2_t} \exp \left(-\frac{2t_t}{\tau_t} \right).
\]
(a) Assuming values of $\tau_r \approx 2s$, $\tau_t \approx 0.2s$, and $v \approx 30\mu\text{ms}^{-1}$, calculate the diffusion coefficient $D$ for the bacterium at long times.

(b) In the presence of a chemical gradient the run times become orientation dependent, and are longer when moving in a favorable direction. For preferred motion up the $z$ axis, let us assume that the average run time depends on its orientation $\hat{n}$ according to $\tau_r(\hat{n}) = \tau_0 + g\hat{n} \cdot \hat{z}$. Calculate the average drift velocity at long times.

3. Steady states: Consider the equation

$$\frac{\partial}{\partial t} p(x, t) = -\frac{\partial}{\partial x} (v(x)p) + D \frac{\partial^2 p}{\partial x^2} - rxp,$$

with a position dependent velocity.

(a) For $r = 0$ and $D \neq 0$, find a general expression for the steady state (time independent) solution $p_r^*(x)$, in terms of integrals involving $v(x)$.

(b) For $r \neq 0$ and $D = 0$, find a general expression for the steady state (time independent) solution $p_D^*(x)$, in terms of integrals involving $v(x)$.

(c) Find and sketch solutions for $v = v_0$ (constant), and $v = -ax$, and comment on their differences.