AN ANALYSIS OF PHASE DISTORTION IN AMPLITUDE LIMITERS

by

ALLAN CARTER SCHELL

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
at the
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
JUNE, 1956

Signature of Author . . . Signature Redacted
Department of Electrical Engineering, May 21, 1956

Certified by . . . Signature Redacted
Thesis Supervisor

Accepted by . . . Signature Redacted
Chairman, Departmental Committee on Graduate Students

EE
Thesis
1956

AN ANALYSIS OF PHASE DISTORTION IN AMPLITUDE LIMITERS

by

ALLAN CARTER SCHELL

Submitted to the Department of Electrical Engineering on May 21, 1956 in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

It has been found that amplitude limiters which contain storage elements convert some of the amplitude modulation present at the input into phase modulation at the output. In general, this effect is undesirable in frequency modulation systems.

The analysis of a simple limiter has been carried out in this thesis. The model chosen is a parallel GLC circuit with a clipper voltage limiter composed of ideal diodes and batteries. The input is considered to be the current fed into the device, and the output is the voltage across it. By means of Laplace transform techniques, representations are found for the circuit parameters, and a relation is developed for the phase deviation of the output voltage for a given amplitude modulation present at the input.

In order to determine the amplitude to phase conversion when the nonlinear part of the circuit does not consist of ideal elements, the analysis technique is changed from that of a transient nature to a Fourier decomposition of the voltage waveform. An analysis technique is developed to characterize and relate the current and voltage in the nonlinear element. Then an example is carried out in which voltage-current phase shift is determined in a circuit with a representative limiting element.

Thesis Supervisor: Samuel J. Mason
Title: Associate Professor of Electrical Engineering.

ACKNOWLEDGEMENT

Wohin es geht, wer weiss es?
Erinnert er sich doch kaum,
woher er kam!

\- *Egmont*

Before the gyres whirl apart and the academic career is over, it is perhaps proper to credit those who donated their time and ability to the preparation of this thesis.

The honor is dubious, for mention in a work often implies association with it. Nevertheless, this is the risk run by Messrs. J. B. Maggio and R. L. Madsen at the Bell Telephone Laboratories and Professor S. J. Mason at M.I.T. I would also like to express my thanks to Miss Virginia Lasse and Miss Muriel Morrow, who gave their time and patience to the typing of this work.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

INTRODUCTION AND STATEMENT OF THE PROBLEM

A major part of the development of electrical engineering has been the formulation and solution of linear problems. The application of the principle of superposition has permitted the utilization of various techniques, such as the introduction of the exponential frequency term and the complex transform, and these have paved the way for theoretical solutions to most if not all linear lumped passive structure properties.

The linear circuit is, however, an ideal model for a physical device, and the solutions obtained are useful only so long as a representation of a nonlinear element by its linear approximation is satisfactory. If the nonlinear properties are such that this is not reasonable, then it is necessary to resort to experimental trials or to various approximation techniques unless the problem is sufficiently simple to permit an exact solution.

In the past few years there has been progress toward the general solution of nonlinear problems. The work of Wiener permits the characterization of general nonlinear circuits for stochastic inputs, thus opening a world of analysis and synthesis heretofore barred from those thinking in terms of linear circuitry. However, there still remains the dilemma of solving nonlinear circuit problems for sinusoidal or transient inputs. Many approximation methods have been expounded, but few give useful results except in special cases or classes of problems. Until a method is found by which general nonlinear problems may be solved for arbitrary inputs, we must content ourselves with developing and extending the present techniques in order to gain some insight into each special case considered.

One such problem is that of the action of amplitude limiters, used in frequency modulation systems to remove unwanted amplitude modulation present on the frequency modulated carrier. Here the "noise" need not be considered stochastic; it is usually a combination of tones and signal which has been converted to amplitude modulation by nonlinearities within the system. The purpose of the system limiters is to remove this perversion of the signal, leaving a wave which contains only the wanted information.

However, it has been found experimentally that limiters used for this purpose have converted some of the unwanted amplitude modulation into phase modulation, thus corrupting the desired signal.

In order to gain some knowledge of the mechanism of this amplitude to phase conversion in limiters, Mr. R. L. Madsen of the Bell Telephone Laboratories began a study of static conversion. Using for a model a parallel GLC network with symmetrical clipper voltage limiter connected across it, he obtained relations for the phase difference between the zero crossings of the voltage waveform and those of the sinusoidal input current.[1]

---

[1]Raymond L. Madsen, Amplitude to Phase Conversion in Clipper Lim ters, Technical Memorandum 54-212-44, Bell Telephone Laboratories (1954).

This thesis is an extension of his work. First, the relations he obtained will be given, with some explanation as to their applicability. Then, a limiting case of the problem, namely an LC (non-dissipative) tank circuit with a switch type limiter, will be analyzed. This will be shown to be the severest case of amplitude to phase distortion and may be looked upon as an upper bound to the distortion from this effect. A few graphs of parameters will end Chapter I.

The analysis of the LC limiter will be developed by physical reasoning in Chapter II, and by means of suitable approximations, relations will be determined for the amount of phase shift in the first harmonic of the voltage for a given amplitude modulation of the input current.

The above work considered the action in two states: first, when the switch limiter was conducting and holding the voltage at a fixed value, and second, as a linear circuit when the switch was open. In Chapter III a different approach to the problem will be taken; that of a continuously acting nonlinear device. The limiting action will be represented by considering a circuit element in which the voltage is proportional to the logarithm of the current. From this an analysis technique will be developed which is exact for single frequency periodic inputs, and a generalization for arbitrary nonlinear dissipative functions will be mentioned.

In Chapter IV an approximation for arbitrary periodic inputs to a general nonlinear dissipative device with linear active elements will be given, and, using the techniques of

Chapter III, a solution will be obtained in terms of Bessel functions of a complex argument, for which tables are used to obtain an answer to the problem. As may well be imagined, this technique is subject to strong limitations, and the error grows as the device to be characterized becomes more nonlinear or as its signals contain more frequency components. Some of the limits of the technique will be given in terms of error curves. However, for the limiter problems, this method will be of some worth. The procedures of Chapters I and II are the more useful when the device is in a limited state for the greater part of the signal period. However, they become very complex when the circuit spends over one-fourth of its time in a linear state. The techniques of Chapters III and IV are of a greater use when the circuit approaches a linear manner of operation, but it is not sufficiently close to allow any linear approximations.

CHAPTER I

STATIC AMPLITUDE TO PHASE CONVERSION

In his technical memorandum R. L. Madsen has made a static analysis of the amplitude to phase conversion problem. Using a parallel GLC network with a clipper voltage limiter, he has found a relation for the voltage across the circuit when a sinusoidal current is applied. Figure 1 shows the circuit and typical input and output waveforms.

Applying Laplace transform techniques to the equation

$$i(t) = I_o \sin(\omega t+\theta) = \frac{1}{L}\int_{-\infty}^{t} V(t)dt + C\frac{dV(t)}{dt} + GV(t) \qquad (1)$$

from the time $t=0$ until $t=t_1$ (while the diodes are not conducting) he obtained the relation

$$\begin{aligned} V_n(t) = \frac{V}{I_oR} &= W\sin \omega t + X\cos \omega t \\ &+\frac{(Y+1/m)}{(4Q^2-1)^{\frac{1}{2}}} e^{-\frac{\omega t}{2QA}} \sin(4Q^2-1)^{\frac{1}{2}}\frac{\omega t}{2QA} \\ &+(1/m -X)\, e^{-\frac{\omega t}{2QA}}\cos(4Q^2-1)^{\frac{1}{2}}\frac{\omega t}{2QA} \end{aligned} \qquad (2)$$

where

$$X = \frac{A\sin\theta\ [A+Q(1-A^2)\cot\theta]}{[Q^2(1-A^2)^2+ A^2]}$$

$$W = \frac{A\sin\theta\ [A\cot\theta -Q(1-A^2)]}{[Q^2(1-A^2)^2+ A^2]}$$

$$Y = \frac{A\sin\theta\ [2Q^2A(1-A^2)-A-Q(1+A^2)\cot\theta]}{[Q^2(1-A^2)^2 +A^2]}$$

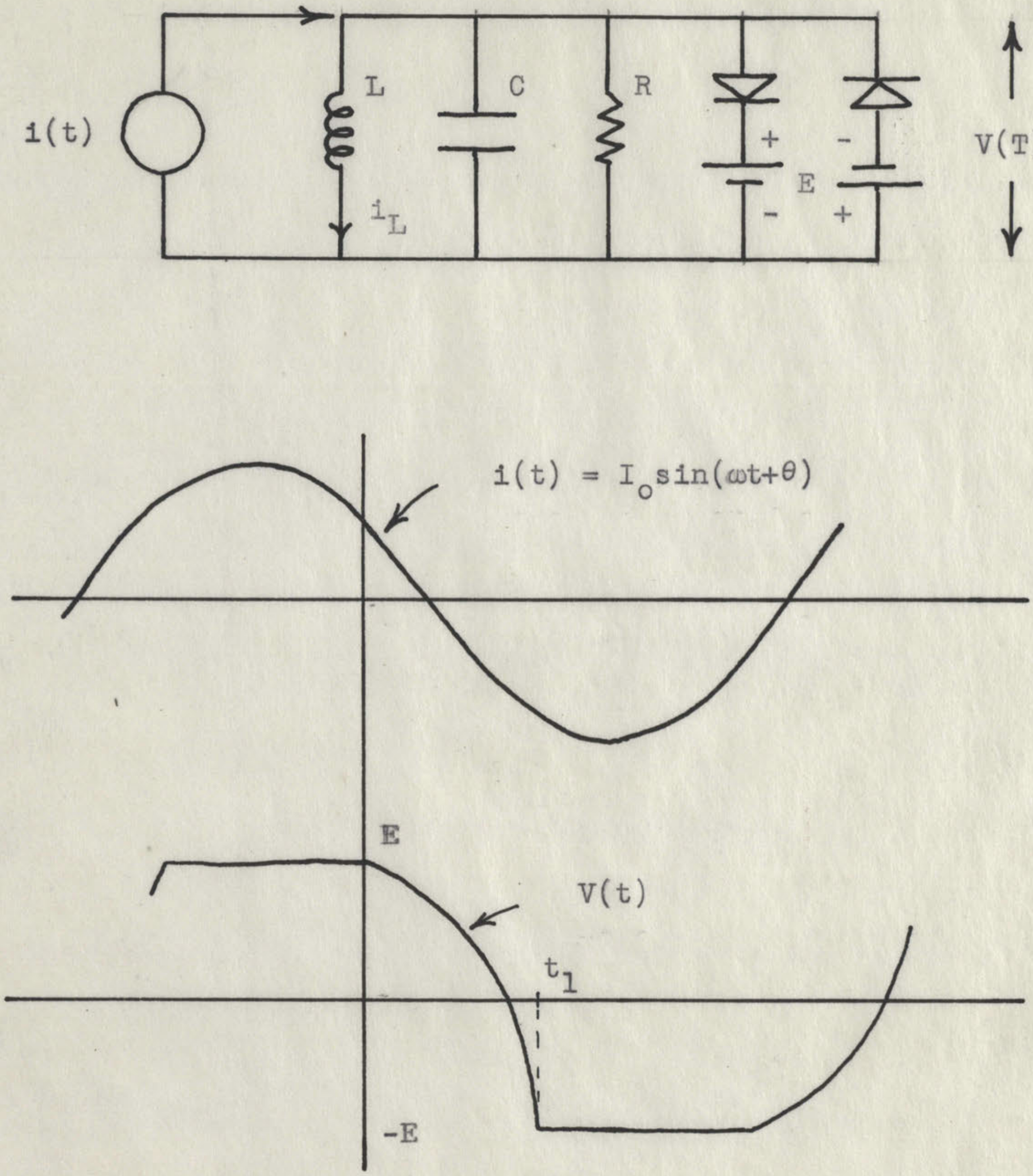


FIGURE 1

THE LIMITER CIRCUIT

and $\frac{1}{m} = \frac{E}{I_o R}$, $Q = R(C/L)^{1/2}$, $A = \omega(LC)^{1/2}$.

He then used the two breakpoints of the diodes to find θ, the current phase angle, and ωt_1, the angle during each half cycle that the voltage is not limited.

At t_1

$$\frac{V(t_1)}{I_o R} = \frac{-E}{I_o R} = -\frac{1}{m} \quad , \qquad (3)$$

and at $\omega t = \pi$

$$-\frac{E}{R} + I_o \sin(\pi+\theta) = \frac{1}{L}\int_0^{t_1} V(t)dt + I_1(0) - \frac{E}{\omega L}(\pi - \omega t_1)$$

$$-2I_o \sin\theta - 2\frac{I_o}{m} = \frac{1}{L}\int_0^{t_1} V(t)dt - \frac{QI_o}{mA}(\pi - \omega t_1) \quad .$$

Integrating V(t) we find

$$\frac{1}{I_o R}\int_0^{t_1} V(t)dt = \frac{W}{\omega}(1-\cos \omega t_1) + \frac{X}{\omega}\sin \omega t_1$$

$$+\frac{A[-Y-X(4Q^2-1)+1/m\ (4Q^2-2)]}{2\omega Q(4Q^2-1)^{1/2}} e^{-\frac{\omega t_1}{2QA}} \sin(4Q^2-1)^{\frac{1}{2}}\frac{\omega t_1}{2QA}$$

$$-\frac{Y+2/m\ -X}{2\omega Q} A e^{-\frac{\omega t_1}{2QA}} \cos(4Q^2-1)^{\frac{1}{2}} \frac{\omega t_1}{2QA}$$

$$+\frac{A[Y+2/m\ -X]}{2\omega Q}$$

and from this we gain the relation

$$-2\sin\theta - \frac{2}{m} = \frac{QW}{A}(1-\cos \omega t_1) + \frac{QX}{A}\sin \omega t_1$$

$$+\frac{[-Y-X(4Q^2-1)+1/m\ (4Q^2-2)]}{2(4Q^2-1)^{1/2}} e^{\frac{\omega t_1}{2AQ}} \sin(4Q^2-1)^{\frac{1}{2}}\frac{\omega t_1}{2QA}$$

$$-\frac{[Y+2/m\ -X]}{2} e^{-\frac{\omega t_1}{2QA}} \cos(4Q^2-1)^{\frac{1}{2}}\frac{\omega t_1}{2QA} \qquad (4)$$

$$+\frac{Y+2/m\ -X]}{2} \quad -\frac{Q}{mA}(\pi - \omega t_1)$$

Using the other condition (3) we have

$$-\frac{1}{m} = W\sin \omega t_1 + X\cos \omega t_1 + \frac{(Y+1/m)}{(4Q^2-1)} e^{-\frac{\omega t_1}{2QA}} \sin(4Q^2-1)^{\frac{1}{2}} \frac{\omega t_1}{2QA}$$

$$+(1/m - X)\, e^{-\frac{\omega t_1}{2QA}} \cos(4Q^2-1)^{\frac{1}{2}} \frac{\omega t_1}{2QA} \quad . \qquad (5)$$

Using relations (4) and (5), values for θ and ωt_1 may be determined for any given circuit and input sinusoid. These, coupled with equation (2), uniquely determine the voltage across the circuit. As may readily be seen, these relations are very difficult to use. However, under certain conditions they are simplified somewhat. If it is assumed:

(1) The driving frequency equals the LC resonant frequency, and

(2) The Q of the linear part of the circuit is high,

then

$$A = 1 \quad , \quad (4Q^2-1)^{1/2} \approx 2Q \quad , \quad e^{-\frac{\omega t}{2QA}} \approx 1 - \frac{\omega t}{2QA}$$

and the relations become

$$-\sin\theta - \frac{3}{m} = \frac{1}{2}\cos\theta \sin \omega t_1 + \sin\theta \cos \omega t_1$$

$$-\frac{1}{2}\omega t_1 \cos(\omega t_1 + \theta) - \frac{Q}{m}(\pi - \omega t_1) \qquad (6)$$

$$+\frac{Q}{m}\sin \omega t_1 - \frac{\omega t_1}{2m} \sin \omega t_1 - \frac{1}{m}\cos \omega t_1$$

and

$$-\frac{1}{m} = \frac{1}{m}\cos \omega t_1 + \frac{\omega t_1}{2Q} \sin(\omega t_1 + \theta) \qquad (7)$$

$$-\frac{\omega t_1}{2Qm}\cos \omega t_1 + \frac{1}{2Qm}\sin \omega t_1 - \frac{1}{2Q}\sin\theta \sin \omega t_1$$

By using the property $\frac{Q}{m} = R(C/L)^{1/2}\frac{E}{I_oR} = \frac{E}{I_o}(C/L)^{1/2}$

we may rewrite these equations in a more useful form:

$$-I_o\sin\theta - \frac{3E}{R} = \frac{I_o}{2}\cos\theta\sin\omega t_1 + I_o\sin\theta\cos\omega t_1$$
$$-\frac{I_o}{2}\omega t_1\cos(\omega t_1+\theta) + E(C/L)^{1/2}\sin\omega t_1 \qquad (8)$$
$$-\frac{E\omega t_1}{2R}\sin\omega t_1 - \frac{E}{R}\cos\omega t_1 - E(C/L)^{1/2}(\pi-\omega t_1)$$

and

$$-E = E\cos\omega t_1 + I_o(L/C)^{1/2}\frac{\omega t_1}{2}\sin(\omega t_1+\theta)$$
$$-\frac{I_o}{2}(L/C)^{1/2}\sin\theta\sin\omega t_1 + \frac{E}{2R}(L/C)^{1/2}\sin\omega t_1 \qquad (9)$$
$$-\frac{E}{2R}(L/C)^{1/2}\omega t_1\cos\omega t_1 \quad .$$

Although the above relations are rather involved, they represent a very close approximation to the values in the actual circuit and lend themselves readily to calculations.

A typical waveform may be determined by using relations (2), (8), and(9). Choosing a representative circuit with the following parameters:

$$\left.\begin{aligned} L &= 40\ \mu\text{henrys} \\ C &= 1000\ \mu\mu\text{farads} \qquad Q = 100 \\ R &= 20{,}000\ \text{ohms} \end{aligned}\right\} \qquad (10)$$

$$i(t) = 30\sin(\omega t+\theta)\ \text{milliamperes}$$
$$\omega = 1/(LC)^{\frac{1}{2}} = 5\text{x}10^6\ \text{rad./sec.}$$
$$E = 1\ \text{volt} \quad ,$$

it may be found by substitution

$$\omega t_1 = \frac{\pi}{4}\ \text{radians} = 45°$$
$$\theta = 180° - 10.4° = 169.6° \quad .$$

This means that the voltage requires 45 degrees during each half cycle to pass between the extremes of plus and minus one volt. Also, it begins the transition between the two states at a point 10.4 degrees before the input current passes through zero. A graph of the voltage for this case is shown in Figure 2.

It is of interest to determine the effect of decreasing the resistance in the above example. Intuition would lead one to believe that this would have the effect of making the voltage appear more nearly sinusoidal and in phase with the input current. This is indeed the case. As the resistance is decreased, the effect of the active elements upon the voltage is lessened, and the amount of current passing through the diodes is decreased. Thus the transit angle between limiting states, ωt_1, is increased, and the current phase angle, θ, is decreased, bringing the two zero crossings closer together.

A graph of the effect of resistance upon ωt_1 and θ for the circuit described by the relations (10) is given in Figure 3, in which the aforementioned effects are shown. The importance of this graph, however, is the illustration of the very slight change of the two quantities for a wide range of circuit Q. The change in ωt_1 is so small as to be negligible compared to its approximate value of 45°. Thus, for calculations in a limiter circuit with an appreciable unloaded Q, it is unnecessary to consider the effect of the resistance. The factors which have the greatest effect on the angles are E/I_o and (L/C).

Since the resistance has slight effect upon voltage calculations, and the effect that it does exert is to lessen the

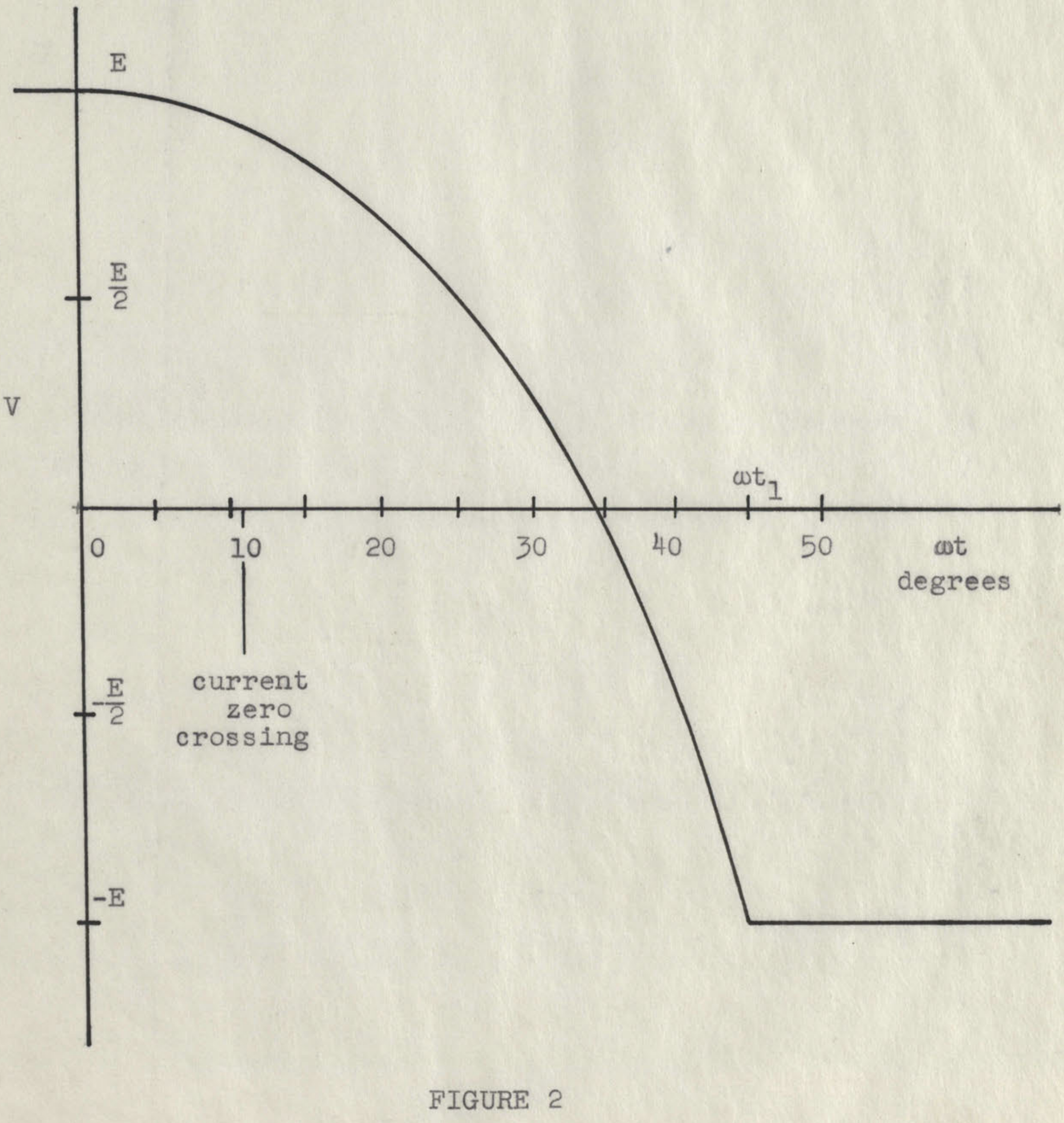


FIGURE 2

A TYPICAL VOLTAGE WAVEFORM

INCREMENTAL CHANGE IN TRANSIT AND CURRENT PHASE

ANGLES VERSUS CONDUCTANCE

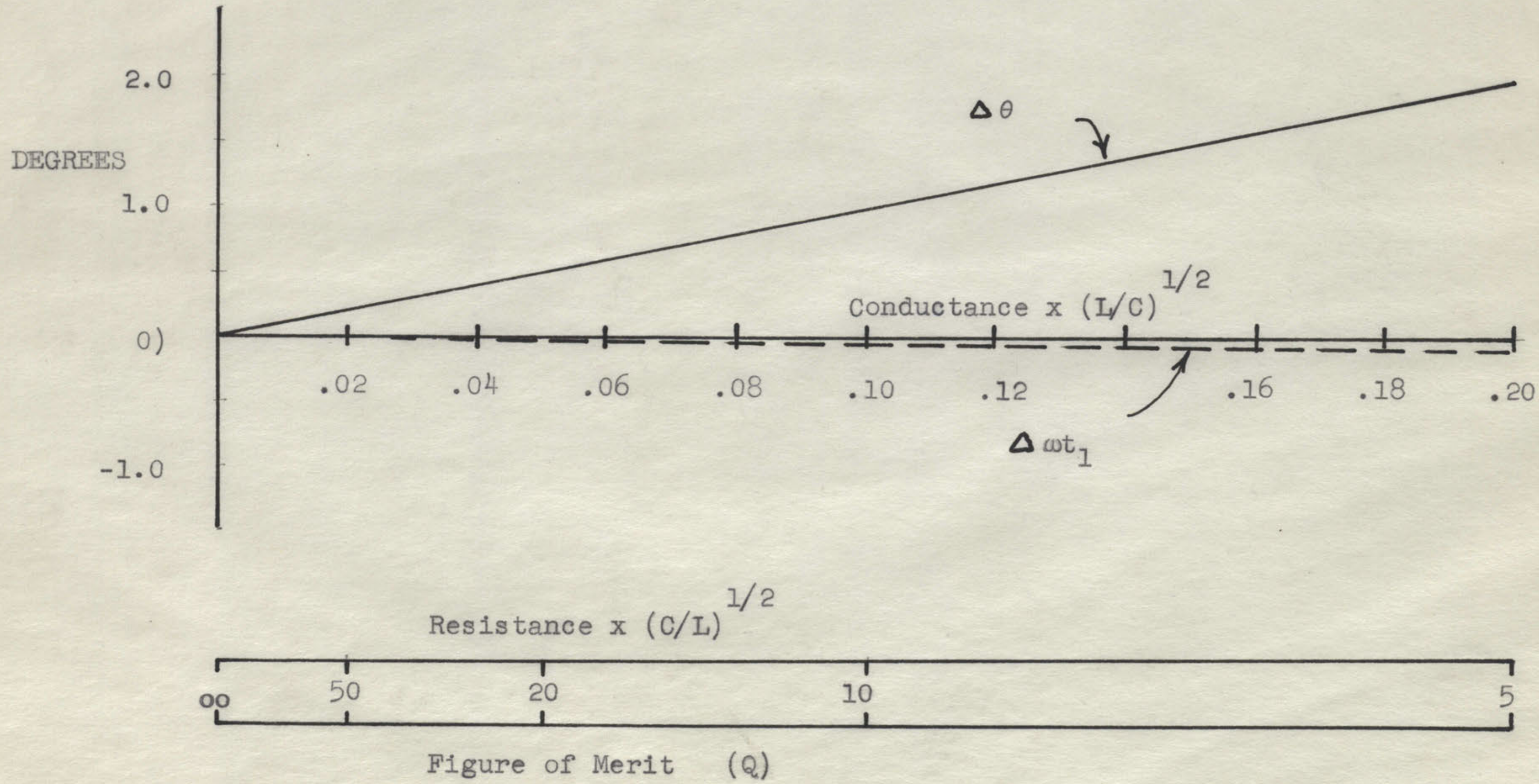


FIGURE 3

discrepancy between current and voltage zero crossings, one may consider the lossless LC circuit coupled to a symmetrical ideal diode limiter as the limiting case when determining voltage parameters. Under the condition $R=\infty$, equations (8) and (9) become

$$-I_o \sin\theta = \frac{I_o}{2}\cos\theta \sin\omega t_1 + I_o \sin\theta \cos\omega t_1 - \frac{I_o}{2}\omega t_1 \cos(\omega t_1+\theta) + E(C/L)^{1/2}\sin\omega t_1 + E(C/L)^{1/2}(\pi-\omega t_1) \qquad (11)$$

and

$$E = E\cos\omega t_1 + I_o(L/C)^{1/2}\frac{\omega t_1}{2}\sin(\omega t_1+\theta) - \frac{I_o}{2}(L/C)^{1/2}\sin\theta \sin\omega t_1 \quad . \qquad (12)$$

To obtain an estimate of the range of values of ωt_1 and θ several values have been determined by direct substitution. Curves of ωt_1 and θ have been plotted as a function of the ratio $\frac{E}{I_o}(C/L)^{1/2}$ and are given in Figure 4. The values of milliamperes given are for the input current to the circuit described by the relations (10).

As might be expected, the difference between zero crossings and between limiting states decreases as the ratio of $\frac{I_o}{E}(L/C)^{1/2}$ is increased It is the case of small transit angles that will be considered in Chapter II.

A PLOT OF TRANSIT AND PHASE ANGLES VERSUS LIMITING RATIO

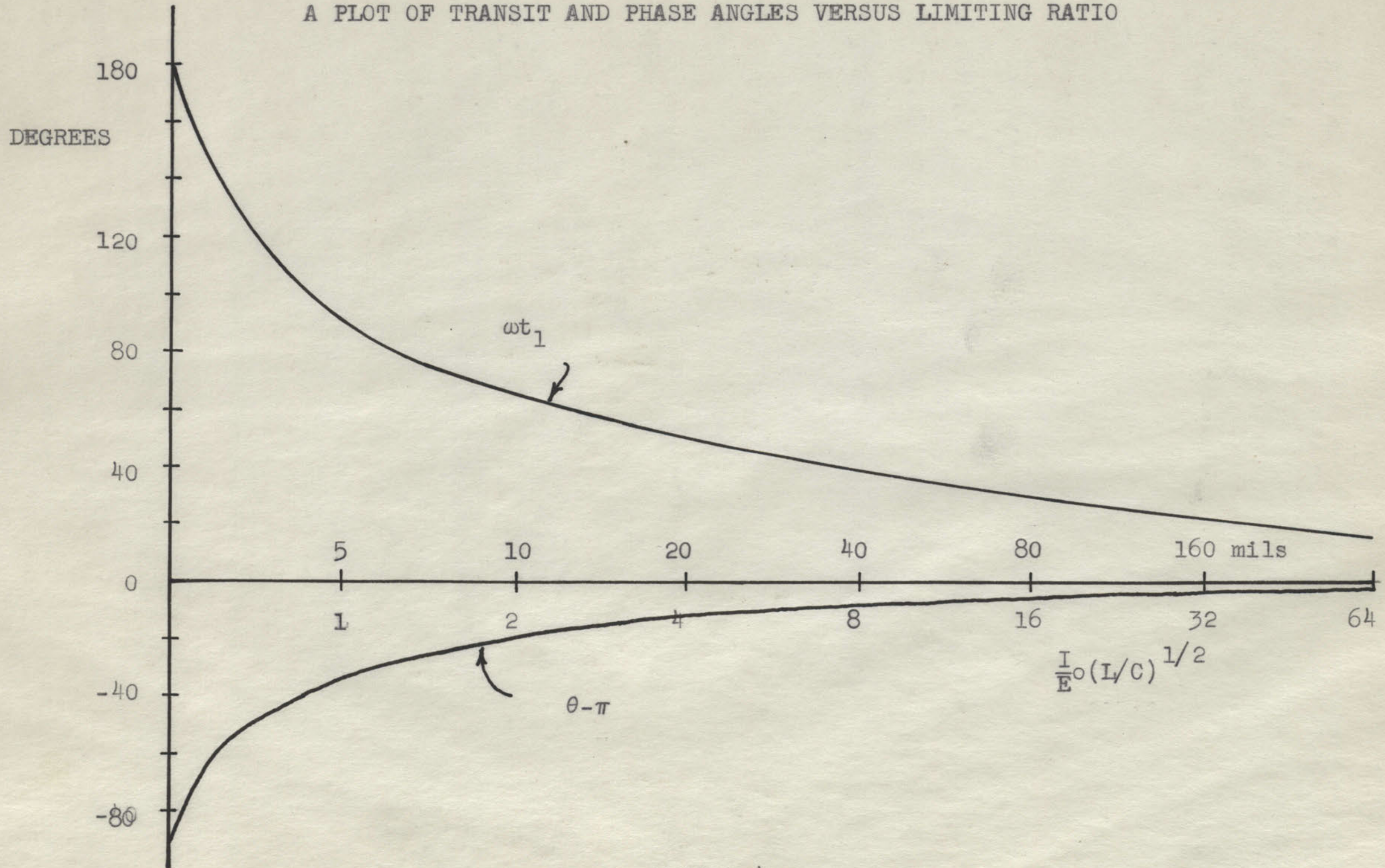


FIGURE 4

CHAPTER II

DYNAMIC AMPLITUDE TO PHASE CONVERSION

The results obtained from the exact expressions determined by Laplace transform techniques are quite involved, and any attempt to use them to find fundamental harmonic phase shift is doomed to derangement. However, the circuit equations may be solved approximately by physical reasoning and by applying the results of Chapter I.

It may be noticed that the waveform of Figure 2 between $t=0$ and $t=t_1$ resembles a parabola which is determined by

$$v(0) = E$$

$$\frac{d\,v(0)}{dt} = 0$$

$$v(t_1) = -E$$

The approximate equation for the voltage between $t = 0$ and $t = t_1$ is, therefore

$$v(t) = E - 2\frac{t^2}{t_1^{\,2}}E \qquad 0 \leq t \leq t_1 \tag{13}$$

The zero crossing of this is

$$t_o = \frac{1}{\sqrt{2}}t_1$$

which, for the example of Figure 2 is

$$\omega t_o \equiv \frac{\pi}{4\sqrt{2}} = 31.8°$$

The actual zero crossing is about 34 degrees; thus this rough approximation has caused nearly 7 percent error in this case. Later in this chapter a relation for the error caused by this approximation will be given.

Now let us reconsider the exact expression for the voltage given by equation (2). With the conditions $R = \infty$, $\omega = \frac{1}{\sqrt{LC}}$, the relation becomes greatly simplified:

$$V(t) = \frac{I_o}{2}(L/C)^{1/2}\omega t \sin(\omega t+\theta) - \frac{I_o}{2}(L/C)^{1/2}\sin\theta \sin \omega t + E\cos \omega t \qquad (14)$$

We shall now approximate $\sin \omega t$ by ωt and $\cos \omega t$ by $1-\frac{(\omega t)^2}{2}$. A similar approximation will be used for $\sin \theta$ and $\cos \theta$, and any terms of order higher than two will be neglected. Then equation (14) becomes

$$V(t) \cong E - \frac{(\omega t)^2}{2} \, [E+I_o(L/C)^{1/2}] \qquad 0 \leq t \leq t_1$$

Comparing this with equation (13), we find

$$\frac{E}{t_1^2} \cong \frac{\omega^2 E}{4} + \frac{\omega^2 I_o}{4}(L/C)^{1/2}$$

$$\omega t_1 \cong 2\left[\frac{1}{1+\frac{I_o}{E}(L/C)^{1/2}}\right]^{1/2} \simeq 2\left[\frac{E}{I_o}(C/L)^{1/2}\right]^{1/2} \qquad (16)$$

One other relation is needed in order tto determine θ and ωt_1. Let us examine the circuit and waveforms of Figure 5, which represent the quantities now being considered. With the above approximate relation for the voltage, the inductor current $i_L = \frac{1}{L}\int v dt$ may be written, and the symmetry of the limiters, which requires that $v(t+\frac{\pi}{\omega}) = -v(t)$, enables us to determine the zero crossings of $i_L(t)$. Thus we may obtain

$$i_L(t) = -\frac{2Et_1}{3L} + \frac{E\pi}{2\omega L} + \frac{Et}{L} + \frac{2t^3 E}{3t_1^2 L} \qquad 0 \leq t \leq t_1$$

$$i_L(t) = -\frac{Et_1}{3L} + \frac{E\pi}{2\omega L} - \frac{E}{\omega L}(\pi - \omega t_1) \qquad t_1 < t \leq \frac{\pi}{\omega}$$

Writing the Kirchoff current relations for the time t =0,

$$i(0) = I_o \sin\theta = i_L(0)$$

$$I_o \sin\theta \cong -\frac{2Et_1}{3L} + \frac{E\pi}{2\omega L}$$

$$\sin\theta \cong \frac{E}{I_o}(C/L)^{1/2}\left[\frac{\pi}{2} - \frac{2\omega t_1}{3}\right] \tag{17}$$

Substituting equation (16) into (17),

$$\sin\theta \cong \frac{E}{I_o}(C/L)^{1/2}\left[\frac{\pi}{2} - \frac{4}{3}\left(\frac{E}{I_o}\sqrt{\frac{C}{L}}\right)^{1/2}\right] \tag{18}$$

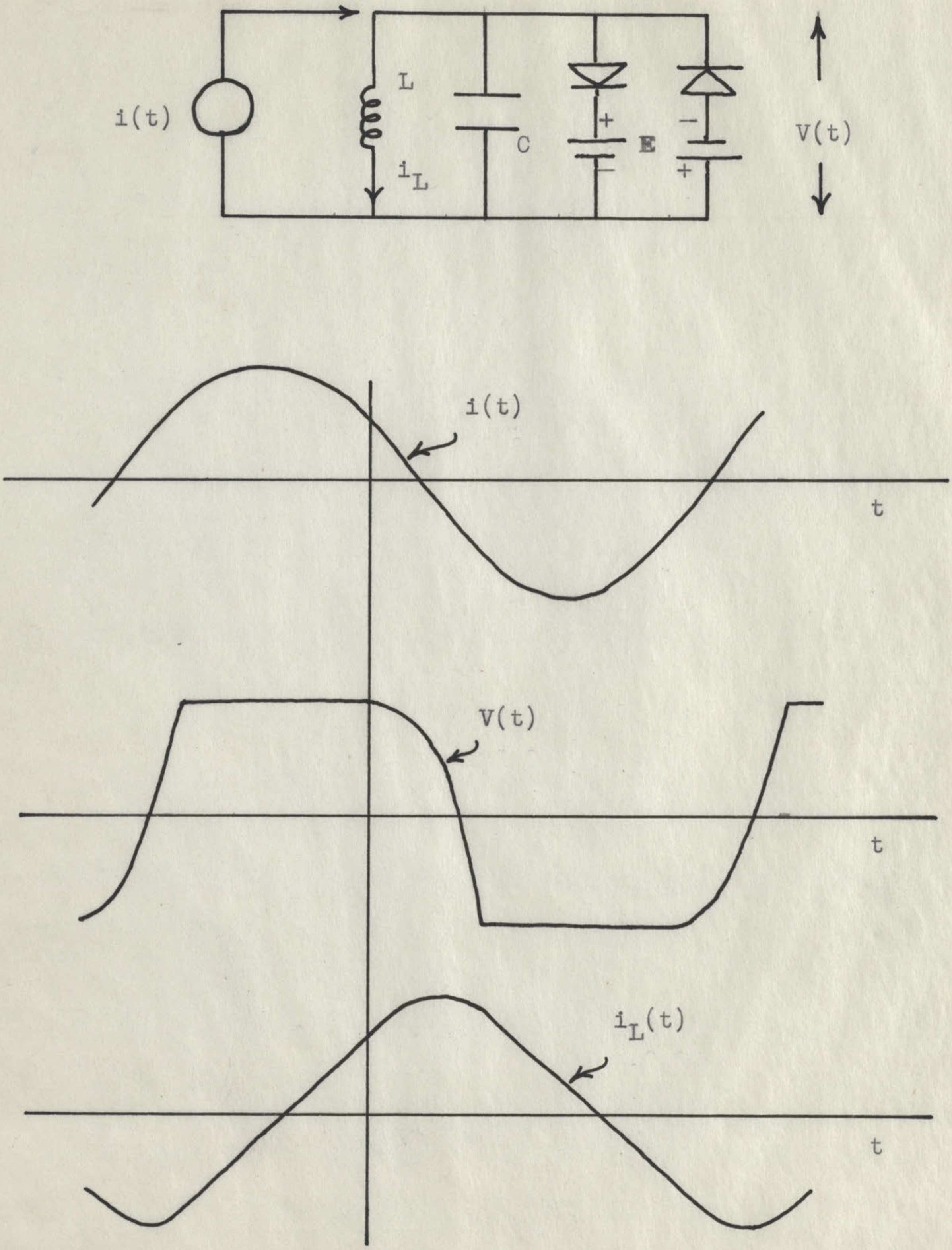
The two relations (16) and (18) yield a convenient method for calculating the parameters of a given circuit and input. They are, however, useful only for small values of θ and ωt_1. By using the more precise values of θ and ωt_1 obtained in Chapter I (see Figure 4) the error caused by the approximation may be found. This has been done and is shown in Figure 6. The current values on the $\frac{E}{I_o}(C/L)^{1/2}$ axis are for the input current maximum, I_o, of the circuit described by the relations (10).

With the above equations for θ and ωt_1 the fundamental frequency components of the voltage may be determined. The amplitude of the fundamental component of the voltage that is in phase with the input current is given by

$$V_{1S} = \frac{\omega}{\pi}\int_0^{2\pi/\omega} V(t)\sin(\omega t + \theta)\,dt$$

$$V_{1S} = \frac{2\omega}{\pi}\int_0^{t_1} V(t)\sin(\omega t+\theta)\,dt + \frac{2\omega}{\pi}\int_{t_1}^{\pi/\omega}(-E)\sin(\omega t+\theta)\,dt \tag{19}$$

FIGURE 5



THE LC CLIPPER LIMITER

TRANSIT AND CURRENT PHASE ANGLES VERSUS LIMITING RATIO

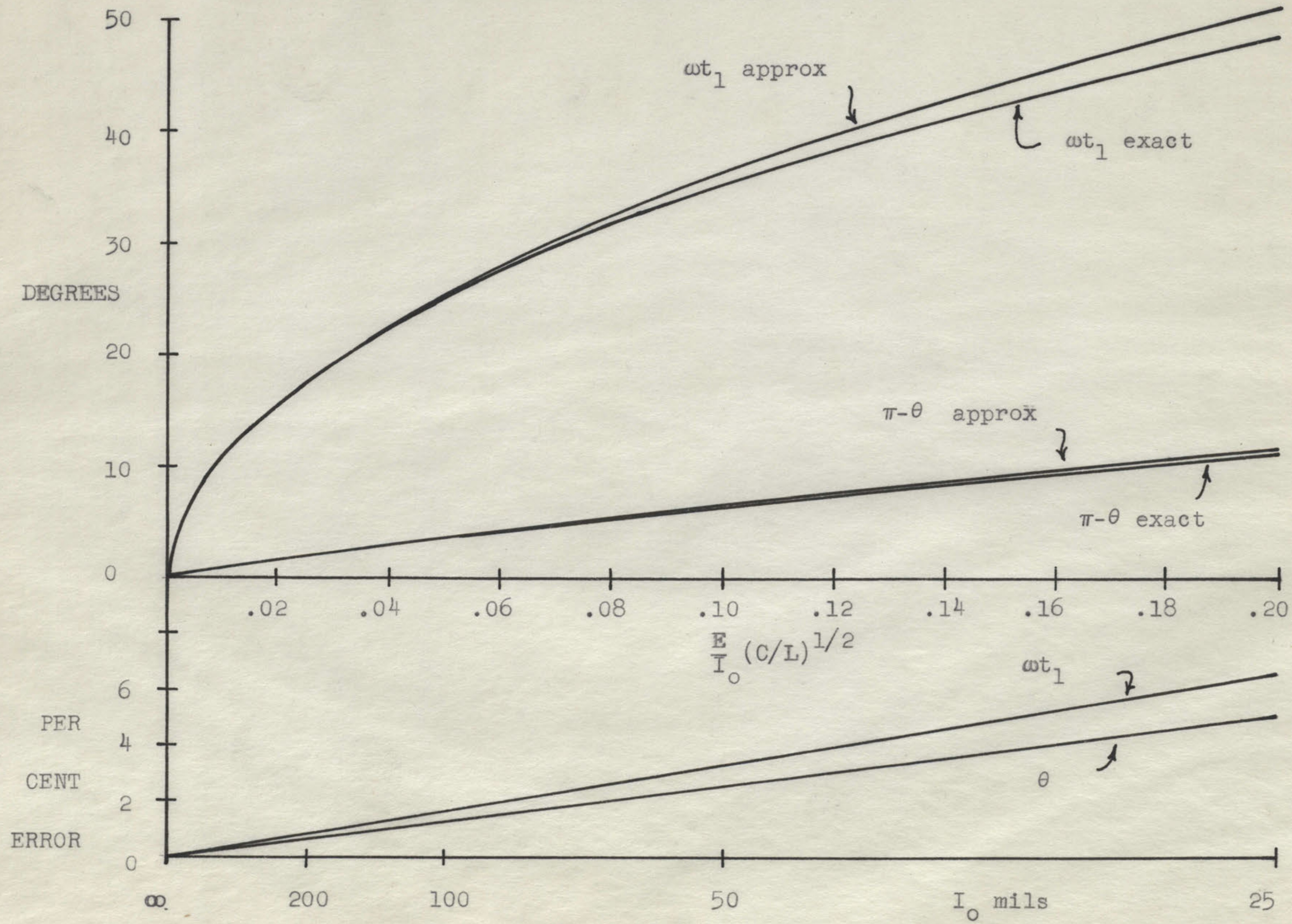


FIGURE 6

Introducing the notation

$$\frac{E}{I_o}(C/L)^{1/2} \triangleq a^2 \qquad (20)$$

then

$$\omega t_1 \cong 2a$$

and

$$\sin\theta \cong a^2[\frac{\pi}{2} - \frac{4}{3}a]$$

and the in phase component is found to be

$$V_{1S} \cong \frac{4E}{\pi}[1-a^2 +2.1a^3 -3.12a^4] \qquad (21)$$

where terms of order greater than a^4 have been neglected.

Similarly, the out of phase fundamental component of V may be determined from the relation

$$V_{1C} = \frac{\omega}{\pi}\int_0^{2\pi/\omega} V(t)\cos(\omega t+\theta)dt$$

yielding

$$V_{1C} \cong \frac{4E}{\pi}a[-1.33 +1.57a -.82a^2 +.25a^4] \qquad (22)$$

The first harmonic of the voltage may be represented as

$$V_1 = V_{1M}\sin(\omega t+\theta-\Phi)$$

where

$$V_{1S} = V_{1M}\cos\Phi$$

and

$$V_{1C} = -V_{1M}\sin\Phi$$

Thus

$$\tan\Phi = -\frac{V_{1C}}{V_{1S}} \cong \frac{a[1.33 -1.57a +.82a^2 -.25a^4]}{[\ 1 - a^2 + 2.1a^3 -3.12a^4\ ]} \qquad (23)$$

where Φ is the phase angle between the current and the voltage fundamental. Notice that for any small value of a the voltage lags the current.

This equation enables one to plot the voltage and current phase difference for values of the limiting ratio $\frac{E}{I_o}(C/L)^{1/2} \triangleq a^2$.

This has been done in Figure 7. There is also on the graph a plot of the phase difference of the zero crossings of the current and the complex voltage waveshape. As might be expected for the case of small a , the difference of the two curves is slight.

It is now possible to develop an approximate relation for dynamic amplitude to phase conversion. Throughout this and the preceding development it has been necessary to assume small transition angles (ωt_1) in order to avoid equations of staggering complexity. The error curve of Figure 6 is a good indication of representative errors in calculations. Now it becomes necessary to further approximate equation (23) by

$$\tan \phi \cong \phi \cong \frac{a[1.33 - 1.57a + .82a^2]}{[1 - a^2]} \qquad (24)$$

The error caused by this is about 5 percent at $\omega t_1 = \frac{\pi}{4}$ and considerably less as ωt_1 becomes smaller.

If the input current is amplitude modulated by a wave $m\cos \omega_m t$ then the equations involving

$$i(t) = I_o \sin(\omega t + \theta)$$

must be modified to account for

$$i(t) = I_o \sin(\omega t + \theta) = I_o(1 + m\cos \omega_m t)\sin(\omega t + \theta)$$

The operations which are not exact are $\frac{di(t)}{dt}$ and $\int i(t)dt$. Considering the derivative as an example, we find

$$\frac{di(t)}{dt} = \omega I_o(1 + m\cos\omega_m t)\cos(\omega t + \theta) - \omega_m I_o m \sin\omega_m t \sin(\omega t + \theta)$$

and if

$$\frac{\omega_m}{\omega} << 1$$

we may use

THE PHASE DIFFERENCE BETWEEN CURRENT AND VOLTAGE

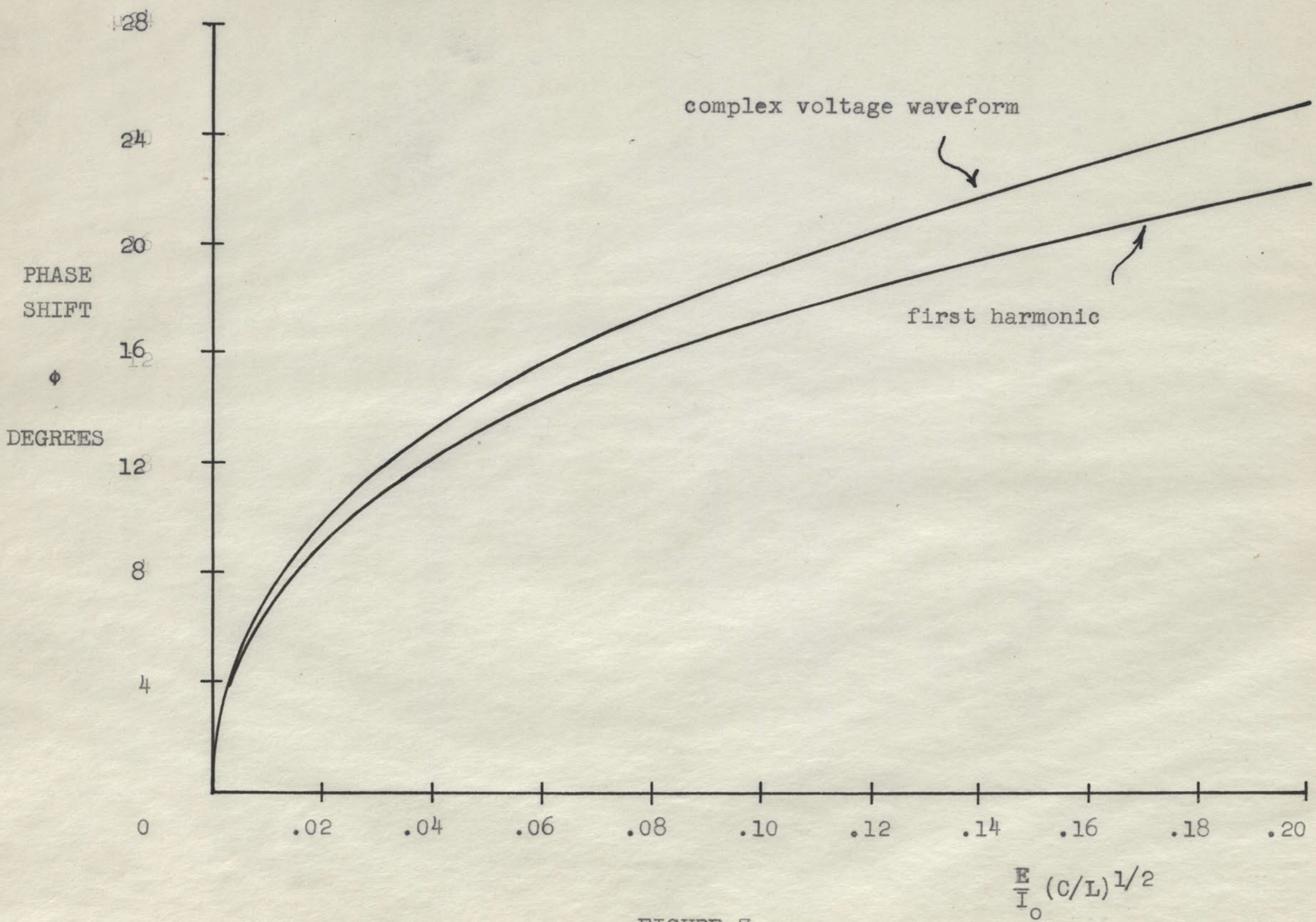


FIGURE 7

we may use

$$\frac{di(t)}{dt} = I_o(t)\frac{d}{dt}\sin(\omega t+\theta)$$

and

$$\int i(t)dt = I_o(t)\int \sin(\omega t+\theta)$$

With the above statement and equation (24) an approximate expression may be determined for the phase deviation of the voltage fundamental harmonic component. The phase deviation is a complicated function of time, but the higher frequency harmonics drop off rapidly as m decreases. Thus the first harmonic of the phase deviation for $m < \frac{1}{2}$ is found to be

$$\phi \cong \frac{a[-1.33 + 3.14a - 3.79a^2]}{2[1 - a^2]} m\cos\omega_m t \qquad (25)$$

The phase modulated wave is often represented as

$$v(t) = V_c\sin(\omega t + k_p m\cos\omega_m t)$$

where k_p is the phase deviation ratio. For the limiter,

$$k_p \cong \frac{a[-1.33 + 3.14a - 3.79a^2]}{2[1 - a^2]} \qquad (26)$$

A plot of k_p as a function of a for small values of amplitude modulation ($m < \frac{1}{2}$) is shown in Figure 8.

THE PHASE DEVIATION FUNCTION

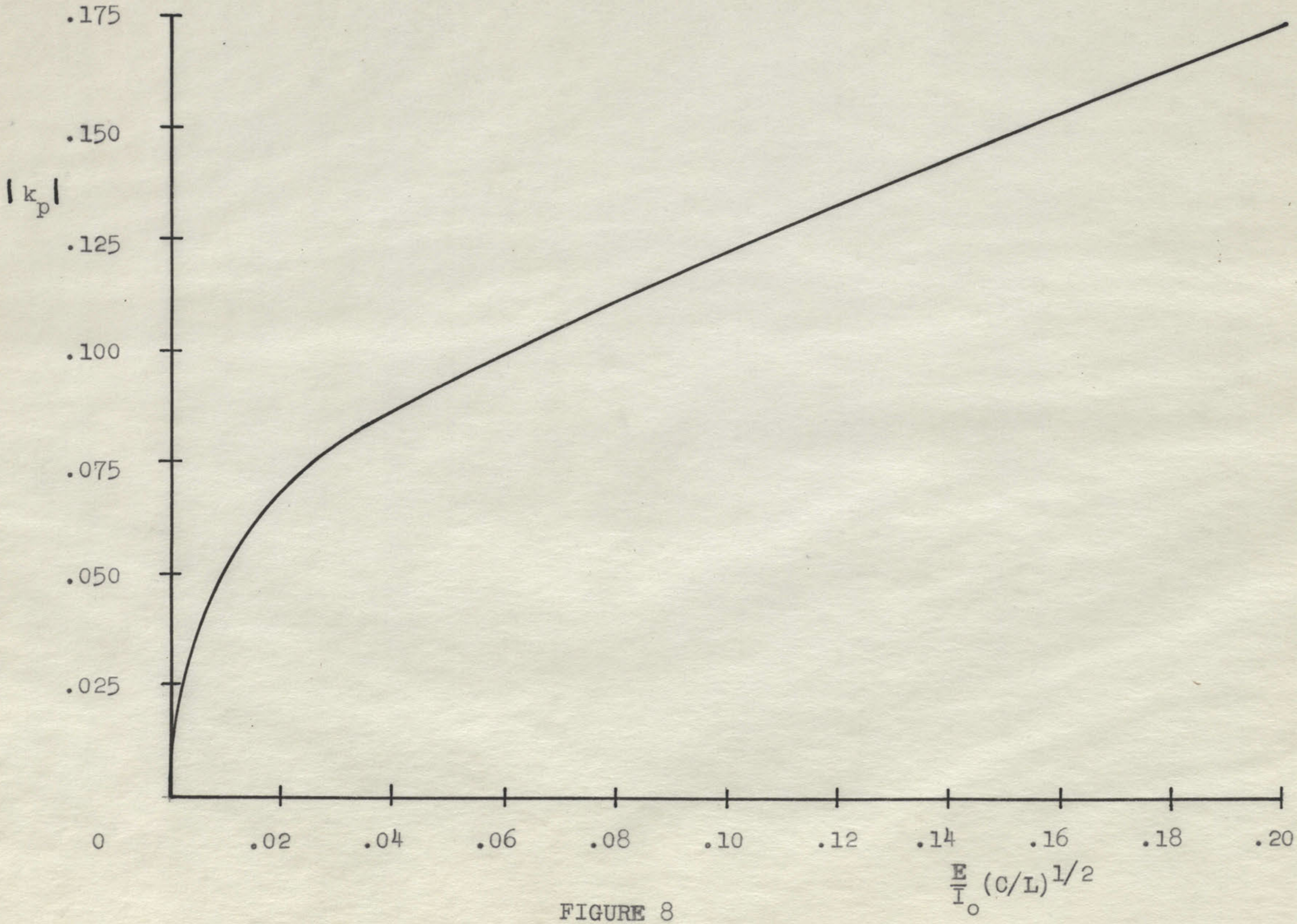


FIGURE 8

CHAPTER III

A NONLINEAR ANALYSIS TECHNIQUE

In the preceding part of this thesis it was necessary to assume large limiting ratios of $\frac{I}{E}$o in order to obtain simple expressions for the voltage parameters. However, this was not due to any radical behavior on the part of the voltage; rather, it was a consequence of the mathematical technique. The clipper or switch type limiter and the Laplace transform method, while yielding perfectly good expressions, completely obscured their meaning.

This is but one of the failings of the switch type limiter as a representation of a typical circuit. The other serious drawback is the assumption of ideal diodes. For any physical structure used for fairly small limiting ratios the nonlinear element is a smooth curve with a finite conductance at all points.

The need, then, is for a method which will permit an accurate representation of actual circuit parameters and enable one to determine these without resorting to overly involved computation.

To this end, Figure 1 may again be examined. To better represent this circuit, it is necessary to find a suitable relation for the diode current and voltage, and this has been done by Shockley[2]

$$i_d = I_s[\, e^{\frac{qv}{kT}} - 1\,]$$

With this expression the diode current of Figure 1 may be represented by an hyperbolic sine equation. For the moment, however, let us concern ourselves with the solution of the equation

$$y(t) = G\, e^{px(t)}$$

for if we are able to determine the relations among the Fourier coefficients of x and y we may apply this result to more general problems.

Let us begin by considering

$$x = x_o \sin \omega t$$

Then

$$y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{jk\omega t} = G\, e^{px_o \sin \omega t} \tag{28}$$

This relation suggests the generating function of the Bessel coefficients which is given by Watson[3] as

$$e^{\frac{1}{2}z(r-\frac{1}{r})} = \sum_{n=-\infty}^{\infty} J_n(z)\, r^n$$

for if we let

$$r = e^{j\omega t} \quad , \qquad z = -jpx_o \quad ,$$

then

$$e^{\frac{1}{2}z(r-\frac{1}{r})} = e^{px_o \sin \omega t} = \sum_{n=-\infty}^{\infty} e^{jn\omega t} J_n(-jpx_o)$$

Thus, from equation (28) we find

$$Y_k = G\, J_k(-jpx_o)$$

---

[2]W. Shockley, Electrons and Holes in Semiconductors, New York (1950).

[3]G. N. Watson, Theory of Bessel Functions (1922).

This relation may also be deduced from the defining equation of the Bessel function:

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-jz\sin \omega t} e^{jn\omega t} d\omega t$$

Here the Bessel function $J_n(z)$ is seen to be the nth Fourier frequency component of $e^{jz\sin n\omega t}$,determined by integration over a period.

For the two terms Y_k and Y_{-k} ,

$$Y_k e^{jk\omega t} + Y_{-k} e^{-jk\omega t} = G[e^{jk\omega t} J_k(-jpx_o) + e^{-jk\omega t} J_{-k}(-jpx_o)$$

But
$$J_{-n}(z) = (-)^n J_n(z)$$
and
$$J_n(ze^{jk\pi}) = e^{jkn\pi} J_n(z)$$

Using these relations one finds

for k even
$$Y_k e^{jk\omega t} + Y_{-k} e^{-jk\omega t} = 2GJ_k(jpx_o)\cos k\omega t$$

for k odd
$$Y_k e^{jk\omega t} + Y_{-k} e^{-jk\omega t} = 2jGJ_k(jpx_o)e^{jk\pi}\sin k\omega t$$

The Bessel function of a complex argument most frequently tabulated is

$$I_n(z) = e^{-\frac{1}{2}n\pi j} J_n(jz) \quad .$$

Writing the above expressions in terms of I_n, the final form is obtained:

k even
$$Y_k e^{jk\omega t} + Y_{-k} e^{-jk\omega t} = 2G(-)^{\frac{k}{2}} I_k(px_o)\cos k\omega t$$

k odd
$$Y_k e^{jk\omega t} + Y_{-k} e^{-jk\omega t} = 2G(-)^{\frac{k-1}{2}} I_k(px_o)\sin k\omega t \qquad (29)$$

$$k = 0 \qquad Y_o = GI_o(px_o) \qquad (29)$$

In this manner we are able to find the harmonics of y(t). If we denote the kth harmonic frequency of y by y_k, then

$$y(t) = y_o + y_1 + y_2 + y_3 + \dots$$

and from the relations (29),

$$y_o = GI_o(px_o)$$

$$y_1 = 2GI_1(px_o)\sin \omega t$$

$$y_2 = -2GI_2(px_o)\cos 2\omega t$$

$$y_3 = -2GI_3(px_o)\sin 3\omega t$$

$$y_4 = 2GI_4(px_o)\cos 4\omega t$$

etc.

The values of $I_k(x)$ are tabulated in Watson and Jahnke and Emde over a sufficient range to cope with most problems. For an illustration, consider the simple example of Figure 9, in which

$$\frac{q}{kT} = 1 \quad , \quad I_s = 1 \quad , \; e(t) = \sin t$$

Then

$$i(t) = e^{\sin t} - 1$$

and the harmonics of i(t) are found to be

$$i_{dc} = 0.265 \text{ amps}$$

$$i_1 = 1.125 \sin t \text{ amps}$$

$$i_2 = -0.271 \cos 2t \text{ amps}$$

$$i_3 = -0.044 \sin 3t \text{ amps}$$

etc.

There is no approximation involved in this solution;

A BASIC DIODE CIRCUIT

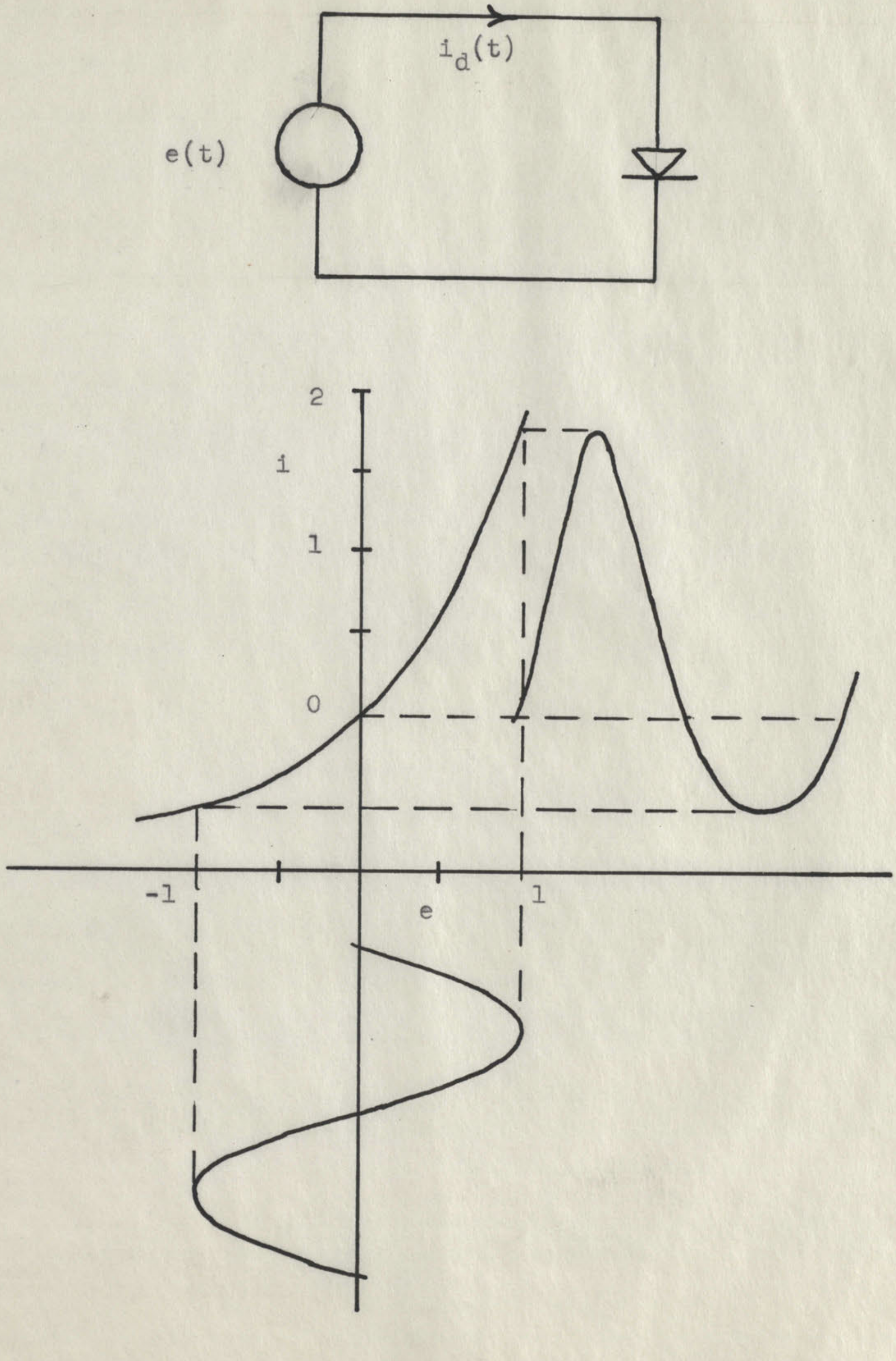


FIGURE 9

the series found represents the exact Fourier series. The drawback to this method is that for involved circuitry the equations determining the parameters become transcendental, and in general the solution is difficult to find.

Since nonlinear dissipation functions usually have current-voltage relations other than $i = e^{kv}$, it is necessary to use some technique to adapt the above method to arbitrary no storage functions. This may be done by resolving the dissipation function

$$i = f(v)$$

into a series of exponential terms. In certain cases it may be possible to use an orthogonal expansion; or, if the function becomes infinite for infinite values of v, which is usually the case, then often the inverse function

$$i' = f^{-1}(v)$$

has the property

$$\int_{-\infty}^{\infty} i'^2 \, dv < \infty$$

and may be expanded in a series of sums of exponentials in v by accepted techniques. The reciprocal may then be approximated by division of numerator by denominator.

In any event, some manner of curve fitting is needed to represent nonlinear dissipative functions by series of exponentials. For the nonlinear element of Figure 1, each diode may be assumed to obey relation (27). Then the diode circuit which conducts for positive voltage may be represented by

$$i_+ = I_s[\, e^{\frac{q}{kT}(v-E)} - 1\,]$$

and the diode and battery which act during negative voltages obey the relation

$$i_{-} = -I_s[\, e^{-\frac{q}{kT}(v+E)} - 1\,]$$

The total nonlinear element current is

$$i_{nonlin}(t) = I_s e^{-\frac{qE}{kT}}[\, e^{\frac{q}{kT}v(t)} - e^{-\frac{q}{kT}v(t)}\,] \qquad (30)$$

or

$$i_{nonlin}(t) = 2I_s e^{-\frac{qE}{kT}} \sinh\frac{q}{kT}v(t) \qquad (31)$$

which is the expansion of this diode network current in exponential voltage terms.

CHAPTER IV

A DIFFERENT ANALYSIS OF THE PROBLEM

The results of the previous chapter may now be applied to the problem of amplitude to phase conversion in limiters. Before proceeding with the solution, however, it is necessary to examine the behavior of e^V for voltages other than sinusoids.

An arbitrary periodic voltage may be represented by a Fourier series

$$v(t) = V_o + V_{1o}\sin(\omega t+\phi_1) + V_{2o}\sin(2\omega t+\phi_2) + \cdots$$

which for simplicity may be written

$$v(t) = V_o + V_1(t) + V_2(t) + \cdots$$

If this voltage is applied to the nonlinear element represented by $i = Ge^V$, the current is

$$i = G\,e^{V_o + V_1 + V_2 + \cdots} \qquad (32)$$

for which there is no simple relation by which the Fourier series of $i(t)$ may be determined. In fact, the harmonic coefficients of the current are related to the voltage by a set of infinite determinental equations.

To escape this dilemma, it is necessary to approximate equation (32) in some manner which leads to a solution. The approximation chosen for a voltage with k frequency components is:

$$e^{\sum_{i=0}^{k} V_i} = \sum_{i=0}^{k} e^{V_i} + \sum_{\substack{i=0 \\ i\neq j}}^{k}\sum_{j=0}^{k} V_i(e^{V_j} - V_j) - \sum_{j=0}^{k} kV_j - k \qquad (33)$$

Note that this expression gives equal weight to each of the components, and that the current changes exponentially with each voltage component.

This approximation may now be applied to the current in the circuit of Figure 1. Observing the symmetrical action of the nonlinear elements, elementary circuit analysis leads to

$$v(t+\frac{\pi}{\omega}) = -v(t)$$

which implies that there are no even frequency harmonics present in v(t) . Thus for the circuit of Figure 1,

$$v(t) = V_1(t) + V_3(t) + V_5(t) + \ldots$$

If the first term of v(t) is the only one considered in the analysis, no phase shift in the voltage will be found if the input current frequency equals the active element resonant frequency. This is an erroneous result, for the action of the nonlinear element is to introduce higher frequency terms which shift the fundamental voltage term away from the driving current, as was shown in Chapter I. By considering a second term, V_3 , in the analysis, the effect of this term upon the phase difference between current and voltage may be found. It would be desirable to use several harmonic terms; however, the mechanics of solution of the equations obtained become much too great and involved for its worth. Consequently, the circuit will be examined over the range in which harmonics higher than V_3 are negligible in terms of phase shift determination.

For the limiter, equation (33) becomes

$$e^{V_1+V_3} = e^{V_1} + e^{V_3} + V_1 e^{V_3} + V_3 e^{V_1} - V_1 - V_3 - 1$$

In order to estimate the errors caused by this approximation, two numerical examples have been calculated and plotted. The first example is

$$V_1 = \sin t$$

$$V_3 = \tfrac{1}{2}\cos 3t$$

The exact relation, its approximation, and the error have been plotted in Figure 10. The second example is

$$V_1 = 4\sin t$$

$$V_3 = \tfrac{1}{2}\cos 3t$$

which is plotted in Figure 11. These examples were chosen because they represent voltage coefficients similar to those found in the limiter analysis. As may be seen from the two figures, the error grows as the voltage coefficients become larger.

Returning to the analysis of the limiter circuit, Kirchoff's current relation may be written

$$2I_s e^{-aE} \sinh aV + C\frac{dV}{dt} + \frac{1}{L}\int V dt + GV = I(t) = I_o \sin \omega t$$

where equation (31) has been used to represent the nonlinear element. Let

$$\omega = 1$$

$$V(t) = V_1 \sin(t+\phi_1) + V_3 \sin(3t+\phi_3)$$

APPROXIMATION EXAMPLES

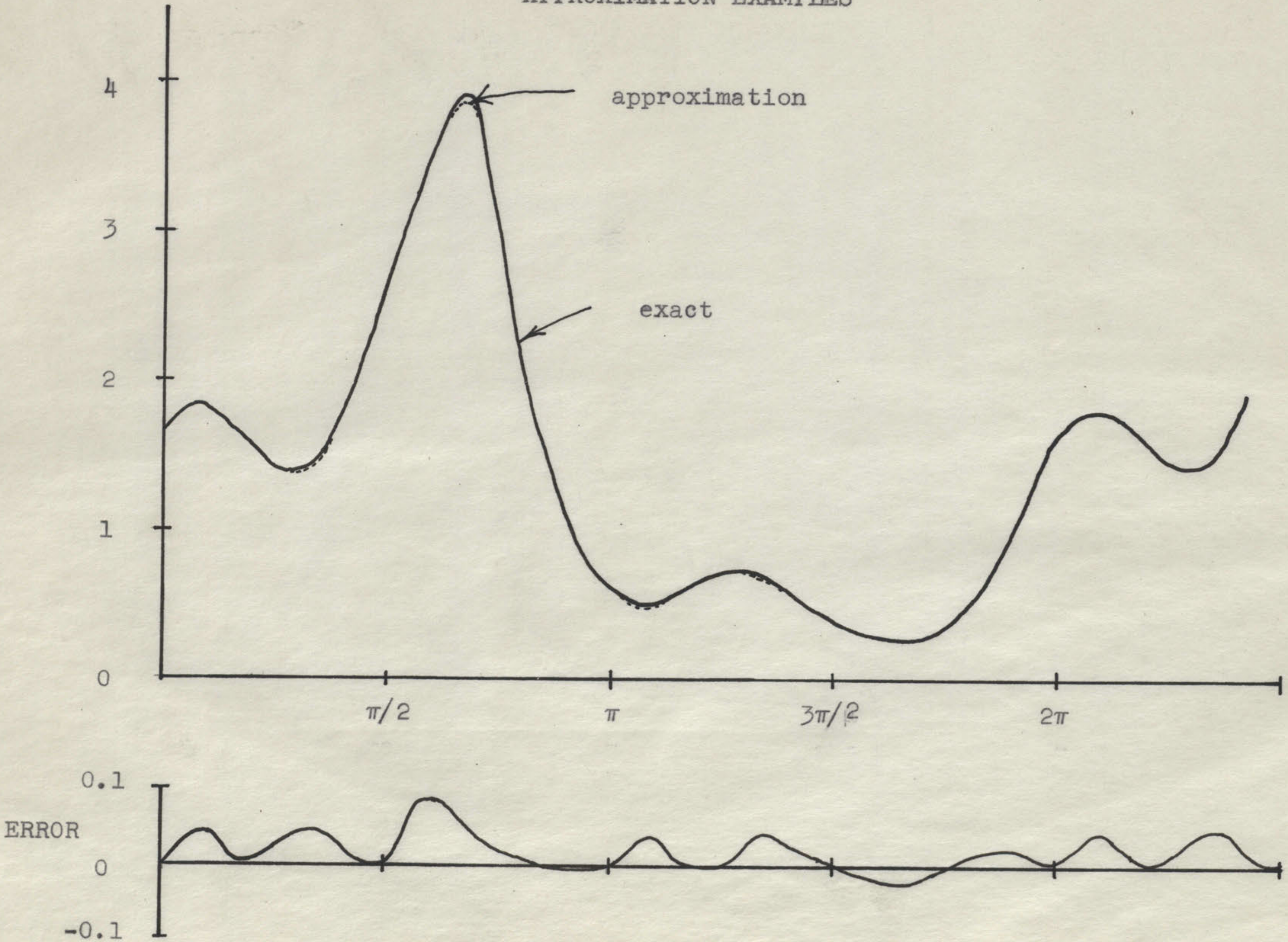


FIGURE 10

APPROXIMATION EXAMPLES

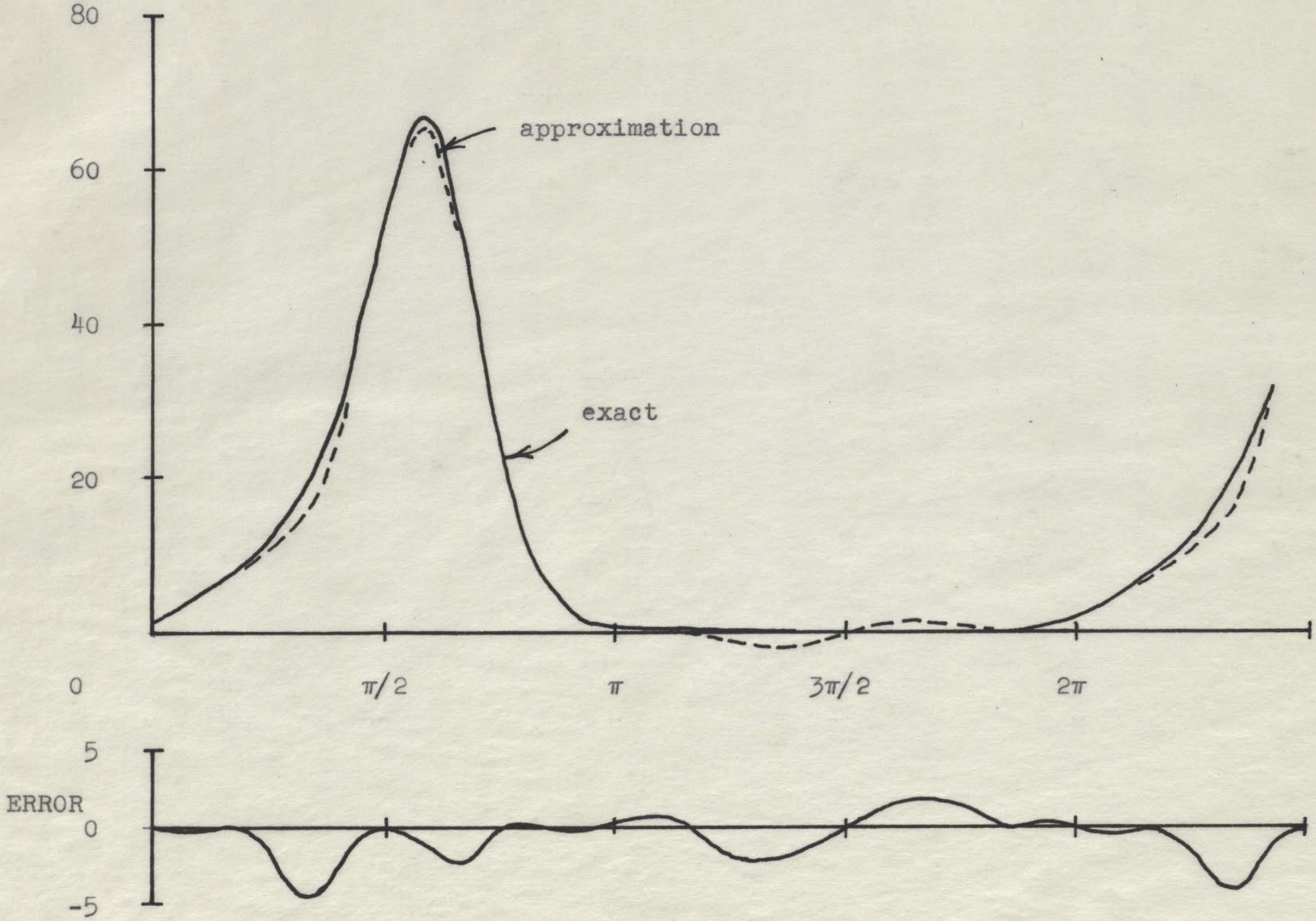


FIGURE 11

Then the equation for the fundamental harmonic is

$$2I_s e^{-aE}[\sinh aV]_{1st\ har.} + CV_1\cos(t+\phi_1) - \frac{V_1}{L}\cos(t+\phi_1) +$$

$$+GV_1\sin(t+\phi_1) = I_o \sin t$$

and the equation for the third harmonic is

$$2I_s e^{-aE}[\sinh aV]_{3rd\ har.} + 3CV_3\cos(3t+\phi_3) -$$

$$-\frac{V_3}{3L}\cos(3t+\phi_3) + GV_3\sin(3t+\phi_3) = 0$$

For computational purposes, the following values were assigned to the circuit parameters:

$$L = \frac{1}{C} = 200$$

$$a = 1 \quad , \quad E = 1 \quad , \quad I_s = \frac{e}{200} \quad , \quad G = 0 \quad .$$

A number was then assigned to the magnitude of the fundamental harmonic, and the remaining unknowns were determined by straightforward manipulation and substitution, using the tables of $I_n(v)$ compiled in Watson's Theory of Bessel Functions. The results are shown in Figure 12. The phase angles obtained are considerably less than those of Chapter II. This is primarily due to two causes. The first is the assumption of only one harmonic other than the fundamental. However, of principal importance is the fact that the nonlinear element is continuously acting; there are no sharp breaks in the voltage-current curve to produce strong higher harmonics which will shift the phase of the fundamental. This effect may also be regarded from the standpoint of the stored energy in the capacitor. The primary

VOLTAGE - CURRENT PHASE SHIFT

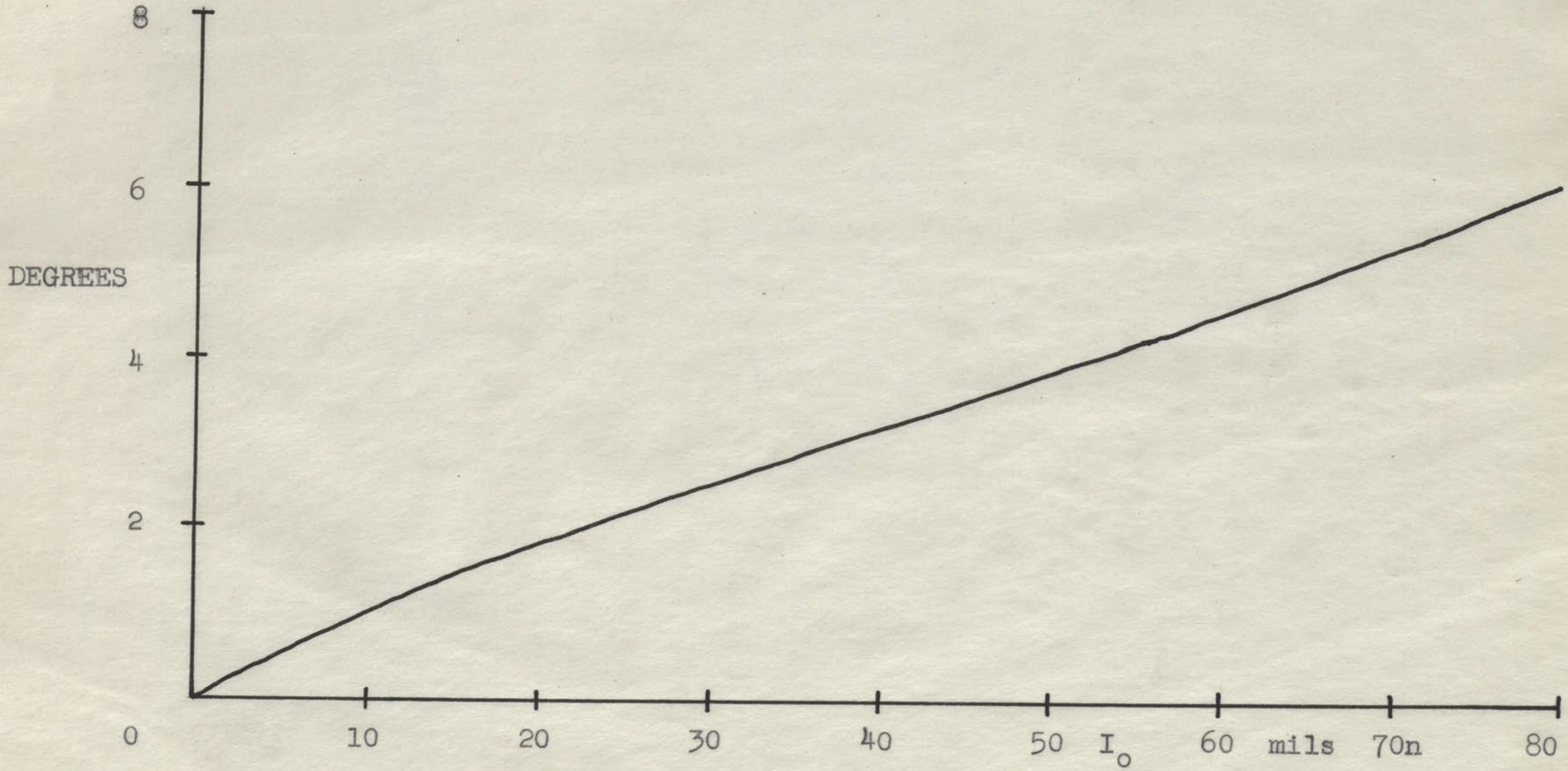


FIGURE 12

purpose of the input current is to change the capacitor voltage from +E to -E twice during each cycle. In an LC clipper limiter such as that analyzed in Chapters I and II, this transition must be accomplished almost entirely by the input current. However, in the limiter considered in this chapter, the non-linear function possessed a low impedance over most of the transition from +E to-E. Therefore, the energy stored in the capacitor could be dissipated through the diodes as the transition took place. Had the limiting action considered here been sharper (larger a), the phase shift would have been greater. In fact, as $a \to \infty$, this model reverts to that considered in Chapter I.

SUMMARY AND CONCLUDING REMARKS

The problem of amplitude to phase conversion is one for which there are no simple solutions leading to a handy graph or table. Yet, by the proper formulation of the problem and the associated method of analysis, an answer may be obtained for any given circuit and input.

This thesis has attempted to show methods by which a quantitative determination of the conversion is possible. Some of the limitations of each method have also been mentioned. However, no attempt has been made to give the reader a set of graphs from which distortion may be read or error calculated. In many places the description of errors incurred in approximations has been limited to the barest essentials of the determination. It appeared to the author that any long and involved sets of calculations would make this paper an overly tedious work to read, and would be of little value to anyone with a slightly different problem. As a result, this thesis contains general results where possible, and one or two examples where specific calculations must be made.

The first two chapters contain an analysis based on the assumption of perfect diodes and batteries in the nonlinear device. Consequently, the voltage is fixed when the diodes conduct, and the transition states are independent of the diodes except for initial conditions. The approach of Chapter I is that of conventional transform techniques. The relations obtained are exceedingly difficult to use, and a solution by substitution completely obscures any meaning of the results and leaves one very prone to computational error. To avoid this, considerable simplification of the relations is necessary. The error incurred by the approximations used has a maximum of about five per cent at the limits of the curves given. By a close examination of the circuit element behavior, a certain amount of insight may be gained into the action during the transition from one voltage state to the other, and from this a relation may be developed which yields the voltage-current phase shift. Then, assuming a slow amplitude modulation present at the input, the phase modulation coefficient is determined as a function of the circuit parameters. Here again restrictions are necessary; namely, that of low index modulation. However, for the situation which created this problem, the need for better amplitude slope correction limiters in radio relay systems, the modulation index is low, and the relations given in Chapter II are pertinent. For the case of very widely varying input levels, the phase deviation function is both large and complex; however, it may be handled by techniques similar to those presented in Chapter I.

The procedures just described have as their basis ideal diodes and batteries. In order to show the difference between such an assumption and the reality of continuously acting nonlinear elements with smooth v-i curves, a different approach to the problem was taken. Instead of the transient analysis of Chapters I and II, the voltage waveform is broken into its frequency harmonics, and these are determined monics, and these are determined by an analysis technique developed in Chapter III. An example is carried out to give an idea of a typical result. The method has the advantage that the fundamental voltage component is obtained directly, without the need for any further Fourier analysis. However, the calculation procedure is tedious, and becomes overpowering if more than two frequency components are considered. For a specific problem, however, an answer may be found by straightforward although brutal substitution and calculation.

It would be possible to assemble a computer by which such calculations could be automatically performed. Although the analysis technique gives results for almost any lumped circuit with linear storage elements, there is a question as to the error incurred in certain types of problems, namely those in which there are many very large exponential terms. If a suitable approximation or an alteration of the one given could be found, such a machine would be worthwhile. This needs further investigation.

The amplitude to phase conversion in the limiters described in this work may be reduced to any prescribed amount by the proper alteration and addition of circuit elements. Work is being continued in this direction, and a method of reduction which has been developed will be released soon.

BIBLIOGRAPHY

Gardner, M., and Barnes, J. Transients in Linear Systems. New York: John Wiley and Sons, 1942.

Jahnke, E., and Emde, F. Tables of Functions. New York: Dover, 1945.

Madsen, R. Amplitude to Phase Conversion in Clipper Limiters. Technical Memorandum 54-212-44, Bell Telephone Laboratories, 1954.

Shockley, W. Electrons and Holes in Semiconductors. New York: D. Van Nostrand Co., 1950.

"The Measurement of Incredibly Limited Signals," Wireless Fantasy, November, 1928.

Watson, G. N. Theory of Bessel Functions. Cambridge: University Press, 1952.

Wiener, N. "Mathematical Problems of Communication Thèory." Summer session lecture notes, M.I.T. 1953-1954.