

The Compression Theorem of Rourke and Sanderson

by

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Submitted to the Department of Mathematics in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

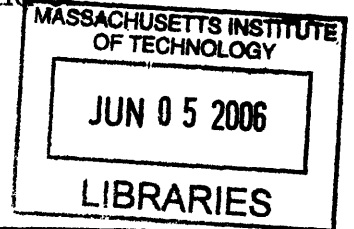
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## Abstract

This is an exposition of the Compression Theorem of Colin Rourke and Brian Sanderson, which leads to a new proof of the Immersion Theorem.

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*His colleagues appeared to him partly as inexorably maniacal public prosecutors and chief detective-inspectors of logic, partly as opium-addicts and eaters of a strange pallid drug that populated their world with a vision of figures and abstract relations. 'Good heavens!' he thought to himself, 'surely I can never have meant to spend the whole of my life as a mathematician?'*

—Robert Musil, *The Man Without Qualities*

*All these numbers. What do they have to do with life? Does the sun have numbers for itself? The bears and the caribou? The fish? The snow that greets us, stays with us, goes away crying?*

—An Eskimo Grandmother in *The Last and the First Eskimos*



# 1 Introduction

The Compression Theorem of Colin Rourke and Brian Sanderson<sup>1</sup> leads to a proof of the Immersion Theorem. The latter gives a correspondence between homotopy classes of immersions of a manifold  $M$  into a manifold  $Q$  and homotopy classes of tangential maps. A tangential map from  $M$  to  $Q$  by definition can be extended to one between  $TM$  and  $TQ$  such that the each fiber of the former is linearly embedded in one of the latter. Differentiating an immersion transforms it into a bundle monomorphism. The Compression Theorem gives a method to go the other direction.

The Multi-Compression Theorem is a constructive procedure for creating an isotopy between an embedding of manifold  $M$  in  $Q \times \mathbb{R}^n$ , endowed with  $n$  independent normal vector fields, and one where the  $n$  vector fields remain normal to  $M$  and independent and are parallel to the coordinate axes. Projection along those axes then gives an immersion of  $M$  into  $Q$ . Given a bundle monomorphism,  $f: TM \rightarrow TQ$ , it can be paired with an embedding of  $M$  into  $\mathbb{R}^n$ , for some  $n$ , to give an embedding of  $TM$  into  $Q \times \mathbb{R}^n$ . A simple trick then produces an embedding of  $M$  with  $n$  linearly independent vector fields. The Multi-Compression Theorem can now be applied to create the desired immersion.

The precise statement of the Multi-Compression theorem is:

**Multi-Compression Theorem:** *Suppose  $M^m$  is compact and embedded in  $Q \times \mathbb{R}^n$  with  $n$  independent normal vector fields and  $q > m$ . Then  $M$  is isotopic (by a  $C^0$ -small isotopy) through normal vector fields to a parallel embedding.*

Rourke and Sanderson call this the *Multi-Compression Theorem* because they deduce it as a corollary from the same statement with  $n = 1$ , which they call the Compression Theorem. To prove the Multi-Compression Theorem they use an inductive argument which requires the  $C^0$ -small condition. Following Rourke and Sanderson we will first prove a simpler global version of the Compression Theorem with no control over the size of the isotopy. This is conceptually useful because it introduces the general framework of the longer and more complicated proof. Furthermore, it is by itself is a pleasing and elegant argument.

## 2 The Global Proof

### 2.1 The Idea

The proof has three steps. The first two straighten the normal field as much as possible to point in the positive  $\mathbb{R}$  direction (henceforth we will call this upwards) without moving  $M$  in  $Q \times \mathbb{R}$ . This allows us to extend it to a vector field defined on  $Q \times \mathbb{R}$ .

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<sup>1</sup>This paper is largely an exposition of [Rourke & Sanderson 2001] and [Rourke and Sanderson 2003] and hence it should be understood that they underlie the following discussion.

For the third step, we flow  $M$  along this vector field until our normal vector field is constant and directed upwards.

The first naïve approach, given the embedding of  $M$  and the vector field, would be to rotate the vector field pointwise towards the upwards direction. However, this approach fails if  $TM$  lies between the initial vector and upwards (or in the worst case, if the vertical direction lies along  $M$ ). It also cannot be done in a canonical fashion if the initial vector field points straight downwards at any time. Therefore the first step is to position the vector field to never be straight downwards and to be perpendicular to  $M$ . The latter condition gives us the control to avoid rotating through  $M$ . The former, the ability to canonically rotate it. After this first step we rotate the vector field, so each vector has a upwards component bounded away from zero. After this we extend the vector field throughout  $Q \times \mathbb{R}$  to be vertical outside a compact set and then flow  $M$  along that vector field until it leaves that set.  $M$  will now have a straightened normal vector field.

## 2.2 The Formal Proof

### Preliminaries

We begin with some preliminary notions and assumptions. We assume that  $Q$  is endowed with a Riemannian metric and we call a normal vector field perpendicular if it is everywhere orthogonal to  $TM$ . A compressible embedding is one in which the vector field is normal and everywhere pointing in the positive  $\mathbb{R}$  (upwards) direction. It is called this because then the manifold  $M$  lies above an immersion by projecting onto  $Q$ . A vector field is grounded if it never points downwards, and  $\varepsilon$ -grounded if it always make an angle of at least  $\varepsilon$  with the negative  $\mathbb{R}$ -axis.

**The Compression Theorem (Global Version):** *Let  $M^m$  be a compact manifold embedded in  $Q \times \mathbb{R}$  equipped with a normal vector field,  $\alpha$ . If  $q > m$  then  $M$  is isotopic through normal vector fields to a compressible embedding.*

First we need a lemma ensuring we can assume our vector field  $\alpha$  is perpendicular and  $\varepsilon$ -grounded. We can make  $\alpha$  perpendicular at  $m \in M$  by considering its projection onto the complement of  $T_m M \subset T_m(Q \times \mathbb{R})$ . We can then rotate  $\alpha(m)$  onto the direction of its projection canonically by thinking of the directions as defining non-antipodal points on the unit sphere in  $T_m(Q \times \mathbb{R})$  and rotating along the unique great circle connecting them. Nowhere does this rotation pass through  $T(Q \times \mathbb{R})|_M$  and since the prescription is canonical we can do it on  $M$  simultaneously and smoothly.

We can assume  $\alpha$  is grounded. We note that the set of points where vertically down is perpendicular to  $M$ , denoted by  $C$  has codimension  $m$  and hence is a collection of points. Roughly, this is because we can reduce to the case when  $Q = \mathbb{R}^q$ , then consider the map  $F: M \rightarrow \mathcal{G}_{q+1,m}$ , where  $\mathcal{G}_{n,m}$  is the Grassmanian of  $m$ -planes in  $\mathbb{R}^n$ , defined by:  $x \in M$  maps to  $T_x M$  as a subset of  $\mathbb{R}^{q+1}$ .  $C$  is the inverse image



of  $\mathcal{G}_{q,m} \subset \mathcal{G}_{q+1,m}$  because at those points  $TM \subset TQ$  and hence the  $\mathbb{R}$  factor is perpendicular to  $M$ . Finally, the codimension of  $\mathcal{G}_{q,m}$  in  $\mathcal{G}_{q+1,m}$  is  $m$  and hence so is the codimension of  $C$  in  $M$ . Therefore, since the normal bundle has dimension at least 2, we can find a small isotopy of  $\alpha$  keeping it perpendicular, so that it does not point vertically down on  $C$ , a set of isolated points.

We defer the formal proof of this general position argument to the appendix. If  $q = m$ , the orthogonal complement of  $T(M)$  in  $T(Q \times \mathbb{R})$  is one dimensional at every point so there is no ‘room’ to slip  $\alpha$  off the downwards direction. Since  $M$  is compact if  $\alpha$  is grounded then  $\alpha$  must be  $\varepsilon$ -grounded for some  $\varepsilon > 0$ . Finally, we note that  $\alpha$  can be isotoped to be of unit length.

### Proof of the Theorem

Now that  $\alpha$  is  $\varepsilon$ -grounded and perpendicular we can rotate it unambiguously upwards while simultaneously avoiding rotating it into  $M$ . This will straighten our vector field ‘as much as possible’ without an isotopy of  $M$  in  $Q \times \mathbb{R}$ .

First we choose a  $\mu \in \mathbb{R}$  such that  $0 < \mu < \varepsilon$  and then, for each  $p \in M$ , rotate each vector  $\alpha(p)$   $\pi/2 - \mu$  towards vertically up. If the angle of  $\alpha(p)$  with upwards is less than  $\pi/2 - \mu$  then we leave it at vertically upwards. Because  $\alpha$  is perpendicular to  $TM$  then we know the rotated vector field, call it  $\beta$ , does not coincide with  $TM$ . Furthermore we know that the angle  $\beta$  makes with the upwards direction is always at most  $\pi/2 - \varepsilon + \mu$ . So  $\beta$  has a minimal vertical component of at least  $\sin(\varepsilon - \mu)$ .

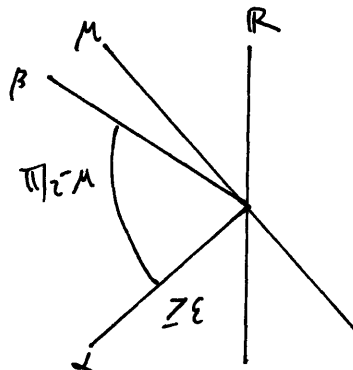


Figure 1: Upwards Rotation

However, the vector field  $\beta$  as defined may only be continuous and not smooth. The reason being the amount of rotation can abruptly become 0. Figure [2a,b] makes this clear. The first graph shows the angle of  $\alpha$  with vertically up along a one dimensional path, the second shows the angle of  $\beta$  with vertically up. The problem is the corner that can arise when the angle  $\alpha$  makes with the vertical  $\mathbb{R}$ -axis becomes  $\pi/2 - \mu$ . We can fix this problem by rotating  $\alpha$  towards vertically up an amount that varies with its angle with the vertical  $\mathbb{R}$ -axis. We pick a smooth function  $f(\theta): \mathbb{R} \rightarrow \mathbb{R}$  that is equal to  $\pi/2 - \mu$  for  $\theta > \pi/2 - \mu$ , less than  $\theta$  for  $\theta \leq \pi/2 - \mu$  and 0 for  $\theta \leq 0$  (see Figure [2c]). Now instead of rotating  $\alpha$   $\pi/2 - \mu$  towards vertically up, we rotate it by  $f(\theta)$  where  $\theta$  is the angle  $\alpha$  makes with the vertical axis. The result will be a smooth

vector field again denoted by  $\beta$ . However, this modified  $\beta$  will still have a vertical component of at least  $\sin(\varepsilon - \mu)$ .

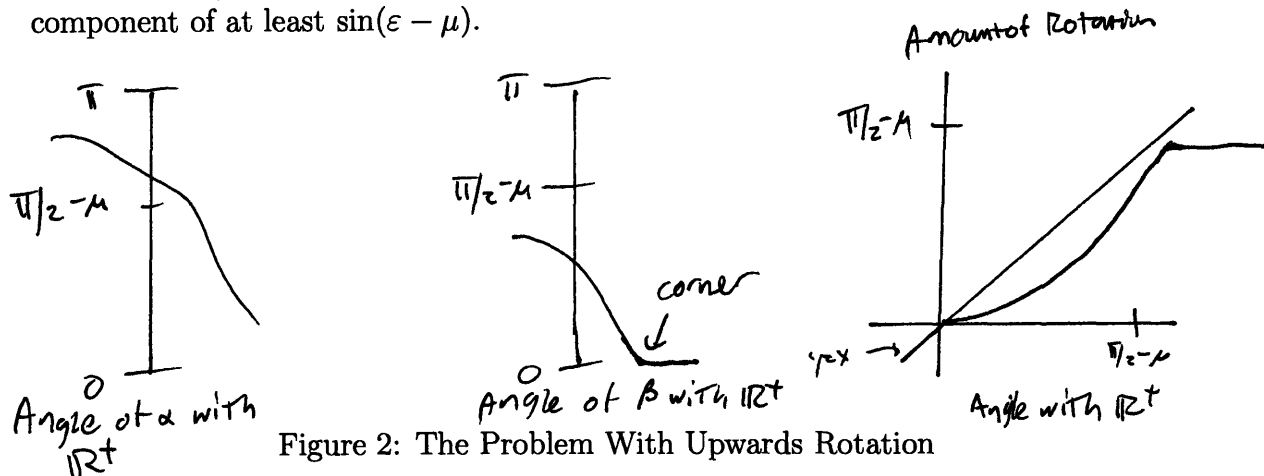


Figure 2: The Problem With Upwards Rotation

We take a tubular neighborhood of  $M$  in  $Q \times \mathbb{R}$  and extend  $\beta$  to a vector field  $\gamma$  over  $Q \times \mathbb{R}$  by radially interpolating  $\beta$  along the fibers to a vertically up vector field of unit length. We set  $\gamma$  equal to the unit upwards vector outside the tubular neighborhood. Therefore  $\gamma$  points straight up outside of a compact set. We flow  $M$  along  $\gamma$  and because the upwards component of  $\gamma$  is bounded away from zero, every point in  $M$  will flow upwards at some minimal speed. Furthermore, since  $\gamma|_M$  was normal,  $\gamma$  will remain normal to  $M$  throughout the flow. Eventually,  $M$  will enter the region of  $Q \times \mathbb{R}$  where  $\gamma$  is constantly pointed upwards. We have then isotoped  $M$  to a compressible embedding. ■

### 2.3 Addenda

The first thing to note is that the theorem can be extended to the case where  $q = m$  provided that the vector field is already  $\varepsilon$ -grounded for some  $\varepsilon$ . The only place in the theorem where the dimension condition was important was the general position argument to assure  $\varepsilon$ -groundedness.

The second thing to note is that the flow may be modified to one which leaves stationary those points at which  $\beta$  points directly upwards. If  $x \in Q \times \mathbb{R}$  and  $s \in \mathbb{R}$  then we denote by  $x + s$ ,  $x$  translated  $s$  in the  $\mathbb{R}$  direction. If the global flow can be described by  $x \rightarrow f_t(x)$  then the modified flow is described by  $x \rightarrow f_t(x) - t$ . This is analogous to a Galilean transformation in basic physics. So here the normal vector field attached to our manifold  $M$  at a point  $x$  is equal to  $\gamma$  at  $x + t$ . Therefore, because we identify the vector spaces at  $x$  and  $x + t$  in the canonical way, our isotoping vector field remains normal to  $TM$ . Furthermore we can regard our modified global flow as being given by a time dependent one. We denote the unit everywhere vertical vector field by  $u$  and  $\gamma$  our vector field that defines the original global flow then we define our time dependent vector field  $\gamma^*$  by

$$\gamma^*(x, t) = \gamma(x + t) - u$$

The only thing left to note is that the normal vector field that is isotoped by this flow is  $\gamma^* + u = \gamma(x + t)$  translated by  $t$  units downwards.

These adjustments to the global flow are in preparation for the local proof. The original flow moves the manifold until it enters a region with a uniformly upwards vector field. Each point moves at the same speed regardless of how close it is to being straightened. The modified flow moves points only as necessary. It allows the local proof to ignore the part of the manifold that is already straightened.

### Limitations of the Global Proof

The procedure for the global proof is in some sense local. The first two steps before isotoping  $M$  take place pointwise and the vector field  $\gamma$  is defined by an interpolation near a given point. It would seem that it would be possible to extend the proof to immersions of  $M$  in  $Q \times \mathbb{R}$  rather than just embeddings. The problem is that a global vector field cannot be defined near double points. However, given an immersion  $f: M \rightarrow Q$  and a point  $m \in M$  we can choose a neighborhood  $U$  of  $m$  such that  $f|U$  is an embedding and then apply the Compression Theorem in  $U$ . We can then cover  $M$  by such open sets and apply the theorem in each of them. The procedures of the theorem are canonical and will coincide on overlaps. However, this procedure can fail because the flow will not necessarily be smooth for all time. Figure [3] shows an immersion of  $S^1$  into  $\mathbb{R}^2$  which is  $\pi/4$ -grounded but the global flow will pull the bottom loop tight and produce a non-smooth kink. In fact, it is impossible to isotope this immersion to a compressible embedding because  $S^1$  cannot be immersed into  $\mathbb{R}$ . If we considered this as an immersion in  $\mathbb{R}^3$  by the natural inclusion of  $\mathbb{R}^2$  then it is isotopic to a compressible embedding, but one which the global proof cannot attain. Namely, we can just untwist the loop, tilt it away from the  $xy$ -plane and straighten the vector field. The global proof cannot 'see' the extra room of the added dimension. However, the local version of the Compression Theorem will allow us to isotope immersions to compressible embeddings in general provided we have that extra room, i.e. that  $q > m$ .

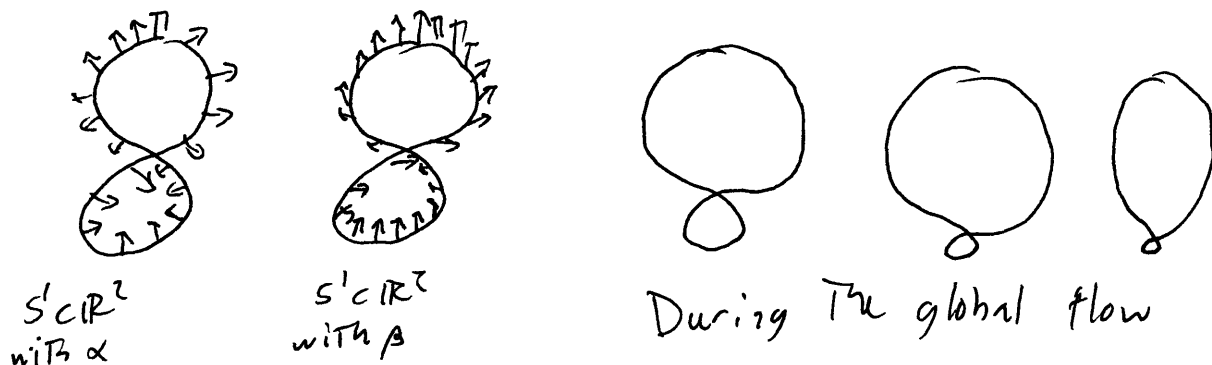


Figure 3: The Global Proof Can Fail For Immersions

Not only does the global proof fail for immersions, even when  $q > m$ , it also may result in an arbitrarily large isotopy, even with the modified global flow. Given

$N > 0$ , consider the segment of the line  $y = x$  in  $\mathbb{R}^3$  from the origin to the point  $(N\sqrt{2}, N\sqrt{2})$ . We consider the  $y$ -axis to be vertical and we work in  $\mathbb{R}^3$  to satisfy the dimension hypothesis, however without loss we can assume to be working in  $\mathbb{R}^2$ . Let the vector field  $\alpha$  be  $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ . We claim that for any choice of  $\mu$  and the size of the tubular neighborhood, the isotopy may move points a distance greater than  $N$ . For simplicity we assume the fibers of our tubular neighborhood are the images under the exponential map of  $\alpha$ . Now we perform upwards rotation to get  $\beta$ . Now we extend  $\beta$  to  $\gamma$  over all of  $\mathbb{R}^2$  by interpolation. By symmetry, the flow will preserve our line segment and its angle with the  $y$ -axis. It will only undergo vertical and horizontal translations. We will focus on the bottom endpoint and so we conveniently ignore the singularity at the upper endpoint. We can imagine the segment continuing smoothly in some fashion and even coming around to the bottom endpoint in  $\mathbb{R}^3$ . In any case the vector at the bottom endpoint can only be straightened when the point leaves the tubular neighborhood. It cannot leave the tubular neighborhood on the side since then  $\gamma$  will have become parallel to our segment at some point. Therefore it must exit the neighborhood at the top. Therefore it will have moved a distance of at least  $N$ , the horizontal distance between the bottom and top points of the segment, in the modified global flow. The modified flow is only changed by a vertical factor and hence the horizontal motion is unchanged. The reader is encouraged to generalize this result to any tubular neighborhood of our initial segment.

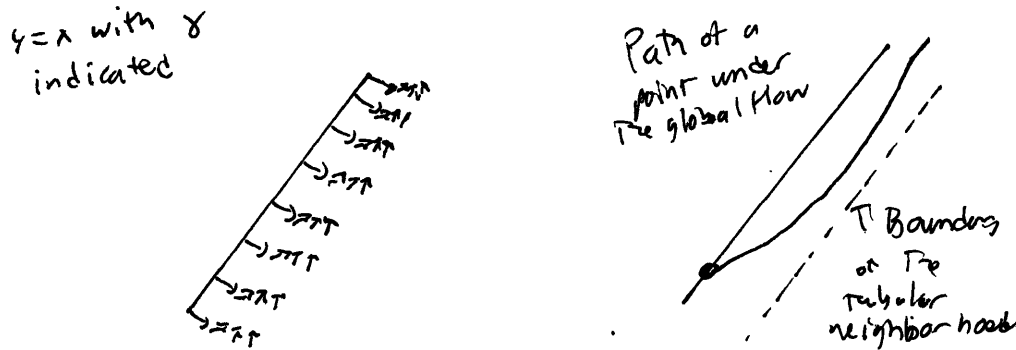


Figure 4: An Arbitrarily Large Isotopy Given by the Global Proof

Finally we conclude with a corollary of the global Compression Theorem.

**Corollary:** *If  $M^m$  cannot be immersed into  $Q^q$  and  $q = m$ , then every perpendicular vector field of any embedding of  $M$  into  $Q \times \mathbb{R}$  must have a vector pointing downwards and one pointing upwards.*

**Proof** If there is no downwards pointing vector then  $M$  is  $\varepsilon$ -grounded for some  $\varepsilon > 0$  and so we can apply the global Compression Theorem to isotope  $M$  to a compressible embedding and then compose this with projection onto the  $Q$  factor to get an immersion of  $M$  into  $Q$ . In particular, if  $M$  does not immerse into  $\mathbb{R}^m$  then the Gauß map of any embedding of  $M$  into  $\mathbb{R}^{m+1}$  is surjective. ■

### 3 The Local Proof

We wish to prove the following version of the Compression Theorem, which is almost identical to the global version:

**The Compression Theorem (Local Version):** *Let  $M^m$  be a compact manifold embedded in  $Q \times \mathbb{R}$  equipped with a normal vector field,  $\alpha$ . If  $q > m$  then  $M$  is isotopic (through a  $C^0$ -small isotopy through normal vector fields) to a compressible embedding.*

#### 3.1 The Idea

The local proof refines the ideas of the global one. In the global proof, the vector field was straightened before isotoping  $M$  in  $Q \times \mathbb{R}$ . However, the straightening process was not subtle enough to give any control of the isotopy of  $M$ . The local proof, by contrast, uses a more refined approach to the initial straightening of the vector field. The local proof identifies a subset of  $M$ , the ‘downset’, outside of which the vector field can be completely straightened without isotoping  $M$ . Furthermore, the vector field on the downset can be straightened within an arbitrary neighborhood in  $Q \times \mathbb{R}$ , and picking this neighborhood to be small enough we can straighten the vector field with a  $C^0$ -small isotopy. The ability to straighten a vector field in an arbitrarily small neighborhood allows us to prove the Compression Theorem for immersions as well as for multiple vector fields. As we said at the beginning, the latter statement allows us to prove the Immersion Theorem.

In the global proof, the first step was to adjust  $\alpha$  to be  $\varepsilon$ -grounded and perpendicular to  $M$ . Although this was sufficient for the proof, by making  $\alpha$  perpendicular we may have, indeed, moved it further away from the upwards direction. The local proof further adjusts  $\alpha$  after it has been made perpendicular. The first key idea of the local proof is that after having made  $\alpha$  perpendicular we can rotate it, off of the downset, to have a positive upwards component. After this, upwards rotation will straighten  $\alpha$ . The problem is then localized to a neighborhood of the downset. The next two moves take place in that neighborhood. The first eliminates the downset except near its boundary, called the horizontal set. The horizontal set is where the  $\mathbb{R}$ -direction is contained in  $TM$ . It is also where the (perpendicular) normal bundle is contained within  $TQ \times \mathbb{R}|M$ . We think of the  $Q$  direction as horizontal, opposed to the vertical  $\mathbb{R}$ -direction. Near the horizontal set, once  $\alpha$  is rotated upwards it will be almost vertical and so the modified global flow will be correspondingly small.

#### 3.2 Preliminaries

We first define the horizontal set and the downset. The latter depends on  $\alpha$ , while the former depends only on our embedding of  $M$  into  $Q \times \mathbb{R}$ .

## The Horizontal Set

The *horizontal set*, denoted  $H(M)$ , prevents a straightening of  $\alpha$  without an isotopy of  $M$  in  $Q \times \mathbb{R}$ . It is defined to be the set of points in  $M$  at which the vertical direction, that parallel to the  $\mathbb{R}$ -axis, is contained in the tangent space of  $M$  at those points. That is  $H(M) = \{(x, y) \in M | T_{(x,y)}(x \times \mathbb{R}) \subset T_{(x,y)}(M)\}$ . Its name comes from the fact that if we consider  $\mathbb{R}$  as vertical and  $Q$  as horizontal, and if  $Q$  has a Riemannian metric that we extend to  $Q \times \mathbb{R}$ , then  $H(M)$  is the set of points of  $M$  where the normal bundle is *horizontal*. The horizontal set is precisely the obstruction to the composition of the embedding of  $M$  into  $Q \times \mathbb{R}$  with the projection map onto the  $Q$  factor being an immersion. Projection at points in  $H(M)$  kills a dimension of  $TM$  at those points, preventing the injectivity of the differential.

**Proposition:** *We can assume  $H(M)$  is a submanifold of  $M$ , of codimension  $q-m+1$ , after a small isotopy of  $M$  in  $Q \times \mathbb{R}$*

This proof again relies on general position arguments and we again defer its proof to the appendix.

## The Downset

The definition of the downset depends on the embedding of  $M$  in  $Q \times \mathbb{R}$ , but unlike  $H(M)$  also depends on  $\alpha$ . At every point of  $M$  we have a unit sphere in the normal bundle. Now at every point of  $M - H$ , there will be a unique vector in that normal sphere with a maximum  $\mathbb{R}$ -component. At every point of  $H(M)$ , every vector in the normal unit sphere has zero  $\mathbb{R}$ -component. We define a vector field  $\psi$  on  $M - H$  to be, the vector in the normal unit sphere with a maximal  $\mathbb{R}$  component. Assuming  $\alpha$  to be perpendicular, we define the downset  $D = \{x \in M | \alpha(x) = -\psi\}$ . So  $D$  is composed of those points of  $M$  with  $\alpha$  pointed the most downwards in the normal bundle.

**Proposition:** *After a small isotopy of  $\alpha$  we can assume that it is transverse to  $-\psi$  and so  $D$  is a manifold. Also we can assume that the closure of  $D$  is a manifold with boundary  $H$ .*

Again we defer the proof to the appendix.

## Localization

Here we explain how we can straighten our vector field except in a neighborhood of the downset. After we have adjusted  $\alpha$  to be perpendicular, then away from the downset we can canonically isotope it to have a positive  $\mathbb{R}$ -component. We can do this because at each point of  $M$ ,  $\alpha$  and  $\psi$  define two points on the unit normal sphere. Except for those points in  $D$ ,  $\alpha$  is not antipodal to  $\psi$  and so there is a unique shortest path, along a great circle, connecting them. This arc is in the unit normal sphere and so it does not intersect  $TM$  at any point. So outside a neighborhood of  $D$  we can

isotope  $\alpha$  to have a strictly positive vertical component. After this move we can now perform the upwards rotation of the global proof to straighten the field away from the downset. To have a globally defined isotopy of  $\alpha$  and not just outside a neighborhood of  $D$ , we pick a tubular neighborhood of  $D$  and phase out our isotopy along its fibers. This is similar to the way we constructed our global vector field  $\gamma$  in the global argument. We have now localized the problem to a neighborhood of the downset. The rest of the proof will involve a  $C^0$ -small straightening in this neighborhood.

### 3.3 After Localization

After localization, there are two moves left to straighten  $\alpha$ . The first, which we call the *local* move is the most complicated. It will eliminate the downset except in a neighborhood of the horizontal set. The second move is the modified global flow, but as we have said before, because  $\alpha$  is nearly perpendicular to  $M$  in an neighborhood of the horizontal set, the upward rotated vector field  $\beta$  will be almost vertical and hence so will be the extended field  $\gamma$ . Hence we can make the modified global flow as small as we like by choosing our parameters carefully.

Before we perform the local move, first we need to introduce a gradient vector field,  $\varphi$ , on  $M$ . The local flow will take place in a local foliation of  $Q \times \mathbb{R}$  defined with respect to  $\varphi$ . This enables us to restrict our attention to a generic leaf of the foliation and so control the amount each point flows as well as control how our vector field changes.

We define  $\varphi$  to be the gradient field on  $M$  given by the projection  $p: M \subset Q \times \mathbb{R} \rightarrow \mathbb{R}$ . I.e. it is the vector field on  $M$  such that for each point  $p \in M$   $\varphi_p$  points upmost in  $T_p M$ . If  $p$  is a local minimum or maximum, then  $\varphi_p$  is zero. Furthermore, our normal field is grounded if and only if the zeroes of  $\varphi$  are not on  $D$ . This is because if  $p \in D$  then  $\alpha_p$  is pointing the most downward in the unit normal bundle at  $p$ . If  $\varphi_p$  is 0 that is precisely the unit vector in the downwards  $\mathbb{R}$ -direction. So  $\alpha_p$  makes an angle of 0 with the  $\mathbb{R}$ -axis precisely at those points. If none exist,  $\alpha$  is grounded.

Now we state a lemma which gives us control over the size of flow lines of  $\varphi$ . Since we can make their size arbitrarily small, we will be able to control the size of the local move. Here we see the importance of being able to localize the problem to the downset, a submanifold of codimension  $q - m \geq 1$  in  $M$ . This non-zero codimension condition is exactly what is needed to be able to place  $\overline{D - U}$  in general position with respect to  $\varphi$ . However, we again defer the proof to the appendix.

**Lemma:** *Suppose that  $\alpha$  is perpendicular and grounded. Let  $U$  be tubular neighborhood of  $H$ . Then there is a small isotopy of  $\alpha$  such that given  $\delta > 0$  there is a neighborhood  $V$  of  $\overline{D - U}$  in  $M$  such that each component of intersection of an integral curve of  $\varphi$  with  $V$  has length  $\leq \delta$ . After this isotopy of  $\alpha$ , we say that  $\overline{D - U}$  is in general position with respect to  $\varphi$ .*

## The Local Move

To perform the local move, first we apply our general position arguments to make  $D$  a manifold with boundary  $H$ . Now we choose two neighborhoods of  $H$ ,  $U$  and  $U'$  with  $U' \subset U$ . Now we apply our lemma to adjust  $\alpha$  so that  $\overline{D - U'}$  is in general position with respect to  $\varphi$ . We can still assume  $D$  to be a manifold with boundary  $H$  because these are general position arguments. We pick the  $\delta$  in the lemma small enough so that  $U'$  is at least  $\delta$  away from  $\overline{M - U}$ . And call  $V$  the neighborhood of  $D - U'$  given by the lemma with the choice of  $\delta$ . Furthermore, we can assume that  $\delta$  is small compared to the scale of  $\alpha$ , the distance over which its direction changes. See the Figure [4] for a representation of the arrangement of the sets.

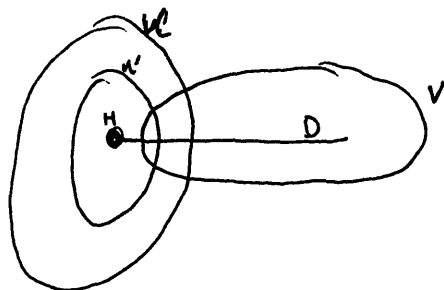


Figure 5: A Schematic of  $D$ ,  $H$ ,  $U$ ,  $U'$  and  $V$

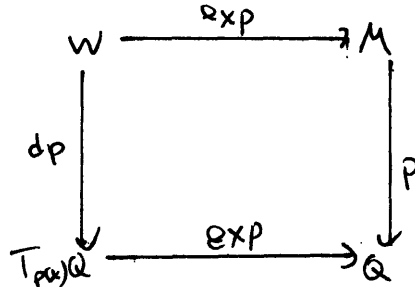
Now we can localize  $\alpha$  with respect to  $U \cup V$ , so outside  $U \cup V$ , we isotope  $\alpha$  to point the most upwards in the unit normal bundle. Now we perform upwards rotation to  $\alpha$  to get the vector field  $\beta$ . We can pick  $\mu$  small enough so that  $\beta$  is straightened outside  $U \cup V$  because after localization  $\alpha$  has a strictly positive upwards component on  $M - U \cup V$ . So on  $M - U \cup V$ ,  $\alpha$  makes an angle of  $\pi/2$  with  $TM$  and at most  $\pi/2 - \eta$  with the positive  $\mathbb{R}$  direction, for some  $\eta > 0$ . Picking  $\mu < \eta$  assures us that the straightening is complete except on  $U \cup V$ . Now as in the global proof, we pick a  $\nu$ -tubular neighborhood of  $M$  in  $Q \times \mathbb{R}$ , where  $\nu$  is the size of the neighborhood. We then extend  $\beta$  to  $\gamma$  on  $Q \times \mathbb{R}$  which is vertical except in that neighborhood. Then we construct the time dependent vector field for the modified global flow,  $\gamma^*$ . The local move is simply flowing along a slightly modified  $\gamma^*$ .

As we have observed before, outside of  $H$  vertical projection,  $p$ , is an immersion on  $M$ , in particular on  $\overline{D - U'}$ . This means that there is a number  $\omega$ , depending on  $U'$  such that if any two points, one in  $\overline{D - U'}$  and the other in  $M$ , project to the same point in  $Q$ , then there are at least  $3\omega$  apart with respect to the metric on  $Q \times \mathbb{R}$ . This minimum distance will allow us to perform the local move without having to worry about interference generated above or below the points we are interested in.

The existence of  $\omega > 0$  follows from the compactness of  $\overline{D - U'}$ . For each point  $x \in \overline{D - U'}$ , let  $r(x)$  be the infimum of distances from  $x$  to other points with the same image in  $Q$ . Since  $\overline{D - U'}$  is compact  $r(x)$  achieves a minimum value,  $\xi$ . If  $\xi = 0$  then there exists a point  $x \in \overline{D - U'}$  and a sequence  $\{m_n\}$  such that  $p(m_n) = x$



for all  $n$  and  $\lim_{n \rightarrow \infty} m_n = x$ . But consider a neighborhood of the origin in  $T_x M$ ,  $W$ , such that  $p \circ \exp: W \rightarrow Q$  is an embedding. Then there is a  $v \in W$  such that  $\exp(v) = m_n$  for some  $n$ . Therefore  $\exp \circ dp(v) = p \circ \exp(v)$  but the RHS equals  $p(x)$  while the LHS clearly does not. This is a contradiction, so  $\xi$  must be strictly positive.



Using this  $\omega$  we modify the time dependent vector field  $\gamma^*(x, t)$  by multiplying it by a bump function  $\rho(t)$ , which is 1 when  $t \leq \omega$  and 0 for  $t \geq 2\omega$ . We call this *phasing out* the flow. This will end the flow once we have straightened the vector field near  $D - U'$  but before it can affect any other points of  $M$ . This flow from this modified  $\gamma^*$  is the local move.

### The Size of the Local Flow

What remains to be proven is that we can ensure that the local move moves each point of  $M$  a distance of less than  $\varepsilon/2$  and that the field will be straightened except for in a neighborhood of  $H$ . As we observed before, the flow near  $H$  can then be made to be  $C^\infty$ -small. So we need to further examine the flow in  $N|V$ , i.e. the tubular neighborhood  $N$  near  $V$ .

The first step is to notice that since the tubular neighborhood near  $V$  is away from  $H$ , vertical projection is an immersion. This allows us to use three kinds of local coordinates: coordinates in  $M$ , horizontal (i.e. parallel to  $Q$ ) coordinates perpendicular to  $M$ , and one vertical coordinate. Without loss of generality we can assume that the fibers of  $N$  over  $V$  are compatible with these coordinates. This implies that the foliation of  $N$  given by restricting to flow lines of  $\varphi$  is compatible with the local coordinates. So the flow lines of  $\varphi$  are invariant under vertical and sideways translation. We call this foliation  $\Upsilon$ . Furthermore we can assume that the upwards rotation, turning  $\alpha$  to  $\beta$ , is described in these local coordinates.

We now come to the reason why we introduced  $\varphi$ : the flowlines of the modified global flow lie in leaves of  $\Upsilon$  or vertical translates. To see this first consider that  $\alpha$  has no components in  $TM$  since it is perpendicular to  $M$ . Secondly we note that the projection of the unit upwards vector,  $\nu$ , onto  $TM$ , outside  $H$ , is in the direction of  $\varphi$ , else  $\varphi$  would not point the most upwards in the tangent space. Therefore the rotation of  $\alpha$  towards  $\nu$  into  $\beta$  takes place in the vector space generated by  $\varphi$  and  $\nu$ . So on  $M - H$ ,  $\beta$  is tangent to the leaves of  $\Upsilon$ . Similarly this is true of  $\gamma$  and therefore initially of

$\gamma^*$ . Thus the modified global flowlines will lie in the leaves of  $\Upsilon$  or its vertical translations. The vertical translations come into it because of the extra vertical factor we introduced when we modified the global flow. Since we have constrained the flow to these leaves, it will allow us to calculate how much each point will move under the flow.

Before continuing it will be useful to have a picture of a generic leaf and of the modified global flow. We can think of the leaf generated by a flowline  $\eta$  of  $\varphi$ ,  $L(\eta)$ , as a vertical river. The river has a depth resulting from the horizontal translates of  $\eta$ . Vertical translation is equivalent to flowing upstream or downstream. The banks are formed from the vertical and horizontal translations of the endpoints of  $\eta$ , either where  $\varphi = 0$  or where it intersects  $\partial V$ . The local move is generated by a time dependent vector field on the leaf  $L(\eta)$ . This vector field, a disturbance in the river, moves downstream at unit speed because of the extra term in the modified flow. However, as it moves downwards its magnitude goes to zero because of the phasing out. A point moving in the local flow will always flow downwards. The flow is stationary on the banks, so when a point reaches a bank, the vector field will be straightened there. It is straightened if the disturbance is below the point as well. Finally, it is also straightened if it reaches the ‘surface’ or ‘bed’ of the river. These are the boundaries of the tubular neighborhood  $N$  in the sideways direction. We need to check that the time dependent vector field that will straighten  $\eta$  will not meet with some part of  $M$  directly below  $\eta$ . Similarly, we might also worry that the disturbance generated from some  $\eta'$  above  $\eta$  might interfere with the vector field once we have straightened it. This is precisely why we phased out the local flow, the vector field is damped to nothing before that can happen. Now let’s examine things a bit more closely to show that we can arrange it so no point moves more than  $\varepsilon/2$ .

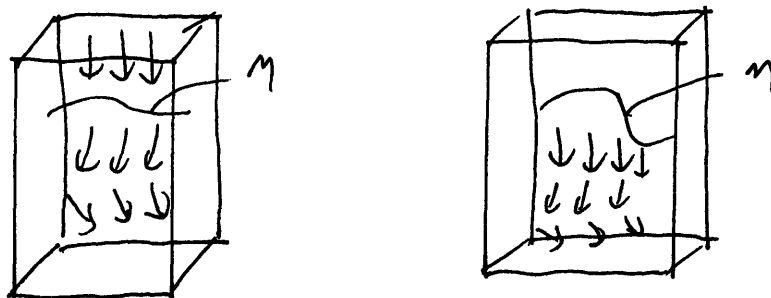


Figure 6:  $L(\eta)$ , the river, with the disturbance before and after it has passed by  $\eta$ .

First we estimate the size of a vector in  $\gamma^*$ . Given a vector  $v$  of  $\gamma$  and suppose its vertically upwards component is less than  $\sqrt{2}/2$ , equivalently that it makes an angle of  $\pi/4$  or less above the horizontal. It is the rotation of a vector of  $\alpha$  of an angle at least  $\pi/2 - \mu$  and this vector has sideways component and a component under  $\varphi$  of at most  $\pi/2 - \mu$  below the horizontal. Then the combined sideways component and component under  $\varphi$  of  $v$  must have size at least  $\sqrt{2}/2$ . Subtracting the unit vector to get the corresponding vector of  $\gamma^*$ , we get that one of the following is always true of a vector of  $\gamma^*$ :

1 Its vertically downwards component is less than  $1 - \sqrt{2}/2$ , i.e. it is moving upwards with respect to the disturbance at a speed bigger than  $\sqrt{2}/2$ .

2 Its combined sideways component and component under  $\varphi$  is at least  $\sqrt{2}/2$ .

In other words, if the vector field points downstream then it always points towards the boundary of the river. Either the points will flow towards the edges of the river where there they will be straightened or will slow in the middle of river and so will be straightened when the disturbance passes them by.

Now the height  $h$  of the disturbance is determined by the size of the leaf which is itself determined by  $\delta$  and  $\nu$ , the size of  $N$ . The horizontal size of the leaf  $L(\eta)$  is also determined by  $\delta$  and  $\nu$ . Lastly we notice that choosing  $\delta$  to be small compared to the variation of  $\alpha$  means that the sideways movement of a point is roughly in the same direction. So our point will have either reached the boundary of the river or a place where the disturbance has passed after a time of at most  $\sqrt{2}(h + \delta + \nu)$ . Since if 1 were always true  $\sqrt{2}h$  time would be needed and if 2 were always true  $\sqrt{2}(\delta + \nu)$  time would be needed. So since always either of the conditions is true the sum of the times will be sufficient for the point to reach a place where it is straightened. We now note that the points move at a speed  $< \sqrt{2}$ , so we can, by shrinking  $\delta$  and  $N$ , ensure that each point moves less than  $\varepsilon/2$ . Furthermore we can also ensure that all this motion takes place before time  $\omega$ , when we started to damp out the disturbance. This is just to make sure that our observations 1 and 2 remain valid.

Finally, we need to make sure that this straightening process is truly local; that it does not interfere with any points below it and is not affected by any points above. But we know from above that if  $\eta$  is outside  $U'$  the river meets  $M$  again at most about  $3\omega$  from  $\eta$ . We can make this approximation as good as we like by shrinking  $\delta$ . We may also have to shrink  $N$  as well if  $\eta$  is almost but not directly above another part of  $M$  and so the river will overlap if not a vertical translate of  $\eta$ . But in any case, the phasing out will prevent the disturbance from affecting any other part of  $M$  after it has straightened  $\eta$ . Similarly we do not have to worry about any disturbance above  $\eta$  affecting it. Finally we note that by choice of  $\delta$  any point of  $\overline{V - U}$  lies on such a flowline outside  $U'$ , because we picked  $\delta$  so that  $U'$  was at least  $\delta$  away from  $M - U$ .

We have just shown that the local move straightens the field on  $\overline{V - U}$  while moving points there less than  $\varepsilon/2$ . However, we still need to ensure that points in  $U$  move less than  $\varepsilon/2$ . But as we noted before  $\alpha$  is almost horizontal in  $U$ , so  $\beta$  will be almost vertical. So we can choose the parameters for the global flow,  $N$  and  $\mu$  to be so small that the local move moves  $M - \overline{V - U}$  a  $C^\infty$ -small amount.

All that remains is to straighten the field on  $U$ . Localization and the local move will have only straightened the field  $\beta$ . So after those two moves  $\beta$  will still be almost vertical on  $U$ . So as we have noted before we can choose  $U$  and  $\mu$  to be small enough

so that the global flow will move points in  $U$  less than  $\varepsilon/2$ . ■

### 3.4 Summary of the Local Proof

We briefly recap the steps involved:

1. Adjust  $\alpha$  to be perpendicular and  $\varepsilon$ -grounded.
2. Adjust  $\alpha$  so that  $D$  is a manifold with boundary  $H$ .
3. Choose small neighborhoods of  $H, U' \subset U$ .
4. Adjust  $\alpha$  so that  $\overline{D - U'}$  is in general position with respect to  $\varphi$ .
5. Pick a sufficiently small  $V$ .
6. Localize  $\alpha$  with respect to  $U \cup V$ .
7. Perform upwards rotation of  $\alpha$  to get  $\beta$ .
8. Choose  $N$ , a tubular neighborhood of  $M$ , and globalize  $\beta$  to  $\gamma$ .
9. Perform the phased out modified global flow to straighten the field near  $D - U'$ .
10. Straighten the field using the modified global flow near  $H$ .

## 4 Multiple vector fields

Now we can use the results of the *local* Compression Theorem to extend it when there are multiple vector fields on  $M$ . The proof is inductive. Once one vector field has been straighten we project along an  $\mathbb{R}$ -factor. The result is an immersion with  $n - 1$  independent normal vector fields. The local Compression Theorem can be applied to immersions so we can proceed. We can not use the global proof because as we showed before, it can fail for immersions.

Let  $M$  be embedded in  $Q \times \mathbb{R}^n$  and suppose that  $M$  has  $n$  linearly independent vector fields. We call  $M$  *parallel* if the  $n$  fields are parallel to the  $n$  coordinate directions of  $\mathbb{R}^n$ .

**Multi-compression Theorem:** *Suppose that  $M^m$  is embedded in  $Q \times \mathbb{R}^n$  with  $n$  independent normal vector fields and  $q > m$ . Then  $M$  is isotopic by a  $C^0$ -small isotopy to a parallel embedding.*

**Proof:** The first step is apply the Compression Theorem. To do that find an embedding of  $M \times D^{n-1}$  in  $(Q \times \mathbb{R}^{n-1}) \times \mathbb{R}$  by exponentiating along the other  $n - 1$  vector fields slightly. Then we straighten the remaining field to get an immersion of

$M \times D^{n-1}$  into  $Q \times \mathbb{R}^{n-1}$ , or rather an immersion of  $M$  with  $n - 1$  independent vector fields into  $Q \times \mathbb{R}^{n-1}$ . Now we pull back a neighborhood of  $M$  by the immersion to an induced neighborhood of  $M$  in  $Q \times \mathbb{R}^n$ . This is made of patches of open sets of  $Q \times \mathbb{R}^{n-1}$  such that  $M$  restricted to each patch is an embedding. We can then apply the local compression theorem to each patch. This determines a regular homotopy of  $M$  in  $Q \times \mathbb{R}^{n-1}$  which lifts to a isotopy of  $M$  in  $Q \times \mathbb{R}^n$  which straightens the second vector field. We can thus continue in this fashion until all the fields have been straightened. ■

We can also note here that the same argument works if  $M$  is immersed in  $Q \times \mathbb{R}$  and  $q > m$ . We can cover  $M$  with open sets such that the restriction of the immersion is an embedding and then straighten the vector field locally but consistently on the overlaps of those neighborhoods. Naturally, the Multi-Compression is true when  $M$  is immersed in  $Q \times \mathbb{R}^n$ . The example noted above, the circle immersed into  $\mathbb{R}^2$  with a grounded normal vector field, demonstrates that the local result cannot be extended to when  $q = m$ . Else, we could immerse  $S^1$  into  $\mathbb{R}$ .

## 5 The Immersion Theorem

The Immersion Theorem (Smale-Hirsch) gives a 1-1 correspondence between regular homotopy classes of immersions from one manifold into another and homotopy classes of tangential maps. A tangential map  $f: M \rightarrow Q$  is one that can be extended to  $f: TM \rightarrow Q$  such that it linearly embeds each fiber of  $TM$  into  $TQ$ . We call this extension a bundle monomorphism. The power of this correspondence is that for a given map  $f: M \rightarrow Q$ , whether it can be extended to a tangential one is a homotopy theoretic question. It involves finding a cross-section of the appropriate Stiefel manifold bundle. In fact, immersions of  $S^n$  into  $R^q$  for  $q \geq n + 1$  are classified by the group  $\pi_n(V_{q,n})$ , where  $V_{q,n}$  is the Stiefel manifold of  $n$ -frames in  $\mathbb{R}^q$ . This leads to the oft-remarked fact that since  $\pi_2(V_{3,2}) = \pi_2(SO(3)) = 0$ , the 2-sphere can be turned inside out in  $\mathbb{R}^3$ . The interested reader is referred to [Smale], [Hirsch], [Rourke & Sanderson, 1999] for more detailed information and to [Spring] for a historical overview of the Immersion Theorem and its vast generalizations.

The baby version of the Immersion Theorem takes a tangential map and constructs a corresponding immersion. To go the other way is easy: differentiating an immersion gives a bundle monomorphism. Conversely given a tangential mapping,  $F: M \rightarrow Q$  we can use the Compression Theorem to construct explicitly the required immersion. The key idea is to use the tangential mapping combined with an embedding of  $M$  into  $\mathbb{R}^n$ , for some  $n$ , to construct an embedding of  $TM$  into  $Q \times \mathbb{R}^n$ . Then we use an easy trick to outfit  $M \subset TM \subset Q \times \mathbb{R}^n$  with  $n$  independent normal vector fields. Finally we use the Multi-Compression Theorem to construct the desired immersion.

The first step is to understand the tangent bundle of  $TM$  restricted to  $M$ . We can

think of  $M$  as embedded inside  $TM$ , by taking the zero section of the tangent bundle. We claim that  $T(TM)|_M = TM_z \oplus TM_f$  ( $z$  for zero section and  $f$  for fiber). Given any point  $v_x \in TN$ , there is a neighborhood about that point isomorphic to  $U \times \mathbb{R}^n$ , where  $U \subset M$ . So there is a direction along the  $M$  factor and one along the fiber factor. But since the fibers are linear they can be identified with their tangent space, so indeed  $T(TM) = TM_z \oplus TM_f$ .

Now we can prove the simplest version of the Immersion Theorem:

**Immersion Theorem (first version):** *Given a tangential map  $f: M^m \rightarrow Q^q$  with  $m < q$ , then  $f$  is homotopic to a smooth immersion of  $M$  into  $Q$ .*

The first step is to embed  $M$  into  $Q \times \mathbb{R}^n$  for some  $n$ . First we extend our tangential map  $f$  to  $f: TM \rightarrow TQ$ . We can then compose this with exponentiation to get a map  $g: TM \rightarrow Q$ . Now we choose an embedding  $h: M \rightarrow \mathbb{R}^n$  for some  $n$ . We now define a map  $g \times h: TM \rightarrow Q \times \mathbb{R}^n$  given by  $g \times h(v_m) = (g(v_m), h(m))$ , where  $v_m$  is an element of the fiber at  $m$ . The only thing to check is injectivity. If  $m \neq n$ , clearly  $g \times h(v_m) \neq g \times h(v_n)$ . We assumed that  $f$  was a tangential mapping so  $g \times h$  is injective on the fibers of  $TM$ . We notice also that the tangent directions along each fiber are perpendicular to the  $\mathbb{R}^n$  directions, because each fiber of  $TM$  is sent to a point in  $\mathbb{R}^n$ . We now only need to exploit this perpendicular relation to get  $n$ -linearly independent normal vector fields on  $M \subset TM$ . To do this we need a lemma whose proof we will momentarily defer.

**Lemma:** *Given a tangent bundle  $TM$ , there is a family of linear transformations taking each factor of  $T(TM) = TM_z \oplus TM_f$  to the other.*

Now pick a complement of  $TM_z \oplus TM_f \subset T(Q \times \mathbb{R}^n)|_M$ . We now use the lemma to define a family of linear transformations  $R_t$  on  $T(Q \times \mathbb{R}^n)|_M$  by declaring that  $R_t$  be the identity on the complement and that  $R_0$  is the identity and  $R_1$  interchanges the factors of  $TM_z \oplus TM_f$ . That is

$$R_1(TM_z) = TM_f \text{ and } R_1(TM_f) = TM_z$$

$R_t$  is an orthogonal transformation of  $T(Q \times \mathbb{R}^n)$ , so since  $TM_f \perp \mathbb{R}^n$  then

$$R_1(TM_f) \perp R_1(\mathbb{R}^n) \iff TM_z \perp R_1(\mathbb{R}^n)$$

So  $R_1(\mathbb{R}^n)$  are  $n$  independent vector fields perpendicular to  $M \subset TM$ . We have embedded  $M$  into  $Q \times \mathbb{R}^n$  with  $n$  linearly independent normal vector fields, so we can apply the Multi-Compression Theorem to get an immersion of  $M$  into  $Q$ . This results in a homotopy of  $f$  to an immersion because our map  $g \times h|_M$  projected onto the  $Q$  factor is our original map, and the Multi-Compression theorem will homotope this image of  $M$  in  $Q$  to an immersion. ■

**Proof of the Lemma** As we showed above  $T(TM)|_M = TM_z \oplus TM_f$ . So each vector  $v$  in  $TM$  corresponds to two vectors,  $v_1$  in the first factor and  $v_2$  in the second. The direction along the zero section is  $v_1$ , while the fiberwise direction is  $v_2$ . These two vectors span a plane in  $TM$  and we rotate one in the other by the following linear transformations

$$R_t(v_1) = \cos\left(\frac{\pi}{2}t\right)v_1 + \sin\left(\frac{\pi}{2}t\right)v_2$$

$$R_t(v_2) = \cos\left(\frac{\pi}{2}t\right)v_2 - \sin\left(\frac{\pi}{2}t\right)v_1$$

Performing the transformation on every plane in  $TM_z \oplus TM_f$  generated by a vector in  $TM$  will rotate the directions along the fibers to be along the zero section, that along  $M$  inside  $TM$ . ■

Now we want to extend this proof to a more substantial version of the Immersion Theorem. We will show that a  $K$ -parametrized set of tangential maps is homotopic to one of immersions through tangential maps. In particular this shows that there is a 1-1 correspondence between homotopy classes of immersions and tangential maps when  $K = I$ .

We need to define what we mean by  $K$ -parametrized and extend the proof of the Compression Theorem to this circumstance. In the  $K$ -parametrized Compression Theorem there is no additional dimensional constraint if the vector field are grounded. Therefore, in order to apply the Multi-Compression Theorem we will need to check very carefully that each step of our inductive straightening process leaves the remaining vector fields grounded.

***K-Parametrized Compression Theorem:*** *Suppose  $M_t^m \subset Q_t^q \times \mathbb{R}$  is a parametrized family of embeddings with a normal vector field for each  $t \in K$ , where  $K$  is a manifold of dimension  $k$ . Then if  $q > m + k$  then there is a parametrized family of small isotopies to a parametrized family of compressible embeddings. Furthermore if all the fields are perpendicular and grounded then there is no extra dimensional constraint, i.e the statement is true if  $q > m$ .*

**Proof** The proof is straightforward. In the appendix we prove that the transversality conditions, i.e. that  $D$  and  $H$  are manifolds and that  $\partial D = H$ , we used hold in the parametrized case. Here  $D$  and  $H$  are subsets of  $K \times M$  and are the unions of the downsets and horizontal sets belonging to each fiber. We also prove that after a small isotopy the fields can be assumed to be perpendicular and grounded if  $q > m + k$ . We can localize the field in the same way and can move  $D$  into general position with respect to the flow  $\varphi$  on  $K \times M$ , which is the same as  $\varphi$  restricted to each fiber. Now the proof works as before. The only additional dimensional constraint is added by the isotopy of our vector field which makes it grounded and perpendicular. So if these conditions are already met then the statement is true if  $q > m$ . ■

The proof of the parametrized version of the Multi-Compression Theorem will be true

if  $q > m + k$ . However, if the  $n$  vector fields are initially grounded we are not assured that they will remain so while we straighten the other vector fields. When  $q > m + k$  we can ground each vector field as we come to it in the inductive step. To prove the full Immersion Theorem without the extra constraint, we need that the vector fields will remain grounded as we come to them.

Now let  $F: K \times M \rightarrow Q$  be a smooth map such that, for each  $t \in K$ ,  $F|_{\{t\} \times M} \rightarrow Q$  is an immersion. Then we call  $F$  a  $K$ -family of immersions of  $M$  into  $Q$ . Similarly we can define the notion of a  $K$ -family of tangential maps and bundle monomorphisms. If we differentiate a  $K$ -family of immersions we get a  $K$ -family of bundle monomorphisms. Now we can state and prove the full Immersion Theorem.

**Full Immersion Theorem:** *Let  $K$ ,  $M$ , and  $Q$  be smooth manifolds with  $M$  compact and suppose  $q > m$ . Suppose  $G: K \times TM \rightarrow TQ$  is a  $K$ -family of bundle monomorphisms then there is a  $K$ -family of immersions  $F: K \times M \rightarrow Q$  such that  $dF$  is homotopic to  $G$  through bundle monomorphisms.*

**Proof** We construct  $F$  as in the first version of the Immersion Theorem but using the parametrized version of the Multi-Compression Theorem. As have noted above, in order that no additional dimensional constraint is imposed by the  $K$  factor, we need to ensure that our vector fields are initially grounded and stay so during each step of the induction.

We modify the construction of the  $n$ -normal independent vector fields in our first version. We want to start off with  $n$  *nearly straight* normal vector fields, that is each of the  $n$  vector fields to make a small angle with one of the  $\mathbb{R}^n$  axes. We will then show that each straightening a vector field only moves the other vectors a small amount. Therefore we can always proceed with the induction and will finish with a compressible embedding of  $M$  in  $Q \times \mathbb{R}^n$ .

Let  $\mu > 0$ . Then we can choose a scale factor of  $\mathbb{R}^n$  to make distances there so large compared to  $Q$  that the embedding of  $M$  inside  $Q \times \mathbb{R}^n$  is almost parallel to  $\mathbb{R}^n$ . Furthermore we can also ensure the following is true for all  $x \in M$ . Consider the rotation of  $T_x M$ , the fiberwise factor, given by  $R_{1-\frac{2\mu}{\pi}}$ , (rotation through an ‘angle’ of  $\frac{\pi}{2} - \mu$ ). If  $\{v_1, \dots, v_m\}$  is a basis for the image of  $R_{1-\frac{2\mu}{\pi}}$  on this factor, and  $\{e_1, \dots, e_n\}$  is a standard basis for  $T_x \mathbb{R}^n$ , then the vectors  $\{v_1, \dots, v_m, e_1, \dots, e_n\}$  will form an  $(n + m)$ -frame at  $x$ .

Now we extend the rotation  $R_{\frac{2\mu}{\pi}}$  as before by choosing a complement to  $TM_x \oplus TM_f \subset T(Q \times \mathbb{R}^n)|_M$  and declaring it to be the identity on that complement. Call the image of  $\{e_1, \dots, e_n\}$  under this transformation  $\{f_1, \dots, f_n\}$ . Then the  $\{f_1, \dots, f_n\}$  are independent, normal to  $M$  at  $x$  and nearly parallel to  $\{e_1, \dots, e_n\}$ . We do this at every point in  $M$  to get our  $n$  nearly straight, normal, independent vector fields. We now have to check that this property of being nearly straight is not changed when we



apply the local Compression Theorem to one vector field.

We will show that the field we use to isotope  $M$  to straighten a given vector field is roughly nearly straight. This means that the effect on the remaining vector fields will not disturb their nearly straight status and we can continue the induction process.

Now suppose we are straightening the nearly straight vector field  $\rho$ . Let  $\alpha$  be the corresponding field that is perpendicular to  $M$ , and let  $\beta$  be the field obtained from  $\alpha$  by upwards rotation. Since both  $\rho$  and  $\alpha$  make angles of less than  $\pi/2$  with  $M$  and  $\alpha$  is perpendicular to  $M$ , then by spherical geometry we see that the straight line isotopy connecting  $\rho$  to  $\beta$  does not pass through  $M$ . Furthermore we note that  $\beta$  is at least as close to vertical as  $\rho$ . Therefore by an isotopy of nearly straight fields we can assume that our nearly straight field is obtained from upwards rotation of a perpendicular field.

Now consider the proof of the Local Compression Theorem. Except for localization, all the isotopies applied to  $\alpha$  are general position moves and therefore . However, the effect of localization is to increase the vertical component of  $\alpha$  and hence of  $\beta$  when upwards rotation is performed. Thus the vector field on  $Q \times \mathbb{R}^n$  used to generate the isotopy of  $M$  to straighten the vector field in question is almost as straight as the original field. Therefore the remaining vector fields yet to be straightened will still be nearly straight and therefore grounded; they will not become ungrounded by small isotopies in a direction perpendicular to them. Thus we do not have to worry about any extra dimensional constraint. So for every element of  $K$  we can straighten the field.

Finally we just need to check that  $dF$  is homotopic to  $G$  through bundle monomorphisms. We first note that our proof provides a homotopy through bundle monomorphisms of  $TM$  into  $T(Q \times \mathbb{R}^n) = T(Q) \oplus \mathbb{R}^n$ . We cannot simply project this homotopy onto the first factor to get a homotopy through bundle monomorphisms of  $G$  and  $dF$ ,  $F$  being the resulting  $K$ -family of immersions. This is because the image of a subspace of  $TM$  may lie in  $\mathbb{R}^n$ . The following adjustment to the homotopy will prevent this from occurring.

If we consider the normal fields as well we get a homotopy,  $h$ , through bundle monomorphisms of  $TM \oplus \mathbb{R}^n$  into  $T(Q) \oplus \mathbb{R}^n$ . Now  $h_0 = G \oplus id$  since our  $n$  normal fields began life parallel to the  $\mathbb{R}^n$  axes. Also  $h_1 = J \oplus id$ , where  $J$  projected onto  $Q$  gives  $dF$ . In our proof we kept the normal fields close to  $\mathbb{R}^n$  so there is a canonical homotopy of  $h|_{\mathbb{R}^n}$  to the identity. We can extend this to a homotopy of  $h$  by considering a complement of the image of  $h|_{\mathbb{R}^n}$  and declaring it to be constant there. So we now have a new homotopy through bundle monomorphisms,  $h'$  from  $G \oplus id$  to  $J \oplus id$  which is constant on the second factor. This means that the aforementioned problem will not occur and so that projection onto  $TQ$  will yield the desired homotopy through bundle monomorphisms of our original  $K$ -family  $G$  and our  $K$ -family of immersions  $F$ . ■



## Appendix

We here prove the technical transversality and general position arguments that we deferred in the previous proofs.

Let  $M^m$  and  $K^k$  be manifolds and  $F: K \times M \rightarrow \mathbb{R}^n$  be a smooth map such that  $F|_{\{t\} \times M}$  is a smooth embedding into  $\mathbb{R}^n$  for each  $t$ . We introduce the manifold  $K$  because we can prove the Compression Theorem for a set of  $K$ -parametrized embeddings. This is required for the full Immersion Theorem which gives a correspondence between a  $K$ -family of tangent maps and one of immersions. We say that  $F$  is a  *$K$ -family of embeddings*. We call  $F(\{t\} \times M)$   $M_t$ . Let  $E_n$  be the group of isometries of  $\mathbb{R}^n$ . Define a new map  $J: E_n \times K \times M \rightarrow \mathbb{R}^n$  by  $J(u, t, x) = uF(t, x)$ . We can think of  $J$  as a two parameter family of maps from  $M$  into  $\mathbb{R}^n$ . Finally let  $\mathcal{G}_{n,m}$  denote the Grassmannian of  $m$ -planes in  $\mathbb{R}^n$ . If  $u \in E_n$ , then we define  $F_u: K \times M \rightarrow \mathcal{G}_{n,m}$  by  $F_u(t, x) = dJ_{J(u,t,x)}(\{u\} \times \{t\} \times M)$ . I.e. we embed  $M$  into  $\mathbb{R}^n$  according to the parameters  $u$  and  $t$  and then consider the tangent space to  $M$  at  $x$  lying inside  $\mathbb{R}^n$ .

**Proposition A.1:** *Given a neighborhood  $N$  of the identity of  $E_n$ . and a submanifold  $W$  of  $\mathcal{G}_{n,m}$  then there is an element  $u \in N$  such that  $F_u: K \times M \rightarrow \mathcal{G}_{n,m}$  is transverse to  $W$ .*

**Proof** We define  $G: E_n \times K \times M \rightarrow \mathcal{G}_{n,m}$  by  $G(u, t, x) = F_u(t, x)$ . Now to prove the existence of such a  $u$  it is enough to show that the  $dG$  is surjective everywhere. This is because the surjectivity of the derivative means that  $G$  is a transverse map to  $W$ . This in turn implies that for almost every  $u \in E_n$   $F_u$  is transverse to  $W$ . The proof of this last implication can be found in [Guillemin and Pollack, pg. 68]. In fact, it is enough to consider the derivative on tangents to  $E_n \times \{t\} \times \{x\}$ . It is not surprising that we can restrict our attention to tangent vectors of  $E_n$  because we have no information about  $K, M$ , or  $F$  except that  $F_t$  is an embedding.

We represent a tangent vector to  $\mathcal{G}_{n,m}$  by  $\dot{\alpha}(0)$ , where  $\alpha: (-\varepsilon, \varepsilon) \rightarrow \mathcal{G}_{n,m}$  and  $\alpha(0) = G(u, t, x)$ . Now let  $q: O_n \rightarrow \mathcal{G}_{n,m}$  be the fibration given by  $q(v) = v(\alpha(0))$ . So  $q(\text{Id}) = \alpha(0)$  and the elements of  $O_n$  rotate the  $m$ -planes around in  $\mathbb{R}^n$ . Choose  $\beta: (-\varepsilon, \varepsilon) \rightarrow O_n$  such that  $q\beta = \alpha$ . So the picture is that  $\alpha$  is a one parameter family of  $n$ -planes and we realize this by a one parameter family of rotations of the ambient space  $\mathbb{R}^n$ . Now we define  $\gamma: (-\varepsilon, \varepsilon) \rightarrow E_n$  by  $\gamma(s) = tr_{J(u,t,x)} \circ \beta(s) \circ tr_{-J(u,t,x)}$  where  $tr_z$  is translation by  $z$ . If we identify  $E_n$  with  $E_n \times \{t\} \times \{x\}$  then  $G\gamma = \alpha$ , since  $\gamma$  rotates  $M$  in  $\mathbb{R}^n$  about the point given by  $(u, t, x)$  in the way determined by  $\alpha$  and  $G$  simply translates this into the domain of the Grassmannian. So finally we see that  $dG(\dot{\gamma}(0)) = \dot{\alpha}(0)$ . ■

Now suppose  $Q^q$  is a manifold and that we have a family of embeddings  $F: K \times M \rightarrow Q \times \mathbb{R}^r$  with  $m \leq q$ . Define the *horizontal set*, denoted  $H(K \times M)$ , to be those points  $(t, x) \in K \times M$  such that the tangents to  $\mathbb{R}^r$  at  $F(t, x)$  are contained in the image of  $dF$ . The definition is such that if  $M_t$  has a perpendicular normal field then the field

is horizontal (i.e. contained in the tangent space to  $Q$ ) at those points  $F(t, x)$  when  $(t, x) \in H(K \times M)$ .

We define the *critical set*,  $C(K \times M)$  to be those points  $(t, x) \in K \times M$  such that the image of the tangents at those points are tangent to  $Q$ . The name is because if  $r = 1$  and  $(t, x) \in C(K \times M)$  then  $x$  is a critical point of the function  $M \rightarrow \mathbb{R}$  which is the composition of  $F$  with projection onto the  $\mathbb{R}$  factor. That is,  $F(t, x)$  is either a local minimum or maximum with respect to height.

Finally we note that if  $Q = \mathbb{R}^q$ , then by considering  $\mathcal{G}_{q, m-r} \subset \mathcal{G}_{q+r, m}$  and  $\mathcal{G}_{q, m} \subset \mathcal{G}_{q+r, m}$ , then by proposition A.1, after small Euclidean motion we can assume that  $H(K \times M)$  and  $C(K \times M)$  and submanifolds of codimension  $r(q + r - m)$  and  $rm$  respectively.

For the horizontal set this is because after a small Euclidean motion, i.e. for a some  $u$  close to the identity,  $F_u$  will be transverse to  $\mathcal{G}_{q, m-r} \subset \mathcal{G}_{q+r, m}$  and so the inverse image of  $\mathcal{G}_{q, m-r}$  will be a manifold. This manifold is the horizontal set. It is set of points  $(t, x)$  where  $m - r$ -dimensions of the tangent space of the image of  $M$  at  $F_u(t, x)$  are parallel to  $Q = \mathbb{R}^q$ . So all the  $\mathbb{R}^r$  factors will be in the image of  $dF$ . Another way to think about it is that the  $m$  directions of  $M$  have to be distributed among the  $\mathbb{R}^q$  and  $\mathbb{R}^r$  factors and the horizontal set is composed of those points which have exactly  $m - r$  directions along  $\mathbb{R}^q$ . Similarly the inverse image of  $\mathcal{G}_{q, m}$  represents those points where all  $m$  directions of  $M$  are along  $\mathbb{R}^q$ . The codimensions are such because codimension is preserved when taking the inverse image of a transverse map and the dimension of  $\mathcal{G}_{n, m}$  is  $m(n - m)$ .

Now we extend this result to when  $Q$  is an arbitrary manifold:

**Proposition A.2:** *Given family of embeddings  $F: K \times M^m \rightarrow Q^q \times \mathbb{R}^r$  with  $m \leq q$  then there is an homotopy of  $F$  through families of embeddings such that  $F_0 = F$  and the horizontal set determined by  $F_1$  is a submanifold of  $K \times M$ . The same is also true for the critical set.*

**Proof** As the proofs for both the horizontal set and critical set are similar, we will prove the proposition for the horizontal set. As before the case when  $Q = \mathbb{R}^q$  follows from proposition A.1. To prove the general case we use a patch by patch argument. First we choose a locally finite cover of  $K \times M$  by disks of the form  $D = D_1 \times D_2$  such that the corresponding half disks, i.e.  $\frac{1}{2}D = \frac{1}{2}D_1 \times \frac{1}{2}D_2$ , cover  $K \times M$  and such that the image of each disk under  $F$  projects onto a Euclidean patch of  $Q$ .

Since it is a locally finite cover we can use induction. First the base case. Suppose that  $F(D) \subset U \times \mathbb{R}^r$  where  $U$  is a Euclidean patch. By A.1 we can find a small Euclidean motion of  $U \times \mathbb{R}^r$ ,  $e$ , such that the horizontal set in neighborhood of  $\frac{1}{2}D$  is a manifold with respect to  $e \circ F|D$ . We can extend this first to  $D$  by picking a path from  $e$  to the identity of  $E_{r+q}$  and phasing out the Euclidean motion along a collar

of  $\frac{1}{2}D$  in  $D$  and thence by the identity on the rest of  $K \times M$ .

Now suppose the horizontal set in a neighborhood of the union of the first  $k$  disks is a manifold. Now we can choose an  $e$  and a sufficiently short path to the identity to adjust our map such that the horizontal set in a neighborhood of the  $(k+1)$ st  $\frac{1}{2}$ -disk is a manifold. We can do this without disturbing the property that the horizontal set is a manifold in a neighborhood of the first  $k$   $\frac{1}{2}$ -disks. This is because for each disk the set of possible choices is a set of full measure. Now since our cover is locally finite, we have defined the required isotopy of  $F$ . ■

Now we consider the special case when  $r = 1$  for application to the local and global proofs. We also suppose that  $F: K \times M \rightarrow Q \times \mathbb{R}$  is a  $K$ -family of embeddings with normal vector fields. So for each  $t \in K$  there is a vector field  $\alpha_t$  and it varies continuously with  $t$ . We can identify  $K \times M$  with its image in  $K \times Q \times \mathbb{R}$  with respect to the embedding  $(t, x) \mapsto (t, F(t, x))$ . Now we can think of the union of each  $\alpha_t$  on  $M_t$  as forming a vector  $\alpha$  on the whole  $K \times M \subset K \times Q \times \mathbb{R}$ .

If we choose a metric we can assume that  $\alpha_t$  is perpendicular to  $M_t$  in  $Q \times \mathbb{R}$  for all  $t$ . We define the *downset* as follows, in a completely analogous way as in non-parametrized Compression Theorem. Let  $\nu$  be the normal bundle on  $K \times M$  in  $K \times Q \times \mathbb{R}$  be given as the union of the normal bundles on each  $M_t$  for each  $t \in K$ . Now define  $\psi$  to be the unit vector field on  $K \times M - H(K \times M)$  in  $K \times Q \times \mathbb{R}$  such that  $\psi_t(x)$  points upmost in  $\nu_{(t,x)}$ . Now we define the downset  $D \subset K \times M - H(K \times M)$  to be those points  $(t, x)$  where  $-\psi_t(x) = \alpha_t(x)$ . We note that  $D$  is the union of downsets for each  $\alpha_t$ .

**Proposition A.3:** *We can assume that  $\alpha$  is transverse to  $-\psi$  after a small isotopy. This in turn implies that  $D$  is a manifold, and we can also assume that its closure is a manifold with boundary  $H$ .*

**Proof** By A.2 we can assume that  $H$ , the horizontal set is a manifold with codimension  $c = q+1-m$  in  $K \times M$  (since we have assumed that  $r = 1$ ). As in the previous proof, it will be enough to prove the proposition when  $Q = \mathbb{R}^q$  and then use an analogous patch by patch argument.  $H$  is the tranverse preimage of  $\mathcal{G}_{q,m-1} \subset \mathcal{G}_{q+1,m}$ . We can think of this as the tranverse preimage of  $\mathcal{G}_{q,c} \subset \mathcal{G}_{q+1,c}$  using the metric. Analogously as above, we want those points in which the normal bundle is contained in the  $Q$  direction, thus the vertical direction will be contained in  $TM$ . We need note that to realize  $H$  as a tranverse preimage in this way, we adjust our map  $F$  so that its image is the orthogonal complement of  $dJ$ . Call this adjusted map  $N$ .

We can identify the restriction of the canonical disk bundle  $\bar{\gamma}_{q+1,c}$  to  $\mathcal{G}_{q,c}$  with a closed tubular neighborhood of  $\mathcal{G}_{q,c}$  in  $\mathcal{G}_{q+1,c}$ . Given  $P_p$ , where  $P \in \mathcal{G}_{q,c}$  and  $p \in P \subset \mathbb{R}^q$ , then the corresponding element  $P_p \in \mathcal{G}_{q+1,c}$  is the  $c$ -dimensional subspace spanned by  $(p, -|p|) \in \mathbb{R}^q \times \mathbb{R} = \mathbb{R}^{(q+1)}$  and the subspace of  $P$  orthogonal to  $p$ . Each fiber of the tubular neighborhood has dimension  $c$  because the codimension of  $\mathcal{G}_{q,c}$  is  $c$  and

the dimension of the canonical disk bundle is  $c$  by definition. The subspace  $P_p$  has dimension  $c$  because we replaced the vector  $p$  with  $(p, -|p|)$ .

Now we consider  $\alpha$  near  $H$ . Let  $D_x$  be a fiber of a tubular neighborhood  $U$  of  $H$  in  $K \times M$ . Let  $\pi: U \rightarrow H$  be projection. Now we can identify the fibers of the normal bundle  $\nu$  over  $D_x$  with  $\nu_x$  by picking an isomorphism of  $\pi^*\nu$  with  $\nu|_U$ . We can also homotope  $\alpha$  to be constant over  $D_x$ . Now consider the map from  $\partial D_x$  to the unit sphere given by the image of  $-\psi$ . Since  $N$  is transverse to  $\mathcal{G}_{q,c}$ , then for a small enough  $U$ , we want to show that this map is a diffeomorphism. Given a point  $\xi \in \partial D_x$  we want to understand  $-\psi_\xi$ . Now  $N(\xi)$  will be a point in the tubular neighborhood of  $\mathcal{G}_{q,c}$ . It will also be the fiber of the normal bundle of  $M$  at  $\xi$  considered as a subset of  $\mathbb{R}^{P_{q+1}}$ . By the preceding paragraph  $N(\xi) = P_p$ , which is the subspace spanned by  $(p, -|p|)$  and the orthogonal complement of  $p$  in  $P$ . Clearly  $(p, -|p|)$  normalized to unit length is the unit vector in  $P_p$  with smallest  $\mathbb{R}$  component. We now see that our correspondence between the tubular neighborhood of  $\mathcal{G}_{q,c}$  and the canonical disk bundle implies that the map  $-\psi \mapsto p$  is a diffeomorphism. This in turn, since  $\alpha$  is constant over  $D_x$ , implies that  $-\psi$  and  $\alpha$  meet in one point on  $\partial D_x$ . So now by shrinking the radius of our tubular neighborhood and varying  $x$  we see that  $\alpha$  is transverse to  $-\psi$  on  $U - H$  and that  $\partial D = H$ . Outside  $U$ ,  $\alpha$  and  $-\psi$  are sections of  $S^n$  bundle for  $n \geq 1$  so we can further homotope  $\alpha$  near the complement of  $U$  so that it is transverse to  $-\psi$  on  $K \times M - H$ . So  $D$  is a manifold with boundary  $H$ .

For the general case when  $Q$  is any manifold and not  $\mathbb{R}^q$  then we can do a patch by patch argument in the same fashion as A.2. ■

**Lemma A.4:** *Any small isotopy of  $D$  fixed near  $H$  can be realized by a small isotopy of  $\alpha$ .*

It suffices to prove this for a small neighborhood of  $D$ . Since  $D$  is the transverse intersection of  $\alpha$  and  $-\psi$  then any small isotopy of  $D$  can be realized by a small one of  $\alpha$  which will remain transverse. This process can be repeated to get the desired small isotopy of  $D$  fixed near  $H$ . ■

**Corollary:** *If  $q - m - k \geq 1$  then by a small isotopy of  $F$  and  $\alpha$  we can assume that  $\alpha_t$  is perpendicular and grounded for each  $t \in K^k$ .*

**Proof** A.2 allows us to adjust  $F$  so that the critical set of  $F$  is a manifold of dimension  $k$ . We now recall that  $\alpha_t$  is grounded if and only if its downset is disjoint from the critical set of  $M_t$ . But the dimension of  $D = \dim(H) + 1 = k + 2m - q$ . So  $\dim(K \times M) - \dim(C) - \dim(D) = q - k - m \geq 1$ . So there is a small isotopy of  $D$  and hence of  $\alpha$  by the lemma so that the two manifolds do not intersect. ■

Finally, we need to prove the lemma about general position of  $\overline{D - U}$  with respect to  $\alpha$ . Once again we generalize to our  $K$ -parametrized version. Let  $\varphi_t$  be the gradient vector field of vertical projection on  $M_t$  for each  $t \in K$  and let  $\varphi$  be the vector field

on  $K \times M$  so that  $\varphi|_{\{t\}} \times M = \varphi_t$ .

**Corollary:** *Suppose that  $\alpha$  is perpendicular and grounded. Let  $U$  be a tubular neighborhood of  $H$  in  $K \times M$ . There is a small isotopy of  $\alpha$  such that given  $\delta > 0$  there is a neighborhood  $V$  of  $\overline{D - U}$  in  $M$  such that every component of an integral curve of  $\varphi$  with  $V$  has length  $< \delta$ . We then say that  $\overline{D - U}$  is in general position with respect to  $\varphi$ .*

The first step is to show that we can adjust  $\alpha$  so that the integral curves of  $\varphi$  intersect  $\overline{D - U}$  in a discrete set. Assuming this, given  $\delta$  we consider a continuous real values function on  $\overline{D - U}$ . We define the function as follows. First pick a tubular neighborhood of  $\overline{D - U}$  in  $M$ ,  $T$ . Now for each point of  $\overline{D - U}$  choose a positive real number  $\epsilon$  such that the intersection of the integral curve passing through the point and the subtubular neighborhood of  $T$  with size  $\epsilon$  has length less than  $\delta$ . Now if  $K$  and  $M$  are compact, this function achieves a minimum on  $\overline{D - U}$  and so the open tubular neighborhood of that radius will be the desired open set  $V$ . If  $K$  and  $M$  are not compact we can use a similar patch by patch argument as in A.2.

To show we can define the aforementioned function consider a parametrization by arc length of an integral curve,  $\eta(t)$ , though a point in  $\overline{D - U}$ . We define a continuous function  $h: (-\epsilon, \epsilon) \rightarrow \mathbb{R}$  by defining  $h(t)$  to be the distance of  $\eta(t)$  to  $\overline{D - U}$  with respect to the tubular neighborhood  $T$ . Because each integral curve intersects  $\overline{D - U}$  in a discrete set, we know that  $h(0) = 0$  and that near zero  $h$  is strictly positive. We can choose a value  $\epsilon$  such that  $0 < \epsilon < \min\{h(\delta), h(-\delta)\}$ . Therefore the intersection of the subtubular neighborhood of  $\overline{D - U}$  of radius  $\epsilon$  with the integral curve passing through our point will be less than  $\delta$ . The idea is that before the integral curve can get to the ‘height’ it achieves when it is of length  $\delta$ , it gets cut off by the tubular neighborhood. Although the choice of  $\epsilon$  is not canonical, it is not hard to see that we can choose it continuously using a partition of unity argument.

Now we prove that after small isotopy of  $\alpha$ , the integral curves of  $\varphi$  will intersect  $\overline{D - U}$  in a discrete set. The lemma A.4 allows us to consider small isotopies of  $\overline{D - U}$  instead of  $\alpha$ . We just need to see that in general position  $\overline{D - U}$  satisfies the statement. Since the condition is, in essence, a local one we consider a small flowbox of  $\varphi$ . That is we can consider a open set  $W$  isomorphic to  $I^m$  with coördinates such that the integral curves are parallel to the first  $I$  factor. Now the statement is equivalent to showing that the inverse image of the map projecting  $\overline{D - U}$  onto  $0 \times I^{m-1}$  has discrete fibers. Here we appeal to a theorem of John Mather, which says that a generic projection is transverse with respect to the Thom-Boardman varieties, here see [Mather] and [Boardman]. Thus we can move  $\overline{D - U}$  slightly so that our condition is satisfied and the corollary is proved. ■





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