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DETECTION OF GEOLOGIC ANOMALIES
BY GRID LINE SEARCH

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The first problem addressed is to determine the probability that a geologic anomaly as represented by its projective area on the surface will be crossed by at least one of a pattern of rectangular (seismic survey) grid lines partitioning a subregion of a petroleum basin into rectangles.

The solution to this problem is used in the design of a maximum likelihood method for making inferences about the size distribution of anomalies in the region from observation of the sizes of a set of detected anomalies and about the number of undetected anomalies remaining as well. (17 pages)

1. INTRODUCTION

The problem of detecting a geologic anomaly by line search was studied by Agocs (1955) for the case where lines run parallel to one another and in one direction only. The probability that an anomaly with convex projective shape on the surface will be crossed by one or more parallel grid lines is the solution to a generalized Buffon needle problem. Agocs briefly mentions the more general problem: determine the probability that an anomaly will be detected by a grid of lines that partitions the region of search into rectangles. A solution for the case of circular anomalies is given.*†

McCammon (1976) reviews these results and, for the case of parallel line search, provides a formula for the probability that two parallel lines will intersect a randomly placed and randomly oriented line of given length. He places bounds on intersection probabilities for parallel-line and square grid search by considering the limiting cases of a circular target and a target composed of a line asserting that, "In exploration, after all, rarely

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*Agoc's solution is incorrect; the correction is given by P. Boisard (1966). It is a simple special case of the a more general result proven in this paper (cf. formula (9)).

†Detection probabilities for continuous grid search for targets whose projective area on the surface can be approximated by a convex figure, or more particularly, by an ellipse, are similar in functional form but not identical to detection probabilities for grid patterns generated by drill holes. For a given partition of a region into rectangles or rhomboids by grid lines, the corresponding drill hole pattern consists of search points at the intersections of grid lines; hence the probability of detection of a target of fixed areal extent is always less than the corresponding detection probability for grid-line search. Detection probabilities for drill hole patterns have been extensively studied by Drew (1966, 1967), Savinskii (1965), Singer (1969, 1972, 1975), Tsaregradskii (1970), Slichter (1955) and others. Savinskii provides tables of detection probabilities for elliptical targets; Singer (1972) presents a computer program, Elipgrid, that computes detection probabilities.

are the shapes of targets being sought known with much precision." McCammon summarizes his study in the form of six conclusions:*

> (1) To a first approximation, the probability of intersection of a simple plane geometric figure is directly proportional to the ratio of the largest dimension of the plane projection of the figure to the minimum spacing between lines along which the search is conducted.
>
> (2) When the largest plane dimension of the hidden target is small compared with the smallest spacing between the lines of search, target shape does not greatly affect the probability of intersection.
>
> (3) The probability of intersecting a target twice for a particular type of search can be used as a lower bound if there is an element of uncertainty of detection for a particular type of geophysical tool.
>
> (4) The geometry of the search pattern becomes more critical as the largest dimension of the target approaches the minimum line spacing of the search. When the largest dimension is less than the minimum line spacing, the probability of intersection is greater for parallel-line search than for an equibalent square-grid type search, whereas the opposite is true when the largest dimension exceeds the minimum line spacing.
>
> (5) The probability of intersection of an elliptical target for a rectangular grid can be approximated by considering the limiting cases of a line and a circle for a parallel-line and square-grid type of search, respectively.
>
> (6) Nonorthogonal grids do not greatly affect the probability of intersection, provided target orientation is unknown.

A principal objective of the work of Slichter (1955) and Agocs is to deduce characteristics of grid-line search useful in optimizing the design of a search strategy when the cost of search increases with an increase in the density of grid-lines crossing a region of fixed area (cf. Slichter (1955)).

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*McCammon (1976), p. 381.

An excellent example is the work of Pachman (1966); he provides a model for optimal allocation of seismic grid-line search effort in one or more regions subject to know cost-effectiveness functions and an overall budget contraint. Targets are elliptical with <u>fixed</u> ratio of major to minor axes, target areas are generated by a lognormal random process, and each target is located in a region by assigning kinematic density to it. Detection probabilities for a range of grid-line configurations and expected payoff generated by them are computed by monte carlo simulation. According to McCammon,

> What is desired, however, is not an optimum spacing but rather some knowledge of the uncertainty associated with a particular search strategy, which is dictated most often by the costs of exploration.

A logical extension of this thought leads to the following inference problem:

In a geographic region bounded by a simple curve of finite length, Nature deposes N+M geologic anomalies, each of which may or may not be a mineral deposit. A search for deposits is carried out by seismic or magnetic surveying along linear grid lines which partition the region into rectangles.

Suppose that an anomaly is <u>detected</u> if at least one grid line crosses the surface area of the anomaly. (More realistically, let the probability that an anomaly will be detected is a monotone increasing function of the number of grid lines crossing it.)

What can be said a priori about the number of anomalies that will be detected? About the surface area and shape of a generic anomaly that remains undetected? If the number of anomalies is not known with certainty what inferences about it can be made from observation of the surface area, shape, and orientation of anomalies detected by the survey? What inference can be made ex post about the surface areas and shapes of undetected anomalies?

In order to answer these questions we need to make specific assumptions about:

(1) the spatial density of anomalies; i.e., the probability law that characterizes the number of anomalies in each element dA of area of the region,

(2) the probability law for the surface area and shape of a generic anomaly,

and

(3) the probability law that describes jointly the location of a fixed point within a generic anomaly and the angle formed by a line fixed in it with respect to a reference coordinate system.

2. SIZE DISTRIBUTION AND DETECTION PROBABILITIES

We study the problems of detection and inference about the parameters of the underlying size distribution in two dimensions, viewing and anomaly. as a rigid figure K on the x-y plane; i.e. an anomaly is represented by its projection onto the x-y plane. Its position is determined by the location of a point $p(x,y)$ fixed in K and the angle θ formed by a line fixed in K and a fixed line in the x-y plane. We consider only figures whose boundaries are simple closed curves.

To fix ideas, first suppose that the figure K is randomly located on the plane in the following fashion: all angles $\theta \varepsilon [0,2\pi]$ are equally likely and the probability that the point $p(x,y)$ is contained in a rectangle with sides of length dx and dy is the same irrespective of the location of this rectangle. We then say that the figure K "has kinematic density on the plane."

A rectangular lattice with sides of lengths 2A and 2B is superposed on the plane and K is <u>detected</u> if at least one lattice line crosses K. Let $D(K|\#)$ be the probability that K is detected given the lattice dimensions. Since the shape and area of K is determined by its boundary curve C , we will write $D(K|\#)$ as $D(C|\#)$ as well.

We further assume that the boundary curve of a generic K can be represented by an equation of specific functional form indexed by a "small" number of parameters, so that numerical values for $D(K|\#)$ can be computed. In the geostatistical literature on search and detection, it is often assumed that an ellipse is a reasonable approximation to the boundary curve of the surface area of an anomaly or deposit, so we study this particular case in detail.

Suppose that N anomalies, $K_1,\ldots,K_N$ are mutually independently distributed in the plane, each with kinematic density and in addition that the size and shape (boundary curve) of each anomaly is random. That is, the boundary curves $C_1,\ldots,C_N$ of $K_1,\ldots,K_N$ are a realization of a random process characterized by a probability law of the following kind: $\tilde{C}_1,\ldots,\tilde{C}_N$ are mutually independent and identically distributed with common density $g(\cdot|\underline{\theta})$ indexed by a parameter $\underline{\theta} \varepsilon \Theta$ of finite (and small) dimension. For example, if the boundary curve of a generic anomaly is elliptical in shape, assigning a joint density to the lengths of the principal axes of an ellipse will determine the density of its boundary curve as well as the density for its area.

In practice, the parameter $\underline{\theta}$ of the "size" distribution of anomalies is not known with certainty and one wishes to make inferences about it based on observation of <u>detected</u> anomalies. In order to do so we must compute the joint probability of observations yielded by superposing a given grid design on the plane. Given a point estimate of $\underline{\theta}$, the probability distribution of "sizes" of undetected anomalies can be estimated as well.

The joint density of the random variables $\tilde{C}_1,\ldots\tilde{C}_N$ is, since they are independent, $\prod_{i=1}^{N} g(C_i|\underline{\theta})$. Let $g(C_i|\underline{\theta})dC_i$ denote (an approximation to) the probability that the rv $\tilde{C}_i$ lies in the infinitesimal set dC_i of curves and for the moment ignore measure-theoretic issues. Defining $d\underline{C} = (dC_1,\ldots,dC_N)$ and $\underline{C} = (C_1,\ldots,C_N)$, the joint probability of $\underline{\tilde{C}} \varepsilon d\,\underline{C}$, $K_1,\ldots,K_r$ being detected, and $K_{r+1},\ldots,K_N$ not detected is

$$\prod_{j=1}^{r} D(C_j|\#)g(C_j|\underline{\theta})dC_j \prod_{k=r+1}^{N} [1-D(C_k|\#)]g(C_k|\underline{\theta})dC_k \,. \tag{1}$$

The probability of <u>observing</u> $\tilde{C}_1 \varepsilon dC_1,\ldots,\tilde{C}_r \varepsilon dC_r$ is

$$[1-\bar{D}(\#;\underline{\theta})]^{N-r} \prod_{j=1}^{r} D(C_j|\#)g(C_j|\underline{\theta})dC_j \tag{2}$$

where, letting $\mathcal{C}$ denote the range set of $\tilde{C}$,

$$\bar{D}(\#;\underline{\theta}) = \int_{\mathcal{C}} D(C|\#)g(C|\underline{\theta})dC. \tag{3}$$

If we wish to ignore labelling of the C_is we append a combinatorial factor $\binom{N}{r}$ to (2).

To recapitulate: (2) is the probability of observing $\underline{\tilde{C}}\varepsilon d\underline{C}$ given the lattice # when

(a) we know that there are N anomalies, each of which has kinematic density;

(b) each C_i is of a given functional form indexed by a small number of parameters;

(c) the $\tilde{C}_i$s are independent rvs, identically distributed with density $g(\cdot|\underline{\theta})$ having parameter $\underline{\theta}\varepsilon\Theta$.

Even in this relatively simple case, computation may be reasonably complicated. In what follows we compute $D(C_i|\#)$ and $\bar{D}(\#;\underline{\theta})$ explicitly for the case of random ellipses.

In practice, the number N of anomalies is not known with certainty and it is reasonable to conceive of the number $N(A)$ of occurrences in a region A as being a realization of a stochastic point process. Since each anomaly is of finite area, modelling the "number of occurrences in a region A" as a point process makes sense only if the location of an "occurrence" is defined with respect to a point fixed within the boundary curve of an anomaly. We might choose the center of gravity of each anomaly; when the boundary curve of each anomaly is symmetric, there is a natural center.

The process may be viewed as unfolding in two steps:

(i) a center is located at a point (x,y) according to the probability law governing the location of centers;

(ii) given that a center is located at (x,y), a size and shape is generated according to the probability law governing the $\tilde{C}_i$s.

In addition to assuming mutual independence of the $\tilde{C}_i$s, we shall also assume independence of the point process locating centers from that generating the $\tilde{C}_i$s.

A reasonable model is to regard spatial occurrence of centers as a Poisson process with inhomogenous intensity $\lambda(x,y)$, (x,y) being coordinates of a point in the plane. The intensity $\lambda(x,y)$ may depend on variables other than location as well.

Imposition of a rectangular lattice of lines on the plane divides it into disjoint rectangular regions which we may number 1,2,.... Letting R_i be the ith rectangular region and setting $R = \bigcup_{i=1}^{m} R_i$, $\lambda(R_i)/\lambda(R)$ is the probability of a center occurring in R_i given that $\tilde{N}(R)=N$ centers occur in the region R. This can be seen as follows: suppose $\tilde{N}(R)=N$. Then for any non-negative integers $k_1,\ldots,k_m$ for which $\sum_{i=1}^{m} k_i = N$ the joint probability that $\tilde{N}(R_i)=k_i$, $i=1,2,\ldots,m$ given $\tilde{N}(R)=N$ is

$$\frac{P\{\tilde{N}(R_1)=k_1,\ \tilde{N}(R_2)=k_2,\ldots,\tilde{N}(R_m)=k_m\}}{P\{\tilde{N}(R)=N\}} \tag{4}$$

The process generating the $\tilde{N}(R_i)$s is Poisson with the properties (a) the $\tilde{N}(R_i)$s are independent unconditional as regards $\tilde{N}(R)$ and (b) the probability that $\tilde{N}(R_i)=k_i$ unconditional as regards $\tilde{N}(R)$ is

$$\exp\{-\lambda(R_i)\}\ [\lambda(R_i)]^{k_i}/k_r! \tag{5}$$

where

$$\lambda(R_i) = \int_{R_i} \lambda(x,y)\,dxdy. \tag{6}$$

Consequently, substituting (6) and recognizing that $\sum_{i=1}^{m} \lambda(R_i) = \lambda(R)$ since $R_i \cap R_j = \phi$ for $i \neq j$ and $\bigcup_{i=1}^{m} R_i = R$, we have

$$P\{\tilde{N}(R_1)=k_1,\ \tilde{N}(R_2)=k_2,\ldots,\tilde{N}(R_m)=k_m|\tilde{N}(R)=N\}$$
$$= \binom{N}{k\ ,\ldots k_m} \left[\frac{\lambda(R_1)}{\lambda(R)}\right]^{k_1} \left[\frac{\lambda(R_2)}{\lambda(R)}\right]^{k_2} \cdots \left[\frac{\lambda(R_m)}{\lambda(R)}\right]^{k_m} \tag{7}$$

For a non-homogenous Poisson process, this last probability is the analogue of the "uniform" distribution for the $\tilde{N}(R_i)$s in the homogenous case when $\tilde{N}(R)=N$ is given. Using it we can compute the analogue of (2) in the non-homogenous case. In place of kinematic density, the location of the center of an anomaly and the angle θ now have joint density $\lambda(x,y)dxdyd\theta/\lambda(R)$ for $(x,y)\epsilon R$ and zero for $(x,y)\notin R$.

3. PROBABILITY OF DETECTION WHEN C IS AN ELLIPSE

If the x-y plane is divided into rectangles by imposing a rectangular grid on it, what is the probability that an ellipse with major axis of length 2α and minor axis of length $2\beta<2\alpha$ will not be crossed by at least one grid line (detected) when the ellipse is equipped with kinematic density?

Let the rectangles formed by the grid have sides of length $2A$ in the direction of the x-axis and length $2B$ in the direction of the y-axis. Then if $\alpha \geq \sqrt{A^2+B^2}$, the ellipse is certain to be detected. Suppose that the origin of the x-y coordinate system is at the center of one of the rectangles. In order to motivate our calculations, consider Figure 1. As an ellipse with principal axis of length $2\alpha < 2\sqrt{A^2+B^2}$ inclined at an angle θ with respect to the x-axis is kept entirely within the rectangle and slid around the perimeter of the rectangle while maintaining at least one point of tangency with this perimeter, the center of the ellipse inscribes a rectangle with corners (x_1,y_1), (x_{-1},y_1), (x_{-1},y_{-1}), and (x_1,y_{-1}). The center of the ellipse may lie anywhere within this inscribed rectangle without crossing a grid line. Hence the probability that the ellipse is not detected is the ratio of area of the inscribed rectangle $R(\alpha,\beta,\theta)$ to $4AB$.

By virtue of symmetry, the area of the inscribed rectangle $R(\theta,\alpha,\beta)$ is $4x_1y_1$. The probability we wish to compute is the expectation, with respect to uniform measure for θ on $[0,2\pi]$, of $4x_1y_1$ over <u>allowable</u> angles of the principal axis for given, fixed α and β. By "allowable angle" we mean an angle θ such that when the center of the ellipse is at (x,y) the ellipse lies entirely within the rectangle R_{AB}. Again, by virtue of symmetry, we need only consider angles $\theta \varepsilon [0,\frac{1}{2}\pi]$.

Consequently, for fixed α and β the expectation of the area of all allowable inscribed rectangles in R_{AB} is

$$\frac{2}{\pi AB}\int_{\theta_1}^{\theta_2} x_1y_1 d\theta \tag{8}$$

where $[\theta_1,\theta_2]$ is the allowable interval for θ.

In the appendix we show that

$$x_1 = A - \{\beta^2\sin^2\theta+\alpha^2\cos^2\theta\}^{\frac{1}{2}},$$

$$y_1 = B - \{\beta^2\cos^2\theta+\alpha^2\sin^2\theta\}^{\frac{1}{2}},$$

whereupon the probability $D((\alpha,\beta)|\#)$ that an ellipse with axes of length 2α and 2β will be crossed by at least one grid line of the lattice when the ellipse is distributed with kinematic density on the x-y plane is

$$\begin{aligned} 1 - \{\frac{2}{\pi}[\theta_2-\theta_1] &- \frac{2\beta}{\pi A}\int_{\theta_1}^{\theta_2}\{1+\frac{\alpha^2-\beta^2}{\beta^2}\cos^2\theta\}^{\frac{1}{2}}d\theta \\ &- \frac{2\beta}{\pi B}\int_{\theta_1}^{\theta_2}\{1+\frac{\alpha^2-\beta^2}{\beta^2}\sin^2\theta\}^{\frac{1}{2}}d\theta \\ &+ \frac{2\alpha\beta}{\pi AB}\int_{\theta_1}^{\theta_2}\{1+\tfrac{1}{4}(\frac{\alpha}{\beta}-\frac{\beta}{\alpha})^2\sin^2\theta\}^{\frac{1}{2}}d\theta\}. \end{aligned} \tag{9}$$

The definition of the allowable interval $[\theta_1,\theta_2]$ depends on α, β, and whether A<B or B<A, and it is presented in the appendix. As a check of the formula for $D((\alpha,\beta)|\#)$, notice that when $\alpha=\beta=r$, the ellipse is a circle of radius r, $\theta_1=0$, $\theta_2=\frac{1}{2}\pi$, and

$$D((\alpha,\beta)|\#) = \frac{r}{A}+\frac{r}{B}-\frac{r^2}{AB}.$$

As $r\to\min\{A,B\}$, the probability of the circle not being detected approaches zero.

4. INFERENCE ABOUT N AND $\underline{\theta}$

Once equipped with an explicit formula for the probability that an anomaly with elliptical boundary curve will be detected given #, we can compute maximum likelihood estimators (MLE) for N and $\underline{\theta}$ using the joint probability of $\tilde{\underline{C}} \epsilon d\underline{C}$, $K_1,\ldots,K_r$ being detected and $K_{r+1},\ldots,K_N$ not being detected. If we ignore labellings of the C_is and assume that the ellipses are spatially distributed with kinematic density, this probability is, defining $C_i \equiv (\alpha_i,\beta_i)$

$$\binom{N}{r}[1-\bar{D}(\#;\underline{\theta})]^{N-r}\prod_{i=1}^{r} D((\alpha_i,\beta_i)|\#)g((\alpha_i,\beta_i)|\underline{\theta})d\alpha_i d\beta_i$$
$$= \binom{N}{r}[\bar{D}(\#;\underline{\theta})]^r[1-\bar{D}(\#,\underline{\theta})]^{N-r}\prod_{i=1}^{r}\left\{\frac{D((\alpha_i,\beta_i)\ \#)g((\alpha_i,\beta_i)|\underline{\theta})d\alpha_i d\beta_i}{\bar{D}(\#;\underline{\theta})}\right\}. \tag{9}$$

Since $\bar{D}(\#;\underline{\theta})$ is the marginal probability that a generic K_i whose shape and area is not known with certainty will be discovered,

$$\frac{D((\alpha_i,\beta_i)|\#)g((\alpha_i,\beta_i)|\underline{\theta})}{\bar{D}(\#;\underline{\theta})} \tag{10}$$

is the density for the size and shape of a generic discovery; $\bar{D}$ in the ratio $D((\alpha_i,\beta_i)|\#)/\bar{D}(\#;\underline{\theta})$ plays the same role here as the tail probability normalizing a sampling density in the presence of truncation.

The joint likelihood function for N and $\underline{\theta}$ is

$$L(N,\underline{\theta}|\text{data}) \propto \binom{N}{r}[1-\bar{D}(\#;\underline{\theta})]^{N-r} \times \prod_{i=1}^{r}[g((\alpha_i,\beta_i|\underline{\theta})] \tag{11}$$

since the probability of detection given (α_i,β_i) does not depend on $\underline{\theta}$ or N . In principal it is possible to compute a joint maximizer $(\hat{N},\hat{\underline{\theta}})$ of $L(N,\underline{\theta}|\text{data})$, but since $\bar{D}(\#;\underline{\theta})$ is an average of terms composed of elliptic integrals, this computation is difficult to carry out. A feasible strategy is to evaluate $\bar{D}(\#,\underline{\theta})$ numerically and then compute numerical values for $L(N,\underline{\theta}|\text{data})$ over $N = r, r+1, \ldots$ and the range of $\underline{\theta}$ values.

Alternatively, conditional maximum likelihood values for N and for $\underline{\theta}$ may be computed using (9):

(i) compute a maximizer $\underline{\theta}^*$ of the likelihood function

$$L(\theta|\text{data}) \propto \prod_{i=1}^{r} g((\alpha_i,\beta_i)|\underline{\theta})/\bar{D}(\#;\underline{\theta})$$

for the size and shape of r observed discoveries;

(ii) given the estimate $\underline{\theta}^*$ of $\underline{\theta}$, compute a conditional (on $\underline{\theta}$) MLE $\underline{N}^*$ for N using

$$L(N|\underline{\theta}^*,\text{data}) \propto \binom{N}{r}[1-\bar{D}(\#;\underline{\theta})]^{N-r}$$

Chapman (1951) has shown that a necessary condition for N^* to be a maximizer of $L(N|\underline{\theta}^*,\text{data})$ is $N^* = [r/\bar{D}(\#;\underline{\theta}^*)]$; if $N^* = r/\bar{D}(\#;\underline{\theta}^*)$ for some integer N^*, then both N^* and N^*-1 maximize $L(N|\underline{\theta}^*,\text{data})$. Numerical computation of $\underline{\theta}^*$ is clearly easier than a corresponding calculation of $(\hat{N},\hat{\underline{\theta}})$. However, conditional MLE and joint MLE are not equivalent in general.

Precise conditions guaranteeing the existence and uniqueness of such estimators, and an investigation of differences between conditional and joint MLE are topics for future study.

APPENDIX

In order to integrate (8) we must express the coordinates (x_1,y_1) in terms of α, β, and θ. To this end we first express (x_1,y_1) in terms of the coefficients a, b, and c of an ellipse

$$a^2(x-x_1)^2 + b^2(y-y_1)^2 + c(x-x_1)(y-y_1) = 1 \tag{A.1}$$

centered at (x_1,y_1) and tangent to the perimeter of the rectangle R_{AB} at points S and T (cf. Figure 1), and then compute a, b, and c in terms of α,β, and θ.

At point S, $dy/dx = 0$, and at point T $dx/dy = 0$, so

$$x_S-x_1 = -[c/2a^2](B-y_1) \quad \text{and} \quad y_S = b, \tag{A.2}$$

and

$$y_T-y_1 = -[c/2b^2](A-x_1) \quad \text{and} \quad x_T = A. \tag{A.3}$$

Substituting (A.2) and then (A.3) back into (A.1) and solving for x_1 and y_1 gives

$$x_1 = A - 2b[4a^2b^2-c^2]^{-\frac{1}{2}}$$

and

$$y_1 = B - 2a[4a^2b^2-c^2]^{-\frac{1}{2}}.$$

Next, transform from (x,y) coordinates to (α,β) coordinates, the α coordinate at an angle θ with respect to the x-coordinate and β perpendicular to α:

$$\begin{aligned} (x-x_1)\cos\theta + (y-y_1)\sin\theta &= u \\ -\ (x-x_1)\sin\theta + (y-y_1)\cos\theta &= v \end{aligned} \tag{A.4}$$

For $(u,v) = (\alpha,0)$, (11) is $a^2\cos^2\theta + b^2\sin^2\theta + c\sin\theta\cos\theta = 1/\alpha^2$ and for $(u,v) = (0,\beta)$, (A.1) is $a^2\sin^2\theta + b^2\cos^2\theta - c\sin\theta\cos\theta = 1/\beta^2$, so $a^2+b^2 = (1/\alpha^2) + (1/\beta^2)$. Since the equation of the ellipse in (u,v) coordinates is $(u^2/\alpha^2) + (v^2/\beta^2) = 1$,

$$\begin{aligned}&\frac{1}{\alpha^2}[(x-x_1)^2\cos^2\theta + (y-y_1)^2\sin^2\theta + xy\sin 2\theta]\\ &+ \frac{1}{\beta^2}[(x-x_1)^2\sin^2\theta + (y-y_1)^2\cos^2\theta - xy\sin 2\theta] = 1\end{aligned} \tag{A.5}$$

and comparing coefficients of (A.5) and (A.1)

$$a^2 = \frac{\cos^2\theta}{\alpha^2} + \frac{\sin^2\theta}{\beta^2} \quad , \quad b^2 = \frac{\sin^2\theta}{\alpha^2} + \frac{\cos^2\theta}{\beta^2}$$

and

$$c^2 = \left[\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right]\sin 2\theta. \tag{A.6}$$

Using (A.6), we find

$$\begin{aligned}x_1 &= A - \{\beta^2\sin^2\theta + \alpha^2\cos^2\theta\}^{\frac{1}{2}},\\ y_1 &= B - \{\beta^2\cos^2\theta + \alpha^2\sin^2\theta\}^{\frac{1}{2}}.\end{aligned} \tag{A.7}$$

Substituting (A.7) in the integral (8) gives the explicit representation for the probability $D((\alpha,\beta)|\#)$ that an ellipse with axes of length 2α and 2β having kinematic density will be crossed by at least one grid line of the lattice #.

$$\begin{aligned}1 - \frac{2}{\pi AB}\int_{\theta_1}^{\theta_2} x_1y_1\,d\theta = 1-\{\frac{2}{\pi}[\theta_2-\theta_1] - \frac{2\beta}{\pi A}\int_{\theta_1}^{\theta_2}\{1 + \frac{\alpha^2-\beta^2}{\beta^2}\cos^2\theta\}^{\frac{1}{2}}d\theta\\ - \frac{2\beta}{\pi B}\int_{\theta_1}^{\theta_2}\{1 + \frac{\alpha^2-\beta^2}{\beta^2}\sin^2\theta\}^{\frac{1}{2}}d\theta\\ + \frac{2\alpha\beta}{\pi AB}\int_{\theta_1}^{\theta_2}\{1 + \tfrac{1}{4}(\frac{\alpha}{\beta} - \frac{\beta}{\alpha})^2\sin^2 2\theta\}^{\frac{1}{2}}d\theta\}\end{aligned} \tag{A.8}$$

As a check of this formula, notice that when $\alpha=\beta=r$, the ellipse is a circle of radius r , $\theta_1 = 0$, $\theta_2 = \frac{1}{2}\pi$ and

$$\frac{2}{\pi AB}\int_0^{\frac{1}{2}\pi} x_1 y_1 d\theta = 1 - \frac{r}{A} - \frac{r}{B} + \frac{r^2}{AB}$$

As $r \to \min\{A,B\}$ the probability of the circle not being detected approaches zero.

Which angles are allowable depend on whether $A<B$ or $B<A$. If $B<A$, then allowable intervals for θ are

(1) $[0,\frac{1}{2}\pi]$ if $\beta<\alpha<B$,

(2) $[0, \arcsin[B^2-\beta^2/\alpha^2-\beta^2]$ if $\beta<B<\alpha<A$,

and

(3) $[\arcsin[\alpha^2-A^2/\alpha^2-\beta^2], \ \arcsin[B^2-\beta^2/\alpha^2-\beta^2]]$ if $\beta<B<A<\alpha$.

If $B>A$ then allowable intervals for θ are

(1) $[0,\frac{1}{2}\pi]$ if $\beta<\alpha<A$,

(2) $[\arcsin[\alpha^2-A^2/\alpha^2-\beta^2], \frac{1}{2}\pi]$ if $\beta<A<\alpha<B$,

and

(3) $[\arcsin[\alpha^2-A^2/\alpha^2-\beta^2], \arcsin[B^2-\beta^2/\alpha^2-\beta^2]]$ if $\beta<A<B<\alpha$.

Defining the allowable interval for θ as $[\theta_1,\theta_2]$, the above conditions may be expressed more compactly, with the understanding that $\alpha>\beta$, as

$$\theta_1 = \begin{cases} 0 \text{ if } \beta<B \text{ and } \alpha<A \text{ when } B<A \text{ or} \\ \qquad \beta<\alpha<A \text{ when } B>A \\ \arcsin[\alpha^2-A^2/\alpha^2-\beta^2] \text{ if } \beta<B<A<\alpha \text{ or if} \\ \qquad \beta<A \text{ and } \alpha>A \text{ when } B>A \end{cases}$$

and

$$\theta_2 = \begin{cases} \frac{1}{2}\pi \text{ if } \beta<A \text{ and } \alpha<B \text{ when } A<B \text{ or} \\ \qquad \beta<\alpha<B \text{ when } A>B \\ \arcsin[B^2-\beta^2/\alpha^2-\beta^2] \text{ if } \beta<A<B<\alpha \text{ or if} \\ \qquad \beta<B \text{ and } \alpha>B \text{ when } A>B \end{cases}$$

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