Question 1

a) Yes, for example, $A \& A^c$ are disjoint and exhaustive. Two sets, $A \& B$, need not be either disjoint or exhaustive. For example, consider rolling a die; the event "roll a 3" and "roll an odd number" are neither disjoint nor exhaustive.

b) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$

Proof:
As in class, we want to divide the sets into mutually exclusive sub-sets:

Venn diagram

Now:

$P(A \cup B \cup C) = P(B^c \cup c^c \cap A) + P(B^c \cap A \cap B) + P(A \cap B \cap c \cap C)$

$+ P(A \cap B \cap c^c) + P(A \cap c \cap B^c) + P(B \cap c \cap A^c)$

$+ P(A \cap B \cap c)$
(3) Let's look at (4), (5) & (6):

\[ P(\overline{A} \cap \overline{B} \cap \overline{C}) = P(\overline{A} \cap \overline{B}) - P(\overline{A} \cap \overline{B} \cap \overline{C}) \]

\[ P(\overline{A} \cap \overline{B} \cap \overline{C}) = P(\overline{A} \cap \overline{C}) - P(\overline{A} \cap \overline{B} \cap \overline{C}) \]

\[ P(\overline{B} \cap \overline{C} \cap \overline{A}) = P(\overline{B} \cap \overline{C}) - P(\overline{B} \cap \overline{C} \cap \overline{A}) \]

Now, let's look at (1), (2) & (3):

\[ P(\overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D}) = P(A) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C) \]

\[ P(\overline{B} \cap \overline{A} \cap \overline{C} \cap \overline{D}) = P(B) - P(A \cap B) - P(B \cap C) + P(A \cap B \cap C) \]

\[ P(\overline{C} \cap \overline{B} \cap \overline{A} \cap \overline{D}) = P(C) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \]

Combining & cancelling terms:

\[ P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \]

\[ \Box \]

Q.E.D.
Question 3

a) \[ p = \frac{N(A)}{N(S)} = \frac{3}{10} \]

b) "The first 9 draws are non-red. The 10th is red."

\[ p = \left( \frac{6}{10} \right)^9 \cdot \frac{4}{10} \]

c) Let \( A = \) red or yellow never chosen
then \( A^c = \) red or yellow chosen.

\[ P(A) = 1 - P(A^c) \]

\[ P(A^c) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \cdots = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n \]

\[ P(A) = 1 - P(A^c) = 1 - \frac{\infty}{2^n} \left( \frac{1}{2} \right)^n \]

Question 3

a) \( \binom{15}{3} \) ways

b) one jump per day per student. So 6 students already
jumped by third jump. So now: \( \binom{15-6}{3} \) ways = \( \binom{9}{3} \)

c) \( \binom{13}{3} + \binom{13}{1} \) ways.

d) \( \binom{10}{3} + \binom{10}{1} \)
Question 4

A bundle of 30 shirts that is 10% defective has 27 good shirts & 3 defective shirts.

A bundle is rejected if the sample from it has 2, 3, or 4 defective shirts; however, 4 defective shirts are not possible if the whole bundle has 3 defective shirts.

\[ P(\text{reject}) = P(2 \text{ or } 3 \text{ defective shirts chosen}) \]

\[ \frac{\binom{30}{3} \times \binom{27}{2} + \binom{30}{3} \times \binom{27}{1}}{\binom{30}{4}} \]

\[ # \text{ of possible groups of 4 shirts from 30} \]

*Note: Can also solve this problem in reverse:  \[ P(\text{reject}) = 1 - P(\text{Accept}) \]

Calculate \[ P(\text{Accept}) = P(0 \text{ or } 1 \text{ defective shirt's chosen}) \]

\[ p(\text{Reject}) = 1 - \frac{\binom{30}{0} \times \binom{27}{4} + \binom{30}{3} \times \binom{27}{3}}{\binom{30}{4}} \]
Let's first write down the information given in the question:

\[ P(\text{Sick}) = \frac{1}{20} = \frac{5}{100} = 0.05 \implies P(\text{Healthy}) = 0.95 \]

\[ P(\text{Positive} \mid \text{Sick}) = 0.8 \implies P(\text{Negative} \mid \text{Sick}) = 0.2 \]

\[ P(\text{Positive} \mid \text{Healthy}) = 0.1 \implies P(\text{Negative} \mid \text{Healthy}) = 0.9 \]

\[ P(\text{Sick} \mid \text{Negative}) = \frac{P(\text{Negative} \mid \text{Sick}) \cdot P(\text{Sick})}{P(\text{Negative} \mid \text{Sick}) \cdot P(\text{Sick}) + P(\text{Negative} \mid \text{Healthy}) \cdot P(\text{Healthy})} \]

\[ = \frac{0.2 \cdot 0.05}{0.2 \cdot 0.05 + 0.9 \cdot 0.95} = 0.0118 \approx 0.012 \]
Define the following events:

\[ A_t = \text{Ann washes at date } t \]
\[ B_t = \text{Bob washes at date } t \]
\[ N\bar{W}_t = \text{No one washes at date } t \]

a) \[ P(A_0) = P(3 \text{ or } 4 \text{ heads}) \]

\[ = \left( \frac{1}{2} \right)^4 \left( \frac{4}{3} \right) + \left( \frac{1}{2} \right)^4 \left( \frac{4}{3} \right) = \frac{5}{16} \]

Notice \[ P(A_0) = P(A_1|A_0) = P(A_1|B_0) \]

\[ P(B_0) = P(3 \text{ or } 4 \text{ tails}) = \frac{5}{16} \quad \text{(by symmetry)} \]

Notice \[ P(B_0) = P(B_1|A_0) = P(B_1|B_0) = P(B_1|N\bar{W}_0) \]

\[ P(N\bar{W}_0) = \text{min} \{1 - P(B_0) - P(A_0)\} = \frac{6}{16} \]

b) \[ P(A_1|N\bar{W}_0) = 1 - P(B_1|N\bar{W}_0) = 1 - \frac{5}{16} = \frac{11}{16} \]

c) \[ P(N\bar{W}_0|A_1) = ? \]

for this we first need \[ P(A_1) \]

\[ \Rightarrow P(A_i) = \frac{11}{16} \left( \frac{6}{16} \right) + \frac{5}{16} \left( \frac{5}{16} \right) + \frac{5}{16} \left( \frac{5}{16} \right) = \frac{116}{16^2} \]

Now,
\[ P(N \& \omega | A_i) = \frac{P(A_i | N \& \omega). P(N \& \omega)}{P(A_i)} \]
\[ \Rightarrow \frac{\frac{11}{16} \cdot \frac{6}{16}}{\frac{116}{16^2}} = \frac{66}{116} = \frac{33}{58} \]