You must answer ALL questions in order to fulfill the course requirement. For each student, the answers will be ordered by score and the top FIVE answers will then comprise the final problem set grade.

Question 1:
Let \( X_1, ..., X_n \) be a random sample (i.i.d.) of size \( n \) from a population \( f(x) \) with mean \( \mu \) and variance \( \sigma^2 \) (both finite). Prove the following:
  a) \( E(S^2) = \sigma^2 \), where \( S^2 \) is the sample variance.
  b) \( S^2 \overset{p}{\to} \sigma^2 \) (\( S^2 \) converges in probability to \( \sigma^2 \)).

Question 2:
Let \( S_0 \) denote the price of a given stock today. Suppose the price of the stock evolves over time as follows:
\[ S_t = S_{t-1} + X_t \]
where
\[ X_t = \begin{cases} 
1 \text{ with probability 0.39} \\
2 \text{ with probability 0.20} \\
3 \text{ with probability 0.41} 
\end{cases} \]

  a) Express the change in the price of the stock over the first 700 periods, \( \Delta S = S_{700} - S_0 \), in terms of the \( X_t \) s.
  b) What is the approximate distribution of the average daily change in the stock’s price?
  c) What is the probability that the stock is up at least 10 after 700 time periods?

Question 3:
A manufacturer of booklets packages them in boxes of 100. It is known that, on average, each book weighs 1oz., with a standard deviation of 0.05oz. The manufacturer is interested in calculating the following probability: \( P(100 \text{ booklets weigh more than 100.4oz.}) \). Approximate this probability, mentioning any relevant theorems you use.

Question 4:
The lifetime of a certain critical electrical part is a random variable with mean 100 hours and standard deviation 20 hours.
  (i) If 16 such parts are tested, find the probability that the sample mean is:
     a) less than 104 hours.
b) Now suppose that instead of 16, 100 such parts are tested. Find the probability that the sample mean is between 98 and 104 hours.

(ii) Given that immediate replacement of a faulty part is strictly necessary for operations, how many of the electrical parts must be in stock so that the probability that the system is in continual operation for the next 2000 hours is at least 0.95?

**Question 5:**
Suppose $X_1, ..., X_n$ are a random sample from a negative binomial $(r, p)$ distribution.

a) How would you apply the Central Limit Theorem as $n$ becomes sufficiently large?

b) If $r = 10$, $p = 0.5$, and $n = 60$, first describe (do not perform) how you would do an exact calculation to find $P(\bar{X} \leq 11)$. Next, apply the Central Limit Theorem to find the same probability.

**Question 6:**
Provide short answers to the following:

a) Under what circumstances is the sample mean an unbiased estimator for the mean of a distribution? Under what circumstances does have a normal distribution?

b) What is a "standardized" random variable? The limit as $n \to \infty$ of a standardized binomial random variable has what distribution?

c) Describe how choosing the minimum mean squared error estimator is a way to balance bias and efficiency of an estimator?

d) A random variable $X$ has mean 0 and variance 20. How large can $P(|X| > 30)$ be? How large is it if $X$ has a normal distribution?

**Question 7:**
Let $X_1, ..., X_n$ be a random sample of size $n$ from a population with mean $\mu$ and variance $\sigma^2$. Consider a statistic formed by taking a linear combination of the $X_i$s, $c_1X_1 + ... + c_nX_n$, where $c_i \geq 0$. For example, the sample mean, $\bar{X}$, is the linear combination with $c_i = \frac{1}{n}$.

a) Under what condition(s), is the statistic an unbiased estimator of $\mu$?

b) What is the variance of the statistic?