A COUNTEREXAMPLE TO A CONJECTURE OF SERRE

by

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Abstract

Let X be a finite simply-connected CW-complex. Serre and others have conjectured that the Poincaré series of the loop space on X, $\sum_{n=0}^{\infty} \text{Rank}(H_*(\Omega X;Q))Z^n$, would always be rational. In this thesis we present a counterexample to this conjecture.

There are three major results in this thesis. The first (Theorem 3.7) gives a formula relating the Poincaré series of ΩX and ΩY, where Y is the mapping cone of a map from a wedge of spheres to X. The second (Theorem 6.1) shows how to construct finitely presented Hopf algebras with transcendental Hilbert series. This result has as a corollary a counterexample to Serre's conjecture. The last (Example 7.1) gives a local ring with an irrational Poincaré series.

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INTRODUCTION AND SUMMARY

Let X be a finite 1-connected CW-complex. Is the Poincaré series $\sum_{n=0}^{\infty} \mathrm{Rank}(H_n(\Omega X;Q))Z^n$ a rational function of Z?

This thesis answers this question negatively by exhibiting an explicit counterexample. The demonstration is divided into two major parts. The first part shows that a counterexample exists if a finitely presented Hopf algebra exists with an irrational Hilbert series. In the second part, we show how such algebras may be constructed and their series computed.

Let $\bigvee_{j=1}^{m} S^{d_j} \xrightarrow{f = \vee f_j} X \to Y$ be a cofibration, X simply connected, each $d_j \geq 2$. We are interested in expressing the Poincaré series of ΩY in terms of the series for ΩX. Let $\mathbb{F}$ be any field and let $H_*(\cdot)$ denote homology with coefficients in $\mathbb{F}$. $H = H_*(\Omega X)$ is a connected Hopf algebra over $\mathbb{F}$ whose structure is assumed to be known.

Our starting point for the computation of $H_*(\Omega Y)$ is the cobar construction of Adams and Hilton [1]. This construction gives us a free differential graded algebra whose homology ring is identical to $H_*(\Omega Y)$. Let (A_0,d_0) be the algebra corresponding to X and (A,d) the algebra corresponding to Y. Since X is a subcomplex of Y, it is possible to choose A to be a free extension of A_0, $A = A_0\langle\gamma_1,\ldots\gamma_m\rangle$, with d an extension of d_0. Here $R\langle x_1,\ldots x_m\rangle$ denotes the free associative algebra over the ring R generated by $x_1,\ldots x_m$. In our case, the γ_j correspond to the attached cells with $|\gamma_j| = d_j$ and $d(\gamma_j) \in A_0$.

We may express $H_*(A,d)$ as the homology of a double complex. We have a spectral sequence E^r_{pq} with $\bigoplus_{p+q=n} E^{\infty}_{pq} \approx H_*(A,d) = H_*(\Omega Y)$. We compute the E^1 term to find that $E^1 = H\langle\gamma_1,\ldots\gamma_m\rangle$ and $d^1: E^1 \to E^1$ satisfies $d^1|_H = 0$, and $d^1(\gamma_j) = \beta_j \in H$. The β_j are the images of the Hurewicz homomorphism applied to $[f_j]$: $S^{d_j-1} \to \Omega X$. Thus $E^2 = H_*(H\langle\gamma_1,\ldots\gamma_m\rangle, d^1)$.

The size of E^2 can be computed explicitly if certain assumptions about the set $\beta = \{\beta_1,\ldots\beta_m\}$ and H are made. Let $H\beta H$ be the two-sided ideal of H generated by β and let $N = H/H\beta H$ be the quotient Hopf algebra. If H has global dimension ≤ 2, or if $H\beta H$ is a free H-module, we get a formula for E^2.

For a graded module $M = \bigoplus_{n\geq 0} M_n$, let M(Z) denote the Hilbert series $\sum_{n=0}^{\infty} \mathrm{Rank}(M_n)Z^n$. Let $\gamma(Z)$ denote $(\mathrm{Span}\{\gamma_1,\ldots\gamma_m\})(Z) = \sum_{j=1}^{m} Z^{d_j}$. Under the above assumptions we obtain the formula

$$E^2(Z)^{-1} = (1+Z)N(Z)^{-1} - ZH(Z)^{-1} - \gamma(Z). \tag{i}$$

We can compute E^2 another way. We construct an explicit set of generators for the subalgebra of E^2 generated by the $E^2_{0,*}$ and $E^2_{1,*}$ columns. The d^2 and higher differentials vanish on this subalgebra. If the same assumptions about H or β as above are made, we find that the Hilbert series of this subalgebra satisfies formula (i). That is, this subalgebra must be the whole of E^2. Thus all d^r, $r \geq 2$, vanish, and $H_*(\Omega Y) \approx E^\infty = E^2$.

We have proved

Theorem A. *Suppose $H = H_*(\Omega X)$ has global dimension ≤ 2. For example, suppose X is a suspension or a product of two suspensions. Or suppose that $H\beta H$ is a free H-module. Then*

$$H_*(\Omega Y)(Z)^{-1} = (1+Z)N(Z)^{-1} - ZH(Z)^{-1} - \gamma(Z), \tag{ii}$$

where $N = H/H\beta H$. In particular, if X is a finite wedge of spheres, then $H_*(\Omega Y)(Z)$ is rational if and only if $N(Z)$ is rational.

The last remark follows from the well-known fact [6] that $H = H_*(\Omega \bigvee_{i=1}^{k} S^{c_i}) = \mathbb{F}\langle\alpha_1,\ldots\alpha_k\rangle$ with $|\alpha_i| = c_i - 1$ and $H(Z) = (1 - \sum_{i=1}^{k} Z^{c_i-1})^{-1}$ is rational.

The remainder of the thesis is dedicated to the construction of examples of finitely presented Hopf algebras N with N(Z) irrational. All examples have X = a wedge of spheres. By Theorem A, they immediately yield finite complexes whose loop spaces have irrational Poincaré series.

Let L be a free graded connected Lie algebra with generators $\{\alpha_1,\ldots\alpha_k\}$. We consider a homomorphism ϕ: $U(L) \to L$, where $U(L) = \mathbb{F}\langle\alpha_1,\ldots\alpha_k\rangle$ is the universal enveloping algebra of L. ϕ is defined by $\phi(\alpha_i) = b_i\alpha_i$ and $\phi(\alpha_{i_1} \ldots\ldots \alpha_{i_n}) = [\phi(\alpha_{i_1} \ldots \alpha_{i_{n-1}}), \alpha_{i_n}]$, where $b_i \in \mathbb{F}^*$ are fixed constants. ϕ is surjective when char $\mathbb{F} \neq 2$ and it satisfies various nice formulas. The real importance of ϕ, however, is that under certain weak conditions it can be defined for a quotient Hopf algebra $H/H\beta H = U(L/[\beta])$, where $[\beta]$ is the Lie ideal of L generated by a set $\beta = \{\beta_1,\ldots\beta_m\} \subseteq L$.

Let $L' = L/[\beta]$ and $G = H/H\beta H$. ϕ: $G \to L'$ is surjective if char $\mathbb{F} \neq 2$. Furthermore, let $\{\beta'_j\} = \{\phi(\delta'_j)\} \subseteq L'$ be any subset. Then $\phi(\delta' G) = G\beta' G \cap L'$.

For a graded module $M = \bigoplus_{n\geq 0} M_n$, let $\mathcal{P}(M)$ denote the tensor product of the tensor algebra on $\bigoplus_{n\geq 0} M_{2n}$ with the exterior algebra on $\bigoplus_{n\geq 0} M_{2n+1}$. By the Poincaré-Birkhoff-Witt theorem,

$$\text{(iii)} \qquad G/G\beta' G \approx \mathcal{P}(\phi(G)/[\beta']) = \mathcal{P}(\phi(G)/\phi(\delta' G)).$$

if char $\mathbb{F} \neq 2$, and a similar formula holds if char $\mathbb{F} = 2$. Thus the problem of evaluating $(G/G\beta' G)(Z)$ is entirely reduced to the problem of determining the Hilbert series of the quotient module $\phi(G)/\phi(\delta' G)$.

We can actually evaluate $\phi(G)/\phi(\delta' G)$ fairly easily when G belongs to a class of algebras called "generalized products". A generalized product G is a semi-tensor product of two free Hopf algebras, $H_1 = \mathbb{F}\langle T_1\rangle$ and $H_2 = \mathbb{F}\langle T_2\rangle$. Letting $H = H_1 \amalg H_2$, G can be written as $H/H\hat{\beta}H$, where $\hat{\beta} = \{[\alpha_i,\alpha_j] - h_{ij} \mid \alpha_i \in T_1,\ \alpha_j \in T_2, \text{ and } h_{ij} \in \phi(H_2)\}$. G is isomorphic as a vector space to the ordinary tensor product $H_1 \otimes H_2$. As an algebra, it is different in that each non-zero h_{ij} introduces a "twist" in the multiplication.

An explicit calculation may be done for the following example:

Let $H_1 = \mathbb{F}\langle\alpha_1,\alpha_2\rangle$, $H_2 = \mathbb{F}\langle\alpha_3,\alpha_4,\alpha_5\rangle$. $H = H_1 \amalg H_2$. All the α_i's have dimension 1.

$$\beta_1 = [\alpha_1,\alpha_3] - [\alpha_3,\alpha_4] \qquad \beta_2 = [\alpha_1,\alpha_4] \qquad \beta_3 = [\alpha_1,\alpha_5]$$

$$\text{(iv)} \qquad \beta_4 = [\alpha_2,\alpha_3] \qquad \beta_5 = [\alpha_2,\alpha_4] \qquad \beta_6 = [\alpha_2,\alpha_5] - [\alpha_3,\alpha_4]$$

$$\beta_7 = [\alpha_3,\alpha_5] \qquad \beta_8 = [\alpha_3,\alpha_3]$$

Here $G = H/\sum_{j=1}^{6} H\beta_j H$ is a generalized product with $h_{13} = h_{25} = [\alpha_3,\alpha_4]$ and $h_{14} = h_{15} = h_{23} = h_{24} = 0$. Also, $\{\beta_7,\beta_8\} \subseteq \phi(H_2)$, so $G/(G\beta_7G+G\beta_8G)$ can be computed with the help of the previous remarks.

Our conclusion is, for char $\mathbb{F} \neq 2$,

$$H/H\beta H \approx H_1 \otimes \mathbb{F}\langle\alpha_4,\alpha_5\rangle \otimes \mathcal{P}(\{\phi(\alpha_3,\alpha_4^k) \mid k \geq 0\}) \ .$$

We deduce immediately

$$N(Z) = \left(\frac{1}{1 - 2Z}\right)\left(\frac{1}{1 - 2Z}\right) P(Z),$$

where

$$\text{(v)} \qquad P(Z^d) = \prod_{k=1}^{\infty} \left(\frac{1}{1 - Z^{2k}}\right) \prod_{k=1}^{\infty} \left(1 + Z^{(2k-1)}\right) .$$

A similar formula is valid when char $\mathbb{F} = 2$.

The infinite products is a transcendental function. We have shown:

Theorem B. *Let V be the complex obtained from* $\bigvee_{i=1}^{5}(S^2)$ *by attaching eight 4-cells corresponding to the Whitehead products given in (iv). Then* ΩV *has an irrational Poincaré series.*

The so-called Serre-Kaplansky problem asks whether the Poincaré series $\sum_{n=0}^{\infty} \text{Rank}(\text{Tor}_n^R(\mathbb{F},\mathbb{F}))Z^n$ of a local ring R is always rational, where $R/\mathfrak{m} = \mathbb{F}$. Jan-Erik Roos has recently demonstrated that this question when $\mathfrak{m}^3 = 0$ is equivalent to the rationality of $H_*(\Omega X)(Z)$ when dim $X \leq 4$. Our space V of Theorem B has dimension four. The equivalence of the two questions is through the cohomology ring of the offending complex.

Theorem C. *Let* $R = H^*(V;\mathbb{F})$, *where V is the space of Theorem B.* $R = \mathbb{F}(x_1,\ldots x_5)/J$, *where J is the ideal generated by* $\mathfrak{m}^3$ *and the relations*

$$x_1^2 = x_2^2 = x_4^2 = x_5^2 = 0 \text{ and } x_1x_2 = x_4x_5 = x_1x_3 + x_3x_4 + x_2x_5 = 0.$$

Then $\sum_{n=0}^{\infty} \text{Rank}(\text{Tor}_n^R(\mathbb{F},\mathbb{F}))Z^n$ *is a transcendental function.*

This follows directly from Roos' work and our Theorem B. R is found explicitly by dualizing (iv).

We close with a brief discussion of just what the possibilities are for $H_*(\Omega X)(Z)$. We have given an example of a finitely presented Hopf algebra whose Hilbert series was a rational function times $\mathcal{P}(\overline{\mathbb{F}(y)})(Z)$, where $|y| = 1$. $\mathcal{P}(\overline{M})(Z)$ will be an infinite product like (v) for any connected module M. In general, however, there are exponents equal to Rank (M_k) instead of unity on the individual factors of the product. It turns out that we can construct an N for which N(Z) is a rational multiple of $\mathcal{P}(\overline{M})$ whenever M is a finitely presented connected (not necessarily Hopf!) algebra. Thus the possibilities for N(Z) are quite rich and can be highly transcendental.

I. THE HOMOLOGY OF $\Omega(X \cup_f C \bigvee_{i=1}^{m} S^{d_i})$

Let $\bigvee_{i=1}^{m} S^{d_i} \xrightarrow{f} X \to Y$ be a cofibration, with each $d_i > 1$ and X a simply connected CW-complex. In Part I we will analyze the homology of ΩY. Under suitable conditions we give a formula for the Poincaré series of ΩY in terms of the series for ΩX and for a certain quotient algebra depending on f. In particular, our formula will hold whenever X is a suspension or a product of two suspensions.

Let $\mathbb{F}$ denote any field. $H_*(\cdot)$ will denote homology with coefficients in $\mathbb{F}$. All tensor products will be over $\mathbb{F}$. Let $H = H_*(\Omega X)$. H is a Hopf algebra with commutative coproduct Ψ. In general, H will be non-commutative. Let $|\cdot|$ denote "dimension of" for elements of a graded module. Let [,] denote the usual $[x,y] = xy - (-1)^{|x|\cdot|y|}yx$. Finally, let $R\langle\alpha_1,\ldots\alpha_n\rangle$ denote the free associative non-commutative algebra over the ring R with generators $\alpha_1,\ldots \alpha_n$.

1. The Adams-Hilton Construction

Our starting point for the study of ΩY is the cobar construction first described by P.J. Hilton and J.F. Adams [1,2]. We assume that X has a CW structure with a single 0-cell and no 1-cells. The cobar construction gives us a graded differential algebra (A,d) whose homology ring is identical with $H_*(\Omega Y)$. We may assume that X is a subcomplex of Y. By a remark [1, p. 310] we may choose A to be an extension of the differential graded algebra A_0, where A_0 is the differential graded algebra constructed for X. $H_*(A_0,d) =$ $= H_*(\Omega X)$ and $H_*(A,d) = H_*(\Omega Y)$.

Let $\{e_i\}_{i\varepsilon I}$ be the set of positive-dimensional cells of X. We may take $\{e_i\}_{i\varepsilon I} \cup \{\hat{e}_j\}_{1\leq j\leq m}$ to be the positive-dimensional cells of Y, where the $\{\hat{e}_j\}$ are the cells attached to X by f. The algebra A_0 is the free associative algebra over $\mathbb{F}$ with generators $\{\alpha_i\}_{i\varepsilon I}$ in one-to-one correspondence with the $\{e_i\}_{i\varepsilon I}$. Their dimensions are given by $|\alpha_i| = \dim(e_i) - 1$. Likewise, $A = \mathbb{F}\langle\{\alpha_i\}_{i\varepsilon I} \cup \{\gamma_j\}_{1\leq j\leq m}\rangle$, where the $\{\gamma_j\}$ correspond to $\{\hat{e}_j\}$ and satisfy $|\gamma_j| = d_j$. Note that $A = A_0\langle\gamma_1, \ldots \gamma_m\rangle$.

The differential d is defined on all of A. d satisfies the product rule

$$d(a_1 \ldots a_n) = \sum_{i=1}^{n} (-1)^{|a_1 \ldots a_{i-1}|} a_1 \ldots d(a_i) \ldots a_n ,$$

so it is enough to specify d on the generators. Let $\beta_i = d(\gamma_i)$. Since each of the cells $\hat{e}_j$ is attached directly to X, we have $\beta_i \varepsilon A_0$. $d^2 = 0$ on A means that each β_i is a cycle in (A_0,d) with $|\beta_i| = d_i - 1$. We will use the same symbol β_i to denote the corresponding cycle in $H_*(A_0,d)$ and $H_*(\Omega X)$.

Let $f_i\colon S^{d_i} = S(S^{d_i-1}) \to X$ be the attaching map for $\hat{e}_i$. f_i may be identified with $[f_i]\colon S^{d_i-1} \to \Omega X$, which may be sent via the Hurewicz homomorphism to a cycle $\beta_i \varepsilon H_{d_i-1}(\Omega X)$. Up to sign, these two definitions of β_i agree. The ambiguity of sign will not matter for our purposes and may be cleared up by orienting each S^{d_i} suitably. Since β_i is the image under $[f_i]$ of the generator of the homology of the sphere S^{d_i-1}, we know that β_i is primitive as an element of the Hopf algebra $H = H_*(\Omega X)$.

We define a filtration on A by setting $A_0 = A_0$ and $A_{p+1} = \sum_{j=1}^{m} A_0\gamma_j A_p$. A_p is generated additively by those monomials of A which include precisely p γ_j's (and any number of α_i's). We obtain a bigrading by specifying that $a \in A_{pq}$ if and only if $a \in A_p$ and $|a| = p+q$. Note that as $\mathbb{F}$-modules, $A = \bigoplus_{p\geq 0, q\geq 0} A_{pq}$. Let $d': A_{pq} \to A_{p,q-1}$ be the extension of $d|_{A_0}$ to A which satisfies the product rule and $d'(\gamma_j) = 0$. Let $d'': A_{pq} \to A_{p-1,q}$ be defined by $d''|_{A_0} = 0$, $d''(\gamma_j) = \beta_j$, and the product rule. Then $d = d' + d''$.

Using this bigradation we may construct a spectral sequence which converges to $H_*(A,d)$ (see, e.g., [3], pp. 330-332). As this spectral sequence is suggested by the work of Eilenberg and Moore [5] (or see [14], chapter 3), we will refer to it as the "Eilenberg-Moore spectral sequence for ΩY", or simply, the "E-M s.s.". We know that $E^0_{p,q} = A_{pq}$ and that $\bigoplus_{p+q=n} E^\infty_{pq} = H_n(A,d)$. Our next task is to evaluate the E^1 and E^2 terms.

We compute the E^1 term by taking the d' homology first. We obtain $E^1_{pq} = (H_*(A_p,d'))_q$. (A_p,d') may be identified with the complex $(\bigoplus_S \underbrace{A_0 \otimes \dots \otimes A_0}_{p+1\ A_0\text{'s}}, d'_p)$, where the set S consists of all p-tuples $(\gamma_{i_1},\dots,\gamma_{i_p})$ with $1 \leq i_j \leq m$. The identification is given by

$\theta(a_0 \otimes \dots \otimes a_p) = a_0\gamma_{i_1}a_1 \dots \gamma_{i_p}a_p$ and $d'_p(a_0 \otimes \dots \otimes a_p) = \sum_{j=0}^{p} (-1)^{|a_0\gamma_{i_1}\dots\gamma_{i_j}|} a_0 \otimes \dots \otimes d(a_j) \otimes \dots \otimes a_p$. It is well known that $H_*(A_0 \otimes \dots \otimes A_0, d'_p) = \bigotimes_{j=0}^{p} H_*(A_0,d') = \bigotimes_{j=0}^{p} H$ (see, e.g., [3], pp. 64-69). Let $\hat{H} = H\langle\gamma_1, \dots \gamma_m\rangle$ and let $\hat{d}: \hat{H} \to \hat{H}$ be given by $\hat{d}(H) = 0$, $\hat{d}(\gamma_i) = \beta_i$, and the product rule. Let $\hat{H}_p$ be spanned by those monomials of $\hat{H}$ containing exactly p γ_j's. Then

$$H_*(\bigoplus_S A_0 \otimes \ldots \otimes A_0, d_p') = \bigoplus_S H_*(A_0 \otimes \ldots \otimes A_0, d_p') = \bigoplus_S (\bigotimes_{i=0}^{p} H) = \hat{H}_p.$$

Thus $E^1_{pq} = (\hat{H}_p)_q$.

The E^2 term is found by taking the homology of E^1_{pq} with respect to the d'' differential. It is clear that the induced d'' on E^1 agrees with the $\hat{d}$ we have already defined on $\hat{H}$. Thus $E^2_{p,q} = H_*(\hat{H},\hat{d})_{p,q}$. We have proved

Theorem 1.1. Let Y be the mapping cone of a finite wedge of spheres, $Y = X \cup_f C \bigvee_{i=1}^{m} S^{d_i}$, where $d_i \geq 2$ and X is 1-connected. Let $H = H_*(\Omega X)$. Then there is a first quadrant homology spectral sequence E^r_{pq} such that $E^2 = H_*(\hat{H},\hat{d})$ and $\bigoplus_{p+q=n} E^\infty_{pq} = H_n(\Omega Y)$.

2. Computation of E^2

Our natural next step is to try to say something stronger about $E^2 = H_*(\hat{H},\hat{d})$. In this section we show that E^2 can be computed explicitly if one additional assumption is made.

Let $K = \ker \hat{d}$, $B = \operatorname{im} \hat{d}$. For M a submodule of $\hat{H}$, let γM denote $\sum_{j=1}^{m} \gamma_j M$ and let $H\gamma M$ denote $\sum_{j=1}^{m} H\gamma_j M$; likewise for βM and $H\beta M$. Let $C = H\beta K$. Let N be the quotient algebra $H/H\beta H$. We are interested in finding a formula for $E^2 = K/B$.

To simplify notation we let $\tilde{\beta}$ be the vector $(\beta_1, \ldots \beta_m)$ and $\tilde{\gamma}$ the vector $(\gamma_1, \ldots \gamma_m)$. $\sum_{j=1}^{m} \beta_j a_j$ will be denoted as the dot product $\tilde{\beta}\cdot\tilde{a}$, where $\tilde{a} = (a_1, \ldots a_m)$; likewise for $\tilde{\gamma}\cdot\tilde{a}$.

Lemma 2.1. *There is an isomorphism*

$$\eta: \quad N \otimes \gamma B \overset{\approx}{\to} B/C$$

given by $\eta(\bar{a} \otimes \tilde{\gamma}\cdot\hat{d}(\tilde{b})) = \overline{a\hat{d}(\tilde{\gamma}\cdot\tilde{b})}$.

Proof. To begin with, $C \subseteq B$ because any $x = a\tilde{\beta}\cdot\tilde{b}$, where $\tilde{\beta}\cdot\tilde{b} \in \beta K$, can be written as $x = (-1)^{|a|}\hat{d}(a\tilde{\gamma}\cdot\tilde{b}) \in B$.

To see that η is well-defined, suppose $\bar{a} = 0$. Then $a \in H\beta H$, so $a\hat{d}(\tilde{\gamma}\cdot\tilde{b}) \in H\beta HB = H\beta B \subseteq H\beta K = C$. We must also show that the definition of η does not depend upon our choice of $\tilde{b}$. This entails verifying that $a\hat{d}(\tilde{\gamma}\cdot\tilde{b}) \in C$ if each component of $\tilde{b}$ lies in K. This holds because then $a\hat{d}(\tilde{\gamma}\cdot\tilde{b}) = a\tilde{\beta} \cdot \tilde{b} + a\tilde{\gamma}\cdot(\pm\hat{d}(\tilde{b})) = a\tilde{\beta}\cdot\tilde{b} \in H\beta K$, where the "±" symbol is introduced to indicate the otherwise cumbersome signs $(-1)^{|\gamma_j|}$.

η is onto by definition of B. To check that η is one-to-one, let $\{a_j\} \subseteq H$ be chosen so that their images $\{\bar{a}_j\}$ in N form a basis for N as an $\mathbb{F}$-module. Suppose $x = \sum_j \bar{a}_j \otimes \tilde{\gamma}\cdot\hat{d}(\tilde{b}_j) \in \ker \eta$ for some $\{b_j\} \subseteq \hat{H}$. Then $\sum_j a_j\hat{d}(\tilde{\gamma}\cdot\tilde{b}_j) = \sum_j a_j\tilde{\gamma}\cdot(\pm\hat{d}(\tilde{b}_j)) + \sum_j a_j\tilde{\beta}\cdot\tilde{b}_j \in C = H\beta K$. Because $\{a_j\}$ are linearly independent of each other and of $H\beta H$ in H, we must have each $(\pm\hat{d}(\tilde{b}_j)) = 0$. But this means that $\hat{d}(\tilde{b}_j) = 0$ and $x = 0$ to begin with, i.e., $\ker \eta = 0$.

Lemma 2.2. *Suppose* K *is a free left* H*-module or* $H\beta H$ *is a free right* H*-module. Then as* $\mathbb{F}$*-modules*, $C \otimes H \approx H\beta H \otimes K$.

Proof. If K is free, let $\phi: \; H \otimes K' \to K$ be the given isomorphism of left H-modules. $C = H\beta K = H\beta(\phi(H \otimes K')) = \phi(H\beta H \otimes K')$. Since ϕ is one-to-one, it is one-to-one when restricted to $H\beta H \otimes K'$, giving $H\beta H \otimes K' \approx C$ and $H\beta H \otimes K \approx H\beta H \otimes H \otimes K' \approx H \otimes H\beta H \otimes K' \approx H \otimes C$.

If $H\beta H$ is free, let ϕ: $S \otimes H \to H\beta H$ be the isomorphism of right H-modules. $\hat{\phi}$: $S \otimes \hat{H} \to H\beta\hat{H}$ is an isomorphism since $\hat{H}$ is a free left H-module. The restriction $\hat{\phi}_K$: $S \otimes K \to H\beta H$ is also an isomorphism of $S \otimes K$ with im $\hat{\phi}_K = H\beta K = C$. We obtain $H\beta H \otimes K \approx S \otimes H \otimes K \approx S \otimes K \otimes H \approx C \otimes H$.

<u>Notation</u>. For a graded module $M = \bigoplus_{n \geq 0} M_n$, let $M(Z)$ denote the series $M(Z) = \sum_{n=0}^{\infty} \text{Rank}_{\mathbb{F}}(M_n)Z^n$. When a module has more than one gradation, the series is taken with respect to the dimension grading. Let $\gamma(Z) = \sum_{j=1}^{m} Z^{d_j}$.

<u>Proposition</u> 2.3.

(1a) $K(Z) + ZB(Z) = \hat{H}(Z)$.

(1b) $\hat{H}(Z) = H(Z)(1 - \gamma(Z)H(Z))^{-1}$.

(1c) $N(Z)\gamma(Z)B(Z) = B(Z) - C(Z)$.

<u>If</u> K <u>is</u> H-<u>free</u> <u>or</u> $H\beta H$ <u>is</u> H-<u>free</u> <u>we</u> <u>also</u> <u>have</u>

(1d) $C(Z) = K(Z)(1 - N(Z)H(Z)^{-1})$.

<u>Proof</u>. (a) From the exact sequence $0 \to K \to \hat{H} \xrightarrow{\hat{d}} B \to 0$, in which $\hat{d}$ lowers dimension by one.

(b) Because $\hat{H} \approx H \oplus H\gamma\hat{H}$, giving $\hat{H}(Z) = H(Z) + H(Z)\gamma(Z)\hat{H}(Z)$. Solve for $\hat{H}(Z)$.

(c) From 2.1.

(d) From 2.2. Solve for $C(Z)$, using $(H\beta H)(Z) = H(Z) - N(Z)$.

<u>Proposition</u> 2.4. <u>Suppose</u> K <u>or</u> $H\beta H$ <u>is</u> H-<u>free</u>. <u>Then</u>

(2) $$E^2(Z)^{-1} = (1 + Z)N(Z)^{-1} - ZH(Z)^{-1} - \gamma(Z)$$

<u>Formula</u> (2) <u>is</u> <u>valid</u> <u>if</u> <u>and</u> <u>only</u> <u>if</u> $C \otimes H \approx H\beta H \otimes K$ <u>as</u> $\mathbb{F}$-<u>modules</u>.

<u>Proof</u>. We think of (1a) through (1d) as a system of four linear equations in the four unknowns K, B, $\hat{H}$ and C, where H, γ, and N are "known". The system is non-degenerate and easily solved by substitutions.

Inverting $K(Z) - B(Z)$ gives formula (2).

For the converse, we note that (1d) can be obtained as a consequence of the relations (1a), (1b), (1c), and (2).

Corollary 2.5. Suppose H has global dimension ≤ 2. Then K is H-free, and formula (2) holds.

Proof. Note that $\hat{H}$ is free over H and consider the projective resolution of $\hat{H}/B$ which starts

$$\cdots \to M \to \hat{H} \xrightarrow{\hat{d}} \hat{H} \to \hat{H}/B \to 0$$

for a suitable M. $\mathrm{Tor}_3^H(\mathbb{F}, \hat{H}/B)$ must vanish because gl.dim.$(H) \leq 2$. It follows that the resolution may be constructed to be zero beyond M. Then $M = \ker \hat{d} = K$. But M is projective, hence free, because H is connected, and K is free.

3. Computation of E^∞

In Section 3 we determine a generating set for a subalgebra of E^2. We show that the E-M.s.s. degenerates when Formula (2) holds. Formula (2) is then also a formula for $H_*(\Omega Y)$.

Let $K_p = K \cap \hat{H}_p = \ker\{\hat{d}\colon \hat{H}_p \to \hat{H}_{p-1}\}$ and $B_p = B \cap \hat{H}_p = \hat{d}(\hat{H}_{p+1})$. Let $\rho\colon N \to H$ denote any right-inverse to the projection $\pi_N\colon H \to N$. As $\mathbb{F}$-modules, $H \approx \rho(N) \oplus H\beta H$.

Lemma 3.1. (a) There is a surjection $\zeta\colon N \otimes \beta\mathbb{F} \otimes H \to H\beta H$ given by $\zeta(a \otimes \beta_j \otimes b) = \rho(a)\beta_j b$. (b). $B = \hat{d}(\rho(N)\gamma\hat{H})$.

Proof. (a) Clearly $H_0\beta H = \beta H \subseteq \mathrm{im}\,\zeta$. Suppose inductively that $H_i\beta H \subseteq \mathrm{im}\,\zeta$ for $i < n$. We want to show that $h\beta_j b \in \mathrm{im}\,\zeta$ if $|h| = n$. Let $a = \pi_N(h)$. Note that $\pi_N(h - \rho(a)) = a - \pi_N\rho(a) = 0$, so $h - \rho(a) \in H\beta H$. $h - \rho(a) \in \sum_{i<n} H_i\beta H$ and $h\beta_j b - \rho(a)\beta_j b \in \sum_{i<n} H_i\beta H \subseteq \mathrm{im}\,\zeta$.

Since $\rho(a)\beta_j b = \zeta(a \otimes \beta_j \otimes b)$, we have $h\beta_j b \in \text{im}\,\zeta$, as desired,

(b) Let $x \in B$ and write $x = \hat{d}(y)$. Let $y = y_1 + y_2$, where $y_1 \in \rho(N)\gamma\hat{H}$ and $y_2 \in H\beta H\gamma\hat{H}$. y_2 is a sum of terms of the form $\rho(a)\tilde{\beta}\cdot\tilde{b}\;\tilde{\gamma}\cdot\tilde{h}$, by part (a). Any such term may be written as $(-1)^{|a|}\hat{d}(\rho(a)\tilde{\gamma}\cdot\tilde{b})\tilde{\gamma}\cdot\tilde{h} =$
$= (-1)^{|a|}\hat{d}(\rho(a)\tilde{\gamma}\cdot\tilde{b}\;\tilde{\gamma}\cdot\tilde{h}) \pm \rho(a)\tilde{\gamma}\cdot\tilde{b}\;\hat{d}(\tilde{\gamma}\cdot\tilde{h})$. $\hat{d}(y_2)$ is a sum of terms of the form $(-1)^{|a|}\hat{d}\hat{d}(\rho(a)\tilde{\gamma}\cdot\tilde{b}\;\tilde{\gamma}\cdot\tilde{h}) \pm \hat{d}(\rho(a)\tilde{\gamma}\cdot\tilde{b}\;\hat{d}(\tilde{\gamma}\cdot\tilde{h}))$. Since $\hat{d}\hat{d} = 0$, we have shown that $\hat{d}(y_2) \in \hat{d}(\rho(N)\gamma\hat{H})$. Thus $x = \hat{d}(y_1) + \hat{d}(y_2) \in \hat{d}(\rho(N)\gamma\hat{H})$, as desired.

Choose a set $\{g_i\} \subseteq H\gamma H$ such that $\{\hat{d}(g_i)\}$ is a basis for $H\beta H$ as a free $\mathbb{F}$-module. By 3.1 we may do this with each $g_i \in \rho(N)\gamma H$. Let $D_0 = \text{Span}\,\{g_i\} \subseteq H\gamma H$ and let $D_1 = \hat{d}(D_0\gamma) \subseteq B_1 \subseteq K_1$. Note that $D_1 \approx H\beta H\gamma$ as $\mathbb{F}$-modules. Using this isomorphism we see easily that $D_1 \otimes H \approx D_1 H \subseteq K_1$. Thus $D_1 H$ is a free H-submodule of $B_1 \subseteq K_1$. Let $D_2 = (\rho(N)\gamma H) \cap K_1$.

<u>Lemma</u> 3.2. $K_1 = D_2 \oplus D_1 H$.

<u>Proof</u>. $D_1 \cap (\rho(N)\gamma H) = 0$, so $D_2 \cap D_1 H = 0$. We need only show that $K_1 = D_2 + D_1 H$. Let $x \in K_1$ and write $x = x_1 + x_2$, where $x_1 \in \rho(N)\gamma H$ and $x_2 \in H\beta H\gamma H$. Write x_2 as a sum $x_2 = \sum_i \hat{d}(g_i)\tilde{\gamma}\cdot\tilde{b}_i$, where $\{g_i\}$ is the set described above. Let $y = \hat{d}(\sum_i g_i\tilde{\gamma}\cdot\tilde{b}_i)$ and note that $x_2 - y = \sum_i(\pm)g_i\hat{d}(\tilde{\gamma}\cdot\tilde{b}_i) \in \rho(N)\gamma H$. $y \in D_1 H \subseteq K_1$ and $x \in K_1$, so $x - y \in K_1$. But $x - y = x_1 + (x_2 - y) \in \rho(N)\gamma H$, so $x - y \in D_2$. Thus $x = (x - y) + y \in D_2 + D_1 H$. Since $x \in K_1$ was arbitrary, $K_1 = D_2 \oplus D_1 H$.

If K_1 is a free right H-module, 3.2 implies that D_2 is projective; H is connected, so D_2 is free. Let $W \subseteq D_2$ be a right H-basis for D_2, i.e., $D_2 = WH \approx W \otimes H$. Then $K_1 = D_2 \oplus D_1 H \approx (W \otimes H) \oplus (D_1 \otimes H) \approx$
$\approx (W \oplus D_1) \otimes H_1$, so $W \oplus D_1$ is a right H-basis for K_1.

For the next four results (3.3 to 3.6) assume K_1 is right-H-free. Let $\{w_j\}_{j\in J}$ be a basis for W. Note that $W \subseteq D_2 \subseteq \rho(N)\gamma H$.

Lemma 3.3. Let $\{x_j\}_{j\in J} \subseteq \hat{H}$. (a) $\sum_j w_j x_j = 0$ implies each $x_j = 0$. (b) $\sum_j w_j x_j \in B$ implies each $x_j \in B$.

Proof. (a) This follows from the fact that K_1 is free, hence $W \otimes H \approx WH$ in $\hat{H}_1$. It follows that $W \otimes \hat{H} \approx W\hat{H}$ which is the stated result.

(b) We may assume that the x_j's are all in the same $\hat{H}_p$, $p \geq 0$. By 3.1 write $\sum_j w_j x_j = \hat{d}(y)$, where $y \in \rho(N)\gamma\hat{H}_{p+1}$. Write $y = \sum_i z_i \tilde{\gamma}\cdot\tilde{b}_i$, where $z_i \in \rho(N)\gamma H$ and $\tilde{\gamma}\cdot\tilde{b}_i \in \gamma\hat{H}_p$. By combining z_i's if necessary we may assume that the $\tilde{\gamma}\cdot\tilde{b}_i$ are linearly independent in $\gamma\hat{H}_p$.

$\sum_j w_j x_j = \hat{d}(y) = \sum_i \hat{d}(z_i)\tilde{\gamma}\cdot\tilde{b}_i + \sum_i (-1)^{|z_i|} z_i \hat{d}(\tilde{\gamma}\cdot\tilde{b}_i)$. Since $\sum_j w_j x_j \in W\hat{H}_p \subseteq$ $\subseteq \rho(N)\gamma\hat{H}_p$ and each $z_i\hat{d}(\tilde{\gamma}\cdot\tilde{b}_i) \in \rho(N)\gamma\hat{H}_p$, but $\sum_i \hat{d}(z_i)\tilde{\gamma}\cdot\tilde{b}_i \in H\beta H\gamma\hat{H}_p$, we must have $\sum_i \hat{d}(z_i)\tilde{\gamma}\cdot\tilde{b}_i = 0$. Because the $\tilde{\gamma}\cdot\tilde{b}_i$ are linearly independent, however, this can only happen if each $\hat{d}(z_i) = 0$, implying $z_i \in K_1$. Because $z_i \in \rho(N)\gamma H$ as well, we have $z_i \in D_2 = WH$. Write $z_i = \sum_j w_j h_{ij}$ for suitable $h_{ij} \in H$. Thus

$\sum_j w_j x_j = \hat{d}(\sum_i z_i\tilde{\gamma}\cdot\tilde{b}_i) = \sum_i\sum_j (-1)^{|z_i|} w_j h_{ij}\hat{d}(\tilde{\gamma}\cdot\tilde{b}_i) = \sum_j w_j(-1)^{|w_j|}\sum_i \hat{d}(h_{ij}\tilde{\gamma}\cdot\tilde{b}_i)$.

By part (a) this can only happen if each $x_j = (-1)^{|w_j|}\sum_i \hat{d}(h_{ij}\tilde{\gamma}\cdot\tilde{b}_i) \in B$.

Proposition 3.4. The map κ: $W \otimes (K/B) \to K/B$ given by $\kappa(w_j \otimes \overline{x}) = \overline{w_j x}$ is monomorphic.

Proof. $W \subseteq K_1 \subseteq K$, K is an algebra, and $K\cdot B \subseteq B$, so the map is well-defined. For injectivity, note that $\sum_j w_j \otimes \overline{x}_j \in \ker \kappa$ would require $\sum_j w_j x_j \in B$. By 3.3(b) this would mean that each $x_j \in B$, or $\overline{x}_j = 0$. Thus $\ker \kappa = 0$.

Proposition 3.5. *There is an embedding of modules* ξ: $TW \otimes N \to E^2$, *where* TW *denotes the tensor algebra on* W. ξ *preserves the left action of* TW *and the right action of* N *on each module. Furthermore, all the higher differentials* d^r, *for* $r \geq 2$, *vanish on* im ξ.

Proof. Recall that $K_0/B_0 = H/H\beta H = N$. By 3.4 and induction on p we have injections $\underbrace{W \otimes \ldots \otimes W}_{p \text{ times}} \otimes N \to K_p/B_p$ for each p. Thus ξ: $TW \otimes N \to K/B$ exists and is monomorphic.

ξ preserves left multiplication by elements of W, so we know that im ξ is generated multiplicatively by $N = K_0/B_0$ and $\xi(W) \subseteq K_1/B_1$. But these generators lie in the 0^{th} and 1^{st} columns of the spectral sequence for E^2, and d^r, $r \geq 2$, vanishes on these first two columns. Since d^r obeys the product rule, d^r vanishes on all of im ξ.

Proposition 3.6. $(TW \otimes N)(z)^{-1} = (1 + z)N(z)^{-1} - zH(z)^{-1} - \gamma(z)$.

Proof. By 3.2 and the remarks immediately before and after it,

$$W(z) = D_2(z)H(z)^{-1} = K_1(z)H(z)^{-1} - D_1(z).$$

$$D_1(z) = (H\beta H\gamma)(z) = (H(z) - N(z))\gamma(z) .$$

From the exact sequence $0 \to K_1 \to H\gamma H \overset{\hat{d}}{\to} B_0 = H\beta H \to 0$, we have $K_1(z) = H(z)\gamma(z)H(z) - z(H(z) - N(z))$. Together, we obtain $W(z) = \gamma(z)N(z) - z(1 - N(z)H(z)^{-1})$.

$$\begin{aligned}(TW \otimes N)(z)^{-1} &= TW(z)^{-1}N(z)^{-1} = N(z)^{-1}(1 - W(z)) \\ &= N(z)^{-1}[1 - \gamma(z)N(z) + z(1 - N(z)H(z)^{-1})] \\ &= N(z)^{-1} - \gamma(z) + zN(z)^{-1} - zH(z)^{-1} \\ &= (1 + z)N(z)^{-1} - zH(z)^{-1} - \gamma(z), \text{ as desired.}\end{aligned}$$

Theorem 3.7. *Suppose* K *or* HβH *is* H-*free*. *Then the* E-M.s.s. *degenerates and* ξ: $TW \otimes N \to H_*(\Omega Y)$ *is an isomorphism preserving the left action of* TW *and the right action of* N. *Furthermore,*

$$(3) \qquad H_*(\Omega Y)(z)^{-1} = (1 + z)N(z)^{-1} - zH(z)^{-1} - \gamma(z).$$

If H(z) *is rational, then* $H_*(\Omega Y)(z)$ *is rational if and only if* N(z) *is rational.*

Proof. K being free includes K_1 being free as a special case (right- and left-free agree here), so the results 3.3 to 3.6 are valid. If HβH is free, K_1 is automatically free because it appears in the resolution $0 \to K_1 \to \hat{H}_1 \overset{\hat{d}}{\to} H\beta H$ and $\hat{H}_1$ is free. By 2.4 and 3.5 and 3.6, ξ is a monomorphism between two modules of equal rank in each dimension, hence an isomorphism. By 3.6, the d^r, $r \geq 2$, vanish on all of E^2, hence $E^\infty = E^2$. Formula (3) follows at once, as does the statement about the rationality of $H_*(\Omega Y)(z)$. In general, there is no guarantee that $H_*(A,d) = E^\infty$ as algebras. In this case, however, each w_j corresponds to a cycle in A. Using this correspondence we may check easily that ξ has the stated properties.

Corollary 3.8. *Suppose* $H = H_*(\Omega X)$ *has global dimension* ≤ 2. *For example, suppose* X *is a suspension or a product of two suspensions. Then* 3.7 *and Formula* (3) *apply.*

Proof. This follows from 2.5. $H_*(\Omega SX_1)$ is known to be free [6], hence, has global dimension one. A product $H_*(\Omega(SX_1 \times SX_2)) =$ $= H_*(\Omega SX_1) \otimes H_*(\Omega SX_2)$ has global dimension ≤ 2.

Proposition 3.9. *Assume* $\beta = \{\beta_1, \ldots \beta_m\}$ *is a linearly independent set. Then the following are equivalent.*

(a) $H \approx N\langle\beta\rangle$ as $\mathbb{F}$-modules

(b) The surjection ζ of 3.1 is an isomorphism

(c) Theorem 3.7 applies and $H_*(\Omega Y) = N$

(d) $K_p = B_p$ for all $p > 0$

(e) $K_1 = B_1$.

Proof. (a) iff (b). $N\langle\beta\rangle \approx N \otimes T(\beta\,\mathbb{F} \otimes N)$, so

$N\langle\beta\rangle(z) = N(z)(1 - \beta(z)N(z))^{-1}$, where $\beta(z) = \sum_{j=1}^{m} z^{d_j - 1} = z^{-1}\gamma(z)$.

The next five lines are equivalent statements.

Condition (a)

$H(z) = N(z)(1 - \beta(z)N(z))^{-1}$

$H(z) - H(z)\beta(z)N(z) = N(z)$

$H(z)\beta(z)N(z) = H(z) - N(z) = (H\beta H)(z)$

$(H \otimes \beta\,\mathbb{F} \otimes N) \approx H\beta H$ as $\mathbb{F}$-modules.

Since ζ is always a surjection, the last statement is equivalent to ζ being an isomorphism.

(b) implies (c). ζ itself demonstrates $H\beta H$ to be free, so 3.7 applies. By the above, $H(z) = N(z)(1 - z^{-1}\gamma(z)N(z))^{-1}$. Substituting this into Eq. (3) gives $H_*(\Omega Y)(z) = N(z)$. Since N is a subalgebra of $H_*(\Omega Y)$ by 3.7, we must have $H_*(\Omega Y) = N$.

(c) implies (b). By formula (3) we obtain

$$N(z)^{-1} = (1 + z)N(z)^{-1} - zH(z)^{-1} - \gamma(z) \ ,$$

which is equivalent to

$$H(z) = N(z)(1 - z^{-1}\gamma(z)N(z))^{-1}.$$

(c) implies (d). We have $K/B = E^2 = E^\infty = N = K_0/B_0$. Thus $K_p/B_p = 0$ for $p > 0$, i.e., $K_p = B_p$.

(d) implies (e). Obvious.

(e) implies (c). Construct a free H-resolution of N which begins $\cdots \to \hat{H}_3 \oplus (M \otimes H) \xrightarrow{(\hat{d} \oplus \mu)} \hat{H}_2 \xrightarrow{\hat{d}} \hat{H}_1 \xrightarrow{\hat{d}} H \to N$. Here $M \otimes H$ is any right-free H-module for which $B_2 + \operatorname{im} \mu = K_2$. Condition (e) assures us of exactness at $\hat{H}_1$. Use this resolution to compute $\operatorname{Tor}_2^H(N, \mathbb{F})$. $\operatorname{Tor}_p^H(N, \mathbb{F})$ is given by the homology of the chain complex

$$\cdots \to \hat{H}_2\gamma \oplus M \xrightarrow{\hat{d}_{\mathbb{F}} \oplus \mu_{\mathbb{F}}} \hat{H}_1\gamma \xrightarrow{\hat{d}_{\mathbb{F}}} H\gamma \xrightarrow{(0)} \mathbb{F} \to 0, \text{ where } \hat{d}_{\mathbb{F}}(\tilde{a}\cdot\tilde{\gamma}) =$$

$= \hat{d}_{\mathbb{F}}(\tilde{a}) \cdot \tilde{\gamma}$. $\operatorname{Tor}_2^H(N, \mathbb{F}) = \ker (\hat{d}_{\mathbb{F}})_1/(\operatorname{im}(\hat{d}_{\mathbb{F}})_2 + \operatorname{im} \mu_{\mathbb{F}}) = K_1\gamma/B_1\gamma = 0$. $\operatorname{Tor}_1^H(H\beta H, \mathbb{F}) = \operatorname{Tor}_2^H(N, \mathbb{F}) = 0$, implying that $H\beta H$ is free. 3.7 applies with $W = 0$ because $K_1/B_1 = 0$.

Results 3.8 and 3.9 extend work done previously by Lemaire. Theorem 3.8 when X is a suspension may be deduced easily from Lemaire's thesis [7]. Lemaire also considered in [8] the question of when $H_*(\Omega Y) = N$ for $m = 1$ (only one attaching cell).

II. FINITELY PRESENTED ALGEBRAS

In Part II we construct a class of finitely presented non-commutative algebras whose Poincaré series can be computed fairly easily. Examples where the Poincaré series is irrational exist and may be used to construct counterexamples to Serre's conjecture. We conclude with a consideration of the question of just what kinds of Poincaré series can be expected from such complexes.

4. The Homomorphism ϕ

Our goal in Section 4 is to establish the properties of a homomorphism ϕ whose range is the underlying Lie algebra of a primitive

Hopf algebra. ϕ will be an important tool when we want to calculate quotient algebras later.

If L is a Lie algebra, let U(L) denote the universal enveloping algebra of L. Let $H = H_*(\bigvee_{j=1}^{k} S^{c_j+1}) = \mathbb{F}\langle\alpha_1, \ldots, \alpha_k\rangle$, where $|\alpha_j| = c_j \geq 1$. Let L be the free Lie algebra generated by $\{\alpha_1, \ldots, \alpha_k\}$; then H = U(L). There is a standard basis S for H consisting of monomials in the $\{\alpha_i\}$. Let ℓ: $S \to \mathbb{Z}_+ \cup \{0\}$ give the length of a monomial, i.e., $\ell(\alpha_{i_1} \ldots \alpha_{i_n}) = n$.

<u>Definition. A function</u> g: $S \to \mathbb{F}$ <u>will be said to be additive if</u> $g(xy) = g(x) + g(y)$. <u>We say that</u> $x \in H$ <u>is homogeneous with respect to ("w.r.t.")</u> g <u>if</u> $x \in \text{Span}(S \cap g^{-1}(n))$ <u>for some</u> n. <u>In such a case we also write</u> $g(x) = n$.

<u>Let</u> g <u>be any additive function on</u> S <u>such that</u> $g(\alpha_j) \neq 0$ <u>for each</u> j. <u>Define a homomorphism</u> ϕ: $H \to L$ <u>by defining it recursively on</u> S, <u>as follows.</u> $\phi(1) = 0$. $\phi(\alpha_j) = g(\alpha_j)\alpha_j$. <u>For</u> $n > 1$, $\phi(\alpha_{j_1} \ldots \alpha_{j_n}) = [\phi(\alpha_{j_1} \ldots \alpha_{j_{n-1}}), \alpha_{j_n}]$.

This definition is inspired by a homomorphism Φ which Serre uses in [13, p. LA. 415] to prove the Baker-Campbell-Hausdorff formula.

ϕ will give us a way to get a handle on the elements of the free Lie algebra L. In practice, the additive function g will usually agree with either length (ℓ) or dimension ($|\cdot|$), but for now it is best to keep things general.

Recall the Jacobi identities

(4a) $$[a,b] + (-1)^{|b|\cdot|a|}[b,a] = 0 \quad .$$

(4b) $$(-1)^{|a|\cdot|c|}[[a,b],c] + (-1)^{|b|\cdot|a|}[[b,c],a] + (-1)^{|c|\cdot|b|}[[c,a],b] = 0.$$

Lemma 4.1. For $a,b \in \overline{H}$, $\phi(a\phi(b)) = [\phi(a),\phi(b)]$.

Proof. It is enough to prove this when $a,b \in S$, since both sides are bilinear in a and b. Use induction on $\ell(b)$. If $\ell(b) = 1$, the lemma holds by definition of ϕ. Suppose the lemma holds for $\ell(b) < n$ and take $\ell(b) = n$. Write $b = uv$, where $\ell(v) = 1$, $\ell(u) = n-1$. We have $\phi(a\phi(b)) = \phi(a\phi(uv)) = \phi(a[\phi(u),v]) =$ $\phi(a\phi(u)v) - (-1)^{|u||v|}\phi(av\phi(u))$.

By our inductive assumption this becomes

$$\begin{aligned}
\phi(a\phi(b)) &= [[\phi(a),\phi(u)],v] - (-1)^{|u|\cdot|v|}[\phi(av),\phi(u)] \\
&= [[\phi(a),\phi(u)],v] - (-1)^{|u|\cdot|v|}[[\phi(a),v],\phi(u)] \\
&= (\text{by (4a)})\ [[\phi(a),\phi(u)],v] + (-1)^{|u|\cdot|v|+|a|\cdot|v|}[[v,\phi(a)],\phi(u)] \\
&= (\text{by (4b)})\ (-1)^{|u|\cdot|a|+|a|\cdot|v|}[[\phi(u),v],\phi(a)] \\
&= (\text{by (4a)})[\phi(a),[\phi(u),v]] = [\phi(a),\phi(uv)] = [\phi(a),\phi(b)].
\end{aligned}$$

Lemma 4.2. If a is homogeneous w.r.t. g, then $\phi(\phi(a)) = g(a)\phi(a)$.

Proof. It is enough to prove this for $a \in S$. If $\ell(a) = 1$, the lemma holds. Suppose the lemma holds for $\ell(a) < n$ and that $\ell(a) = n$. Write $a = uv$, where $\ell(v) = 1$, $\ell(u) = n-1$. Then $\phi(\phi(a)) = \phi(\phi(uv))$

$$\begin{aligned}
&= \phi([\phi(u),v]) = \phi(\phi(u)v) - (-1)^{|u|\cdot|v|}\phi(v\phi(u)) \\
&= [\phi(\phi(u)),v] - (-1)^{|u|\cdot|v|}[\phi(v),\phi(u)] \\
&= (\text{by}(4a))\ [\phi(\phi(u)),v] + [\phi(u),\phi(v)].
\end{aligned}$$

By the inductive assumption this becomes

$$\begin{aligned}
\phi(\phi(a)) &= g(u)[\phi(u),v] + g(v)[\phi(u),v] = (g(u) + g(v))[\phi(u),v] \\
&= g(uv)\phi(uv) = g(a)\phi(a).
\end{aligned}$$

Lemma 4.3. ϕ: $H \to L$ is surjective if char $\mathbb{F} \neq 2$. If char $\mathbb{F} = 2$, then $L = \text{im}\,\phi + (\text{im}\,\phi)^2$.

Proof. Im ϕ contains each α_j because each $g(\alpha_j)$ is a unit in $\mathbb{F}$. By 4.1, im ϕ is closed under brackets. Thus im $\phi = L$ if char $\mathbb{F} \neq 2$.

If char $\mathbb{F} = 2$, L comes with a squaring operation on odd-dimensional elements as well as a bracket operation. A span for L consists of everything we obtain by a sequence of brackets and squarings. Because $[x^2,y] = [x,[x,y]]$, however, we may assume that the squarings occur only at the end of a sequence of operations. Furthermore, since only odd-dimensional elements may be squared, at most one such squaring can occur. Thus $L = \text{im}\,\phi + (\text{im}\,\phi)^2$.

Lemma 4.4. Let I be any two-sided ideal of H. If $\phi(a) \in I$, then $\phi(ab) \in I$ for any $b \in H$.

Proof. By induction on $\ell(b)$. If $\ell(b) = 1$, $\phi(ab) = [\phi(a),b] \in$ $\in IH + HI = I$. For $\ell(b) > 1$ write $b = uv$ with $\ell(v) = 1$. $\phi(ab) = \phi(auv)$. By the inductive assumption $\phi(au) \in I$. By the above, then, $\phi(ab) = \phi(auv) \in I$ as well.

We are concerned next with extending these results to the case where H is a quotient algebra of a free algebra.

Lemma 4.5. Let $\beta = \{\beta_j\} \subseteq \text{im}\,\phi$ and suppose that each β_j is homogeneous w.r.t. g. Let N be the quotient algebra $H/H\beta H$ and let π_N: $H \to N$ be the natural projection. Let L_N denote the quotient Lie algebra $L/(L \cap H\beta H)$. Then $N = U(L_N)$ and there is a well-defined homomorphism ϕ_N: $N \to L_N$ satisfying $\phi_N(\pi_N(x)) = \pi_N(\phi(x))$ for all $x \in H$.

Proof. That $N = U(L_N)$ is easy to check. To show ϕ_N well-defined we need only confirm that $x \in \ker \pi_N$ implies $\phi(x) \in \ker \pi_N$. Write $\beta_j = \phi(\delta_j)$. Because $g(\beta_j)$ exists for each β_j, we may assume that each δ_j is homogeneous w.r.t. g and that $g(\delta_j) = g(\beta_j)$.

Ker $\pi_N = H\beta H = \beta H + \overline{H}\beta H$. Any $x \in \overline{H}\beta H$ is a sum of terms of the form $a\beta_j b$. $\phi(a\beta_j b) = \phi(a\phi(\delta_j)b) = -(-1)^{|a|\cdot|\delta_j|}\phi(\delta_j\phi(a)b) \in H\beta H$ by using 4.1 and 4.4 if $a \in \overline{H}$. So $\phi(\overline{H}\beta H) \subseteq H\beta H$. 4.2 yields $\phi(\beta_j) = \phi(\phi(\delta_j)) = g(\delta_j)\phi(\delta_j) = g(\delta_j)\beta_j$, so $x \in \beta H$ implies $\phi(x) \in H\beta H$ by 4.4. $\phi(H\beta H) = \phi(\overline{H}\beta H) + \phi(\beta H) \subseteq H\beta H$, as desired.

Consider the diagram

$$\begin{array}{ccc} H & \xrightarrow{\pi_N} & N \\ \phi \downarrow & & \downarrow \phi_N \\ L & \xrightarrow{\pi_N} & L_N \end{array} \tag{5}$$

which commutes by the way ϕ_N was defined. All results obtained so far can be extended to N and ϕ_N, as we now observe.

Proposition 4.6. *Let* $\beta' \subset \mathrm{im}\,\phi$ *be a set of elements homogeneous with respect to* g. *Let* $G = H/H\beta' H$ *and* $L_G = L/(L \cap H\beta' H)$ *and* π_G: $H \to G$ *be the natural quotients and projection. Then Lemmas* 4.1 *through* 4.5 *still hold if* H, L, *and* ϕ *are replaced everywhere by* G, L_G, *and* ϕ_G.

Proof. We use diagram (5) for N = G. The fact that π_G is surjective means that any statement about elements of G can be lifted to a corresponding statement about H. After applying the appropriate lemma in H we project back down to G.

For the next three lemmas, let H be a free algebra and $G = H/H\beta' H$, where each $\beta'_j = \phi(\delta'_j)$ is homogeneous with respect to g.

Lemma 4.7. *Let $\beta = \{\beta_j\} \subseteq \text{im}\ \phi_G$ and write $\beta_j = \phi_G(\delta_j)$. $L_G \cap G\beta G = \phi_G(\delta G)$ if char $\mathbb{F} \neq 2$ and $L_G \cap G\beta G = \phi_G(\delta G) + \phi_G(\delta G)^2$ if char $\mathbb{F} = 2$.*

Proof. Let I be the Lie ideal of L_G generated by β. That is, I is the smallest Lie ideal of L_G which contains β. $G/G\beta G = U(L_G/I)$ because $G/G\beta G$ has the requisite universal property. Since $L_G/I \to G/G\beta G$ is an embedding and $L_G \cap G\beta G$ is in the kernel of the composition $L_G \to L_G/I \to G/G\beta G$, we must have $L_G \cap G\beta G \subseteq I$. $I \subseteq L_G \cap G\beta G$, so $I = L_G \cap G\beta G$.

When char $\mathbb{F} \neq 2$, $\phi_G(\delta G)$ is a Lie ideal by 4.1 and 4.3. Since $\phi_G(\delta G) \subseteq I$, $I = \phi_G(\delta G)$. When char $\mathbb{F} \neq 2$, I must be closed under squares as well as brackets with elements of L_G. $\phi_G(\delta G) + \phi_G(\delta G)^2 \subseteq I$. $\phi_G(\delta G) + \phi_G(\delta G)^2$ is a Lie ideal by 4.1, 4.3 and the rule $[x^2,y] = [x,[x,y]]$. Conclude that $I = \phi_G(\delta G) + \phi_G(\delta G)^2$.

Lemma 4.8. *If char $\mathbb{F} = 2$, $L_G = \phi_G(G) \oplus \phi_G(G)^2$ and $(\ )^2_G\colon (\phi_G(G))_{odd} \to \phi_G(G)^2$ is an isomorphism which doubles degrees.*

Proof. If H is free, $\phi(H) \cap \phi(H)^2 = 0$ and $L = \phi(H) \oplus \phi(H^2)$. Recall that $G = H/H\beta' H$, where $\beta'_j = \phi(\delta'_j)$. $\phi(\delta' H) \cap \phi(\delta' H)^2 \subseteq \phi(H) \cap \phi(H)^2 = 0$, so $\phi(\delta' H) + \phi(\delta' H)^2 = \phi(\delta' H) \oplus \phi(\delta' H)^2$. Using 4.7 and 4.3, $L_G = L/(L \cap H\beta' H) = (\phi(H) \oplus \phi(H)^2)/(\phi(\delta' H) \oplus \phi(\delta' H)^2) = (\text{im}\ \phi_G) \oplus (\text{im}\ \phi_G)^2$.

If $y \in L_G$ has $y^2 = 0$, pull back to any $x \in L$ with $\pi_G(x) = y$. $x^2 \in \phi(\delta' H)^2$. Since $(\ )^2$ is an isomorphism in H, this requires $x \in \phi(\delta' H)$, i.e., $y = \pi_G(x) = 0$. Thus $(\ )^2_G$ is an isomorphism as well.

Let M be a graded $\mathbb{F}$-module. If char $\mathbb{F} \neq 2$, let $\mathcal{P}(M)$ denote the commutative algebra generated by M. That is, $\mathcal{P}(M)$ is the tensor product of an exterior algebra on a basis for odd-dimensional M with a tensor algebra on a basis for even-dimensional M. If char $\mathbb{F} = 2$, $\mathcal{P}(M)$ denotes the tensor algebra on a basis for M.

In Lemma 4.9 we drop the subscripts on ϕ_G and L_G and associate ϕ and L with the quotient algebra G.

Lemma 4.9. *As graded* $\mathbb{F}$-*modules, there is an isomorphism* $G \approx \mathcal{P}(\text{im } \phi)$. *Furthermore, suppose* $\beta = \{\beta_j\}$, $\beta_j = \phi(\delta_j)$. *Then as* $\mathbb{F}$-*modules,* $G/G\beta G = \mathcal{P}(\phi(G)/\phi(\delta G))$.

Proof. First take char $\mathbb{F} \neq 2$. That $G \approx \mathcal{P}(\text{im } \phi) = \mathcal{P}(L)$ is simply the graded version of the Poincaré-Birkhoff-Witt theorem [4]. Let $N = G/G\beta G$. The same theorem indicates that $N \approx \mathcal{P}(L_N) =$ $= \mathcal{P}(L/(L \cap G\beta G))$. By 4.7 this may be written as $N \approx \mathcal{P}(L/\phi(\delta G)) =$ $\mathcal{P}(\phi(G)/\phi(\delta G))$.

For char $\mathbb{F} = 2$, let $L_1 = (\text{im } \phi)_{\text{odd}}$ and $L_2 = (\text{im } \phi)_{\text{even}}$. By 4.8, $L = L_1 \oplus L_2 \oplus L_1^2$ and $(\ )^2\colon\ L_1 \to L_1^2$ is an isomorphism. Let $E(\cdot)$ denote an exterior algebra on a basis and $T(\cdot)$ a tensor algebra. $G \approx E(L_{\text{odd}}) \otimes T(L_{\text{even}}) \approx E(L_1) \otimes T(L_1^2) \otimes T(L_2) \approx T(L_1) \otimes T(L_2) =$ $T(L_1 \oplus L_2) = T(\text{im } \phi)$. Finally, using 4.7 and 4.8, $N = G/G\beta G \approx$

$$\approx E((L_N)_{\text{odd}}) \otimes T((L_N)_{\text{even}})$$
$$\approx E(L_1/\phi(\delta G)_{\text{odd}}) \otimes T(L_1^2/\phi(\delta G)^2) \otimes T(L_2/\phi(\delta G)_{\text{even}})$$
$$\approx T(L_1/\phi(\delta G)_{\text{odd}}) \otimes T(L_2/\phi(\delta G)_{\text{even}})$$
$$\approx T(\phi(G)/\phi(\delta G)),$$ as desired.

Remarks. We will find Lemma 4.9 very useful when we do Poincaré series computations.

Lemmas 4.1 through 4.6 will simplify our work considerably when evaluating $\phi(G)$ and $\phi(\delta G)$. The only "loose end" is the somewhat unusual constraint that each β_j' be homogeneous w.r.t. g. Since β_j' will always be homogeneous w.r.t. dimension anyway, we can generally take $g = \pi_{\mathbb{F}} \circ |\cdot|$, where $\pi_{\mathbb{F}}\colon\ \mathbb{Z} \to \mathbb{F}$ is the canonical map of rings.

This approach always works if char $\mathbb{F} = 0$. However, it fails if char $\mathbb{F} = p \neq 0$ and there is a generator α_j whose dimension is divisible by p. Then $\phi(\alpha_j) = 0$, and im ϕ(or im ϕ + (im $\phi)^2$) is no longer all of L.

Using $\pi_{\mathbb{F}} \circ \ell$ for g always results in a suitable ϕ, but there is no guarantee that each β_j will be homogeneous w.r.t. $\pi_{\mathbb{F}} \circ \ell$. (Of course, this may be true in individual cases, such as when all the generators have the same dimension.) For these reasons we have done everything with the flexibility afforded by an arbitrary additive g.

5. Generalized Products

In general, the problem of precisely determining the Lie elements or the Poincaré series of a finitely presented algebra is very difficult. In Section 5 we define a class of such algebras, called "generalized products", whose algebraic structures are particularly well-behaved. At the same time, there is sufficient freedom in the definition to allow quotient algebras of these generalized products to have very interesting properties.

We begin with a discussion of semi-tensor products as described by Massey and Peterson [9] and by Smith [15]. Let H_1 and H_2 be connected algebras over $\mathbb{F}$, H_1 a Hopf algebra. Let χ: $H_1 \otimes H_2 \to H_2$ make H_2 into an algebra over H_1 (see [15], p. 18). The multiplication μ_2: $H_2 \otimes H_2 \to H_2$ is a morphism of H_1-modules. Writing $\psi(x) = \sum_x x' \otimes x''$, where ψ is the coproduct of H_1, this means that

(6a) $$\sum_x (-1)^{|x''||y_1|}\chi(x' \otimes y_1)\chi(x'' \otimes y_2) = \chi(x \otimes y_1 y_2)$$

must hold for all $x \in H_1$ and all $y_1, y_2 \in H_2$.

Let $H = H_1 \amalg H_2$ be the free product or "coproduct" of rings as described by Smith ([14], p. 124). H has a universal property based on its being the push-out of the pair of maps $\mathbb{F} \to H_1$, $\mathbb{F} \to H_2$. Any module over H_1 and H_2 is a module over H.

Let M be a module over both H_1 and H_2, with λ_i: $H_i \otimes M \to M$ giving the actions for i = 1,2. By the above remark, this is the same as saying that M is an H-module. In ([15], p. 22) Smith defines M to be an (H_2, H_1, χ)-module if λ_2 is a morphism of H_1-modules, i.e., if

(6b) $$\sum_x (-1)^{|x''||y|}\lambda_2(\chi(x' \otimes y) \otimes \lambda_1(x'' \otimes z)) = \lambda_1(x \otimes \lambda_2(y \otimes z))$$

for all $x \in H_1$, $y \in H_2$, $z \in M$.

Lastly, the semi-tensor product of H_1 and H_2, denoted $H_2 \odot H_1$, is defined to be an object isomorphic with $H_2 \otimes H_1$, as an $\mathbb{F}$-module. Its algebraic structure, however, is given by μ: $(H_2 \odot H_1) \otimes (H_2 \odot H_1) \to$ $\to H_2 \odot H_1$, where

(6c) $$\mu((y_1 \odot x_1) \otimes (y_2 \odot x_2)) = \sum_{x_1} (-1)^{|x_1''||y_2|} y_1\chi(x_1' \otimes y_2) \odot x_1'' x_2.$$

Theorem 5.1. _Let_ H_1, H_2 _be as above, with_ H_1 _primitive. Let_ T_i _be a set of multiplicative generators for_ H_i, i = 1,2. _For_ $(a,b) \in T_1 \times T_2$, _let_ $h_{ab} = \chi(a \otimes b) \in H_2$. _Let_ $\hat{\beta}_{ab} = [a,b] - h_{ab} \in H =$ $= H_1 \amalg H_2$. _Let_ $\hat{\beta} = \{\hat{\beta}_{ab} | (a,b) \in T_1 \times T_2\}$ _and let_ $G = H/H\hat{\beta}H$. _Then an_ H-_module_ M _is an_ (H_2, H_1, χ)-_module if and only if it is a_ G-_module_.

<u>Proof</u>. We notate the actions of λ_1 and λ_2 simply by juxtaposition. First suppose M is an (H_2, H_1, χ)-module. Taking $x = a$, $y = b$ in (6b) we get $h_{ab}z + (-1)^{|a||b|}baz = abz$ for $a \in T_1$, $b \in T_2$, $z \in M$. Then $\hat{\beta}_{ab}z = (ab - (-1)^{|a||b|}ba - h_{ab})z = 0$, so M is a module over G as well.

Conversely, assume Formula (6b) holds for $(a,b) \in T_1 \times T_2$. We must show that it holds for any $(x,y) \in H_1 \times H_2$. First we show that it holds for any $(a,y) \in T_1 \times H_2$. It is enough to show that (6b) holds for (a,y_1y_2) given that it holds for (a,y_1) and (a,y_2). Since <u>a</u> is primitive, (6b) becomes

$$\chi(a \otimes y_1y_2)z + (-1)^{|a||y_1y_2|}y_1y_2az = ay_1y_2z.$$

By (6a), $\chi(a \otimes y_1y_2) = \chi(a \otimes y_1)y_2 + (-1)^{|a||y_1|}y_1\chi(a \otimes y_2)$. We wish to verify that

$$(\chi(a \otimes y_1)y_2 + (-1)^{|a||y_1|}y_1\chi(a \otimes y_2) + (-1)^{|a||y_1y_2|}y_1y_2a - ay_1y_2)z = 0.$$

Since (6b) is valid for (a,y_2), the second and third terms may be combined, giving

$$(\chi(a \otimes y_1)y_2 + (-1)^{|a||y_1|}y_1ay_2 - ay_1y_2)z = 0, \quad \text{or}$$

$$(\chi(a \otimes y_1) + (-1)^{|a||y_1|}y_1a - ay_1)y_2z = 0.$$

But this last equation follows from the fact that (6b) holds for (a,y_1). We have shown that (6b) is valid for any $(a,y) \in T_1 \times H_2$.

Now let $x_1, x_2 \in H_1$ and suppose that (6b) holds for (x_1,y) and (x_2,y) for any $y \in H_2$. We now show that (6b) holds for (x_1x_2,y). From this it will follow that (6b) is valid for all $(x,y) \in H_1 \times H_2$.

We wish to check that

$$\sum_{x_1x_2} (-1)^{|(x_1x_2)''||y|} \chi((x_1x_2)' \otimes y)(x_1x_2)''z = x_1x_2yz.$$

By the usual formula,

$$\sum_{x_1x_2} (x_1x_2)' \otimes (x_1x_2)'' = \sum_{x_1} \sum_{x_2} (-1)^{|x_1''||x_2'|} x_1'x_2' \otimes x_1''x_2'' .$$

Using this, our expression becomes

$$\sum_{x_1} \sum_{x_2} (-1)^{|x_1''||x_2'|} (-1)^{|x_1'' x_2''||y|} \chi(x_1'x_2' \otimes y)x_1''x_2''z = x_1x_2yz.$$

Since $\chi(x_1'x_2' \otimes y) = \chi(x_1' \otimes \chi(x_2' \otimes y))$, we may obtain

$$\sum_{x_2} \sum_{x_1} (-1)^{|x_1''||x_2'|} (-1)^{|x_2''||y|} (-1)^{|x_1''||y|} \chi(x_1' \otimes \chi(x_2' \otimes y))x_1''x_2''z = x_1x_2yz,$$

which in turn becomes

$$\sum_{x_2} (-1)^{|x_2''||y|} [\sum_{x_1} (-1)^{|x_1''||x_2' \otimes y|} \chi(x_1' \otimes \chi(x_2' \otimes y)x_1'']x_2''z = x_1x_2yz.$$

Because (6b) is valid for each $(x_1, \chi(x_2' \otimes y))$, the expression in the bracket can be replaced by $x_1\chi(x_2 \otimes y)$. We obtain

$$x_1 \sum_{x_2} (-1)^{|x_2''||y|} \chi(x_2' \otimes y)x_2''z = x_1x_2yz.$$

This last equation follows from our assumption that (6b) holds for (x_2,y).

Corollary 5.2. *Under the conditions of 5.1*, $G = H_2 \circledcirc H_1$.

Proof. By ([15], Prop. 2.2), the semi-tensor product is an H-module and is universal among (H_2,H_1,χ)-modules. By 5.1, then, $H_2 \circledcirc H_1$ is the universal G-module. The universal module for any ring is the ring itself, hence, $H_2 \circledcirc H_1 = G$.

For the remainder of Section 5 we will assume that $H_1 = \mathbb{F}\langle T_1\rangle$ and $H_2 = \mathbb{F}\langle T_2\rangle$. Let $h_{ab} \in H_2$ be arbitrary for $(a,b) \in T_1 \times T_2$, subject only to the condition $|h_{ab}| = |a| + |b|$. For each $a \in T_1$, let $\xi_a\colon H_2 \to H_2$ be defined by $\xi_a(1) = 0$, $\xi_a(b) = h_{ab}$ for $b \in T_2$, and $\xi_a(y_1y_2) = \xi_a(y_1)y_2 + (-1)^{|a||y_1|}y_1\xi_a(y_2)$. Because H_2 is free, ξ_a is well-defined. Define an action $\chi\colon H_1 \otimes H_2 \to H_2$ by $\chi(a_1 \ldots a_n \otimes y) = \xi_{a_1} \circ \ldots \circ \xi_{a_n}(y)$. Because H_1 is free, χ is well-defined.

<u>Proposition</u> 5.3. χ <u>makes</u> H_2 <u>into an algebra over</u> H_1.

<u>Proof</u>. Considering the similarity between (6a) and (6b), H_2 is an algebra over H_1 if it is an (H_2,H_1,χ)-module. We would like to apply 5.1. In the proof of 5.1 we assumed only that (6a) holds for $x = a \in T_1$. This assumption follows in this case from ξ_a being a derivation. We need only check that H_2 is a G-module. This means verifying that $\chi(a \otimes bz) = (-1)^{|a||b|}b\chi(a \otimes z) + h_{ab}z$ for any $a \in T_1$, $b \in T_2$, $z \in H_2$. This is the same as the claim

$$\xi_a(bz) = (-1)^{|a||b|}b\xi_a(z) + h_{ab}z, \text{ or } \xi_a(bz) = \xi_a(b)z + (-1)^{|a||b|}b\xi_a(z).$$

This last expression follows directly from the derivation rule also.

<u>Lemma</u> 5.4. <u>View</u> H_2 <u>as a submodule of</u> $G = H_2 \odot H_1$. <u>Then</u> $\xi_a = \mathrm{ad}(a)$.

<u>Proof</u>. There is a unique homomorphism $\lambda\colon H_2 \to H_2$ satisfying $\lambda(1) = 0$, $\lambda(b) = h_{ab}$ for $b \in T_2$, and $\lambda(y_1y_2) = \lambda(y_1)y_2 + (-1)^{|a||y_1|}y_1\lambda(y_2)$. Because $[a,b] = \mathrm{ad}(a)(b) = h_{ab}$ in G, both ξ_a and $\mathrm{ad}(a)$ satisfy these conditions.

<u>Proposition</u> 5.5. <u>Let</u> $\beta \subseteq H_2$ <u>be any subset</u>. <u>Let</u> $N = H_2/H_2\chi(H_1\otimes\beta)H_2$. <u>Then</u> N <u>is an algebra over</u> H_1, <u>and</u> $N \odot H_1 = H/(H\hat{\beta}H + H\beta H)$.

Proof. Let $G = H/H\hat{\beta}H$. The action of H_1 on N is inherited directly from χ: $H_1 \otimes H_2 \to H_2$. Let $I = H_2\chi(H_1 \otimes \beta)H_2$. We must verify that $\chi(x \otimes y) \in I$ if $y \in I$ for any $x \in H_1$. It is enough to check that $\xi_a(y) \in I$ if $y \in I$ for each $a \in T_1$. This follows from the derivation property for ξ_a.

To obtain $N \odot H_1 = H/(H\hat{\beta}H + H\beta H)$, we show that $N \odot H_1 = G/G\beta G$. From 5.2, $N \odot H_1$ is a quotient of $H = H_1 \amalg H_2$ and the set $\hat{\beta} \cup \chi(H_1 \otimes \beta)$ generates all the relations. We must show that any relation $\chi(x \otimes \beta_j) = 0$, $x \in H_1$, is a consequence of the relations $\hat{\beta} = 0$ and $\beta = 0$ in H. Factoring through G, $\chi(H_1 \otimes \beta) \subseteq G\beta G$ by 5.4. Thus $N \odot H_1 = G/G\beta G$.

Recall the homomorphism ϕ of Section 4, defined for an additive function g. Choose such a ϕ for the free algebra $H = \mathbb{F}\langle T_1 \cup T_2\rangle$.

Definition. *Suppose each* $h_{ab} \in \phi(H_2)$; *write* $h_{ab} = \phi(\hat{\delta}_{ab})$. *Suppose also that each* $\hat{\beta}_{ab}$ *is homogeneous w.r.t.* g, *i.e., that* $g(a) + g(b) = g(\hat{\delta}_{ab})$ *for each* $(a,b) \in T_1 \times T_2$. *Then* $G = H/H\hat{\beta}H$ *is called a generalized product*.

By 4.5, ϕ_G is defined on G. We henceforth drop the subscript on ϕ_G and associate ϕ with G.

Define an action σ: $H_1 \otimes H_2 \to H_2$ by $\sigma(x \otimes 1) = 0$ for $x \in H_1$ and $\sigma(a \otimes by) = g(b)\hat{\delta}_{ab}y + (-1)^{|a||b|}b\xi_a(y)$ for $a \in T_1$, $b \in T_2$, $y \in H_2$. H_2 is a module, but not an algebra, over H_1 via σ.

Proposition 5.6. Let G be a generalized product. Suppose $\delta \subseteq H_2$ is any set and let $\beta_j = \phi(\delta_j)$. Let $\Delta = \sigma(H_1 \otimes \delta)$. Then $\phi(\Delta) = \chi(H_1 \otimes \beta)$. Furthermore, as $\mathbb{F}$-modules,

$$G/G\beta G \approx H_1 \otimes \mathcal{P}(\phi(H_2)/\phi(\Delta H_2)). \quad (7)$$

<u>Proof</u>. Let S be the standard basis of monomials for H_1. We prove that $\phi(\sigma(x \otimes \delta_j)) = \chi(x \otimes \beta_j)$ by induction on $\ell(x)$ for $x \in S$. To begin with, $\phi(\sigma(1 \otimes \delta_j)) = \phi(\delta_j) = \beta_j = \chi(1 \otimes \beta_j)$, so the formula holds for $x = 1$, i.e., when $\ell(x) = 0$. Now suppose that it has been verified for $\ell(x) < n$ and that $u \in S$ has $\ell(u) = n$. Write $u = ax$, where $a \in T_1$ and $\ell(x) = n-1$. We wish to verify that

$$\phi(\sigma(ax \otimes \delta_j)) = \chi(ax \otimes \beta_j).$$

By the inductive hypothesis and 5.4 and 5.1 the right-hand side is $\chi(ax \otimes \beta_j) = \xi_a(\chi(x \otimes \beta_j)) = [a,\phi(\sigma(x \otimes \delta_j))]$. Since the left-hand side is $\phi(\sigma(ax \otimes \delta_j)) = \phi(\sigma(a \otimes \sigma(x \otimes \delta_j)))$, our formula will follow if we show that $\phi(\sigma(a \otimes by)) = [a,\phi(by)]$ for any $\underline{by} \in \overline{H}_2$. (Here we are replacing $\sigma(x \otimes \delta_j)$ by a sum of terms of the form $\underline{by}$).

This last equation can readily be confirmed. In fact, it is what motivated the rather unusual definition of σ. Starting with the fact that $[a,b] = \phi(\hat{\delta}_{ab})$ in G, we have

$$\phi(ba) = g(b)[b,a] = -(-1)^{|a||b|}g(b)[a,b] = -(-1)^{|a||b|}g(b)\phi(\hat{\delta}_{ab})$$

and

$$g(b)\phi(\hat{\delta}_{ab}y) = -(-1)^{|a||b|}\phi(bay) \text{ for any } y \in H_2.$$

$$\begin{aligned}
\phi(\sigma(a \otimes by)) &= \phi(g(b)\ \hat{\delta}_{ab}y + (-1)^{|a||b|}b\xi_a(y)) \\
&= -(-1)^{|a||b|}\phi(bay) + (-1)^{|a||b|}\phi(b[a,y]) \\
&= -(-1)^{|a||b|}\phi(bay - bay + (-1)^{|a||y|}bya) \\
&= -(-1)^{|a||b|}(-1)^{|a||y|}\phi(bya) \\
&= -(-1)^{|a||by|}[\phi(by),a] \\
&= [a,\phi(by)] \qquad \text{as desired.}
\end{aligned}$$

To obtain Formula (7), $G/G\beta G = N \odot H_1 \approx N \otimes H_1$ by 5.5, so we must show that $N \approx \mathcal{P}(\phi(H_2)/\phi(\Delta H_2))$. This follows from 4.9 and the now-established relation $\phi(\Delta) = \chi(H_1 \otimes \beta)$.

Remarks. Formula (7) simplifies the work of computing $G/G\beta G$ immensely. We need only to find a basis for $\phi(H_2)/\phi(\Delta H_2)$. This simplification makes quotients of generalized products especially favorable objects to study when we are looking for finitely presented algebras with prescribed properties. The task before us now is to construct one with an irrational Poincaré series. To be sure this will give us what we want, however, we need:

Lemma 5.7. *Let* G *be a generalized product. Let* $\delta \subseteq H_2$ *be any subset. Let* S *be the standard basis of monomials for* $H = \mathbb{F}\langle T_1 \cup T_2\rangle$. *Suppose that each* $\hat{\delta}_{ab}$ *and each* δ_j *is a finite sum of the form* $\sum_{x \in S} c_x x$, *where the coefficients* c_x *and* $\mathrm{im}(g)$ *are in the image of the natural map* $\pi_{\mathbb{F}}\colon \mathbb{Z} \to \mathbb{F}$. *Let* $\tilde{\beta} = \hat{\beta} \cup \phi(\delta)$. *Then there is a complex* Y *which is the mapping cone of two wedges of spheres whose homology is described in 3.7 with* $H = H_*(\Omega X)$ *and* $N = G/G\beta G$. *If* T_1, T_2, *and* δ *are finite, then* Y *is finite and* $H_*(\Omega Y)(Z)$ *is a rational function of* $(G/G\beta G)(Z)$.

Proof. Under the given conditions, each $\hat{\beta}_{ab}$ and each $\beta_j = \phi(\delta_j)$ can be realized by a sum of repeated Whitehead products of generators. Thus we can actually construct a map $\tilde{f}$ from a wedge of spheres to X which gives rise to $\tilde{\beta} \subseteq H_*(\Omega X) = H$. If T_1 and T_2 are finite, $H(Z)$ is rational and $\hat{\beta}$ is finite. Theorem 3.7 applies because X is a wedge of spheres.

6. An Irrational Poincaré Series

We next consider a fairly specific type of generalized product for which we can do an explicit calculation. As a corollary we obtain a finite complex whose loop space has an irrational Poincaré series. At the same time, we illustrate various ideas and methods which can be used to compute the $\phi(H_2)/\phi(\Delta H_2)$ of Formula (7).

Let M be any finitely presented connected (not necessarily Hopf!) algebra. Write $M = W/WrW$, where $W = \mathbb{F}\langle w_1,\ldots w_n\rangle$ and WrW is the two-sided ideal of W generated by the set $r = \{r_1,\ldots r_m\}$.

Our generalized product is constructed as follows. Let T_2 consist of $\{w_1,\ldots w_n\} \cup \{u_1,\ldots u_n\} \cup \{s\}$. Their dimensions are given by $|u_j| = |w_j|$ and $|s|$ is arbitrary as long as $|s| < 2 \min \{|w_j|\}$. Let $U = \mathbb{F}\langle u_1,\ldots u_n\rangle$. Note that $H_2 = \mathbb{F}\langle T_2\rangle$ has various free subrings, including W, W<s>, U, and $W \amalg U$. Let T_1 consist of $\{p_1,\ldots p_n\} \cup$ $\cup \{q_{ij} | 1 \leq i,j \leq n\}$. These should satisfy $|p_j| = |w_j|$ and $|q_{ij}| + |s| =$ $= |w_i| + |w_j|$.

The action of H_1 on H_2 is determined by the set $\{\hat{\delta}_{ab}\}$ for $(a,b) \in T_1 \times T_2$. $\hat{\delta}$ has $\#(T_1)\#(T_2) = (2n+1)(n^2+n)$ elements, but a great simplification is achieved because most of them will be zero. Define $\hat{\delta}_{ab} = 0$ with the following exceptions:

$$\hat{\delta}_{p_i u_j} = u_i w_j \text{ for } 1 \leq i,\ j \leq n$$

$$\hat{\delta}_{q_{ij} s} = u_i w_j \text{ for } 1 \leq i,\ j \leq n.$$

Let $g = \pi_{\mathbb{F}} \circ \ell$, i.e., specify that g(any generator) = 1. We obtain a non-trivial ϕ and each $\hat{\beta}_{ab}$ is homogeneous w.r.t. g. Indeed, each $g(\hat{\beta}_{ab}) = 2$. Thus $G = H_2 \circledast H_1$ is a generalized product.

Submodules of H_1 and H_2 will be denoted according to our usual conventions, for example, uW denotes the subset of H_2 spanned by all $u_i w_{j_1} \ldots w_{j_k}$ and $H_1 q H_1$ is the two-sided ideal of H_1 generated by the $\{q_{ij}\}$.

Next we specify the set $\beta = \phi(\delta) \subseteq H_2$ which we divide out by. Let <u>us</u> denote the set $\{u_i s\}_{1 \leq i \leq n}$ and uu denote $\{u_i u_j\}_{1 \leq i,j \leq n}$. Define a map $\theta\colon \overline{W} \to uW$ by $\theta(w_{j_1} \ldots w_{j_k}) = u_{j_1} w_{j_2} \ldots w_{j_k}$. θ is an isomorphism of right W-modules. Let $\delta = us \cup uu \cup \theta(r)$, where we recall that $r \subseteq \overline{W}$ is our original set of relations used in defining M.

<u>Theorem</u> 6.1. <u>Let</u> M, H_1, H_2, G, δ <u>be as above</u>. <u>Then as</u> $\mathbb{F}$-<u>modules</u>,

$$G/G\beta G \approx H_1 \otimes W\langle s\rangle \otimes \mathcal{P}(\overline{M}). \tag{8}$$

<u>Proof</u>. By 5.6 we must show that $\mathcal{P}(\phi(H_2)/\phi(\Delta H_2)) = W\langle s\rangle \otimes \mathcal{P}(\overline{M})$, where $\Delta = \sigma(H_1 \otimes \delta)$. The proof is given in a series of lemmas.

<u>Lemma</u> 6.2. $\sigma(H_1 \otimes \theta(r)) = \theta(Wr)$.

<u>Proof</u>. Note that $\xi_{p_i}(W\langle s\rangle) = 0$ and consequently $\sigma(p_i \otimes \theta(x)) = \theta(w_i x)$ for $x \in W$. $\xi_{q_{ij}}(x) = 0$ and $\sigma(q_{ij} \otimes x) = 0$ for $x \in W \amalg U$. Thus $\sigma(H_1 q H_1 \otimes \theta(r)) = 0$ and $\sigma(H_1 \otimes \theta(r)) = \theta(Wr)$.

<u>Lemma</u> 6.3. $\sigma(H_1 \otimes (usW + uuW)) = uWsW + uWuW$.

<u>Proof</u>. Let $I = uWsW + uWuW$. Recall our formulas for ξ_a and σ.
$\sigma(p_i \otimes uWsW) \subseteq uWsW$. $\sigma(q_{ij} \otimes uWsW) \subseteq uW\xi_{q_{ij}}(s)W \subseteq uWuW$.
$\sigma(p_i \otimes uWuW) \subseteq \hat{\delta}_{p_i u} WuW + uW\xi_{p_i}(u)W \subseteq uWuW$. $\sigma(q_{ij} \otimes uWuW) = 0$.
From these four inclusions we deduce that $\sigma(H_1 \otimes I) \subseteq I$.

It remains to show that all of I can be obtained by starting with usW and uuW. $\sigma(p_i \otimes u_j xsy) = u_i w_j xsy$ for $x,y \in W$. $uWsW \subseteq \sigma(H_1 \otimes usW)$ by induction on $\ell(x)$.

To obtain $uWuW \subseteq \sigma(H_1 \otimes (usW + uuW))$ is more difficult. Let $J = \sigma(H_1 \otimes (usW + uuW))$. Let $y \in W$ and let x belong to the standard basis for W. If $\ell(x) = 0$, $u_i x u_j y \in J$ because $uuW = \sigma(1 \otimes uuW) \subseteq J$. Suppose that $u_i x_1 u_j y \in J$ is known for all y and for all monomials x_1 shorter than x. Write $x = x_1 w_k$ and start with the assertion, proved above, that $u_i x_1 sy \in J$. $\sigma(q_{jk} \otimes u_i x_1 sy) = (-1)^{|q_{jk}||u_i|} u_i x_1 \phi(u_j w_k) y =$ $= \pm u_i x_1 u_j w_k y \pm u_i x_1 w_k u_j y$, where each "$\pm$" denotes an appropriate sign. $\pm u_i x_1 u_j w_k y \pm u_i x u_j y \in J$ and $u_i x_1 u_j (w_k y) \in J$ by the inductive assumption, so $u_i x u_j y \in J$, as desired.

Lemma 6.4. $\phi(H_2)/\phi(\Delta H_2) = \phi(W\langle s\rangle) \oplus (\phi(uW)/\phi(\theta(WrW)))$.

Proof. $\phi(H_2) = \phi(W\langle s\rangle) \oplus \phi(H_2 u H_2) = \phi(W\langle s\rangle) \oplus \phi(uH_2) =$
$= \phi(W\langle s\rangle) \oplus \phi(uW) \oplus \phi(uWsH_2 + uWuH_2)$.

$$\phi(\Delta H_2) = \phi(\Delta W H_2) = \phi(\theta(Wr)H_2) + \phi(\sigma(H_1 \otimes (us \cup uu))WH_2).$$

Because σ is a right W-morphism, this becomes

$$\phi(\Delta H_2) = \phi(\theta(Wr)W) + \phi(\theta(Wr)WuH_2 + \theta(Wr)WsH_2) + \phi(uWsH_2 + uWuH_2).$$

Since $\theta(Wr) \subseteq uW$, the second summand is contained in the last, yielding

$$\phi(\Delta H_2) = \phi(\theta(Wr)W) + \phi(uWsH_2 + uWuH_2).$$

θ is a right W-morphism and $\theta(WrW) \subseteq uW$. We obtain

$$\phi(H_2)/\phi(\Delta H_2) = \phi(W\langle s\rangle) \oplus (\phi(uW)/\phi(\theta(WrW))) \oplus (0).$$

Lemma 6.5. $\phi \circ \theta: \overline{W} \to \phi(uW)$ is an isomorphism.

Proof. $\phi \circ \theta$ is immediately surjective. For injective we refer the reader to [16, pp. 15-16]. One possible basis for the free Lie algebra $\phi(H_2)$ is the basis of Chen-Fox-Lyndon. Choose an ordering on T_2 which satisfies $u_i < w_j$ for all i and j. For any basis monomial $x \in \overline{W}$, $\theta(x)$ is one of the "basic products" of H_2 which correspond to a basis for $\phi(H_2)$. $\{\phi(\theta(x)) \mid x \text{ a basis monomial of } \overline{W}\}$ is a subset of the Chen-Fox-Lyndon basis for $\phi(H_2)$. This set is therefore linearly independent, implying $\ker \phi \circ \theta = 0$.

Lemma 6.6. $\mathcal{P}(\phi(H_2)/\phi(\Delta H_2)) \approx W\langle s\rangle \otimes \mathcal{P}(\overline{M})$.

Proof. By 6.4, $\mathcal{P}(\phi(H_2)/\phi(\Delta H_2)) = \mathcal{P}(\phi(W\langle s\rangle)) \otimes$ $\otimes \mathcal{P}(\phi(uW)/\phi(\theta(WrW)))$. By 4.9 the first factor isomorphic with $W\langle s\rangle$. By 6.5 the second factor is isomorphic with $\mathcal{P}(\phi \circ \theta(\overline{W}/WrW)) \approx$ $\approx \mathcal{P}(\overline{M})$. This completes the proof of Theorem 6.1.

Example 6.7. Take M to be a polynomial ring on one generator. $M = W = \mathbb{F}\langle w_1\rangle$ with no relations. Take $|p_1| = |q_{11}| = |u_1| = |w_1| =$ $= |s| = 1$. (This is the simplest possible case of 6.1). Because there is only one generator of each type, we drop the subscripts and write $H = \mathbb{F}\langle p,q,u,w,s\rangle$. Let G be the generalized product obtained from H by dividing out by the six relations

$$\beta_1 = [p,u] - [u,w] \qquad \beta_2 = [p,w] \qquad \beta_3 = [p,s]$$

$$\beta_4 = [q,u] \qquad \beta_5 = [q,w] \qquad \beta_6 = [q,s] - [u,w]$$

Let $N = G/G\beta G$ be the quotient algebra obtained by dividing further by $\beta = \{\beta_7,\beta_8\}$, where

$$\beta_7 = [u,s] \qquad \beta_8 = [u,u] .$$

Then N(Z) is not a rational function of Z.

Proof. By 6.1, $N(Z) = H_1(Z) \cdot W\langle s\rangle(Z) \cdot \mathcal{P}(\overline{M})(Z)$. The first factor is $H_1(Z) = (1 - 2Z)^{-1}$ because $H_1 = \mathbb{F}\langle p,q\rangle$ is free with $|p| = |q| = 1$. The second factor is also $(1 - 2Z)^{-1}$ because $W\langle s\rangle = \mathbb{F}\langle w,s\rangle$ is free with $|w| = |s| = 1$.

The last factor is $\mathcal{P}(\overline{\mathbb{F}\langle w\rangle})(Z)$. A basis for $\overline{\mathbb{F}\langle w\rangle}$ consists of $\{w^j | j\geq 1\}$ and hence has one element in each dimension 1,2,3,.... Let $P_2(Z) = \prod_{j=1}^{\infty} (1 - Z^j)^{-1}$ and $P_0(Z) = \prod_{j=1}^{\infty} \left(\frac{1 + Z^{2j-1}}{1 - Z^{2j}}\right)$. We have shown that

(9) $$N(Z) = \begin{cases} (1 - 2Z)^{-2} P_2(Z) & \text{if char } \mathbb{F} = 2 \\ (1 - 2Z)^{-2} P_0(Z) & \text{if char } \mathbb{F} \neq 2. \end{cases}$$

The proof of 6.7 will be complete when we show that $P_2(Z)$ and $P_0(Z)$ are not rational. For this we have

Lemma 6.8. *$P_2(Z)$ and $P_0(Z)$ are not rational functions of Z.*

Proof. First note that both infinite products converge for $|Z| < 1$. From the inequality $e^x \geq 1 + x$ for real x, conclude $x \geq 1 + \log(x)$. Set $x = \frac{1}{1 - |Z|^j}$ to obtain $\frac{|Z|^j}{1 - |Z|^j} \geq -\log(1 - |Z|^j)$.

Then $|\log P_2(Z)| = \left|-\sum_{j=1}^{\infty} \log(1 - Z^j)\right| \leq -\sum_{j=1}^{\infty} \log(1 - |Z|^j) \leq$

$\leq \sum_{j=1}^{\infty} \frac{|Z|^j}{1 - |Z|^j} \leq \sum_{j=1}^{\infty} \frac{|Z|^j}{1 - |Z|} = \frac{|Z|}{(1 - |Z|)^2} < \infty$, so $P_2(Z)$ converges

for $|Z| < 1$. For $P_0(Z)$ simplify each factor of the infinite product by $\left|\frac{1 + Z^{2j-1}}{1 - Z^{2j}}\right| \leq \frac{1 + |Z|^{2j-1}}{1 - |Z|^{2j}} \leq \frac{1 + |Z|^j}{1 - |Z|^{2j}} = \frac{1}{1 - |Z|^j}$ for each $j \geq 1$,

so $|P_0(Z)| \leq P_2(|Z|)$, and $P_0(Z)$ also converges for $|Z| < 1$.

Both $P_0(Z)$ and $P_2(Z)$ are analytic functions which converge for $|Z| < 1$. If they were rational, they could be extended to analytic functions with a pole of at most finite order at $Z = 1$. But $\lim_{Z \to 1^-} (Z - 1)^k P_2(Z)$ and $\lim_{Z \to 1^-} (Z - 1)^k P_0(Z)$ do not exist for any k -- contradiction! So $P_2(Z)$ and $P_0(Z)$ are not rational functions.

Corollary 6.9. *Let V be the four-dimensional complex obtained from $\bigvee_{j=1}^{5} S^2$ by attaching eight cells corresponding to the Whitehead products of Example 6.7. Then $H_*(\Omega V)(Z) = \sum_{n=0}^{\infty} \mathrm{Rank}(H_n(\Omega V; \mathbb{F}))Z^n$ is not a rational function of Z.*

Proof. This follows directly from 5.7 and 6.7.

Remarks. V has only thirteen cells (in addition to a base point) and dim V = 4. If char $\mathbb{F}$ = 2, the last cell (corresponding to β_8) can be omitted since $[u,u] = 0$. In fact, over a field of characteristic different from two, β_8 can be omitted from the description and N(Z) will still be irrational. With this change Eq. (9) would be modified by an additional factor of $(1 - Z^2)^{-1}$ in front of $P_0(Z)$.

7. The Serre-Kaplansky Problem

Let R be a local Artin ring with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{F} = R/\mathfrak{m}$. Is the Poincaré series of R, $P_R(Z) = \sum_{n=0}^{\infty} \text{Rank}\,(\text{Tor}_n^R(\mathbb{F}, \mathbb{F}))Z^n$, a rational function of Z?

Jan-Erik Roos [11] has recently shown that this question, known as the Serre-Kaplansky problem, ties in closely with the question of the rationality of the Hilbert series for a finitely presented Hopf algebra. In particular, suppose $N = H/H\beta H$, where $H = \mathbb{F}\langle\alpha_1, \ldots \alpha_n\rangle$ with each $|\alpha_i| = 1$ and $\beta = \{\beta_1, \ldots \beta_m\} \subseteq \phi(H)$ is linearly independent with each $|\beta_j| = 2$. Then Roos shows [11, pp. 298 - 301] that there is a local ring R with n generators and $\mathfrak{m}^3 = 0$ satisfying

$$(10) \qquad P_R(Z)^{-1} = (1 + Z^{-1})N(Z)^{-1} - Z^{-1}(1 - nZ + mZ^2)$$

Thus $P_R(Z)$ is a rational function of N(Z). Example 6.7 therefore allows us to answer the Serre-Kaplansky problem in the negative. To make this specific, we have

Example 7.1. Suppose char $\mathbb{F} \neq 2$. Let R be the local ring $R = \mathbb{F}(x_1, \ldots x_5)/J$, where J is the ideal generated by $\mathfrak{m}^3$ and the relations

$$x_1^2 = x_2^2 = x_4^2 = x_5^2 = 0$$

$$x_1x_2 = x_4x_5 = x_1x_3 + x_3x_4 + x_2x_5 = 0.$$

If char $\mathbb{F} = 2$, include $x_3^2 = 0$ in J as well. Then $P_R(Z)$ is not rational.

Proof. R is found by dualizing Example 6.7, with β_8 being omitted if char $\mathbb{F} = 2$.

$P_R(Z)$ may be computed explicitly from formulas (9) and (10). In (9), take $n = 5$ and $m = 8$ if char $\mathbb{F} \neq 2$ and take $n = 5$ and $m = 7$ if char $\mathbb{F} = 2$.

8. What Can $H_*(\Omega X)(Z)$ Be?

Let $\mathcal{C} = \{H_*(\Omega X)(Z) \mid X$ a simply-connected finite CW-complex$\}$. We have seen that $\mathcal{C}$ includes more than just rational power series. Is there some other easily characterized, countable set of power series which contains $\mathcal{C}$? We do not have a complete answer, but in this section we take some steps toward a description of $\mathcal{C}$.

Lemma 8.1. Let $H(Z) = H_*(\Omega X)(Z) \in \mathcal{C}$, where X is not homotopically trivial. As a power series in Z, H(Z) has a radius of convergence r about $Z = 0$, where $0 < r \leq 1$.

Proof. Let $H(Z) = \sum_{i=0}^{\infty} c_i Z^i$ and $H_*(X)(Z) = \sum_{i=0}^{\infty} d_i Z^i$. Use the Serre spectral sequence of the fibration for ΩX. Since X is finite and not homotopy equivalent to a point, there are infinitely many dimensions in which $c_i \geq 1$. So $H(1)$ does not converge, i.e., $r \leq 1$.

From the same spectral sequence we have $c_i \leq \sum_{j=1}^{\infty} d_{j+1}c_{i-j}$, with $c_0 = 1$, $c_i = 0$ for $i < 0$. Let $\{b_i\}$ be the coefficients satisfying $(1 - \sum_{i=1}^{\infty} d_i Z^{i-1})^{-1} = \sum_{i=1}^{\infty} b_i Z^i$. $b_0 = 1$, $b_i = 0$ for $i < 0$, and $b_i = \sum_{j=1}^{\infty} d_{j+1}b_{i-j}$. By induction on i, conclude that $0 \leq c_i \leq b_i$ for all i. Because $d_i = 0$ for $i > \dim X$, $(1 - \sum_{i=1}^{\infty} d_i Z^{i-1})^{-1}$ is a rational function of Z. X simply connected means $d_1 = 0$, so this function is continuous and non-zero in a neighborhood of Z = 0. In particular, it has a positive radius of convergence. There is an $\mathcal{r}_0 > 0$ such that $\sum_{i=0}^{\infty} b_i \mathcal{r}_0^i < \infty$, which implies $\sum_{i=0}^{\infty} c_i \mathcal{r}_0^i < \infty$ as well. Thus $\mathcal{r} \geq \mathcal{r}_0 > 0$.

Lemma 8.2. _If dim_ X $\leq$ 3, _then_ $H_*(\Omega X)(Z)$ _is rational_.

Proof. Any simply connected finite X of dimension three may be written as the mapping cone of a map between two wedges of S^2's. It follows that X is the suspension of a finite complex X_1. $H_*(\Omega X)(Z) = (1 - \bar{H}_*(X_1)(Z))^{-1}$ is rational.

Thus four is the minimum dimension X can have for $H_*(\Omega X)(Z)$ to be irrational. In 6.9, the complex V has this minimal dimension.

Let $\mathcal{N} = \{N(Z) \mid N = H/H\beta H$, where $H = \mathbb{F}\langle T\rangle$ and T is finite and $\beta \subseteq \phi(H)$ is finite$\}$. By 3.7, each member of $\mathcal{N}$ is a rational function of something in $\mathcal{C}$. By 9.1, each $N(Z) \in \mathcal{N}$ has a positive radius of convergence.

Definition. _Let_ A,B _be power series in_ Z _with leading coefficient unity_. _The wedge_ $A \vee B$ _of_ A _and_ B _is given by_ $(A \vee B)^{-1} = A^{-1} + B^{-1} - 1$.

This terminology is suggested by the fact that $H_*(\Omega X_1)(Z) \vee \vee H_*(\Omega X_2)(Z) = H_*(\Omega(X_1 \vee X_2))(Z)$ (see [14], p. 130).

Lemma 8.3. $\mathcal{C}$ and $\mathcal{N}$ are each closed under wedges and products.

Proof. For $\mathcal{C}$, let $A = H_*(\Omega X)(Z)$, $B = H_*(\Omega Y)(Z)$. We have $A \vee B = H_*(\Omega(X \vee Y))(Z)$ and $AB = H_*(\Omega(X \times Y))(Z)$. For $\mathcal{N}$, let $H_i = \mathbb{F}\langle T_i\rangle$ and $N_i = H_i/H_i\beta_iH_i$ for $i = 1,2$. $N_1(Z) \vee N_2(Z) = N(Z)$, where N is the free product of N_1 and N_2. Specifically, $H = \mathbb{F}\langle T_1 \amalg T_2\rangle$, $\beta_0 = \beta_1 \amalg \beta_2$, and $N = H/H\beta_0H$. Lastly, the product $N_1(Z) \cdot N_2(Z) = (N_1 \otimes N_2)(Z)$, and $N_1 \otimes N_2 = H/H\beta H$, where $\beta = \beta_0 \amalg \{[\alpha_i,\alpha_j] \mid \alpha_i \in T_1, \alpha_j \in T_2\}$.

Definition. Let $P(Z^d)$ denote $(1 - Z^d)^{-1}$, if d is even or char $\mathbb{F} = 2$ and let $P(Z^d) = 1 + Z^d$ if d is odd and char $\mathbb{F} \neq 2$. Define a function $\mathcal{P}$ from power series with leading coefficient zero to power series with leading coefficient unity by $\mathcal{P}(\overline{M}(Z)) = \mathcal{P}(\overline{M})(Z)$, or $\mathcal{P}(\sum_{i=1}^{\infty} a_iZ^i) = \prod_{i=1}^{\infty} P(Z^i)^{a_i}$. $\mathcal{P}$ takes the coefficients of a power series and uses them as exponents in an infinite product.

Proposition 8.4. Let $N(Z) \in \mathcal{N}$. Write $N = H/H\beta H$, where $H = \mathbb{F}\langle\alpha_1,\ldots\alpha_n\rangle$. Set $\alpha(Z) = \sum_{i=1}^{n} Z^{|\alpha_i|}$. Then

$$(1 - \alpha(Z) - Z^{-1}\alpha(Z)^2)^{-1}\ (1 - Z - \alpha(Z))^{-1}\ \mathcal{P}(N(Z) - 1) \in \mathcal{N}.$$

Proof. This is a direct consequence of Theorem 6.1. Let $M = N$ and take $|s| = 1$ for simplicity. $H_1(Z) = (1 - \alpha(Z) - Z^{-|s|}\alpha(Z)^2)^{-1}$ and $W\langle s\rangle(Z) = (1 - Z^{|s|} - \alpha(Z))^{-1}$ are rational functions of Z. Our hypotheses could actually be weakened in that N(Z) could be the Hilbert series of any finitely presented algebra.

Proposition 8.4 shows that any set containing $\mathcal{C}$ or $\mathcal{N}$ will have to be fairly complicated. For any $N \in \mathcal{N}$, $\mathcal{N}$ contains a rational function of $\mathcal{P}(N - 1)$. Thus $\mathcal{N}$ contains rational functions of $\prod_{i=1}^{\infty} P(Z^i)^{a_i}$, where a_i can be a polynomial in i, a geometric series, or defined by many other finite recursions. Furthermore, these irrational series can themselves be subjected to the operation $\mathcal{P}$, and so on. In this way we obtain some very highly transcendental functions as the Hilbert series of finitely presented Hopf algebras.

To apply these results to local rings, let $\mathcal{C}_1 = \{H_*(\Omega X)(Z) \in \mathcal{C} \mid \dim X \leq 4\}$ and $\mathcal{N}_1 = \{(H/H\beta H)(Z) \in \mathcal{N} \mid H = \mathbb{F}\langle\alpha_1,\ldots\alpha_k\rangle$ and $\beta = \{\beta_1,\ldots\beta_m\}$, where each $|\alpha_i| = 1$ and each $|\beta_j| = 2\}$.

Proposition 8.5. $\mathcal{C}_1$ is closed under wedges. $\mathcal{N}_1$ is closed under wedges and products. Also, 8.4 still holds if $\mathcal{N}$ is replaced by $\mathcal{N}_1$ throughout.

Proof. The proofs from 8.3 are still valid. In the proof that $\mathcal{N}$ is closed under products, the only new relations we introduced are commutators of generators, which always have dimension two in $\mathcal{N}_1$. Also, in the proof of 6.1, all relations introduced have dimension two because each $|w_j| = 1 = |s|$ and each $|r_j| = 2$.

Our research has left several questions unanswered, and we close with just one conjecture about the class $\mathcal{C}$. Recall that a complex X is said to have category $\leq$ n if X can be written as the union of n contractible closed subsets. If cat X $\leq$ n, then any cup products in $\overline{H}^*(X)$ involving n or more factors must vanish.

Conjecture 8.6. Let X be finite with cat X = n > 1 and let $H(Z) = H_*(\Omega X)(Z)$. Let $\mathcal{r}$ be the radius of convergence of H, as in 9.1. Then $\mathcal{r} + 0i$ is a pole of H(Z) whose order is $\leq$ n-1.

REFERENCES

1. Adams, J.F. and Hilton, P.J., "On the chain algebra of a loop space", Comm. Math. Helv. 30 (1958), 305 - 330.

2. Adams, J.F., "On the cobar constructions", Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 409 - 412.

3. Cartan, H. and Eilenberg, S., Homological Algebra, Princeton University Press, Princeton, N.J., 1956.

4. Cohen, F.R., Moore, J.C., and Neisendorfer, J.A., "Torsion in homotopy groups", Annals of Math. 109 (1979), 121 - 168.

5. Eilenberg, S. and Moore, J.C., "Homology and fibrations I", Comm. Math. Helv. 40 (1966), 199 - 236.

6. James, I.M., "Reduced product spaces", Annals of Math. 62 (1955), 170 - 197.

7. Lemaire, J-M., "Algèbres Connexes et Homologie des Espaces de Lacets", Lecture Notes in Mathematics 422, Springer-Verlag, Berlin, 1974.

8. Lemaire, J-M., "Autopsie d'un meurtre dans l'homologie d'une algèbre de chaines", Ann. scient. Éc. Norm. Sup., 4e série, Vol. 11 (1978), pp. 93 - 100.

9. Massey, W.S. and Peterson, F.P., "The cohomology structure of certain fiber spaces - I", Topology, Vol. 4, No. 1 (1965), pp. 47 - 66.

10. Milnor, J. and Moore, J.C., "On the structure of Hopf algebras", Annals of Math. 81 (1965), 211 - 264.

11. Roos, J-E., "Relations between the Poincaré-Betti series of loop spaces and of local rings", Lecture Notes in Mathematics 740, pp. 285 - 322, Springer-Verlag, Berlin, 1979.

12. Serre, J-P., "Algèbre Locale, Multiplicités", Lecture Notes in Mathematics 11, Springer-Verlag, Berlin, 1965.

13. Serre, J-P., Lie Algebras and Lie Groups, "1964 Lectures given at Harvard University", W.A. Benjamin, Inc., New York, 1965.

14. Smith, L., "Lectures on the Eilenberg-Moore Spectral Sequence", Lecture Notes in Mathematics 134, Springer-Verlag, Berlin, 1970.

15. Smith, L., "Split extensions of Hopf algebras and semi-tensor products", Mathematica Scandinavia 26 (1970), pp. 17 - 41.

16. Viennot, G., "Algèbres de Lie Libres et Monoïdes Libres", Lecture Notes in Mathematics 691, Springer-Verlag, Berlin, 1978.