There is good evidence that the deuteron ground state is primarily 1s ($n=1$, $\ell=0$). First, the lowest energy state in practically all the model potentials is an s state. Secondly, the magnetic moment of H^2 is approximately the sum of the proton and the neutron moments, indicating that $\underline{s}_n \parallel \underline{s}_p$ and no orbital motion of the proton relative to the neutron; this is also consistent with the total angular momentum of the ground state, I=1.

We therefore consider only the $\ell=0$ radial wave equation,

$$-\frac{\hbar^2}{m}\frac{d^2u(r)}{dr^2} + V(r)u(r) = \underset{\hookrightarrow -E_B}{E}\,u(r)$$

where $u(r) = rR(r)$ and m is the neutron (or proton) mass.

Notice this equation is mathematically equivalent to the 1-D wave equation except for the B.C., $u(0) = 0$.

Solutions:

$$u(r) = a e^{iKr} + b e^{-iKr} \qquad r<r_o,\ K = [m(V_o - E_B)]^{1/2}/\hbar$$

$$u(r) = a' e^{K_r r} + b' e^{-K_r r} \qquad r>r_o,\ K_r = \sqrt{mE_B}/\hbar$$

Applying the boundary conditions

(i) $u(0) = 0$, so $R(0)$ is finite

(ii) $u(r\to\infty) = 0$ since we are dealing with a bound state

(iii) continuity of u and first-order derivative at $r=r_o$

one obtains

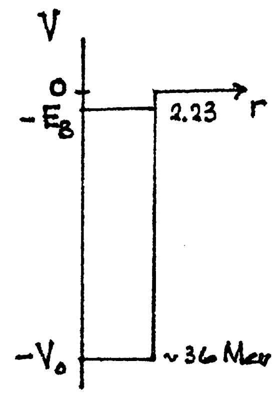
$$u(r) = c\sin Kr \qquad r<r_o$$

$$a' = 0,\ \therefore\ u(r) = b' e^{-K_r r} \qquad r>r_o$$

$$K\cot Kr_o = -K_r$$ this is a relation between V_o, E_B and r_o

$$\text{or } \tan Kr_o = -\frac{K}{K_r} = -\left(\frac{V_o - E_B}{E_B}\right)^{1/2}$$

Suppose $V_o \gg E_B$, then RHS is large and $Kr_o \sim \pi/2$ (we will see below that $V_o \sim 36$ Mev as compared to $E_B \sim 2.23$ Mev).



Then $$K \sim \frac{\sqrt{mV_0}}{\hbar} \sim \frac{\pi}{2r_0}$$

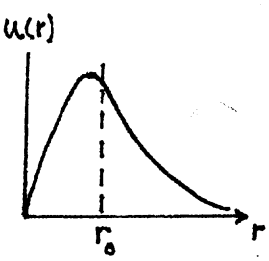
or $$V_0 r_0^2 \sim \left(\frac{\pi}{2}\right)^2 \frac{\hbar^2}{m} \sim 1 \text{ Mev-barn}$$

We see from this result that a knowledge of E_B allows us to only determine the product $V_o r_o^2$. From n-p scattering (discussed below) we will find $r_o \sim 2F$, thus we obtain $V_o \sim 36$ Mev.

Since the interior wave function, sinKr, must match with the exterior wave function, exp($-\kappa$r), the quantity Kr_o must be slightly greater than $\pi/2$ (more accurate estimate gives 116° instead of 90°), so if we write

$$K = 2\pi/\lambda \sim \pi/2r_0$$

then the 'effective wavelength' λ is approximately $4r_o$ which suggests that much of the wave function is not in the interior region.



We can estimate the relaxation constant (or decay length) in the exterior region, $r > r_o$

$$\frac{1}{\kappa} = \frac{\hbar}{\sqrt{mE_B}} \sim 4.3 \text{ F}$$

This means that the two nucleons in H^2 spend a large fraction of their time at $r > r_o$, the classically forbidden reion of negative kinetic energy. We can calculate the root-mean-square radius of the deuteron wave function,

$$R_{rms}^2 \equiv \frac{\int d^3r\, r^2 R^2(r)}{\int d^3r\, R^2(r)} = \frac{\int_0^\infty r^2 dr\, r^2 R^2(r)}{\int_0^\infty r^2 dr\, R^2(r)}$$

If we put $R(r) \sim e^{-\kappa r}$ for all r which should result in an overestimate, one gets

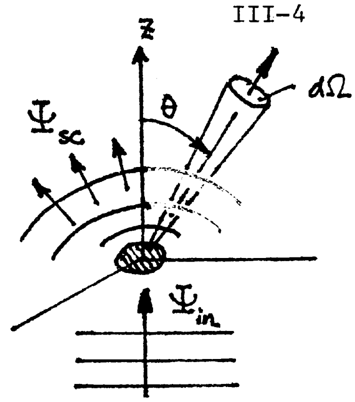
$$R_{rms}^2 = \frac{\hbar}{\sqrt{2mE_B}} = 3 \text{ F}$$

which can be compared with

$$(1.4 \times A^{1/3})^2 \sim 3.11 \quad \text{or} \quad (1.2 \times A^{1/3})^2 \sim 2.29 \qquad A = 2$$

B. Neutron-proton Scattering

One can also obtain information about the neutron-proton interaction by studying the scattering of neutrons by a hydrogen sample. To describe the scattering process we imagine the incident neutrons are represented by a plane wave of the form



$$\Psi_{in} = b\, e^{i(kz-\omega t)}$$

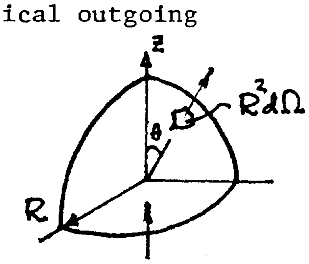
where $k = \sqrt{2\mu T}/\hbar$, μ is the reduced mass of the incoming particle.

T is the kinetic energy in center-of-mass coordinates (CMCS).

Incident flux $J_{in} = v\,\Psi_{in}^{*}\Psi_{in} = v|b|^2$, v is relative speed.

We look for the scattered wave in the form of a spherical outgoing wave,

$$\Psi_{sc} = f(\theta)\, b\, \frac{e^{i(kr-\omega t)}}{r}$$



where f(θ) is called the scattering amplitude.

$$dN = (\Psi_{sc}^{*}\Psi_{sc} v)\, R^2 d\Omega$$ No. particles scattered into area $R^2 d\Omega$ per unit time.

$$\frac{d\sigma}{d\Omega} = \frac{dN/d\Omega}{J_{in}} = |f(\theta)|^2$$ angular differential scattering cross section

$$\sigma = \int d\Omega\, |f(\theta)|^2$$ total scattering cross section

The quantity we want to calculate is f(θ). Stating the problem in another way, we want to solve the wave equation

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + V(r)\psi = T\psi$$

subject to the B.C.

$$\psi \xrightarrow[r\to\infty]{} b\,[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}]$$

We proceed by considering a partial wave analysis:

Since V(r) is spherically symmetric it is most conveninet to discuss the solution in spherical coordinates. We already know that the general solution is of the form $R(r)P_\ell^{(m)}(\cos\theta)\,e^{im\varphi}$

which, because the potential does not depend on φ , reduces to $R_\ell(r)P_\ell(\cos\theta)$. Thus we expand

$$\psi(r,\theta) = \sum_{\ell=0}^{\infty} \psi_\ell(r)\, P_\ell(\cos\theta)$$

one can show that $e^{ikz} = e^{ikr\cos\theta} = \sum_\ell F_\ell(r) P_\ell(\cos\theta)$,

then $F_0(r) = \sin kr / kr$

[For arbitrary ℓ, one finds $F_\ell(r) \sim i^\ell (2\ell+1) \dfrac{\sin(kr - \ell\pi/2)}{kr}$].

Furthermore, we will write $f(\theta) = \sum_\ell f_\ell P_\ell(\cos\theta)$.

Then

$$\psi_0 \underset{r\to\infty}{\sim} b \left[\frac{\sin kr}{kr} + f_0 \frac{e^{ikr}}{r} \right]$$

$$= \frac{b}{2ikr} \left[\underbrace{(1 + 2ikf_0) e^{ikr}}_{\text{outgoing}} - \underbrace{e^{-ikr}}_{\text{incoming}} \right] \equiv R_0(r) = \frac{u_0(r)}{r} \qquad (*)$$

For $r > r_0$, the wave equation is

$$-\frac{\hbar^2}{2\mu} \frac{d^2 u_0}{dr^2} = T u_0 \quad , \qquad k = \sqrt{2\mu T} / \hbar$$

$$u_0(r) = A \sin(kr + \underbrace{\delta_0}_{\text{s-wave phase shift}})$$

$$= \frac{A e^{-i\delta_0}}{2i} \left(e^{ikr} e^{2i\delta_0} - e^{-ikr} \right) \qquad (**)$$

Comparing (*) and (**) one obtains

$$1 + 2ikf_0 = e^{2i\delta_0}$$

$$f_0 = \frac{e^{2i\delta_0} - 1}{2ik} = \frac{e^{i\delta_0} \sin\delta_0}{k} \qquad (\dagger)$$

Thus,

$$\frac{d\sigma}{d\Omega} = \frac{\sin^2\delta_0}{k^2} = \lambda\!\!\!^-{}^2 \sin^2\delta_0 \quad , \qquad \sigma = 4\pi \lambda\!\!\!^-{}^2 \sin^2\delta_0$$

We will define the scattering length as $a \equiv \lim_{k\to 0} (-f_0)$.

Since f_0 can not $\to \infty$ as $k \to 0$ (otherwise $\sigma \to \infty$), $(\dagger)$ shows that δ_0 must $\to 0$ as $k \to 0$, so

$$a = \lim_{k\to 0} \left[- \frac{e^{i\delta_0} \sin\delta_0}{k} \right] = - \frac{\delta_0}{k}$$

We can therefore obtain

$$\lim_{k\to 0} \frac{d\sigma}{d\Omega} = a^2$$

and $$\lim_{k\to 0} \sigma = 4\pi a^2$$

$\delta > 0$ (attractive pot.)

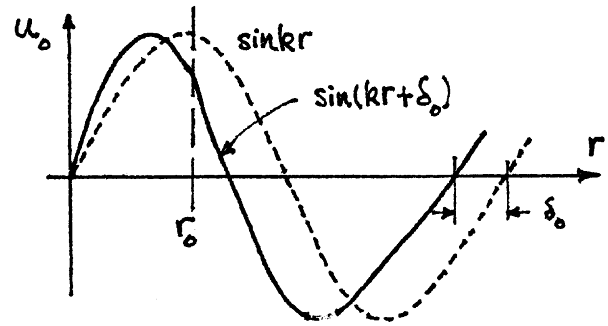
$\delta < 0$ (repulsive)

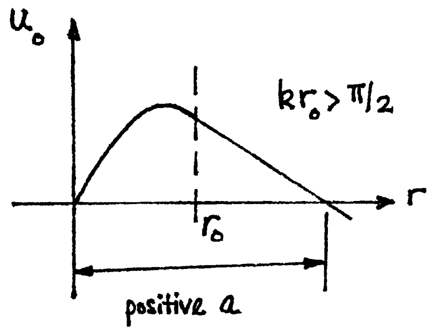
Significance of a as an extrapolation distance:

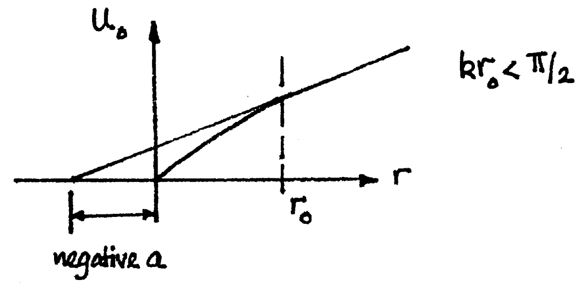
Notice that

$$u_0 = A \sin k(r-a) \quad , \quad r > r_0$$

$$\underset{kr_0 \to 0}{\sim} Ak(r-a)$$







[this case corresponds to a potential which can give rise to a bound state]

[this potential can give rise to a virtual state]

We now apply these results to n-p scattering. In this case we have the same equation as the deuteron calculation except that $E > 0$.

$$r < r_0 \qquad u_0 = C \sin K'r \ , \qquad K' = [2\mu(V_0+T)]^{1/2}/\hbar$$

$$r > r_0 \qquad u_0 = A \sin(kr+\delta_0), \qquad k = \sqrt{2\mu T}/\hbar \ , \quad \mu = \text{reduced mass} = M/2$$

Matching B.C. at $r=r_0$,

$$K' \cot K'r_0 = k \cot(kr_0+\delta_0)$$

this gives δ_0 in terms of V_0, r_0, and T.

To simplify the calculation we will assume

(i) $a \gg r_o$ so that $kr_o \ll \delta_o$
(ii) $T \ll V_o$ so $K' \sim K$

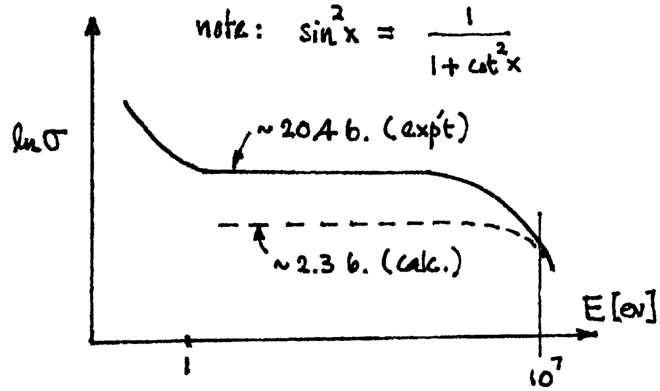
then $k \cot \delta_o \approx K \cot K r_o = -K_1$ ← from the deuteron problem, $K_1 = \sqrt{mE_B}/\hbar$

$$\frac{d\sigma}{d\Omega} = \frac{\sin^2 \delta_o}{k^2} = \frac{1}{k^2} \frac{1}{1 + K_1^2/k^2}$$

note: $\sin^2 x = \dfrac{1}{1+\cot^2 x}$

$$\frac{d\sigma}{d\Omega} \approx \frac{1}{k^2 + K_1^2} = \frac{\hbar^2}{m} \frac{1}{T + E_B}$$

$$\sigma \approx \frac{4\pi\hbar^2}{m} \frac{1}{T + E_B}$$



Since $T < E_B$ (2.23 Mev), $\sigma \sim 2.3$ barns

This does not agree with the experimental value which is ~20.4 barns.

Wigner (1933) has suggested that n-p scattering depends on whether the neutron and proton spins are parallel (triplet state, I=1) or anti-parallel (singlet state, I=0); for these two cases the potential (and ∴ δ_o) is different. Following this idea one can write

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2} \left[\frac{1}{4} \sin^2 \delta_{os} + \frac{3}{4} \sin^2 \delta_{ot} \right]$$

singlet triplet

We already know that the triplet state gives rise to the ground state of the deuteron at $E = -E_B$. If the singlet state produces a state of energy $E=E^*$ (E^* could be positive or negative), then

$$\sigma \approx \frac{\pi\hbar^2}{m} \left(\frac{3}{T+E_B} + \frac{1}{T+|E^*|} \right)$$

With $|E^*| \approx 70$ kev, σ is now in good agreement with experiment. Since $|E^*| \ll V_o$, just like $E_B \ll V_o$ in the triplet state, the singlet wave function inside the nuclear potential is also $\sim \frac{1}{4}$ of a sine wave,

$$K r_o \approx \frac{\pi}{2}, \quad \text{or}$$

$$V_{os} r_{os}^2 \approx V_{ot} r_{ot}^2 \sim 1.0 \text{ Mev-barn}$$

However, using information from neutron scattering from parahydrogen (I=0) one finds a scattering length $a_s = -24$ F, thus indicating that the singlet interaction gives rise to a virtual state ($a_s < 0$).
In summary, one finds from such considerations the following results [cf. Preston, Physics of the Nucleus]

Interaction	Scattering lengths a[F]	r_o[F]	V_o[Mev]
n-p(triplet)	5.4	2	36
n-p(singlet)	-23.7 (n-p)	$\sim$2.5	18
	-17 (p-p)		

Final remark:

Experimentally it is known that the total angular momentum (nuclear spin) of the deuteron ground state is I=1, where

$$\underline{I} = \underline{L} + \underline{S}$$

where $\underline{L}$ is the orbital angular momentum and $\underline{S}$ the intrinsic spin. It is also known that the ground state is mostly 1s (ℓ=0), therefore for this state S=1 (n and p spins parallel). Now we have seen from the discussions of part A that the ground state is barely bound (E_B=2.23 Mev), so all the higher energy states are not bound states. Example, the 1s state with S=0 (n and p spins antiparallel), is a virtual state; it is unbound by $\sim$60 kev. The significance of this is that nuclear interaction is different for different S states, i.e., nuclear forces are spin-dependent.

From 22.111 Lecture Notes by S. Yip, Chap. III.

A. Bound State of the Deuteron

Experimentally it is known that the binding energy E_B of the deuteron is 2.23 Mev. The deuteron is the only stable bound system of two nucleons. We will see later that the di-neutron and di-proton are not stable. The energy E_B is known from the γ energy in the reaction

$$n + H^1 \rightarrow H^2 + \gamma \;(2.23\,\text{Mev})$$

The inverse reaction of using electrons of known energy to produce external bremsstrahlung for (γ,n) reaction on H^2 also has been used. Besides the ground state no stable excited states of H^2 have been found. (There is a virtual state at ~ 2.30 Mev.)

Suppose we assume a square well potential (in 3-D),

$$V(r) = \begin{cases} -V_0 & r < r_0 \\ 0 & r > r_0 \end{cases}$$

Then we ask what is the level structure and what values should V_0 and r_0 take in order to be consistent with a bound state at energy E_B = 2.23 Mev?

*For a general discussion of nuclear forces, see Chapter 6 of Meyerhof.