Problem 1:

This problem considers the rotational mechanical system shown in Figure 1.

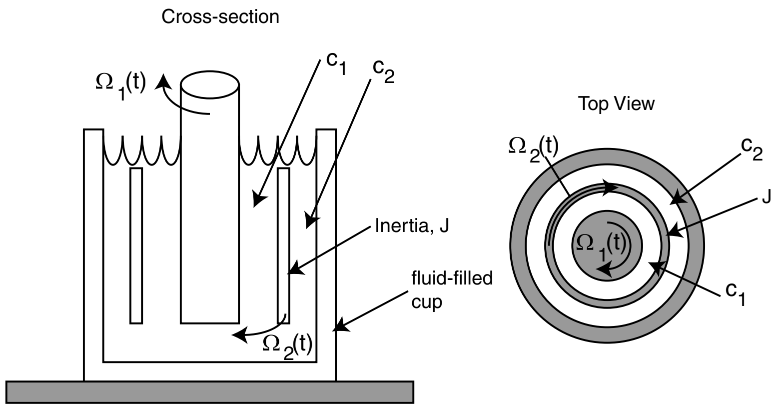

Figure 1: Cross-section and Top view

The setup is similar to the one used in Lab 2. The central shaft rotates at an arbitrary velocity $\Omega_1(t)$ in a fluid filled cup, which is fixed. Unlike Lab 2, there is an intermediate ring, of inertia J. This ring is supported on bearings which are not shown in the figure. The ring rotates with angular velocity $\Omega_2(t)$ as shown.

The fluid-filled annuli create a damper c_1 between the shaft and the ring, and a damper $c2$ between the ring and the cup. Assume that any other damping is negligible.

a.) Draw a free-body diagram for the ring showing the torques acting on the ring.

b.) Use this free-body diagram to derive a differential equation in terms of $\Omega_1(t)$ and $\Omega_2(t)$ which describes this system. Note that $\Omega_1(t)$ is an arbitrary velocity which is externally specified.

c.) Assume that $\Omega_1(t)$ is a step, i.e., $\Omega_1(t) = u_s(t)$, and that $\Omega_2(0) = 0$. solve for the resulting motion $\Omega_2(t)$ for $t \geq 0$.

d.) In steady-state, what torque must be exerted on the input shaft? Why?

Solutions

a.) The free body diagram for the inertia is shown in Figure 2 along with a schematic of the dynamic elements in this problem. As can be seen, the inertia has two torques acting on it: T_1 generated by the damper c_1 between the input and the inertia and T_2 generated by the damper c_2 between the inertia and ground. The schematic drawing is helpful but not required in the solution.

b.) Using the FBD from part a, we can write the following relationship:

$$\Sigma T = J\dot{\Omega}_2 = T_1 + T_2$$

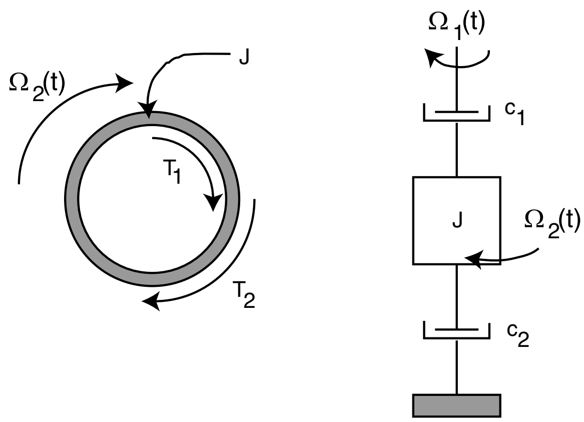

Figure 2: Freebody diagram of Inertia and Schematic of system.

This is where the schematic becomes helpful. By inspection, we can work out the following relationships

$$\begin{aligned} T_1 &= c_1(\Omega_1 - \Omega_2) \\ T_2 &= -c_2\Omega_2 \end{aligned}$$

Substituting for T_1 and T_2 yields the following

$$J\dot{\Omega}_2 = c_1(\Omega_1 - \Omega_2) - c_2\Omega_2 \Rightarrow \boxed{J\dot{\Omega}_2 + (c_1 + c_2)\Omega_2 = c_1\Omega_1}$$

c.) The input to the system has been defined as $\Omega_1(t) = u_s(t)$ thus the differential equation for this system become

$$J\dot{\Omega}_2 + (c_1 + c_2)\Omega_2 = c_1 u_s(t)$$

This differential equation (DE) matches that for a first order forced response. We know the solution to this DE is of the form

$$\begin{aligned} \Omega_2(t) &= \Omega_h(t) + \Omega_p(t) \text{ where} \\ \Omega_p(t) &= \Omega_{ss} = \frac{c_1}{c_1 + c_2} \text{ (Particular solution)} \\ \Omega_h(t) &= Ae^{-st} \text{ (Homogeneous solution)} \\ s &= -\frac{c_1 + c_2}{J} \end{aligned}$$

Using initial conditions

$$\Omega_2(0) = 0 = A + \frac{c_1}{c_1 + c_2} \Rightarrow A = -\frac{c_1}{c_1 + c_2}$$

$$\Omega_2(t) = \frac{c_1}{c_1 + c_2}\left(1 - e^{-\frac{c_1 + c_2}{J}t}\right)$$

d.) We determined earlier that

$$\Omega_{ss} = \frac{c_1}{c_1 + c_2}$$

Looking carefully at out schematic of the system, we know that the torque from the shaft acting on the inertia is T_1. Since, the torques on either side of the damper need to be equal, we know the torque acting on the shaft is equal and opposite to T_1. Since the system is in steady state, we know that the velocity of the shaft is constant which means that the sum of torques acting on the shaft equal 0.

$$\Sigma T_{shaft} = 0 = T_{in} - T_1 \Rightarrow \; T_{in} = T_1$$

We know that

$$\begin{aligned}
T_1 &= c_1(\Omega_1 - \Omega_2) \\
\Omega_1 &= u_s(t) = 1 \\
\Omega_2 &= \Omega_{ss} = \frac{c_1}{c_1 + c_2} \\
\text{Thus } T_1 &= c_1\left(1 - \frac{c_1}{c_1 + c_2}\right) = \frac{c_1 c_2}{c_1 + c_2} \\
\text{More generally } T_1 &= \frac{c_1 c_2}{c_1 + c_2}\Omega_1
\end{aligned}$$

Problem 2 - Toy Flywheel:
A toy consists of a rotating flywheel supported on a pair of bearings as shown in Figure 3. The flywheel is connected to a pulley , around which is wrapped a flexible but inextensible cable connected to a spring. In operation, the flywheel is initially at rest, the string made taut, and at $t = 0$, the input $x_s(t)$ undergoes a step change in position of magnitude x_0.

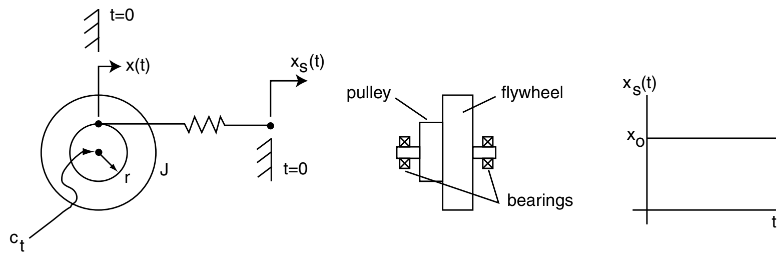

Figure 3:

Assume that the flywheel-shaft-pulley unit has rotational inertia, J. The bearings can be modelled as a viscous rotational damper of coefficient, c_t. the pulley is of radius, r. The spring is an ideal linear spring with spring constant, k.

1. Write the system equation as a differential equation in $x(t)$, the length of cable unwound from the pulley, as well as system parameters, J, r, and k. [Note that $x(0) = 0$]
2. For what range of values of c_t (expressed in terms of system parameters J, r, and k) will the cable never go slack?
3. Assuming that c_t has some non-zero value such that the cable **does** go slack, write an expression (in terms of system parameters J,r, and k) for the response $x(t)$, i.e. the length of cable unwound from the pulley. Sketch the response, x(t). Carefully indicate the time over which the expression and the sketch are valid.
4. Assume that c_t is zero. Write an expression (in terms of system parameters J, r, and k) for t, the time at which the cable first goes slack.

Solutions
1.) Figure 4 shows the free body diagram for this system. Summing the torques

$$\begin{aligned} \Sigma T &= J\ddot{\theta} = T_k - Tc \\ T_k &= rF_k \\ F_k &= k(x_s - x) \\ T_c &= c_t\dot{\theta} \\ \text{so } J\ddot{\theta} + c_t\dot{\theta} &= rk(x_s - x) \end{aligned}$$

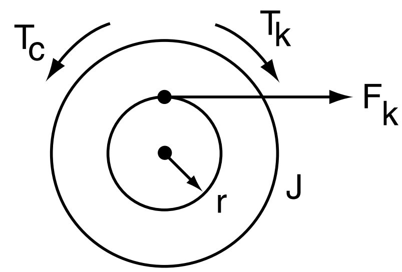

Figure 4:

$$J\ddot{\theta} + c_f\dot{\theta} + rkx = rkx_s$$
$$\text{Recall that } r\theta = x \Rightarrow \theta = \frac{x}{r}$$

so $\boxed{\frac{J}{r}\ddot{x} + \frac{c_t}{r}\dot{x} + rkx = rkx_s(t)}$

This may be rewritten as

$$\ddot{x} + \frac{c_t}{J}\dot{x} + \frac{r^2k}{J}x = \frac{r^2k}{J}x_s(t)$$

b.) For this part we note the following relationships

$$2\zeta\omega_n \equiv \frac{c_t}{J}$$
$$\omega_n^2 \equiv \frac{r^2k}{J}$$

For the string not to go slack the system must not overshoot x_0, this means that $\zeta \geq 1$. For $\zeta \geq 1$ to be true

$$c_t \geq 2\sqrt{r^2kJ}$$

3.) For the string to go slack the system must be allowed to overshoot or $\zeta < 1$. In the case of $\zeta < 1$

$$x(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{\zeta\omega_n t}\cos(\omega_d t - \psi)$$
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$
$$\psi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

This equation can be expressed in terms of the system variables by substituting for ω_n and ζ from 3. Figure 5 shows the time response for this system where t^* indicates the time after which the response is not valid. **4.)** When $c_t = 0$, the

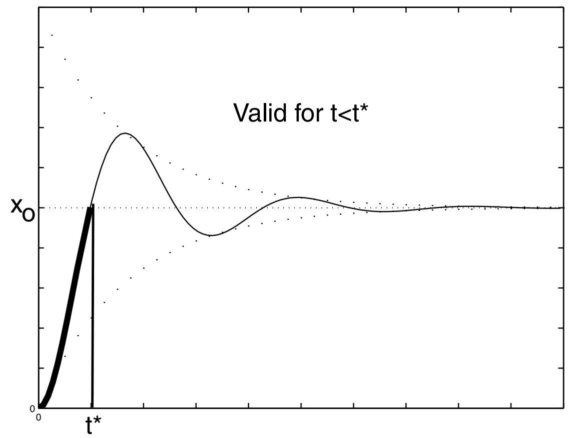

Figure 5:

expression for $x(t)$ becomes

$$x(t) = x_0(1 - \cos\omega_n t)$$

As we saw in part 3, t^* is the time at which the response passes x_0. In this case, it takes a quarter cycle to get to x_0, thus

$$t^* = \frac{1}{4}\frac{2\pi}{\omega_n}$$

Figure 6 shows the response for this system.

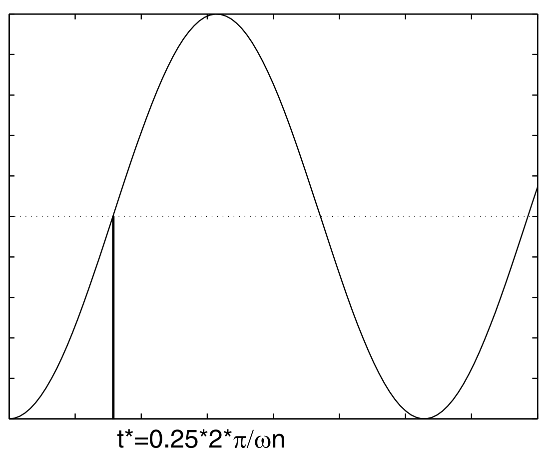

Figure 6: