8.07 – Basic Formulae

$$\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}\,, \qquad \vec{a}\cdot\vec{b} = a_i b_i\,, \qquad (\vec{a}\times\vec{b})_i = \epsilon_{ijk}a_j b_k$$

$$\det A = \epsilon_{i_1 i_2 \cdots i_n} A_{1,i_1} A_{2,i_2} \cdots A_{n,i_n}$$

$$\int_V \nabla\cdot\vec{F} = \int_S \vec{F}\cdot d\vec{a}\,, \qquad \int_S \nabla\times\vec{F} = \int_\Gamma \vec{F}\cdot d\vec{l}$$

$$\nabla\times(\nabla\phi) = 0, \qquad \nabla\cdot(\nabla\times\vec{A}) = 0$$

$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}\,, \quad g(x_i) = 0$$

$$\nabla r = \vec{e}_r \qquad \nabla^2\,\frac{1}{r} = -4\pi\delta(\vec{x})$$

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\vec{e}_\phi$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial\Phi}{\partial r}\Big) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial\Phi}{\partial\theta}\Big) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

$$|\vec{E}(r)| = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \qquad V(r) = \frac{1}{4\pi\epsilon_0}\,\frac{q}{r}$$

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}, \qquad \int_S \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\epsilon_0}\,, \qquad \vec{E} = -\nabla V \qquad \nabla^2 V = -\frac{\rho}{\epsilon_0}$$

$$W = \frac{1}{8\pi\epsilon_0}\sum_{i\neq j}\frac{q_i q_j}{|\vec{x}_i - \vec{x}_j|} \to \frac{1}{8\pi\epsilon_0}\int d^3x d^3x'\,\frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x}_i - \vec{x}_j|} \to w = \frac{\epsilon_0}{2}|\vec{E}|^2$$

$$Q = CV \to Q_i = C_{ij}V_j$$

$$V(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' G(\vec{x},\vec{x}')\rho(x') + \frac{1}{4\pi}\oint_S \Big[G(\vec{x},\vec{x}')\frac{\partial V}{\partial n'} - V(\vec{x}')\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}\Big]da'$$

$$\nabla^2_{\vec{x}'}G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$$

$G(\vec{x},\vec{x}')$ = "potential" at $\vec{x}'$ due to a unit charge at $\vec{x}$, when ....

Image set for sphere: $(q\,,\,y)$, $(q'\,,\,y')$ with $y\,y' = a^2$, $\quad q' = -\frac{a}{y}\,q$

$$\left\{\begin{matrix}\cos\alpha x\\ \sin\alpha x\end{matrix}\right\}\left\{\begin{matrix}\cos\beta y\\ \sin\beta y\end{matrix}\right\}\left\{\begin{matrix}\cosh\gamma z\\ \sinh\gamma z\end{matrix}\right\} \qquad \gamma^2 = \alpha^2+\beta^2$$ a cartesian solution of Laplace's

$\{r^l\,, r^{-(l+1)}\}\,P_l(\cos\theta)$ azimuthal solutions of Laplace's

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\,P_l(\cos\gamma)\,, \qquad \frac{1}{\sqrt{1-2hx+h^2}} = \sum_{l=0}^{\infty}h^l P_l(x)$$

$$P_l(1) = 1 \qquad P_l(-x) = (-1)^l P_l(x) \qquad \int_{-1}^{1} dx\, P_{l'}(x) P_l(x) = \frac{2}{2l+1}\delta_{ll'}$$

$$S'_{i_1\cdots i_n} = (\det R)\, R_{i_1 j_1} \,\cdots\, R_{i_n j_n}\, S_{j_i \cdots j_n} \qquad \text{(pseudo) tensor under rotation}$$

$$V(\vec{x}) = \frac{1}{4\pi\epsilon_0}\Big[\, \frac{Q}{|\vec{x}|} + \frac{\vec{p}\cdot\vec{x}}{|\vec{x}|^3} + \frac{1}{2}\frac{x_i x_j}{|\vec{x}|^5}\, Q_{ij} + \cdots \Big]$$

$$Q = \int d^3x\, \rho(\vec{x})\,, \quad p_i = \int d^3x\, \rho(\vec{x})\, x_i \quad Q_{ij} = \int d^3x\, \rho(\vec{x})(3x_i x_j - \delta_{ij}|\vec{x}|^2)\,,$$

$$\vec{E}_{dip} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}(3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}) \quad \to \quad \vec{E}_{dip} = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})\,, \quad \vec{p} = p\hat{z}$$

$$\vec{J}\, d^3x \;\Leftrightarrow\; \vec{K}\, da \;\Leftrightarrow\; I d\vec{l}\,, \qquad \vec{J} = \rho\vec{v}\,, \qquad \nabla\cdot\vec{J} = 0, \qquad \frac{d\vec{p}}{dt} = q\,\vec{v}\times\vec{B}$$

$$d\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\frac{I d\vec{l}\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\,, \qquad \vec{B} = \frac{\mu_0 I}{4\pi}\nabla\Omega$$

$$\vec{B} = \nabla\times\vec{A}\,, \qquad \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x'\, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\,, \quad \nabla\cdot\vec{A} = 0\,,$$

$$\nabla\times\vec{B} = \mu_0\vec{J}\,, \qquad \int_\Gamma \vec{B}\cdot d\vec{l} = \mu_0 I_{linked}\,, \qquad \nabla\cdot B = 0$$

$$\vec{A} = \frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\vec{x}}{|\vec{x}|^3}\,, \qquad \vec{B} = -\frac{\mu_0}{4\pi}\,\nabla\Big(\frac{\vec{m}\cdot\vec{x}}{|\vec{x}|^3}\Big)\,, \qquad \vec{m} = \frac{1}{2}\int d^3x\, \vec{x}\times\vec{J}(\vec{x})\,, \qquad \vec{m} = I\vec{A}$$

$$\vec{B}_{dip} = \frac{\mu_0}{4\pi}\frac{1}{r^3}(3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\,) \quad \to \quad \vec{B}_{dip} = \frac{\mu_0}{4\pi}\frac{m}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})\,, \quad \vec{m} = m\hat{z}$$

$$\mathcal{E} = \int_\Gamma (\vec{E} + \vec{v}\times\vec{B})\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a}$$

$$W = \frac{1}{2\mu_0}\int \vec{B}\cdot\vec{B}\, d^3x = \frac{1}{2}\int \vec{A}\cdot\vec{J}\, d^3x$$

$$M_{ij} = \frac{\mu_0}{4\pi}\oint_{C_i}\oint_{C_j}\frac{d\vec{l}_i\cdot d\vec{l}_j}{|\vec{x}_i - \vec{x}_j|} = \frac{1}{I_j}\int_{S_i}\vec{B}_{ij}(\vec{x})\cdot d\vec{a}\,, \qquad M_{ij} = M_{ji}$$

$$\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\frac{dE_{mech}}{dt} = \int_V d^3x\, \vec{J}\cdot\vec{E}\,, \qquad \vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

$$\frac{dE_{mech}}{dt} + \frac{d}{dt}\int_V d^3x\,\frac{1}{2}\,(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}) = -\int_S \vec{S}\cdot d\vec{a}$$

$$\frac{dP_{mech}}{dt} = \int_V d^3x\,(\rho\vec{E} + \vec{J}\times\vec{B})$$

$$T_{\alpha\beta} = \epsilon_0\Big[E_\alpha E_\beta + c^2 B_\alpha B_\beta - \frac{1}{2}(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B})\delta_{\alpha\beta}\Big]$$

$$\Big(\frac{dP_{mech}}{dt} + \frac{d}{dt}\int_V d^3x\,\epsilon_0(\vec{E}\times\vec{B})\Big)_i = +\int_S T_{ij}n_j\,da$$

$$\vec{E} = -\nabla\Phi - \frac{\partial A}{\partial t}\,, \quad \vec{B} = \nabla\times\vec{A}$$

$$\Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int d^3x'\;\frac{\rho\big(\vec{x}',t-\frac{|\vec{x}-\vec{x}'|}{c}\big)}{|\vec{x}-\vec{x}'|}\,, \qquad \vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\;\frac{\vec{J}\big(\vec{x}',t-\frac{|\vec{x}-\vec{x}'|}{c}\big)}{|\vec{x}-\vec{x}'|}$$

$$\vec{E}(\vec{x},t) = \vec{\mathcal{E}}e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$
$$\vec{B}(\vec{x},t) = \vec{\mathcal{B}}e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

$$\vec{n}\cdot\vec{\mathcal{E}} = \vec{n}\cdot\vec{\mathcal{B}} = 0\,, \qquad \vec{\mathcal{B}} = \sqrt{\mu\epsilon}\,\vec{n}\times\vec{\mathcal{E}} \qquad \vec{\mathcal{E}} = Z\,\vec{\mathcal{H}}\times\vec{n}\,, \quad Z = \sqrt{\mu/\epsilon}$$

$$\langle\vec{E}\circ\vec{B}\rangle = \frac{1}{2}\mathrm{Re}(\vec{\mathcal{E}}\circ\vec{\mathcal{B}}^*)$$

$$\text{Radiation: } e^{-i\omega t} \qquad \vec{H}(\vec{x}) = \frac{1}{\mu_0}\nabla\times\vec{A}(\vec{x}) \qquad \vec{E}(\vec{x}) = \frac{iZ_0}{k}\nabla\times\vec{H}(\vec{x})$$

$$\vec{A} = -\frac{i\mu_0}{4\pi}\,\omega\,\vec{p}\,\frac{e^{ikr}}{r}\,, \qquad \frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2}\,k^4|(\vec{n}\times\vec{p})\times\vec{n}|^2\,, \quad P = \frac{c^2 Z_0}{12\pi}\,k^4|\vec{p}|^2$$

$$\text{magnetic dipole radiation} \qquad \frac{dP}{d\Omega} = \frac{dP}{d\Omega}(\vec{p}\to\vec{m}/c)\,, \quad P = P(\vec{p}\to\vec{m}/c)$$

$$\vec{D} = \epsilon_0\vec{E} + \vec{P}\,, \quad \nabla\cdot\vec{D} = \rho \qquad \text{linear material: } \vec{D} = \epsilon\,\vec{E}$$
$$\rho_{pol} = -\nabla\cdot\vec{P}\,, \quad \sigma_{pol} = \vec{n}\cdot\vec{P}$$

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}\,, \qquad \nabla\times\vec{H} = \vec{J}\,, \qquad \text{linear material: } \vec{B} = \mu\vec{H}$$

$$\vec{J}_b = \nabla\times\vec{M}\,, \quad \vec{K}_b = \vec{M}\times\vec{n}$$

$$\vec{H} = -\nabla\Phi_M \quad \to \quad \nabla^2\Phi_M = -\rho_M\,, \qquad \rho_M = -\nabla\cdot\vec{M}\,, \quad \sigma_M = \vec{M}\cdot\vec{n}$$