8.07 Fall 2001 – Test 2

Two hours. Closed book and notes – Formula sheet allowed.

Problem 1. (15 points) Exercises with a solenoid.
Consider a solenoid of radius R, N turns per unit length and current I_0, aligned along the z axis.

(i) (5 points) Give the expression for the stress tensor matrix $T_{\alpha\beta}$ at any point inside the solenoid.

(i) (5 points) Calculate the energy stored in the solenoid per unit length.

(iii) (5 points) Calculate the self-inductance per unit length.

Problem 2. (25 points) Consider a capacitor made out of two circular conducting plates both of radius a. One plate is placed on the xy plane centered at the origin, and the second is at some $z > 0$ parallel to the first plate and also centered at $x = y = 0$. There is a time-varying electric field between the plates in the z-direction of the form

$$\vec{E} = E_0 \, cos(\omega_0 t) \, \vec{e}_z \tag{1}$$

Here E_0 is a constant, and $\vec{E}$ is independent of z and of the radial distance ρ to the z-axis. This is the leading term in the calculation we will do of the true fields between the plates.

(i) (10 points) Due to the time varying electric field in (1) there will be an induced magnetic field $\vec{B}(\rho, t)$. Find the magnitude and the direction of this magnetic field (Hint: use the integral form of the Maxwell equation: $\nabla \times \vec{B} = \cdots$).

(ii) (10 points) Due to the time varying magnetic field calculated above, there will be a new electric field $\vec{E}_1$ in the z-direction, which by definition of E_0 above, can be required to satisfy $\vec{E}_1(\rho = 0) = 0$. Use Faraday's law to calculate it. It is convenient to use a closed contour that includes the z-axis.

(iii) (5 points) Assume the electric field in (1) together with $\vec{E}_1$ calculated above are a good approximation to the exact electric field. For what value of the radius a could we close off the capacitor to turn it into a closed cylindrical conducting resonating cavity ? Express your answer as $a = \gamma \frac{c}{\omega_0}$ where γ is a constant you must determine. [In the exact calculation $\gamma \simeq 2.40486$ is the first zero of the Bessel function $J_0(x)$].

Problem 3. (25 points) Retarded Potentials.

Suppose the electrically neutral xy plane carries a time dependent but uniform surface current $\vec{K} = K(t)\vec{e}_y$. Here $K(t)$ is some arbitrary function vanishing in the far past: $K(-\infty) = 0$.

(i) (5 points) By translational invariance the vector potential $\vec{A}$ at a point (x, y, z) only depends on z. Use the retarded potential formula to write an integral expression for the vector potential $\vec{A}(z, t)$ at a point $P = (0, 0, z)$, with $z > 0$. Simplify so that the variable of integration is the radial distance ρ to the origin in the xy plane.

(ii) (10 points) Show that the retarded vector potential can be written as the remakably simple:

$$\vec{A}(z, t) = \frac{\mu_0 c}{2} \, \vec{e}_y \int_0^\infty K\Big(t - \frac{z}{c} - u\Big) du$$

(iii) (10 points) Give integral expressions for $\vec{E}$ and $\vec{B}$, and do the integrals by writing them as total derivatives.

Problem 4. (35 points)
A circularly polarized wave moving in the z direction has a finite extent in the x and y directions. An approximate description of such a wave is given by ($e^{-i\omega t}$ implicit, E_0 real)

$$\vec{E}(x,y,z) \approx \Big(E_0(x,y)(\vec{e}_x + i\vec{e}_y) + \frac{i}{k}\Big[\frac{\partial E_0}{\partial x} + i\frac{\partial E_0}{\partial y}\Big]\vec{e}_z \Big)\, e^{ikz}\,, \quad \vec{B} = -i\sqrt{\mu_0\epsilon_0}\,\vec{E}\,.$$

The finite extent condition means that $E_0(x,y)$ vanishes when either x or y are infinite. The above field is a good approximation when the derivative terms are small.

(i) (10 points) Is the Maxwell equation $\nabla\cdot\vec{E} = 0$ satisfied exactly by this *ansatz* ? (The Maxwell equation $\nabla\times\vec{E} = i\omega\vec{B}$ is only satisfied approximately).

We now try to calculate the time averaged angular momentum $\langle\vec{L}_z\rangle$ carried by this wave in the direction of propagation. The density $\vec{\mathcal{L}}$ of angular momentum is given by

$$\begin{aligned} <\vec{\mathcal{L}}> &= \frac{\epsilon_0}{2}\,\vec{x}\times\mathrm{Re}(\vec{E}\times\vec{B}^*) \\ &= \frac{\epsilon_0}{2c}\,\vec{x}\times\mathrm{Re}(i\vec{E}\times\vec{E}^*) = \frac{\epsilon_0}{2c}\,\vec{x}\times(i\vec{E}\times\vec{E}^*) \end{aligned}$$

and expanding, the z-component of the angular momentum density is

$$<\vec{\mathcal{L}}_z> = \frac{\epsilon_0}{2c}\Big(x(i\vec{E}\times\vec{E}^*)_y - y(i\vec{E}\times\vec{E}^*)_x \Big)\,.$$

We therefore need to calculate the x and y components of $(i\vec{E}\times\vec{E}^*)$.

(ii) (10 points) Show, without explicit computation that, as assumed above, $(i\vec{E}\times\vec{E}^*)$ is real (something is real if it equals its complex conjugate). Calculate the x and y components of $(i\vec{E}\times\vec{E}^*)$. Note that they can be written as total derivatives.

To avoid getting an infinite answer for the angular momentum carried by this wave, consider the volume between $z=0$ and $z=a$.

(iii) (10 points) Show that the (time averaged) total angular momentum carried by the fields on the slab can be written in the form

$$\langle\vec{L}_z\rangle_{slab} = K\int_0^a dz\int dxdy\, E_0^2(x,y)$$

What is the value of K ? (Hint: You will use integration by parts in the x,y directions).

(iv) (5 points) What is the relation between $\langle\vec{L}_z\rangle_{slab}$ and the non-derivative approximation to the energy density $\langle u\rangle$ carried by the wave ?