8.07 Fall 2002 – Test 1

Test duration: One hour and a half.
Closed book and notes – Formula sheet allowed.

Problem 1. (30 points) Three exercises:

(a) (10 points) Consider the three dimensional vector field $\mathbf{F} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}$. Could $\mathbf{F}$ be an electric field ? Could $\mathbf{F}$ be a magnetic field ?

(b) (10 points) Use index notation to derive a formula for $\nabla \cdot (\mathbf{a} \times \mathbf{b})$, where $\mathbf{a}$ and $\mathbf{b}$ are vector fields. Present your answer in vector notation.

(c) (10 points) A completely antisymmetric *tensor* B_{ijk} can be written as $B_{ijk} = \epsilon_{ijk} B$. Give the value of B in terms of one component of B_{ijk}. State the tensor properties of B, and prove your statement.

Problem 2. (30 points) Conducting sphere in an electric field.

A conducting *uncharged* sphere of radius a is placed inside a uniform external electric field $\mathbf{E} = E_0 \hat{\mathbf{z}}$. Choose the origin of the coordinate system to coincide with the center of the sphere.

(a) (10 points) Calculate the potential $V(r, \theta)$ outside the conductor using the appropriate axial solutions, and the boundary conditions at the surface of the conductor.

(b) (10 points) Calculate the surface charge $\sigma(\theta)$ on the surface of the conducting sphere. Find the electric dipole moment $\mathbf{p}$ of this charge distribution.

(c) (10 points) Could you have anticipated the value of $\mathbf{p}$ from your answer for $V(r, \theta)$ in part (a)? Explain. Does the charge distribution on the surface of the sphere have a non-vanishing quadrupole moment ? Explain.

Problem 3. (20 points) Multipole expansion.

A short piece of wire is placed along the z axis, centered at the origin. The wire carries a total charge Q, and the linear charge density λ is an even function of z: $\lambda(z) = \lambda(-z)$. The rms length of the charge distribution in the wire is l_0 ($Q l_0^2 = \int_{wire} z^2 \lambda(z) dz$).

(a) (10 points) Find the dipole and quadrupole moments for this charge distribution.

(b) (10 points) Give an expression for the potential $V(r, \theta)$ for large r, including the quadrupole contribution, and writing your result with the help of the required Legendre polynomial(s). ($P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1)$.)

(*turn over*)

Problem 4. (20 points) Green's function

Consider the Dirichlet boundary value problem where the potential is specified over the xy plane, and the region of interest is $z \geq 0$ (This situation was discussed in lecture).

(a) (5 points) Consider a unit charge at $(0, 0, z)$. Give the position and magnitude of the image charge(s) needed to find the potential in the region of interest, assuming that the xy plane is kept at zero potential.

(b) (5 points) Write an explicit expression for the Green's function $G_D(0, 0, z; x', y', z')$ needed to solve the Dirichlet boundary value problem where the xy plane is the Dirichlet surface.

(c) (5 points) Consider now the boundary value problem where a potential $V_0(x, y)$ is prescribed on the xy plane. Assume $\rho = 0$ over the region of interest. Write an integral expression for the potential $V(0, 0, z)$. Simplify your expression by evaluating all needed derivatives (be neat please!).

(d) (5 points) Consider now the case when $V_0(x, y) = V_0$, is a constant on the disk $x^2 + y^2 \leq a^2$, and vanishes outside this disk. Calculate $V(0, 0, z)$.