8.07 Homework 11

Problem 1. Retarded potentials.

A sheet of charge with density σ lies in the plane $z = 0$, so that $\rho = \sigma\delta(z)$.

(i) What are the electrostatic fields throughout space ?

At time $t = 0$ the sheet begins to move in the y direction with a constant speed v, so that for $t > 0$, $j_y = \sigma v\delta(z)$, $j_x = j_z = 0$.

(ii) Calculate directly from the formula for the retarded potential the value of $A_y(z,t)$. Plot its value as a function of t for fixed z (positive and negative) and also as a function of z for fixed $t > 0$.

(iii) Get B_x and E_y and plot them as functions of time at a fixed z (positive and negative) and as functions of z for a fixed time $t > 0$.

Problem 2. Conservation laws in the setting of Problem 1.

Consider a box bounded by the planes $z = \pm h$, $x = 0, x = 1$, $y = 0, y = 1$. This box extends a distance h above and below the current sheet. Explain *quantitatively* how conservation laws work for the volume in the following cases:

(i) Conservation of energy for $0 < t < h/c$.

(ii) Conservation of energy for $t > h/c$.

(iii) Conservation of the y-component of momentum for $0 < t < h/c$.

(iv) Conservation of the y-component of momentum for $t > h/c$.

The statement of energy conservation, for example, is

$$\frac{d}{dt}(E_{mech} + E_{field}) = -\oint \mathbf{S}\cdot\mathbf{da}\,, \quad \text{with} \quad \frac{dE_{mech}}{dt} = \int \mathbf{j}\cdot\mathbf{E}\,d^3x$$

Problem 3. A plane wave travelling along the positive z-direction is incident normally on a uniform material filling the half-space $z \geq 0$. The material has a constant conductivity $\sigma > 0$ ($\vec{J} = \sigma\vec{E}$), and $\epsilon/\epsilon_0 = \mu/\mu_0 = 1$. The incident field (for $z < 0$) is of the form

$$\vec{E}_{\rm inc}(\vec{r},t) = Re\left\{E_0\, e^{i(kz-\omega t)}\hat{\mathbf{x}}\right\}\,, \tag{1}$$

with $k = \frac{\omega}{c}$ and E_0 a real constant. Consider the simple *ansatz* for the wave in the conductor ($z > 0$) and a reflected wave

$$\vec{E}_{\rm con}(\vec{r},t) = Re\left\{\vec{E}_c(\vec{r})\, e^{-i\omega t}\right\}. \tag{2}$$

$$\vec{E}_{\rm ref}(\vec{r},t) = Re\left\{E_r\, e^{i(-kz-\omega t)}\,\hat{\mathbf{x}}\right\}\,, \tag{3}$$

Here, the possibly complex vector $\vec{E}_c(\vec{r})$, and E_r are to be found.

(i) Show that $\vec{E}_c(\vec{r})$ satisfies the differential equation

$$\left\{\nabla^2 + k^2\left(1 + i\frac{\sigma}{\omega\epsilon_0}\right)\right\}\vec{E}_c(\vec{r}) = 0 \quad . \tag{4}$$

(ii) Assume now that the fields in the conductor are of the form

$$\begin{aligned} \vec{E}_c(\vec{r}) &= E_c\, e^{i\beta z}\hat{\mathbf{x}}\,, \\ \vec{B}_c(\vec{r}) &= B_c\, e^{i\beta z}\hat{\mathbf{y}}\,. \end{aligned} \tag{5}$$

What is the value of β in terms of k, σ, ω ? Find B_c in terms of E_c.

(iii) Using the relevant boundary conditions for $\vec{E}$ *and* $\vec{B}$ at the boundary, set up and solve the system of equations that determine E_c and E_r in terms of E_0, k and β.

(iv) Find the pressure on the conducting wall. Express your answer in terms of E_c, k and β. [Hint: use the stress tensor.]

Problem 4. Calculate the exact electric field for electric dipole radiation. Show that you get

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\Big\{k^2(\mathbf{n}\times\mathbf{p})\times\mathbf{n}\frac{e^{ikr}}{r} + (3\mathbf{n}\,(\mathbf{n}\cdot\mathbf{p}) - \mathbf{p})\Big(\frac{1}{r^3} - \frac{ik}{r^2}\Big)e^{ikr}\Big\}$$

Problem 5. (Jackson 9.3). Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap. An alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V\cos\omega t$. In the long wavelength limit find the radiation fields, the angular distribution of ratiated power and the total radiated power from this sphere.

Problem 6. Griffiths 4.2 (p.163).

Problem 7. Griffiths 4.10 (p.169).

Problem 8. Griffiths 4.26 (p.193).

Problem 9. Griffiths 4.32 (p.198).

Problem 10. Griffiths 6.4 (p.259).

Problem 11. Griffiths 6.5 (p.260).

Problem 12. Bar magnet (based on Jackson 5.19, and Griffiths 6.9).
A bar magnet is in the shape of a right circular cylinder of length L and radius a. The cylinder has a permanent magnetization M_0 uniform throughout its volume and parallel to its axis.
(a) Calculate **H** and **B** at all points on the axis of the cylinder, both inside and outside the magnet.
(b) Sketch in a plot the ratios $\mathbf{B}/\mu_0 M_0$ and $\mathbf{H}/M_0$ for $L/a = 5$.
(c) Find the bound current.
(d) Why is the **B** field far away from the magnet approximately that of a magnetic dipole? What is the effective dipole moment?