Lecture 6
Applications of Nash equilibrium

14.12 Game Theory

Road Map

1. Cournot (quantity) Competition
   1. Nash Equilibrium in Cournot oligopoly
2. Bertrand (price) Competition
3. Commons Problem
4. Quiz
5. Mixed-strategy Nash equilibrium
Cournot Oligopoly

- \( N = \{1,2,\ldots,n\} \) firms;
- Simultaneously, each firm \( i \) produces \( q_i \) units of a good at marginal cost \( c \), and sells the good at price
  \[ P = \max\{0,1-Q\} \]
  where \( Q = q_1 + \ldots + q_n \).
- Game = \((S_1,\ldots,S_n; \pi_1,\ldots,\pi_n)\)
  where \( S_i = [0,\infty) \),

\[
\pi_i(q_1,\ldots,q_n) = q_i[1-(q_1+\ldots+q_n)-c] \quad \text{if} \quad q_1+\ldots+qn < 1,
\]
\[-q_ic \quad \text{otherwise.}\]

![Cournot Oligopoly -- profit](image)
Cournot Oligopoly -- Equilibrium

- \( q > 1 - c \) is strictly dominated, so \( q \leq 1 - c \).
- \( \pi_i(q_1, \ldots, q_n) = q_i[1-(q_1+\ldots+q_n)-c] \) for each \( i \).
- FOC: 
  \[
  \left. \frac{\partial \pi_i(q_1, \ldots, q_n)}{\partial q_i} \right|_{q=q_i^*} = \left. \frac{\partial [q_i(1-q_1-\ldots-q_n-c)]}{\partial q_i} \right|_{q=q_i^*} = (1-q_1^*-\ldots-q_n^*-c) - q_i^* = 0.
  \]

  - That is,
    \[
    2q_1^* + q_2^* + \ldots + q_n^* = 1 - c \\
    q_1^* + 2q_2^* + \ldots + q_n^* = 1 - c \\
    \vdots \\
    q_1^* + q_2^* + \ldots + nq_n^* = 1 - c
    \]
  - Therefore, \( q_1^* = \ldots = q_n^* = (1-c)/(n+1) \).

Cournot oligopoly – comparative statics

![Diagram of Cournot oligopoly](attachment:image.png)
Bertrand (price) competition

- \( N = \{1,2\} \) firms.
- Simultaneously, each firm \( i \) sets a price \( p_i \);
- If \( p_i < p_j \), firm \( i \) sells \( Q = \max\{1 - p_i, 0\} \) unit at price \( p_i \); the other firm gets 0.
- If \( p_1 = p_2 \), each firm sells \( Q/2 \) units at price \( p_1 \), where \( Q = \max\{1 - p_1, 0\} \).
- The marginal cost is 0.

\[
\pi_i(p_1, p_2) = \begin{cases} 
  p_1(1 - p_1) & \text{if } p_1 < p_2 \\
  p_1(1 - p_1)/2 & \text{if } p_1 = p_2 \\
  0 & \text{otherwise.}
\end{cases}
\]

Bertrand duopoly -- Equilibrium

**Theorem:** The only Nash equilibrium in the “Bertrand game” is \( p^* = (0,0) \).

**Proof:**
1. \( p^* = (0,0) \) is an equilibrium.
2. If \( p = (p_1, p_2) \) is an equilibrium, then \( p = p^* \).
   1. If \( p = (p_1, p_2) \) is an equilibrium, then \( p_1 = p_2 \).
      - If \( p_1 > 0 \), for sufficiently small \( \varepsilon > 0 \), \( p_j' = p_j - \varepsilon \) is a better response to \( p_i \) for \( j \). If \( p_1 > p_j > 0 \), \( p_i' = p_i \) is a better response for \( i \).
   2. Given any equilibrium \( p = (p_1, p_2) \) with \( p_1 = p_2 \), \( p = p^* \).
      - If \( p_1 = p_2 > 0 \), for sufficiently small \( \varepsilon > 0 \), \( p_j' = p_j - \varepsilon \) is a better response to \( p_j \) for \( i \).
Commons Problem

- \( N = \{1, 2, \ldots, n\} \) players, each with unlimited money;
- Simultaneously, each player \( i \) contributes \( x_i \geq 0 \) to produce \( y = x_1 + \ldots + x_n \) unit of some public good, yielding payoff
  \[ U_i(x_i, y) = y^{1/2} - x_i. \]

Quiz

Each student \( i \) is to submit a real number \( x_i \). We will pair the students randomly. For each pair \((i, j)\), if \( x_i \neq x_j \), the student who submits the number that is closer to \( (x_i + x_j)/4 \) gets 100; the other student gets 20. If \( x_i = x_j \), then each of \( i \) and \( j \) gets 50.
Stag Hunt

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<th>(2,2)</th>
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<td>(0,4)</td>
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Equilibrium in Mixed Strategies

What is a strategy?
- A complete contingent-plan of a player.
- What the others think the player might do under various contingency.

What do we mean by a mixed strategy?
- The player is randomly choosing his pure strategies.
- The other players are not certain about what he will do.
Stag Hunt

\[
\begin{array}{cc}
(2,2) & (4,0) \\
(0,4) & (5,5)
\end{array}
\]

**Mixed-strategy equilibrium in Stag-Hunt game**

- Assume: Player 2 thinks that, with probability \( p \), Player 1 targets for Rabbit. What is the best probability \( q \) she wants to play Rabbit?
- His payoff from targeting Rabbit:
  \[
  U_2(R; p) = 2p + 4(1-p) = 4 - 2p.
  \]
- From Stag:
  \[
  U_2(S; p) = 5(1-p)
  \]
- She is indifferent iff
  \[
  4 - 2p = 5(1-p)
  \]
- \( p = \frac{1}{3} \).

\[
q_B^R(p) = \begin{cases} 
0 & \text{if } p < \frac{1}{3} \\
q & \text{if } p = \frac{1}{3} \\
1 & \text{if } p > \frac{1}{3}
\end{cases}
\]
Best responses in Stag-Hunt game

Bertrand Competition with costly search

- $N = \{F1,F2,B\}$; $F1$, $F2$ are firms; $B$ is buyer
- $B$ needs 1 unit of good, worth 6;
- Firms sell the good; Marginal cost = 0.
- Possible prices $P = \{3,5\}$.
- Buyer can check the prices with a small cost $c > 0$.

Game:
1. Each firm $i$ chooses price $p_i$;
2. $B$ decides whether to check the prices;
3. (Given) If he checks the prices, and $p_1 \neq p_2$, he buys the cheaper one; otherwise, he buys from any of the firm with probability $\frac{1}{2}$. 
Bertrand Competition with costly search

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Check

Don’t Check

Mixed-strategy equilibrium

- Symmetric equilibrium: Each firm charges “High” with probability q;
- Buyer Checks with probability r.
- $U(\text{check};q) = q^21 + (1-q^2)3 - c = 3 - 2q^2 - c$;
- $U(\text{Don’t};q) = q1 + (1-q)3 = 3 - 2q$;
- Indifference: $2q(1-q) = c$; i.e.,
- $U(\text{high};q,r) = 0.5(1-r(1-q))5$;
- $U(\text{low};q,r) = qr3 + 0.5(1-qr)3$
- Indifference: $r = 2/(5-2q)$. 