Lectures 7
Backward Induction

14.12 Game Theory

Road Map

1. Bertrand competition with costly search
2. Backward Induction
3. Stackelberg Competition
4. Sequential Bargaining
5. Quiz
Bertrand Competition with costly search

- N = {F1,F2,B}; F1, F2 are firms; B is buyer
- B needs 1 unit of good, worth 6;
- Firms sell the good; Marginal cost = 0.
- Possible prices P = {3,5}.
- Buyer can check the prices with a small cost c > 0.

Game:
1. Each firm i chooses price $p_i$;
2. B decides whether to check the prices;
3. (Given) If he checks the prices, and $p_1 \neq p_2$, he buys the cheaper one; otherwise, he buys from any of the firm with probability $\frac{1}{2}$. 

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Mixed-strategy equilibrium

- Symmetric equilibrium: Each firm charges “High” with probability q;
- Buyer Checks with probability r.
- \( U(\text{check}; q) = q^2 + (1-q)^3 - c = 3 - 2q^2 - c; \)
- \( U(\text{Don’t}; q) = q + (1-q)3 = 3 - 2q; \)
- Indifference: \( 2q(1-q) = c \); i.e.,
- \( U(\text{high}; q, r) = 0.5(1-r(1-q))^5; \)
- \( U(\text{low}; q, r) = qr^3 + 0.5(1-qr)^3 \)
- Indifference: \( r = 2/(5-2q). \)

Dynamic Games of Perfect Information
&
Backward Induction
Definitions

**Perfect-Information game** is a game in which all the information sets are singleton.

**Sequential Rationality:** A player is sequentially rational iff, at each node he is to move, he maximizes his expected utility conditional on that he is at the node – even if this node is precluded by his own strategy.

In a finite game of perfect information, the “common knowledge” of sequential rationality gives “**Backward Induction**” outcome.

### A centipede game

```
1   A  2   α  1   a
    D  δ  d
(4,4) (5,2) (3,3) (1,-5)
```
**Backward Induction**

1. Take any pen-terminal node.
2. Pick one of the payoff vectors (moves) that gives ‘the mover’ at the node the highest payoff.
3. Assign this payoff to the node at the hand.
4. Eliminate all the moves and the terminal nodes following the node.

*Any non-terminal node*

- Yes
- No

The picked moves

---

**Battle of The Sexes with perfect information**

```
  1
 /\  
|  |
/\  
2 2
 /\  
L L  R R
```

Payoffs:
- (2,1)
- (0,0)
- (0,0)
- (1,2)
Note

• There are Nash equilibria that are different from the Backward Induction outcome.
• Backward Induction always yields a Nash Equilibrium.
• That is, Sequential rationality is stronger than rationality.

Matching Pennies (wpi)
Stackelberg Duopoly

Game:
N = \{1,2\} firms w MC = 0;
1. Firm 1 produces \(q_1\) units
2. Observing \(q_1\), Firm 2 produces \(q_2\) units
3. Each sells the good at price
   \[P = \max\{0,1-(q_1+q_2)\}\].

\[\pi_i(q_1, q_2) = q_i[1-(q_1+q_2)]\] if \(q_1 + q_2 < 1\),
\[0\] otherwise.

“Stackelberg equilibrium”

• If \(q_1 > 1\), \(q_2^*(q_1) = 0\).
• If \(q_1 \leq 1\), \(q_2^*(q_1) = (1-q_1)/2\).
• Given the function \(q_2^*\), if \(q_1 \leq 1\)
  \[\pi_1(q_1; q_2^*(q_1)) = q_1[1-(q_1+ (1-q_1)/2)]\]
  \[= q_1 (1-q_1)/2;\]
  0 otherwise.
• \(q_1^* = \frac{1}{2}\).
• \(q_2^*(q_1^*) = \frac{1}{4}\).
Sequential Bargaining

- $N = \{1,2\}$
- $X =$ feasible expected-utility pairs $(x,y \in X)$
- $U_i(x,t) = \delta_i^t x_i$
- $d = (0,0) \in D$ disagreement payoffs

Timeline – 2 period

At $t = 1$,
- Player 1 offers some $(x_1,y_1)$,
- Player 2 Accepts or Rejects the offer
- If the offer is Accepted, the game ends yielding $(x_1,y_1)$,
- Otherwise, we proceed to date 2.

At $t = 2$,
- Player 2 offers some $(x_2,y_2)$,
- Player 1 Accepts or Rejects the offer
- If the offer is Accepted, the game ends yielding payoff $\delta(x_2,y_2)$.
- Otherwise, the game ends yielding $d = (0,0)$. 
At $t = 2$,
• Accept iff $y_2 \geq 0$.
• Offer $(0,1)$.

At $t = 1$,
• Accept iff $x_2 \geq \delta$.
• Offer $(1 - \delta, \delta)$.

Timeline – $2n$ period

$T = \{1, 2, \ldots, 2n-1, 2n\}$

If $t$ is odd,
- Player 1 offers some $(x_t, y_t)$,
- Player 2 Accept or Rejects the offer
- If the offer is Accepted, the game ends yielding $\delta(x_t, y_t)$,
- Otherwise, we proceed to date $t+1$.

If $t$ is even
- Player 2 offers some $(x_t, y_t)$,
- Player 1 Accept or Rejects the offer
- If the offer is Accepted, the game ends yielding payoff $(x_t, y_t)$,
- Otherwise, we proceed to date $t+1$, except at $t = 2n$, when the game ends yielding $d = (0,0)$. 

(0,0)