Lectures 10 -11
Repeated Games

14.12 Game Theory

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2. Finitely Repeated Games with observable actions
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3. Infinitely repeated games with observable actions
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Forward Induction

**Strong belief in rationality:** At any history of the game, each agent is assumed to be rational if possible. (That is, if there are two strategies s and s’ of a player i that are consistent with a history of play, and if s is strictly dominated but s’ is not, at this history no player j believes that i plays s.)
Repeated Games

Entry deterrence

1 Enter 2 Acc.

X

(0,2) (-1,-1)

(1,1)

Fight
Entry deterrence, repeated twice, many times

What would happen if repeated n times?

Prisoners’ Dilemma, repeated twice, many times

- Two dates $T = \{0, 1\}$;
- At each date the prisoners’ dilemma is played:

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- At the beginning of 1 players observe the strategies at 0. Payoffs= sum of stage payoffs.
What would happen if $T = \{0, 1, 2, \ldots, n\}$?

**A general result**

- $G = \text{“stage game”} = \text{a finite game}$
- $T = \{0, 1, \ldots, n\}$
- At each $t$ in $T$, $G$ is played, and players remember which actions taken before $t$;
- Payoffs = Sum of payoffs in the stage game.
- Call this game $G(T)$.

**Theorem:** If $G$ has a unique subgame-perfect equilibrium $s^*$, $G(T)$ has a unique subgame-perfect equilibrium, in which $s^*$ is played at each stage.
With multiple equilibria

\[ T = \{0,1\} \]

\[
\begin{array}{c|ccc}
 & L & M2 & R \\
\hline
T & 1,1 & 5,0 & 0,0 \\
M1 & 0,5 & 4,4 & 0,0 \\
B & 0,0 & 0,0 & 3,3 \\
\end{array}
\]

Infinitely repeated Games with observable actions

- \( T = \{0,1,2,\ldots,t,\ldots\} \)
- \( G = \) “stage game” = a finite game
- At each \( t \) in \( T \), \( G \) is played, and players remember which actions taken before \( t \);
- Payoffs = Discounted sum of payoffs in the stage game.
- Call this game \( G(T) \).
Definitions

The Present Value of a given payoff stream \( \pi = (\pi_0, \pi_1, \ldots, \pi_t, \ldots) \) is

\[
PV(\pi; \delta) = \sum_{t=1}^{\infty} \delta^t \pi_t = \pi_0 + \delta \pi_1 + \ldots + \delta^t \pi_t + \ldots
\]

The Average Value of a given payoff stream \( \pi \) is

\[
(1-\delta)PV(\pi; \delta) = (1-\delta)\sum_{t=1}^{\infty} \delta^t \pi_t
\]

The Present Value of a given payoff stream \( \pi \) at \( t \) is

\[
PV_t(\pi; \delta) = \sum_{s=t}^{\infty} \delta^{s-t} \pi_s = \pi_t + \delta \pi_{t+1} + \ldots + \delta^s \pi_{t+s} + \ldots
\]

Infinite-period entry deterrence

1. Enter
2. Acc.

X

(0,2) (1,1) (-1,-1)

Strategy of Entrant:
Enter iff
Accomodated before.

Strategy of Incumbent:
Accommodate iff
accomodated before.
Incumbent:
- $V(\text{Acc.}) = V_A = \ldots$
- $V(\text{Fight}) = V_F = \ldots$
- Case 1: Accommodated before.
  - Fight $\Rightarrow$
  - Acc. $\Rightarrow$
- Case 2: Not Accommodated
  - Fight $\Rightarrow$
  - Acc. $\Rightarrow$
  - Fight $\Leftrightarrow$

Entrant:
- Accommodated
  - Enter $\Rightarrow$
  - X $\Rightarrow$
- Not Acc.
  - Enter $\Rightarrow$
  - X $\Rightarrow$

Infinitely-repeated PD

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A Grimm Strategy: Defect iff someone defected before.

- $V_D = 1/(1-\delta)$;
- $V_C = 5/(1-\delta) = 5V_D$;
- Defected before (easy)
- Not defected

- D $\Rightarrow$
- C $\Rightarrow$
- C $\Leftrightarrow$
Tit for Tat

- Start with C; thereafter, play what the other player played in the previous round.
- Is (Tit-for-tat, Tit-for-tat) a SPE?

**Modified:** Start with C; if any player plays D when the previous play is (C, C), play D in the next period, then switch back to C.

Folk Theorem

**Definition:** A payoff vector \( v = (v_1, v_2, \ldots, v_n) \) is feasible iff \( v \) is a convex combination of some pure-strategy payoff-vectors, i.e.,

\[
v = p_1u(a^1) + p_2u(a^2) + \ldots + p_ku(a^k),
\]

where \( p_1 + p_2 + \ldots + p_k = 1 \), and \( u(a^i) \) is the payoff vector at strategy profile \( a^i \) of the stage game.

**Theorem:** Let \( x = (x_1, x_2, \ldots, x_n) \) be a feasible payoff vector, and \( e = (e_1, e_2, \ldots, e_n) \) be a payoff vector at some equilibrium of the stage game such that \( x_i > e_i \) for each \( i \). Then, there exist \( \delta < 1 \) and a strategy profile \( s \) such that \( s \) yields \( x \) as the expected average-payoff vector and is a SPE whenever \( \delta > \overline{\delta} \).
Folk Theorem in PD

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- A SPE with PV (1.1,1.1)?
  - With PV (1.1,5)?
  - With PV (6.0)?
  - With PV (5.9,0.1)?

Infinitely-repeated Cournot oligopoly

- N firms, MC = 0; P = max{1-Q,0};
- Strategy: Each is to produce q = 1/(2n); if any firm defects produce q = 1/(1+n) forever.

- $V_C =$
- $V_D =$
- $V(D|C) =$
- Equilibrium $\Leftrightarrow$
IRCD (n=2)

- Strategy: Each firm is to produce $q^*$; if any one deviates, each produce $1/(n+1)$ thereafter.
- $V_C = q^*(1-2q^*)/(1-\delta)$;
- $V_D = 1/(9(1-\delta))$;
- $V_{D|C} = \max q(1-q^*-q) + \delta V_D = (1-q^*)^2 / 4 + \delta / 9(1-\delta)$.
- Equilibrium iff
  
  
  $q^*(1-2q^*) \geq (1-\delta)(1-q^*)^2 / 4 + \delta / 9$

- $q^* \geq \frac{3(9-\delta)}{9-5\delta}$
Carrot and Stick

Produce ¼ at the beginning; at ant $t > 0$, produce ¼ if both
produced ¼ or both produced x at t-1; otherwise, produce
x.

Two Phase: Cartel & Punishment

$V_C = 1/8(1-\delta)$. $V_x = x(1-2x) + \delta V_C$.

$V_{DC} = \max q(1-1/4-q) + \delta V_x = (3/8)^2 + \delta V_x$

$V_{DX} = \max q(1-x-q) + \delta V_x = (1-x)^2/4 + \delta V_x$

$V_C \geq V_{DC} \iff V_C \geq (3/8)^2 + \delta^2 V_C + \delta x(1-2x)$

$\iff (1-\delta^2) V_C - (3/8)^2 \geq \delta x(1-2x) \iff (1+\delta)/8 - (3/8)^2 \geq \delta x(1-2x)$

$V_X \geq V_{DX} \iff (1-\delta)V_x \geq (1-x)^2/4 \iff (1-\delta)(x(1-2x) + \delta/8(1-\delta)) \geq (1-x)^2/4$

$\iff (1-\delta)x(1-2x) + \delta/8 \geq (1-x)^2/4$

$$2x^2 - x + 1/8 - 9/64\delta \geq 0$$

$$(9/4-2\delta)x^2 - (3-2\delta)x + \delta/8(1-\delta) \leq 0$$