Lectures 12-13
Incomplete Information
Static Case

14.12 Game Theory

Road Map
1. Examples
2. Bayes’ rule
3. Definitions
   1. Bayesian Game
   2. Bayesian Nash Equilibrium
4. Mixed strategies, revisited
5. Economic Applications
   1. Cournot Duopoly
   2. Auctions
   3. Double Auction
Incomplete information

We have incomplete (or asymmetric) information if one player knows something (relevant) that some other player does not know.

An Example

```
Nature
  / \   / \\
High p  Low 1-p
  \ /   \ / \\
Firm   Work
     /   / \\
     W   Shirk
   /   / \\
Hire  Shirk
  /   / \\
Do not hire  W
     /   / \\
     Shirk  Work
   /   / \\
Do not hire  (-1, 2)
```

(1, 2)  (0, 1)  (0, 0)  (1, 1)  (-1, 2)  (0, 0)
The same example

Another Example

What would you ask if you were to choose \( p \) from \([0,4]\)?
Same “Another Example”

Bayes’ Rule

\[
\text{Prob}(A \text{ and } B) = \frac{\text{Prob}(A|B) \text{Prob}(B)}{\text{Prob}(B)}
\]

\[
\text{Prob}(A|B) = \frac{\text{Prob}(B|A) \text{Prob}(A)}{\text{Prob}(B)}
\]

What would you ask if you were to choose \( p \) from \([0,4]\)?
Example

- \( \text{Prob(Work|Success)} = \frac{\mu p}{\mu p + (1-\mu)(1-p)} \)
- \( \text{Prob(Work|Failure)} = \frac{(1-\mu)p}{\mu(1-p) + (1-\mu)p} \)
Bayesian Game (Normal Form)

A Bayesian game is a list
\[ G = \{A_1, \ldots, A_n; T_1, \ldots, T_n; p_1, \ldots, p_n; u_1, \ldots, u_n\} \]
where
- \( A_i \) is the action space of \( i \) (\( a_i \) in \( A_i \))
- \( T_i \) is the type space of \( i \) (\( t_i \))
- \( p_i(t_i|t_i) \) is \( i \)'s belief about the other players
- \( u_i(a_1, \ldots, a_n; t_1, \ldots, t_n) \) is \( i \)'s payoff.

An Example

<table>
<thead>
<tr>
<th>Nature</th>
<th>Firm</th>
<th>Hire</th>
<th>Work</th>
<th>Shirk</th>
</tr>
</thead>
<tbody>
<tr>
<td>High p</td>
<td>W</td>
<td>(1, 2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low 1-p</td>
<td>W</td>
<td>(1, 1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Do not hire</td>
<td>(0, 0)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( T_{\text{Firm}} = \{t_f\} \);
\( T_W = \{\text{High,Low}\} \);
\( A_{\text{Firm}} = \{\text{Hire, Don’t}\} \);
\( A_W = \{\text{Work,Shirk}\} \);
\( p_F(\text{High}) = p \);
\( p_F(\text{Low}) = 1-p \)
Bayesian Nash equilibrium

A Bayesian Nash equilibrium is a Nash equilibrium of a Bayesian game.

Given any Bayesian game \( G = \{A_1, \ldots, A_n; T_1, \ldots, T_n; p_1, \ldots, p_n; u_1, \ldots, u_n\} \)

a strategy of a player \( i \) in \( a \) is any function \( s_i : T_i \to A_i; \)

A strategy profile \( s^* = (s_1^*, \ldots, s_n^*) \) is a Bayesian Nash equilibrium iff \( s_i^*(t_i) \) solves

\[
\max_{a_i \in A_i} \sum_{t_i \in T_i \cap a_i} u_i(s_i^*(t_i), \ldots, s_{i-1}^*(t_{i-1}), a_i, s_i^*(t_i), \ldots, s_n^*(t_n), t_i)p_i(t_i | t_i)
\]

i.e., \( s_i^* \) is a best response to \( s_{-i}^* \).

An Example

\[
\begin{array}{c|cc}
\text{Nature} & \text{High} & \text{Low} \\
\hline
\text{Firm} & \text{Hire} & \text{Don’t} \\
\text{Shirk} & \text{W} & \text{Shirk} \\
\text{Do not} & \text{W} & \text{Shirk} \\
\text{hire} & \text{W} & \text{Shirk} \\
\end{array}
\]

\[
\begin{array}{c|cc}
\text{Firm} & \text{High} & \text{Low} \\
\hline
\text{Hire} & (1, 1) & (0, 0) \\
\text{Don’t} & (0, 1) & (1, 2) \\
\text{Shirk} & (-1, 2) & (0, 0) \\
\end{array}
\]

\( T_{\text{Firm}} = \{t_f\}; \)
\( T_W = \{\text{High, Low}\} \)
\( A_{\text{Firm}} = \{\text{Hire, Don’t}\} \)
\( A_{W} = \{\text{Work, Shirk}\} \)
\( p_F(\text{High}) = p > 1/2 \)
\( p_F(\text{Low}) = 1-p \)

\( s_F^* = \text{Hire}, \)
\( s_F^* (\text{High}) = \text{Work} \)
\( s_F^* (\text{Low}) = \text{Shirk} \)

Another equilibrium?
Stag Hunt, Mixed Strategy

- (2,2) | (4,0) 
- (0,4) | (6,6) 

Mixed Strategies

- $t$ and $v$ are iid with uniform distribution on $[-\varepsilon, \varepsilon]$.
- $t$ and $v$ are privately known by 1 and 2, respectively, i.e., are types of 1 and 2, respectively.

- Pure strategy:
  - $s_1(t) = \text{Rabbit}$ iff $t > 0$;
  - $s_2(v) = \text{Rabbit}$ iff $t > 0$.

- $p = \text{Prob}(s_1(t)=\text{Rabbit} | v) = \text{Prob}(t > 0) = 1/2$.
- $q = \text{Prob}(s_2(v)=\text{Rabbit} | t) = 1/2$.

- $U_1(R|t) = t + 2q + 4(1-q) = t + 4 - 2q$ 
  $U_1(S|t) = 6(1-q)$ 
  $U_1(R|t) > U_1(S|t) \Leftrightarrow t + 4 - 2q > 6(1-q)$ 
  $\Leftrightarrow t > 6 - 6q + 2q - 4 = 2 - 4q = 0$. 

- $U_1(R|t) = 6,6$ 
  $U_1(S|t) = 0,4 + v$ 
  $U_1(R|t) = 2 + t,2+v$ 
  $U_1(S|t) = 4 + t,0$ 
  $U_1(R|t) > U_1(S|t)$