18.06 Problem Set #2 Solutions

1) A matrix multiplication AB is allowed if the number of columns of A is the same as the same as the number of rows of B. If A is m by n, B is n by k, then AB is m by k. A matrix summation $A+B$ is allowed iff A, B has the same shape. So we know: $BA, ABD, ABABD$ are allowed and has shapes 5 by 5, 3 by 1, 3 by 1 respectively.

2) Assume the inverse of $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ is $\begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$. Then

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} = \begin{bmatrix} AX+BZ & AY+BW \\ DZ & DW \end{bmatrix}$$

So we have matrix equations:

$$AX + BZ = I, AY + BW = 0, DZ = 0, DW = I.$$

Solve to get $W = D^{-1}, Z = 0, X = A^{-1}, Y = -A^{-1}BD^{-1}$. So

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{bmatrix}.$$

Answers to other two matrices are

$$\begin{bmatrix} I & 0 \\ C & -I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ C & -I \end{bmatrix}, \begin{bmatrix} 0 & I \\ -I & D \end{bmatrix}^{-1} = \begin{bmatrix} D & -I \\ I & 0 \end{bmatrix}$$

3)

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, U = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

The condition that A has 4 pivots is $a \neq b \neq c \neq d$.

4) To make A upper triangular via permutation of rows, the second row and the third row need to be swapped. so $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. To make A lower triangular via permutation of rows and columns, one need to move the second row to the first row, the third row to the second row, the first row to the third row; swap the first column and the last column. So

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

5) (a) assume $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 1 \\ -3 & -9 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & 3 & 3 \\ x & -1 & -1 \\ y & z & -1 \end{bmatrix}$, then because

$$I = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 1 \\ -3 & -9 & -1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 3 \\ x & -1 & -1 \\ y & z & -1 \end{bmatrix} = \begin{bmatrix} 4+3x & 0 & 0 \\ 8+5x+y & 1+z & 0 \\ -12-9x-y & -z & 1 \end{bmatrix}$$

Solve to have $x = -1, y = -3, z = 0$.

(b) Use $(A^T)^{-1} = (A^{-1})^T$, we have $\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & -9 \\ 0 & 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & -1 & -3 \\ 3 & -1 & 0 \\ 3 & -1 & -1 \end{bmatrix}$