18.06 Problem Set 5 Solutions

Problem 1. If a parabola fit the data exactly we would get a solution (v_1, v_2, v_3) to the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 7 \\ 1 \end{bmatrix}.$$

Since this system is over determined, we seek the vector $\hat{\mathbf{x}} = (B, C, D)$ which best approximates a solution to this system in the sense of least squares. Then the least squares parabola will be $y = B + Cx + Dx^2$. We know that the vector $\hat{\mathbf{x}}$ satisfies the normal equation

$$A^T A\hat{\mathbf{x}} = A^T b.$$

Here, A denotes the 4×3 matrix above, and $\mathbf{b} = (2, 5, 7, 1)$. We have

$$A^T A = \begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix}; \quad A^T\mathbf{b} = \begin{bmatrix} 15 \\ 37 \\ 101 \end{bmatrix}$$

Solving the system gives $\hat{\mathbf{x}} = (\frac{-29}{4}, \frac{223}{20}, \frac{-9}{4})$.

Problem 2. (a) The line for the European data is $y = C + Dx$ where the vector $\hat{\mathbf{x}} = (C, D)$ satisfies $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$ with

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 5 \\ 1 & 10 \\ 1 & 15 \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} 90 \\ 197 \\ 335 \\ 394 \end{bmatrix}$$

crunching the numbers gives $\hat{\mathbf{x}} = (\frac{193}{2}, 21)$.

For the North American data we use the same matrix A with $\mathbf{b} = (317, 474, 816, 1101)$ which gives $\hat{\mathbf{x}} = (\frac{2729}{10}, \frac{1347}{25})$.

(b) Using the line $y = \frac{193}{2} + 21x$, we plug in $x = 30$ corresponding to the year 2000 and get $y = 726.5$ as the estimated expenditures.

(c) Certainly one expects the difference in expenditures to increase dramatically since the slope of the North American line is more than twice that of the European line.

Problem 3. To show that S is linearly independent, suppose there are constants $c_1, \dots c_n$ such that

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots c_n\mathbf{v_n} = 0.$$

Now, since S is an orthogonal set, we know that $\mathbf{v_i} \cdot \mathbf{v_j} = 0$ for $i \neq j$ and $\mathbf{v_i} \cdot \mathbf{v_i} \neq 0$ $(1 \leq i, j \leq n)$. Hence for each i between 1 and n we get

$$\mathbf{v_i} \cdot (c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots c_n\mathbf{v_n}) = \mathbf{v_i} \cdot \mathbf{0} = 0.$$

Hence

$$c_1(\mathbf{v_1} \cdot \mathbf{v_i}) + \cdots + c_n(\mathbf{v_n} \cdot \mathbf{v_i}) = 0,$$

with all the terms on the left side of this equation zero except the $i-th$ term. Thus the equation reduces to $c_i(\mathbf{v_i}\cdot\mathbf{v_i})=0$. Since $\mathbf{v_i}\cdot\mathbf{v_i}\neq 0$, we must have $c_i=0$. Hence we see that if

$$c_1\mathbf{v_1}+c_2\mathbf{v_2}+\cdots c_n\mathbf{v_n}=0$$

Then $c_1=c_2=\cdots=c_n=0$, and the set S is linearly independent.

Problem 4.(a) Let $\mathbf{v_1},\ldots,\mathbf{v_n}$ denote the columns of B, and write $AB=[Av_1\ Av_2\ \cdots\ Av_n]$. We need to show that the set $S=\{A\mathbf{v_1},A\mathbf{v_2},\ldots,A\mathbf{v_n}\}$ is orthonormal. To do this, consider $A\mathbf{v_i}\cdot A\mathbf{v_j}$ with $1\leq i,j\leq n$. We have

$$A\mathbf{v_i}\cdot A\mathbf{v_j}=(A\mathbf{v_i})^TA\mathbf{v_j}=\mathbf{v_i}^TA^TA\mathbf{v_j}.$$

Now, since A is orthogonal, $A^TA=I$, and $\mathbf{v_i}^TA^TA\mathbf{v_j}=\mathbf{v_i}^T\mathbf{v_j}$. Moreover, since B is orthogonal and the vectors $\mathbf{v_1},\ldots,\mathbf{v_n}$ are its columns, we have $\mathbf{v_i}^T\mathbf{v_j}=0$ if $i\neq j$ and $\mathbf{v_i}^T\mathbf{v_j}=1$ if $i=j$. Hence S is an orthonormal set. Since S consists of the columns of the matrix AB, this matrix is orthogonal.

Alternative Proof: A square matrix C is orthogonal if and only if $C^TC=I$ (Problem set 4). Now, $(AB)^T(AB)=B^TA^TAB$. Since A and B are orthogonal, $A^TA=I$ and $B^TB=I$. Thus $(AB)^TAB=B^TA^TAB=B^TIB=B^TB=I$, and AB is orthogonal.

(b) Since $det(A)=det(A^T)$ and $AA^T=I$, we have $det(AA^T)=det(A)det(A^T)=det(A)^2=1$. Hence $det(A)$ is either 1 or -1.

Problem 6. Using row operations we have:

$$\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}=\begin{vmatrix}1 & a & a^2\\ 0 & b-a & (b^2-a^2)\\ 0 & c-a & (c^2-a^2)\end{vmatrix}=\begin{vmatrix}1 & a & a^2\\ 0 & b-a & (b^2-a^2)\\ 0 & 0 & (c^2-a^2)-\frac{(c-a)}{(b-a)}\end{vmatrix}=\begin{vmatrix}1 & a & a^2\\ 0 & b-a & (b^2-a^2)\\ 0 & 0 & (c-a)(c-b)\end{vmatrix}$$
$$=(b-a)(c-a)(c-b).$$