18.06 Problem Set #4 Solutions

1. $C(A^T)$: Denote the matrices in the problem as $A = LU$. The row space of A is the same as the row space of its row echelon matrix U. So a basis for $C(A^T)$ is $\{(0,1,2,3,4),(0,0,0,1,-1)\}$.

   $N(A)$: Since L is invertible, the null-space of A is equal to the nullspace of U. Look at U, columns 1, 3, 5 give a free variable each. So a basis for $N(A)$ is $\{(1,0,0,0,0),(0,-2,1,0,0),(0,-7,0,1,1)\}$.

   $C(A)$: Each column of A is a linear combination of column vectors of L with coefficients given by the corresponding column of U. For example, the fourth column of A is 3 times first column of L plus 1 times second column of L. But the third row of U is a zero row. When expressing column vectors of A as linear combinations of column vectors of L, the coefficient of the third column vector of L is always zero. Hence, it is enough to use the first two columns of L to express any column of A. Also considering the rank of A is two, a basis of $C(A)$ would be first two column vectors of L. That is $\{(1,-4,8),(0,1,3)\}$.

   $N(A^T)$: $N(A^T)$ is the orthogonal complement of $C(A)$, so a basis of $N(A^T)$ is $\{(-20,-3,1)\}$.

2. (a) Denote the i-th column vector of A as v_i, then the entry of A^TA in the i-th row, j-th column is the inner product of v_i and v_j. $v_i \cdot v_i = |v_i|^2 = 1$. $v_i \cdot v_j = 0$ if $i \neq j$ because v_i and v_j are orthogonal. Hence $A^TA = I$ the identity matrix.

   (b) Just check the conditions.

   (c) For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an orthogonal matrix, but two times it, $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is not orthogonal because the first column vector $(2,0)$ has norm 2 so not a unit vector.

   (d) $A^T = A^{-1} \Leftrightarrow A^TA = I$ and the argument in part (a) is reversible.

3. $S^\perp$: Put those vectors spanning S as row vectors of a matrix A:

$$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 4 & -1 \\ 2 & 2 & 0 & -1 \end{bmatrix}$$

   Now use the row echelon form to find $S^\perp = C(A^T)^\perp = N(A)$ as the span of $\{(2,-2,1,0),(0,1/2,0,1)\}$.

   $(S^\perp)^\perp$: $(S^\perp)^\perp =$ null-space of $\begin{bmatrix} 2 & -2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$, is the span of $\{(-1/2,0,1,0),(-2,-2,0,1)\}$.

   $(S^\perp)^\perp = S$: It is a simple calculation to express the spanning vectors from S as linear combinations of basis vectors of $(S^\perp)^\perp$, or vice versa. So the two vector spaces are the same. $(S^\perp)^\perp = S$ is generally true for any vector space S.

4. $U^\perp$: $U^\perp$ is the null-space of the matrix with row vectors the spanning vectors of U, i.e. $\begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 0 & -1 \end{bmatrix}$.

   A basis for $U^\perp$ is $\{(0,0,1,0),(-1,1,0,1)\}$.

   $V^\perp$: $V^\perp$ is the null-space of the matrix with row vectors the spanning vectors of V, i.e. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

   A basis for $V^\perp$ is $\{(0,0,0,1)\}$.

   $U \cap V$: U is the nullspace of the matrix whose rows are the basis vectors of $U^\perp$ from above. It follows that any vector (x_1,x_2,x_3,x_4) in U will satisfy $x_3 = 0$, and $-x_1 + x_2 + x_4 = 0$. Similarly, a vector (x_1,x_2,x_3,x_4) in V has $x_4 = 0$. For a vector to be in both U and V, it must satisfy all these equations, that is, it must be in the nullspace of

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The usual computation shows that $N(A)$ is span of $\{(1,1,0,0)\}$.

Alternatively: Suppose v is in $U \cap V$. Since v is in U we have $v = a(1,-2,0,3) + b(0,1,0,1) = (a, -2a+b, 0, 3a-d)$ for constants a, b. But v is in V so we must the last component equal to zero, that is, $3a - d = 0$ so $3a = d$. Hence, $v = (c, -2c+3c, 0, 0) = c(1,1,0,0)$, so $U \cap V$ is a subset of $span\{(1,1,0,0)\}$. But any vector in the span of $(1,1,0,0)$ is in $U \cap V$, Hence $U \cap V = span\{(1,1,0,0)\}$.

5. (a) Just follow the procedures, sorry for the poor numbers provided by Wei :).

$$A^Tb = \begin{bmatrix} -4 \\ 3 \end{bmatrix}, A^TA = \begin{bmatrix} 6 & 1 \\ 1 & 10 \end{bmatrix}, \hat{x} = \begin{bmatrix} -\frac{43}{59} \\ \frac{22}{59} \end{bmatrix} .p = \begin{bmatrix} -\frac{43}{59} \\ \frac{108}{59} \\ \frac{23}{59} \end{bmatrix}$$

(b)

$$A^Tb = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, A^TA = \begin{bmatrix} 5 & -5 \\ -5 & 15 \end{bmatrix}, \hat{x} = \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} .p = \begin{bmatrix} \frac{2}{5} \\ \frac{2}{5} \\ \frac{1}{5}\frac{1}{5} \end{bmatrix}$$

6. P multiplies a vector in $\mathbb{R}^m$ to give another vector in $\mathbb{R}^m$. We know P has to be an $m \times m$ matrix. For all $v \in \mathbb{R}^m$, $Pv \in C(A^T)$, so $C(P^T)$ is contained in $C(A^T)$. On the other hand, $Pv = v$ for all $v \in C(A^T)$. So $C(P^T)$ contains all vectors in $C(A^T)$. Thus $C(P^T) = C(A^T)$. So $r(P) = r(A) = n$.