18.06 Problem Set #3 Solutions

1. The set in (a) can't be a basis because two vectors can span an at most 2 dimensional vector space, while $\mathbb{R}^3$ is 3 dimensional. The sets in (b) and (d) can't be bases since these sets cannot be linearly independent. (If the vectors are the columns of a matrix, then the matrix is 3×4, so has a non trivial null space, and the vectors have a non-trivial dependence relation.) For (c), we put the vectors in a matrix and row reduce.

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 3 & -1 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the matrix is invertible, the columns of the original matrix form a basis. (A has full column rank so the columns span $\mathbb{R}^3$, and A has a trivial nullspace so the columns are linearly independent.)

2. We show that the set is a basis by showing that it spans M_2 and is linearly independent. Denote the vectors in this set m_1, m_2, m_3, m_4.

Spanning: Let $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ be any vector in M_2, and we look for constants a, b, c, d so that

$$am_1 + bm_2 + cm_3 + dm_4 = \begin{bmatrix} x & y \\ z & w \end{bmatrix}.$$

This gives

$$\begin{bmatrix} a+c & a+d \\ b+d & b+c+d \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}.$$

This gives rise to a system of four equations in the four unknowns a, b, c, d. We get the augmented matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 & x \\ 1 & 0 & 0 & 1 & y \\ 0 & 1 & 0 & 1 & z \\ 0 & 1 & 1 & 1 & w \end{bmatrix}.$$

Row reducing, we get the unique solution

$$a = x - w + z,\ b = 2z - y + x - w,\ c = w - z,\ d = y - x + w - z.$$

We conclude that the set above spans M_2.

Linear Indepencence: Suppose $am_1 + bm_2 + cm_3 + dm_4 = 0$, where $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. We get the same system of equations as above with x, y, z, w all zero. But the coefficient matrix has full rank, so the nullspace is the zero vector. Hence $am_1 + bm_2 + cm_3 + dm_4 = 0$ only when $a = b = c = d = 0$. We conclude that the set is linearly independent.

3. (a) We have $\{(a, b, c, a+b)\} = span\{(1,0,0,1),(0,1,0,0),(0,0,1,0),(0,1,0,1)\}$. Since the last vector in this set is a linear combination of the other three, we can remove it. The resulting set is a basis, so the dimension is 3.

   (b) The set $\{(a, b, a-b, a+b)\}$ is the span of the linearly independent set $\{(1,0,1,1),(0,1,-1,1)\}$. Hence the dimension is 2.

   (c) The set $\{(a, a, c, d)\}$ is the span of the linearly independent set $\{(1,1,0,0),(0,0,1,0),(0,0,0,1)\}$. Hence the

   dimension is 3.

(d) The set $\{(a+c, a-b, b+c, -a+b)\}$ is the span of the linearly independent set $\{(0,-1,1,1),(1,0,1,0)\}$. Hence the dimension is 2.

4. Row reducing A gives

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \to \cdots \to \begin{bmatrix} 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank is 2. A basis for the row space consists of the non-zero rows of the echelon matrix, so $\mathcal{B}_{row} = \{(0,1,2,0,-2),(0,0,0,1,2)\}$. The nullspace has dimension $5-2=3$. Assigning the value one or zero to the free variables x_1, x_3, x_5 gives our basis $\mathcal{B}_{null} = \{(1,0,0,0,0),(0,-2,1,0,0),(0,2,0,-2,1)\}$. A basis for the column space (which has dimension 2) is given by the columns of the original matrix that correspond to pivot columns in the echelon matrix. Hence $\mathcal{B}_{col} = \{(1,1,0),(3,4,1)\}$. Finally, for the left nullspace, $N(A^T)$, we can use the fact that $N(A^T)$ is the orthogonal complement of the column space in $\mathbb{R}^3$. It's easy to see that the (one dimensional) space of vectors orthogonal to the column space of A has basis $\mathcal{B}_{N(A^T)} = \{(1,-1,1)\}$.

5. We show that this set spans $\mathbb{R}^n$ and is linearly independent.Since A is invertible, given any vector $\mathbf{b}$ in $\mathbb{R}^n$, there is exactly one vector $\mathbf{x}$ in $\mathbb{R}^n$ with $A\mathbf{x} = \mathbf{b}$. But $\mathbf{x}$ can be written as a linear combination of the basis vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$. That is, $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ for some constants $c_1, \ldots, c_n$. But then we have

$$\mathbf{b} = A\mathbf{x} = A(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = c_1 A\mathbf{v}_1 + \cdots + c_n A\mathbf{v}_n.$$

Hence, we have written $\mathbf{b}$ as a linear combination of the vectors $A\mathbf{v}_1, \ldots, A\mathbf{v}_n$. Since $\mathbf{b}$ was an arbitrary vector in $\mathbb{R}^n$, The set $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_n\}$ spans $\mathbb{R}^n$.

Now, suppose that there are constants $d_1, \ldots, d_n$ so that $d_1 A\mathbf{v}_1 + \cdots + d_n A\mathbf{v}_n = 0$. Then

$$0 = d_1 A\mathbf{v}_1 + \cdots + d_n A\mathbf{v}_n = A(d_1\mathbf{v}_1 + \cdots + d_n\mathbf{v}_n).$$

Since A is invertible, it has trivial nullspace, and we conclude that $d_1\mathbf{v}_1 + \cdots + d_n\mathbf{v}_n = 0$. But the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent, so we must have $d_1 = d_2 = \cdots = d_n = 0$. But that says that if $d_1 A\mathbf{v}_1 + \cdots + d_n A\mathbf{v}_n = 0$ then all the constants are zero. That is, the set $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_n\}$ is linearly independent. Since this set is a linearly independent spanning set in $\mathbb{R}^n$, it is a basis.