18.S34 (FALL, 2002)

PROBLEMS ON PROBABILITY

1. Three closed boxes lie on a table. One box (you don't know which) contains a \$1000 bill. The others are empty. After paying an entry fee, you play the following game with the owner of the boxes: you point to a box but do not open it; the owner then opens one of the two remaining boxes and shows you that it is empty; you may now open either the box you first pointed to or else the other unopened box, but not both. If you find the \$1000, you get to keep it. Does it make any difference which box you choose? What is a fair entry fee for this game?

2. You are dealt two cards face down from a shuffled deck of 8 cards consisting of the four queens and four kings from a standard bridge deck. The dealer looks at both of your two cards (without showing them to you) and tells you (truthfully) that at least one card is a queen. What is the probability that you have been given two queens? What is this probability if the dealer tells you instead that at least one card is a red queen? What is this probability if the dealer tells you instead that at least one card (or exactly one card) is the queen of hearts?

3. An unfair coin (probability p of showing heads) is tossed n times. What is the probability that the number of heads will be even?

4. Two persons agreed to meet in a definite place between noon and one o'clock. If either person arrives while the other is not present, he or she will wait for up to 15 minutes. Calculate the probability that the meeting will occur, assuming that the arrival times are independent and uniformly distributed between noon and one o'clock.

5. (58P) Real numbers are chosen at random from the interval $[0, 1]$. If after choosing the nth number the sum of the numbers so chosen first exceeds 1, show that the expected or average value for n is e.

6. (61P) Let α and β be given positive real numbers with $\alpha < \beta$. If two points are selected at random from a straight line segment of length β, what is the probability that the distance between them is at least α?

7. (93P) Two real numbers x and y are chosen at random in the interval $(0, 1)$ with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

8. (92P) Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

9. (89P) Let $(x_1, x_2, \ldots, x_n)$ be a point chosen at random from the n-dimensional region defined by $0 < x_1 < x_2 < \cdots < x_n < 1$. Let f be a continuous function on $[0, 1]$ with $f(1) = 0$. Set $x_0 = 0$ and $x_{n+1} = 1$. Show that the expected value of the Riemann sum

$$\sum_{i=0}^{n} (x_{i+1} - x_i) f(x_{i+1})$$

is $\int_0^1 f(t)\, P(t)\, dt$, where P is a polynomial of degree n, independent of f, with $0 \leq P(t) \leq 1$ for $0 \leq t \leq 1$.

10. (89P) A dart, thrown at random, hits a square target. Assuming any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $(a\sqrt{b} + c)/d$, where a, b, c, d are integers.

11. (89P) If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both 1/2. A game is finite if, with probability 1, it must end in a finite number of moves.)

12. (85P) Let C be the unit circle $x^2 + y^2 = 1$. A point p is chosen randomly on the circumference C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x- and y-axes with diagonal pq. What is the probability that no point of R lies outside of C?

13. (82P) Let p_n be the probability that $c + d$ is a perfect square when the integers c and d are selected independently at random from the set $\{1, 2, \ldots, n\}$. Show that $\lim_{n\to\infty}(p_n\sqrt{n})$ exists, and express this limit in the form $r(\sqrt{s} - t)$ where s and t are integers and r is a rational number.

14. (68P) The temperatures in Chicago and Detroit are x° and y°, respectively. These temperatures are not assumed to be independent; namely, we are given:

    (i) $P(x^\circ = 70^\circ)$, the probability that the temperature in Chicago is 70°,

    (ii) $P(y^\circ = 70^\circ)$, and

    (iii) $P(\max(x^\circ, y^\circ) = 70^\circ)$.

    Determine $P(\min(x^\circ, y^\circ) = 70^\circ)$.

15. In the Massachusetts MEGABUCKS lottery, six distinct integers from 1 to 36 are selected each week. Great care is exercised to insure that the selection is completely random. If N_{max} denotes the largest of the six numbers, find the expected value for N_{max}.

16. (a) A fair die is tossed repeatedly. Let p_n be the probability that after some number of tosses the sum of the numbers that have appeared is n. (For instance, $p_1 = 1/6$ and $p_2 = 7/36$.) Find $\lim_{n\to\infty} p_n$.

    (b) More generally, suppose that a "die" has infinitely many faces marked $1, 2, \ldots$. When the die is thrown, the probability is a_i that face i appears (so $\sum_{i=1}^{\infty} a_i = 1$). Let p_n be as in (a), and find $\lim_{n\to\infty} p_n$.