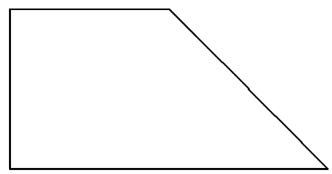
18.S34 PROBLEMS #10

Fall 2002

100. [1] The following shape consists of a square and half of another square of the same size, divided diagonally.



Cut the shape into four congruent pieces.

101. [1.5] Take five right triangles with legs of length one and two, cut one of them, and put the resulting six pieces together to form a square.

102. [1.5] Find an integer n whose first digit is three, such that $3n/2$ is obtained by removing the 3 at the beginning and putting it at the end.

103. (a) [1] Show that for any real x, $e^x > x$.

  (b) [1.5] Find the largest real number α for which it is *false* that $\alpha^x > x$ for all real x.

104. (a) [1] What amounts of postage cannot be obtained using only 5 cent stamps and 7 cent stamps? (For instance, 9 cents cannot be obtained, but $17 = 5 + 5 + 7$ cents can be.)

  (b) [2.5] Let a and b be relatively prime positive integers. For how many positive integers c is it impossible to obtain postage of c cents using only a cent and b cent stamps? What is the largest value of c with this property?

105. [2.5] Two players A and B play the following game. Fix a positive real number x. A and B each choose the number 1 or 2. A gives B one dollar if the numbers are different. B gives A x dollars times the sum of their numbers. For instance, if A chooses 1 and B chooses 2, then A gives B one dollar and B gives A $3x$ dollars. Both players are playing their best possible strategy. What value of x makes the game fair, i.e., in the long run both players should break even?

106. [3] Given positive integers n and b, define the *total b-ary expansion* $T_b(n)$ as follows: Write n as a sum of powers of b, with no power occurring more than $b-1$ times. (This is just the usual base b expansion of n.) For instance, if $n = 357948$ and $b = 3$, then we get

$$3^{11} + 3^{11} + 3^7 + 3^6 + 3^6 + 3^2.$$

Now do the same for each exponent, giving

$$3^{3^2+1+1} + 3^{3^2+1+1} + 3^{3+3+1} + 3^{3+3} + 3^{3+3} + 3^{1+1}.$$

Continue doing the same for every exponent not already a b or 1, until finally only b's and 1's appear. In the present case we get that $T_3(357948)$ is the array

$$3^{3^{1+1}+1+1} + 3^{3^{1+1}+1+1} + 3^{3+3+1} + 3^{3+3} + 3^{3+3} + 3^{1+1}.$$

Now define a sequence $a_0, a_1, \ldots$ as follows. Choose a_0 to be any positive integer, and choose a base $b_0 > 1$. To get a_1, write the total b_0-ary expansion $T_{b_0}(a_0 - 1)$ of $a_0 - 1$, choose a base $b_1 > 1$, and replace every appearance of b_0 in $T_{b_0}(a_0 - 1)$ by b_1. This gives the total b_1-ary expansion of the next term a_1. To get a_2, write the total b_1-ary expansion $T_{b_1}(a_1 - 1)$ of $a_1 - 1$, choose a base $b_2 > 1$, and replace every appearance of b_1 in $T_{b_1}(a_1 - 1)$ by b_2. This gives the total b_2-ary expansion of the next term a_2. Continue in this way to obtain $a_3, a_4, \ldots$. In other words, given a_n and the previously chosen base b_n, To get a_{n+1}, write the total b_n-ary expansion $T_{b_n}(a_n - 1)$ of $a_n - 1$, choose a base $b_{n+1} > 1$, and replace every appearance of b_n in $T_{b_n}(a_n - 1)$ by b_{n+1}. This gives the total b_{n+1}-ary expansion of the next term a_{n+1}.

Example. Choose $a_0 = 357948$ and $b_0 = 3$ as above. Then

$$a_0 - 1 = 357947 = 3^{3^{1+1}+1+1} + 3^{3^{1+1}+1+1} + 3^{3+3+1} + 3^{3+3} + 3^{3+3} + 3 + 3 + 1 + 1.$$

Choose $b_1 = 10$. Then

$$\begin{aligned} a_1 &= 10^{10^{1+1}+1+1} + 10^{10^{1+1}+1+1} + 10^{10+10+1} + 10^{10+10} + 10^{10+10} + 10 + 10 + 1 + 1 \\ &= 10^{102} + 10^{102} + 10^{21} + 10^{20} + 10^{20} + 22. \end{aligned}$$

Then

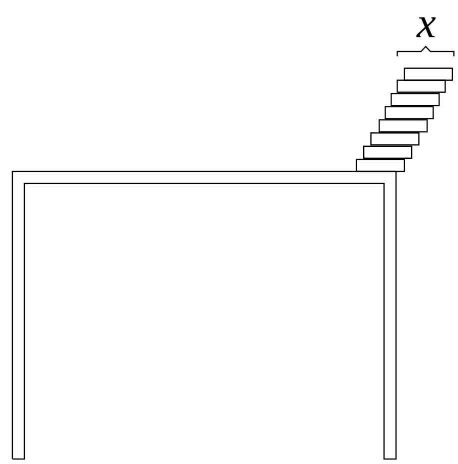
$$a_1 - 1 = 10^{10^{1+1}+1+1} + 10^{10^{1+1}+1+1} + 10^{10+10+1} + 10^{10+10} + 10^{10+10} + 10 + 10 + 1.$$

Choose $b_2 = 766$. Then

$$a_2 = 766^{766^{1+1}+1+1} + 766^{766^{1+1}+1+1} + 766^{766+766+1} + 766^{766+766} + 766^{766+766} + 766 + 766 + 1,$$

etc. Prove that for some n we have $a_n = a_{n+1} = \cdots = 0$. (Note how counterintuitive this seems. How could we not force $a_n \to \infty$ by choosing the b_n's sufficiently large?)

107. [2.5] What is the longest possible overhang x that can be obtained by stacking dominos of unit length over the edge of a table, as illustrated below? (The condition for the dominos not to fall is that the center of mass of all the dominos above any domino D lies directly above D.)



108. [3] Let G be a simple (i.e., no loops or multiple edges) finite graph and v a vertex of G. The *neighborhood* $N(v)$ of v consists of v and all adjacent vertices. Show that there exists a subset S of the vertex set of G such that $\#(S \cap N(v))$ is odd for all vertices v of G.