18.S34 (FALL 2002)

PROBLEMS ON ROOTS OF POLYNOMIALS

NOTE. The terms "root" and "zero" of a polynomial are synonyms. The problems are stated as they appeared on the Putnam Exam verbatim (except for one minor correction).

1. (39P) Find the cubic equation whose roots are the cubes of the roots of

$$x^3 + ax^2 + bx + c = 0.$$

2. (a) (40P) Determine all rational values for which a, b, c are the roots of

$$x^3 + ax^2 + bx + c = 0.$$

   (b) (not on Putnam Exam) Show that the only real polynomials $\prod_{i=0}^{n-1}(x-a_i) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ in addition to those given by (a) are x^n, x^2+x-2, and exactly two others, which are approximately equal to

$$x^3 + .56519772x^2 - 1.76929234x + .63889690$$

   and

$$x^4 + x^3 - 1.7548782x^2 - .5698401x + .3247183.$$

3. (51P) Assuming that all the roots of the cubic equation $x^3 + ax^2 + bx + c$ are real, show that the difference between the greatest and the least roots is not less than $\sqrt{a^2 - 3b}$ nor greater than $2\sqrt{(a^2-3b)/3}$.

4. (56P) The nonconstant polynomials $P(z)$ and $Q(z)$ with complex coefficients have the same set of numbers for their zeros but possibly different multiplicities. The same is true of the polynomials $P(z)+1$ and $Q(z)+1$. Prove that $P(z) = Q(z)$. (On the original Exam, the assumption that $P(z)$ and $Q(z)$ are nonconstant was inadvertently omitted.)

5. (58P) If $a_0, a_1, \ldots, a_n$ are real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0,$$

   show that the equation $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ has at least one real root.

6. (68P) Determine all polynomials of the form

$$\sum_0^n a_i x^{n-i} \quad \text{with } a_i = \pm 1$$

$(0 \leq i \leq n,\ 1 \leq n < \infty)$ such that each has only real zeros.

7. (81P) Let $P(x)$ be a polynomial with real coefficients and form the polynomial

$$Q(x) = (x^2+1)P(x)P'(x) + x(P(x)^2 + P'(x)^2).$$

Given that the equation $P(x) = 0$ has n distinct real roots exceeding 1, prove or disprove that the equation $Q(x) = 0$ has at least $2n-1$ distinct real roots.

8. (91P) Find all real polynomials $p(x)$ of degree $n \geq 2$ for which there exist real number $r_1 < r_2 < \cdots < r_n$ such that

(i) $p(r_i) = 0, \quad i = 1, 2, \ldots, n,$

and

(ii) $p'\left(\frac{r_i + r_{i+1}}{2}\right) = 0, \quad i = 1, 2, \ldots, n-1,$

where $p'(x)$ denotes the derivative of $p(x)$.

9. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a monic polynomial of degree n with complex coefficients a_i. Suppose that the roots of $P(x)$ are $x_1, x_2, \cdots, x_n$, i.e., we have $P(x) = (x-x_1)(x-x_2)\cdots(x-x_n)$. The *discriminant* $\Delta(P(x))$ is defined by

$$\Delta(P(x)) = \prod_{1 \leq i < j \leq n} (x_i - x_j)^2.$$

Show that

$$\Delta(x^n + ax + b) = (-1)^{\binom{n}{2}} \left(n^n b^{n-1} + (-1)^{n-1}(n-1)^{n-1}a^n\right).$$

HINT. First note that

$$P'(x) = P(x)\left(\frac{1}{x-x_1} + \cdots + \frac{1}{x-x_n}\right).$$

Use this formula to establish a connection between $\Delta(P(x))$ and the values $P'(x_i)$, $1 \leq i \leq n$.

10. (a) (relatively easy) Let k be the smallest positive integer with the following property:

    There are distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial $p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$ has exactly k nonzero coefficients.

    Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

    (b) (considerably more difficult) Let $P(x) = x^{11} + a_{10}x^{10} + \cdots + a_0$ be a monic polynomial of degree eleven with real coefficients a_i, with $a_0 \neq 0$. Suppose that all the zeros of $P(x)$ are real, i.e., if α is a complex number such that $P(\alpha) = 0$, then α is real. Find (with proof) the least possible number of nonzero coefficients of $P(x)$ (including the coefficient 1 of x^{11}).

11. Let $P_n(x) = (x+n)(x+n-1)\cdots(x+1) - (x-1)(x-2)\cdots(x-n)$. Show that all the zeros of $P_n(x)$ are purely imaginary, i.e., have real part 0.