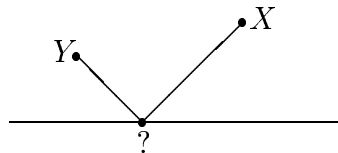


## 18.S34 PROBLEMS #4

FALL 2002

36. [1] Three students  $A, B, C$  compete in a series of tests. For coming in first in a test, a student is awarded  $x$  points; for coming second,  $y$  points; for coming third,  $z$  points. Here  $x, y, z$  are positive integers with  $x > y > z$ . There were no ties in any of the tests. Altogether  $A$  accumulated 20 points,  $B$  10 points, and  $C$  9 points. Student  $A$  came in second in the algebra test. Who came in second in the geometry test?
37. [1] Remove the upper-left and lower-right corner squares from an  $8 \times 8$  chessboard. Show that the resulting board cannot be covered by 31 dominoes. (A domino consists of two squares with an edge in common.)
38. [1] Mr.  $X$  brings some laundry from his house to a nearby river. After washing it in the river, he delivers it to Ms.  $Y$  who lives on the same side of the river.



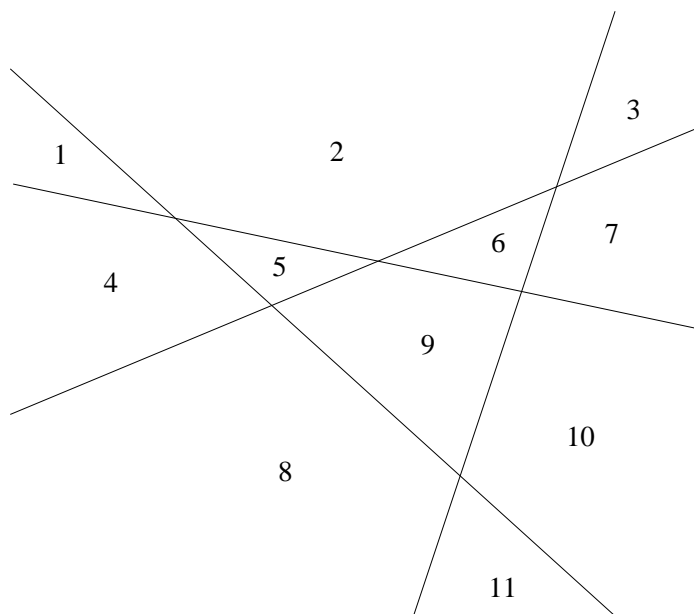
At what point on the river should Mr.  $X$  bring the laundry in order to travel the least possible distance? Try to do this problem without using calculus. (Assume of course that the river is a straight line.)

39. [2] An *antimagic square* is an  $n \times n$  matrix whose entries are the distinct integers  $1, 2, \dots, n^2$  such that any set of  $n$  entries, no two in the same row or column, have the same sum of their elements. For instance,

$$\begin{bmatrix} 14 & 8 & 16 & 6 \\ 9 & 3 & 11 & 1 \\ 10 & 4 & 12 & 2 \\ 13 & 7 & 15 & 5 \end{bmatrix}$$

For what values of  $n$  do there exist  $n \times n$  antimagic squares?

40. [1] Let  $f(n)$  be the number of regions which are formed by  $n$  lines in the plane, where no two lines are parallel and no three meet in a point. E.g.,  $f(4) = 11$ .



Find a simple formula for  $f(n)$ .

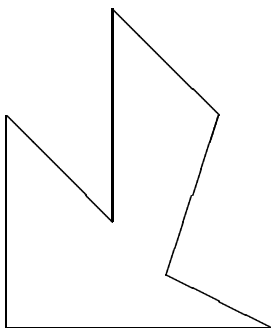
41. [2.5] Define a sequence  $a_0, a_1, a_2, \dots$  of integers as follows:  $a_0 = 0$ , and given  $a_0, a_1, \dots, a_n$ , then  $a_{n+1}$  is the least integer greater than  $a_n$  such that no three distinct terms (not necessarily consecutive) of  $a_0, a_1, \dots, a_{n+1}$  are in arithmetic progression. (This means that for no  $0 \leq i < j < k \leq n+1$  do we have  $a_j - a_i = a_k - a_j$ .) Find a simple rule for determining  $a_n$ . For instance, what is  $a_{1000000}$ ? The sequence begins  $0, 1, 3, 4, 9, 10, 12, \dots$
42. (a) [1] Let  $a, b, m, n$  be positive integers. Suppose that an  $m \times n$  checkerboard can be tiled with  $a \times b$  boards (in any orientation), i.e., the  $a \times b$  boards can be placed on the  $m \times n$  board to cover it completely, with no overlapping of the interiors of the  $a \times b$  boards. Show that  $mn$  is divisible by  $ab$ .
- (b) [3] Assuming the condition of (a), show in fact that at least one of  $m$  and  $n$  is divisible by  $a$ . (Thus by symmetry, at least one of

$m$  and  $n$  is divisible by  $b$ .) For instance, a  $6 \times 30$  board cannot be tiled with  $4 \times 3$  boards.

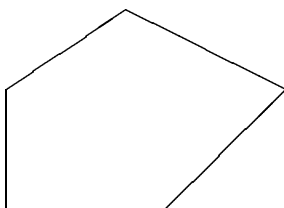
(c) [3] Generalize (b) to any number of dimensions.

43. [2.5] Let  $R$  be a rectangle whose sides can have any positive real lengths. Show that if  $R$  can be tiled with finitely many rectangles all with at least one side of integer length, then  $R$  has at least one side of integer length.

44. [3] A *polygon* is a plane region enclosed by non-intersecting straight line segments, such as



A polygon  $P$  is *convex* if any straight line segment whose endpoints lie in  $P$  lies entirely in  $P$ . For instance, the above polygon is not convex, but



is convex. Can a convex polygon be dissected into non-convex quadrilaterals? (A quadrilateral is a four-sided polygon. The non-convex quadrilaterals in the above question may be of any size and shape, provided none are convex.) This problem was formulated and solved by a Berkeley undergraduate; none of the mathematics professors to whom he showed it were able to solve it.

45. [3.5] Does there exist a polynomial  $f(x)$  with real coefficients such that  $f(x)^2$  has fewer nonzero coefficients than  $f(x)$ ?