18.S34 (FALL, 2002)

GREATEST INTEGER PROBLEMS

NOTE: We use the notation $\lfloor x \rfloor$ for the greatest integer $\leq x$, even if the original source used the older notation $[x]$.

1. (48P) If n is a positive integer, prove that
$$\left\lfloor \sqrt{n} + \sqrt{n+1} \right\rfloor = \left\lfloor \sqrt{4n+2} \right\rfloor.$$

2. (a) Let p denote a prime number, and let m be any positive integer. Show that the exponent of the highest power of p which divides $m!$ is
$$\left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{m}{p^2} \right\rfloor + \cdots + \left\lfloor \frac{m}{p^s} \right\rfloor,$$
where $p^{s+1} > m$.

   (b) In how many zeros does the number 1000! end, when written in base 10?

3. (68IMO) For every natural number n, evaluate the sum
$$\sum_{k=0}^{\infty} \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+2}{4} \right\rfloor + \cdots + \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor + \cdots.$$

4. A sequence of real numbers is defined by the *nonlinear* first order recurrence
$$u_{n+1} = u_n(u_n^2 - 3).$$
   (a) If $u_0 = 5/2$, give a simple formula for u_n.

   (b) If $u_0 = 4$, how many digits (in base ten) does $\lfloor u_{10} \rfloor$ have?

5. Define a sequence $a_1 < a_2 < \cdots$ of positive integers as follows. Pick $a_1 = 1$. Once $a_1, \ldots, a_n$ have been chosen, let a_{n+1} be the least positive integer not already chosen and not of the form $a_i + i$ for $1 \leq i \leq n$. Thus $a_1 + 1 = 2$ is not allowed, so $a_2 = 3$. Now $a_2 + 2 = 5$ is also not

allowed, so $a_3 = 4$. Then $a_3 + 3 = 7$ is not allowed, so $a_4 = 6$, etc. The sequence begins:

$$1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, \ldots.$$

Find a simple formula for a_n. Your formula should enable you, for instance, to compute $a_{1,000,000}$. (HINT. This is a hard problem. The answer involves $\tau = \frac{1}{2}(1 + \sqrt{5})$.

6. (a) (Problem A6, 93P. No competitor solved it.) The infinite sequence of 2's and 3's

$$2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, \ldots$$

has the property that, if one forms a second sequence that records the number of 3's between successive 2's, the result is identical to the first sequence. Show that there exists a real number r such that, for any n, the nth term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m.

   (b) (similar in flavor to (a), though not involving the greatest integer function) Let $a_1, a_2, \ldots$ be the sequence

$$1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7, 8, 8, 8, 8, 9, 9, 9, 9, 9, \ldots$$

of integers a_n defined as follows: $a_1 = 1$, $a_1 \le a_2 \le a_3 \le \cdots$, and a_n is the number of n's appearing in the sequence. Find real numbers $\alpha, c > 0$ such that

$$\lim_{n \to \infty} \frac{a_n}{n^\alpha} = c.$$

7. (Problem B6, 95P. Three competitors solved it.) For a positive real number α, define

$$S(\alpha) = \{\lfloor n\alpha \rfloor : n = 1, 2, 3, \ldots\}.$$

Prove that $\{1, 2, 3, \ldots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$, and $S(\gamma)$.

8. Let m be a positive integer and k any integer. Define a sequence $a_m, a_{m+1}, \ldots$ as follows:

$$\begin{aligned} a_m &= k \\ a_{n+1} &= \left\lfloor \frac{n+2}{n} a_n \right\rfloor, \quad n \geq m. \end{aligned}$$

Show that there exists a positive integer N and polynomials $P_0(n)$,$P_1(n)$, $\ldots$, $P_{N-1}(n)$ such that for all $0 \leq i \leq N-1$ and all integers t for which $tN + i \geq m$, we have

$$a_{tN+i} = P_i(t).$$