18.S34 (FALL 2002)

PROBLEMS ON CONGRUENCES AND DIVISIBILITY

1. (55P) Do there exist $1,000,000$ consecutive integers each of which contains a repeated prime factor?

2. A positive integer n is *powerful* if for every prime p dividing n, we have that p^2 divides n. Show that for any $k \geq 1$ there exist k consecutive integers, none of which is powerful.

3. (56P) Prove that every positive integer has a multiple whose decimal representation involves all ten digits.

4. (66P) Prove that among any ten consecutive integers at least one is relatively prime to each of the others.

5. (70P) Find the length of the longest sequence of equal nonzero digits in which an integral square can terminate (in base 10), and find the smallest square which terminates in such a sequence.

6. (72P) Show that if n is an integer greater than 1, then n does not divide $2^n - 1$.

7. (a) (77P) Prove that $\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p}$ for all integers p, a, and b with p a prime, $p > 0$, and $a \geq b \geq 0$.
   (b) (not on Putnam exam) Show in fact that the above congruence holds modulo p^2.
   (c) (not on Putnam exam) Show that if $p \geq 5$, then the above congruence even holds modulo p^3.

8. (82P) Let $n_1, n_2, \ldots, n_s$ be distinct integers such that
$$(n_1 + k)(n_2 + k) \cdots (n_s + k)$$
is an integral multiple of $n_1 n_2 \cdots n_s$ for every integer k. For each of the following assertions, give a proof or a counterexample:
   (a) $|n_i| = 1$ for some i.
   (b) If further all n_i are positive, then
$$\{n_1, n_2, \ldots, n_s\} = \{1, 2, \ldots, s\}.$$

9. (83P) Let p be in the set $\{3, 5, 7, 11, \ldots\}$ of odd primes, and let
$$F(n) = 1 + 2n + 3n^2 + \cdots + (p-1)n^{p-2}.$$
Prove that if a and b are distinct integers in $\{0, 1, 2, \ldots, p-1\}$ then $F(a)$ and $F(b)$ are not congruent modulo p, that is, $F(a) - F(b)$ is not exactly divisible by p.

10. (85P) Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?

11. (86P) What is the units (i.e., rightmost) digit of

$$\left[\frac{10^{20000}}{10^{100}+3}\right]?$$

Here $[x]$ is the greatest integer $\leq x$.

12. (91P) Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

13. (96P) If p is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.

14. (97P) Prove that for $n \geq 2$,

$$\overbrace{2^{2^{\cdots^{2}}}}^{n \text{ terms}} \equiv \overbrace{2^{2^{\cdots^{2}}}}^{n-1 \text{ terms}} \pmod{n}.$$

15. (99P) The sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 1$, $a_2 = 2$, $a_3 = 24$, and, for $n \geq 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$

Show that, for all n, a_n is an integer multiple of n.

16. (00P) Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.