18.S34 (FALL 2002)

PROBLEMS ON RECURRENCES

1. (90P) Let $T_0 = 2, T_1 = 3, T_2 = 6$, and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are: 2, 3, 6, 14, 40, 152, 784, 5158, 40576, 363392. Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

2. (80P) For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$? (Express the answer in simplest form.)

3. (83P) Prove or disprove that there exists a positive real number u such that $[u^n] - n$ is an even integer for all positive integers n. (Here, $[x]$ is the greatest integer $\leq x$.)

4. Define u_n by $u_0 = 0, u_1 = 4$, and $u_{n+2} = \frac{6}{5}u_{n+1} - u_n$. Show that $|u_n| \leq 5$ for all n. (In fact, $|u_n| < 5$ for all n. Can you show this?)

5. (46P) Show that the next integer above $(\sqrt{3}+1)^{2n}$ is divisible by 2^{n+1}.

6. Let $a_0 = 0, a_1 = 1$, and for $n \geq 2$ let $a_n = 17a_{n-1} - 70a_{n-2}$. For $n > 6$, show that the first (most significant) digit of a_n (when written in base 10) is a 3.

7. Let a, b, c denote the (real) roots of the polynomial $P(t) = t^3 - 3t^2 - t + 1$. If $u_n = a^n + b^n + c^n$, what linear recursion is satisfied by $\{u_n\}$? If a is the largest of the three roots, what is the closest integer to a^5.

8. Solve the first order recursion given by $x_0 = 1$ and $x_n = 1 + (1/x_{n-1})$. Does $\{x_n\}$ approach a limiting value as n increases?

9. If $u_0 = 0, u_1 = 1$, and $u_{n+2} = 4(u_{n+1} - u_n)$, find u_{16}.

10. (a) Define $u_0 = 1, u_1 = 1$, and for $n \geq 1$,

$$2u_{n+1} = \sum_{k=0}^{n} \binom{n}{k} u_k u_{n-k}.$$

Find a simple expression for $F(x) = \sum_{n\geq 0} u_n \frac{x^n}{n!}$. Express your answer in the form $G(x) + H(x)$, where $G(x)$ is even (i.e., $G(-x) = G(x)$) and $H(x)$ is odd (i.e., $H(-x) = -H(x)$).

(b) Define $u_0 = 1$ and for $n \geq 0$,

$$2u_{n+1} = \sum_{k=0}^{n} \binom{n}{k} u_k u_{n-k}.$$

Find a simple expression for u_n.

11. (97P) For a positive integer n and any real number c , define x_k recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k, $1 \leq k \leq n$.

12. (a) Define u_n recursively by $u_0 = u_1 = u_2 = u_3 = 1$ and

$$u_n u_{n-4} = u_{n-1}u_{n-3} + u_{n-2}^2, \quad n \geq 4.$$

Show that u_n is an integer.

(b) Do the same for $u_0 = u_1 = u_2 = u_3 = u_4 = 1$ and

$$u_n u_{n-5} = u_{n-1}u_{n-4} + u_{n-2}u_{n-3}, \quad n \geq 5.$$

(c) (much harder) Do the same for $u_0 = u_1 = u_2 = u_3 = u_4 = u_5 = 1$ and

$$u_n u_{n-6} = u_{n-1}u_{n-5} + u_{n-2}u_{n-4} + u_{n-3}^2, \quad n \geq 6,$$

and for $u_0 = u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = 1$ and

$$u_n u_{n-7} = u_{n-1}u_{n-6} + u_{n-2}u_{n-5} + u_{n-3}u_{n-4}, \quad n \geq 7.$$

(d) What about $u_0 = u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = u_7 = 1$ and

$$u_n u_{n-8} = u_{n-1}u_{n-7} + u_{n-2}u_{n-6} + u_{n-3}u_{n-5} + u_{n-4}^2, \quad n \geq 8?$$