18.S34 (FALL 2002)

LIMIT PROBLEMS

1. Let a and b be positive real numbers. Prove that

$$\lim_{n\to\infty}(a^n+b^n)^{1/n}$$

equals the larger of a and b. What happens when $a=b$?

2. Show that $\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log(n)\right)$ exists and lies between $\frac{1}{2}$ and 1.

NOTE. This number, known as *Euler's constant* and denoted γ, is probably the third most important constant in the theory of complex variables, after π and e. Numerically we have

$$\gamma = 0.5772156649015328606065120900824024310421593359 92\cdots.$$

It is a famous unsolved problem to decide whether γ is irrational.

3. (47P) If (a_n) is a sequence of numbers such that, for $n\geq 1$,

$$(2-a_n)a_{n+1}=1,$$

prove that $\lim_{n\to\infty}a_n$ exists and equals 1.

4. Let K be a positive real number. Take an arbitrary positive real number x_0 and form the sequence

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{K}{x_n}\right).$$

Show that $\lim_{n\to\infty}x_n=\sqrt{K}$. (REMARK. this is how most calculators determine $\sqrt{K}$.)

5. (70P) Given a sequence (x_n) such that $\lim_{n\to\infty}(x_n-x_{n-2})=0$, prove that

$$\lim_{n\to\infty}\frac{x_n-x_{n-1}}{n}=0.$$

6. Let $x_{n+1}=x_n^2-6x_n+10$. For what values of x_0 is $\{x_n\}$ convergent, and how does the value of the limit depend on x_0?

7. (90P) Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$, $(n, m = 0, 1, 2, \ldots)$? Justify your answer.

8. Let $x_0 = 1$ and $x_{n+1} = x_n + 10^{-10^{x_n}}$. Does $\lim_{n\to\infty} x_n$ exist? Explain.

9. (00P) Show that the improper integral

$$\lim_{B\to\infty} \int_0^B \sin(x)\ \sin(x^2)\, dx$$

converges.

PART II

LIMITS. Two useful techniques are:

(a) *L'Hôpital's rule.* If $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$, then

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)},$$

provided the derivatives in question exist. Some limits can be converted to this form by first taking logarithms, or by substituting $1/x$ for x, etc.

(b) If $f(x)$ is reasonably well-behaved (e.g., continuous) on the closed interval $[a, b]$, then

$$\lim \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) = \int_a^b f(x)dx,$$

where the limit is over any sequence of "partitions of $[a, b]$" $a = x_0 < x_1 < \cdots < x_n = b$ such that the maximum value of $x_i - x_{i-1}$ approaches 0. In particular, taking $a = 0$, $b = 1$, $x_i = i/n$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(i/n) = \int_0^1 f(x)dx.$$

Sometimes a limit of products can be converted to this form by taking logarithms.

The next five problems are all from the Putnam Exam.

9. Let $a > 0$, $a \neq 1$. Find

$$\lim_{x\to\infty} \left(\frac{1}{x} \frac{a^x - 1}{a - 1} \right)^{1/x}$$

10. Find

$$\lim_{n\to\infty}\left[\frac{1}{n^4}\prod_{i=1}^{n}(n^2+i^2)^{1/n}\right]$$

11. Let $0 < a < b$. Evaluate

$$\lim_{t\to 0}\left[\int_0^1 (bx+a(1-x))^t dx\right]^{1/t}$$

12. Evaluate

$$\lim_{x\to 0}\frac{1}{x}\int_0^x (1+\sin(2t))^{1/t}dt$$

13. Evaluate

$$\lim_{n\to\infty}\sum_{j=1}^{n^2}\frac{n}{n^2+j^2}$$