18.S34 (FALL, 2002)

PROBLEMS ON GENERATING FUNCTIONS

1. [58P] Let $f(m,1) = f(1,n) = 1$ for $m \geq 1$, $n \geq 1$, and let

$$f(m,n) = f(m-1,n) + f(m,n-1) + f(m-1,n-1) \text{ for } m > 1 \text{ and } n > 1.$$

Also let

$$S(n) = \sum_{a+b=n} f(a,b), \ a \geq 1 \text{ and } b \geq 1.$$

Prove that

$$S(n+2) = S(n) + 2S(n+1) \text{ for } n \geq 2.$$

2. [62P] Let $x^{(n)} = x(x-1)\cdots(x-n+1)$ for n a positive integer, and let $x^{(0)} = 1$. Prove that

$$(x+y)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)}.$$

NOTE: $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdots k}$.

3. [74P] For a set with n elements, how many subsets are there whose cardinality (the number of elements in the subset) is respectively $\equiv 0 \pmod 3$, $\equiv 1 \pmod 3$, $\equiv 2 \pmod 3$? In other words, calculate

$$s_{i,n} = \sum_{k \equiv i \pmod 3} \binom{n}{k} \text{ for } i = 0,1,2.$$

Your result should be strong enough to permit direct evaluation of the numbers $s_{i,n}$ and to show clearly the relationship of $s_{0,n}$ and $s_{1,n}$ and $s_{2,n}$ to each other for all positive integers n. In particular, show the relationships among these three sums for $n = 1000$. [An illustration of the definition of $s_{i,n}$ is $s_{0,6} = \binom{6}{0} + \binom{6}{3} + \binom{6}{6} = 22$.]

4. [39P] Given the power series

$$a_0 + a_1 x + a_2 x^2 + \cdots$$

in which

$$a_n = (n^2+1)3^n,$$

show that there is a relationship of the form

$$a_n + p a_{n+1} + q a_{n+2} + r a_{n+3} = 0,$$

in which p, q, r are constants independent of n. Find these constants and the sum of the power series.

5. [48P] Show that

$$x + \frac{2}{3}x^3 + \frac{2}{3}\frac{4}{5}x^5 + \frac{2}{3}\frac{4}{5}\frac{6}{7}x^7 + \cdots = \frac{\text{arc sin } x}{\sqrt{1-x^2}}.$$

NOTE (not on Putnam Exam): arc $\sin x$ is the same as $\sin^{-1} x$.

6. [83P] Let k be a positive integer and let $m = 6k - 1$. Let

$$S(m) = \sum_{j=1}^{2k-1} (-1)^{j+1} \binom{m}{3j-1}.$$

For example with $k = 3$,

$$S(17) = \binom{17}{2} - \binom{17}{5} + \binom{17}{8} - \binom{17}{11} + \binom{17}{14}.$$

Prove that $S(m)$ is never zero. [As usual, $\binom{m}{r} = \frac{m!}{r!(m-r)!}$.]

7. [92P] For nonnegative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{n} \binom{n}{j} \binom{n}{k-2j},$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a \geq 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$, and $\binom{a}{b} = 0$ otherwise.)

8. Given $a_0 = 1$ and $a_{n+1} = (n+1)a_n - \binom{n}{2} a_{n-2}$ for $n \geq 0$, compute $y = \sum_{n \geq 0} a_n \frac{x^n}{n!}$.

9. Find the coefficients of the power series $y = 1 + 3x + 15x^2 + 184x^3 + 495x^4 + \cdots$ satisfying

$$(27x - 4)y^3 + 3y + 1 = 0.$$

10. Find the unique power series $y = 1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \frac{1}{30240}x^6 + \cdots$ such that for all $n \geq 0$, the coefficient of x^n in y^{n+1} is equal to 1.

11. Let $f(m,0) = f(0,n) = 1$ and $f(m,n) = f(m-1,n) + f(m,n-1) + f(m-1,n-1)$ for $m,n > 0$. Show that

$$\sum_{n=0}^{\infty} f(n,n)x^n = \frac{1}{\sqrt{1 - 6x + x^2}}.$$

12. [97P] Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1 + x + x^2)^m$. Prove that for all $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor 2k/3 \rfloor} (-1)^i a_{k-i,i} \leq 1.$$