18.S34 (FALL, 2002)

PIGEONHOLE PROBLEMS

NOTE: Notation such as (78P) means a problem from the 1978 Putnam Exam.

1. (78P) Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Prove that there must be two distinct integers in A whose sum is 104. [Actually, 20 can be replaced by 19.]

2. Five points are situated inside an equilateral triangle whose side has length one unit. Show that two of them may be chosen which are less than one half unit apart. What if the equilateral triangle is replaced by a square whose side has length of one unit?

3. (71P) Let there be given nine lattice points (points with integral coordinates) in three dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points. (WARNING. It is easy to misread this problem. Take time to make sure you understand what is being asked for.) [To test your understanding, how many lattice points does one need in *four* dimensions to reach the same conclusion?]

4. (72IMO) Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint subsets whose members have the same sum. [Though not stated in the problem, one should assume that not both the subsets are empty, or even that neither of the subsets is empty.]

5. (80P) Let $A_1, A_2, \ldots, A_{1066}$ be subsets of a finite set X such that $|A_i| > \frac{1}{2}|X|$ for $1 \leq i \leq 1066$. Prove that there exist ten elements $x_1, \ldots, x_{10}$ of X such that every A_i contains at least one of $x_1, \ldots, x_{10}$. (Here $|S|$ means the number of elements in the set S.)

6. Given any $n+2$ integers, show that there exist two of them whose sum, or else whose difference, is divisible by $2n$.

7. Given any $n + 1$ distinct integers between 1 and $2n$, show that two of them are relatively prime. Is this result best possible, i.e., is the conclusion still true for n integers between 1 and $2n$?

8. Given any $n + 1$ integers between 1 and $2n$, show that one of them is divisible by another. Is this best possible, i.e., is the conclusion still true for n integers between 1 and $2n$?

9. Let u be an irrational real number. Let S be the set of all real numbers of the form $a + bu$, where a and b are integers. Show that S is dense in the real numbers, i.e., for any real number x and any $\epsilon > 0$, there is a element $y \in S$ such that $|x - y| < \epsilon$. (HINT. First let $x = 0$.)

10. Two disks, one smaller than the other, are each divided into 200 congruent sectors. In the larger disk 100 of the sectors are chosen arbitrarily and painted red; the other 100 sectors are painted blue. In the smaller disk each sector is painted either red or blue with no stipulation on the number of red and blue sectors. The small disk is then placed on the larger disk so that their centers coincide. Show that it is possible to align the two disks so that the number of sectors of the small disk whose color matches the corresponding sector of the large disk is at least 100.

11. A collection of subsets of $\{1, 2, \ldots, n\}$ has the property that each pair of subsets has at least one element in common. Prove that there are at most 2^{n-1} subsets in the collection.

12. (95P) For a partition π of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are two distinct numbers x and y in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$.

    [A *partition* of a set S is a collection of (nonempty) disjoint subsets (parts) whose union is S.]

13. $\mathbb{N}$ is the set of nonnegative integers. For any subset S of $\mathbb{N}$, let $P(S)$ be the set of all *pairs* of members of S. (A *pair* is a set (unordered) with two distinct members.) Partition $P(\mathbb{N})$, arbitrarily, into two sets (of pairs) P_1 and P_2. Prove that $\mathbb{N}$ must contain an infinite subset S such that either $P(S)$ is contained in P_1 or $P(S)$ is contained in P_2.