

Buyout Prices in Online Auctions

by

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B. Tech., Mechanical Engineering (2002)
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Submitted to the Sloan School of Management
in partial fulfillment of the requirements for the degree of

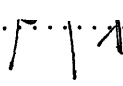
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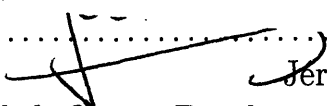
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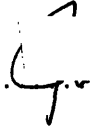
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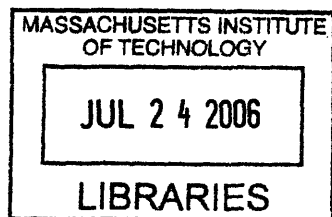
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Abstract

Buyout options allow bidders to instantly purchase at a specified price an item listed for sale through an online auction. A *temporary* buyout option disappears once a regular bid above the reserve price is made, while a *permanent* option remains available until it is exercised or the auction ends. Buyout options are widely used in online auctions and have significant economic importance: nearly half of the auctions today are listed with a buyout price and the option is exercised in nearly one fourth of them.

We formulate a game-theoretic model featuring time-sensitive bidders with independent private valuations and Poisson arrivals but endogenous bidding times in order to answer the following questions: How should buyout prices be set in order to maximize the seller's discounted revenue? What are the relative benefits of using each type of buyout option? While all existing buyout options we are aware of currently rely on a static buyout price (i.e. with a constant value), what is the potential benefit associated with using instead a dynamic buyout price that varies as the auction progresses?

For all buyout option types we exhibit a Nash equilibrium in bidder strategies, argue that this equilibrium constitutes a plausible outcome prediction, and study the problem of maximizing the corresponding seller revenue. In particular, the equilibrium strategy in all cases is such that a bidder exercises the buyout option provided it is still available and his valuation is above a time-dependent threshold. Our numerical experiments suggest that a seller may significantly increase his utility by introducing a buyout option when any of the participants are time-sensitive. Furthermore, while permanent buyout options yield higher predicted revenue than temporary options, they also provide additional incentives for late bidding and may therefore not be always more desirable. The numerical results also imply that the increase in seller's utility (over a fixed buyout price auction) enabled by a dynamic buyout price is small and does not seem to justify the corresponding increase in complexity.

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Chapter 1

Introduction

As they were initially conceived during the last decade of the previous century, online auctions were arguably suffering from two perceived drawbacks relative to posted price mechanisms: waiting time and price uncertainty. In order to render these auctions more attractive to time-sensitive or risk averse participants, many auction sites have since introduced a new feature known as a *buyout option*, which offers potential buyers the opportunity to instantaneously purchase at a specified price an item put for sale through an online auction. Indeed Mathews (2003b) notes that “one of the reasons *eBay* introduced [the buyout] option is that buyers wanted to be able to obtain and sellers wanted to be able to sell items more quickly”. Augmented with this option, an online auction thus becomes a hybrid between an electronic catalogue and a traditional auction.

Buyout options are now widespread and have significant economic importance: in the fourth quarter of 2003 alone, fixed income trading (primarily from the buyout option “Buy It Now”) contributed \$2 billion or 28% of *eBay*’s gross annual merchandise sale¹; other examples of buyout options include *Yahoo*’s “Buy Price”, *Amazon*’s “Take-It” and *uBid*’s “uBuy it!”. Remarkably, buyout options in these large auction sites currently differ in one important aspect: *eBay*’s “Buy It Now” option disappears as soon as a regular bid above the reserve price is submitted, so it is called *temporary*; in contrast *Yahoo*, *Amazon* and *uBid*’s options remain until they are exercised or the

¹Source: <http://investor.ebay.com/>, see also Reynolds and Wooders (2003).

auction in which they are featured ends, so they are called *permanent* (Hidvégi et al. 2003).

These observations motivate in our view the following questions: What is the benefit associated with using a buyout option for a seller in an online auction? How should the buyout price be set when doing so? Should a temporary or a permanent buyout option be used? We design a game-theoretic model to answer these questions in a stylized setting – the market environment and the auction mechanism we consider are specified in §2.1 and §2.2 respectively, and we discuss the realism of the model in §2.3. For this model, for both the temporary and permanent buyout option an equilibrium in bidder strategies is characterized and the associated seller’s optimization problem is discussed. We extend next our analysis with the goal of answering the following question: while all existing buyout options we are aware of rely on a *static* buyout price (i.e. with a constant value), what is the potential benefit associated with using instead a *dynamic* buyout price that varies as the auction progresses? The main contributions of our research can be summarized as following:

1. Analysis for temporary and permanent buyout price auctions – We characterize equilibrium strategies for temporary and permanent buyout price auctions in §3.1.1 and §3.2.1 respectively, and analyze their robustness to perturbations in strategy and payoff space (§3.1.2 for temporary and §3.2.2 for permanent). We also conduct a simple empirical analysis, described in §5.3, to validate the predictions of these strategies using bidding data from actual online auctions. For limiting regimes of bidder arrival rate, and bidder and seller time sensitivity, we derive optimal buyout prices for both options (§3.1.3 and §3.2.3).
2. Comparison of temporary and permanent buyout option – Our numerical experiments, discussed in §5.1, suggest that the seller’s expected discounted revenue derived from an optimal permanent buyout option is larger than that obtained with an optimal temporary option. Furthermore, the relative attractiveness for the seller of a temporary buyout option decreases with the expected number of bidders, whereas it increases in the case of a permanent option. The equilib-

rium analysis however implies that the permanent option promotes late bidding – a fact that is also corroborated by bidding data obtained from online auction websites – which may negatively impact the seller’s revenue.

3. Dynamic buyout price – We analyze temporary and permanent dynamic buyout price auctions – §4.1.1 (resp. §4.2.1) focuses on outcome prediction in the temporary (resp. permanent) case and §4.1.2 (resp. §4.2.2) discusses the resulting optimization problem. Our numerical experiments, discussed in §5.2, suggest that the increase in seller’s utility (over a fixed buyout price auction) enabled by a dynamic buyout price is small and does not seem to justify the corresponding increase in complexity - dynamic prices will be difficult to implement and may be too complex for bidders to understand.

Chapter 6 contains the concluding remarks, and all proofs omitted from the main text are included in the Appendix.

The results (1) and (2) can assist a seller in selecting the appropriate auction mechanism (including the optimal buyout price, if required) based on his preferences and the market environment for the product. The above results could also have important auction design implications: for example, the outcome prediction for a permanent buyout price auction advocates using an auction mechanism which would allow bidders to submit “last-minute” bids in advance hence avoiding the hassle of tracking the auction to place a bid near its end. This could however convert the auction into a sealed bid second-price auction which may not be desirable; see Roth and Ockenfels (2002) for a discussion on introducing the sniping option in an auction. The conclusion (3) suggests that there is little advantage for the seller of introducing a buyout option with time-varying price.

The following section in this Chapter reviews literature on auctions and buyout prices.

1.1 Literature Survey

Auctions, in many different forms, are very widely used; a historical sketch of the use of auctions is provided by Shubik (1983) – one of the most famous being an auction of the entire Roman empire in AD 193. More recently government contracts, United States Treasury bills, cars, arts and antiques have been auctioned (see Klemperer (1999)). With the advent of online auctions, the list of items sold by auction has expanded to include software, collectibles, electronic items, used books, concert tickets, furniture, and almost everything else² (also see Lucking-Reiley (2000)).

As a consequence of their importance and popularity, there is a significant body of theoretical literature on auctions. A comprehensive but somewhat dated bibliography of auction literature is provided in Stark and Rothkopf (1979) while more recent surveys include Milgrom (1985), Milgrom (1989) and Klemperer (1999). McAfee and McMillan (1987) discuss developments in the theory of bidding mechanisms restricting to models analyzing, like we do, a single isolated auction. A critical discussion of available models aiding competitive decision making in auctions– bidding strategy for bidders and auction design for sellers – is presented by Rothkopf and Harstad (1994).

1.1.1 Literature on Online Auctions

Introduced in 1995³, online auctions have gained tremendous popularity – *eBay*, arguably the biggest online auction website, had 135 million registered users and a gross merchandize volume, which is the total value of everything sold on *eBay*, of \$34.2 billion in 2004⁴. Lucking-Reiley (2000) traces the development of online auctions describing transaction volumes, types of auction formats used, type of goods auctioned, fee structure and the business model of various auction websites. There has been much recent research activity seeking to answer the many new questions posed by the emergence of online auctions; Pinker et al. (2003) characterize the state

²<http://en.wikipedia.org/wiki/Ebay>

³Lucking-Reiley (2000)

⁴*eBay* 2004 Annual Report – <http://investor.ebay.com/annual.cfm>

of research and present many open problems in this field.

Online auctions have several unique characteristics – for instance, unlike traditional auctions, a bidder in an online auction usually faces a random number of bidders; in addition, online auctions are typically longer in duration. These features impact bidder behavior raising new theoretical issues, which many papers seek to answer. Taking a dynamic programming approach, Bertsimas et al. (2002) develop a computationally-feasible algorithm to determine the optimal bidding strategy for a potential bidder in a single unit auction assuming that bids from other bidders are generated from a probability distribution which can be estimated using publicly available bidding data. They also extend their results to multi-unit auctions. Arieli and Simonson (2003) analyze bidding behavior focusing on two key aspects affecting the decision making process: value-assessment and decision dynamics. They discuss the effect of these two factors on bidder behavior at three key stages of an auction: (a) beginning of the auction (bidder decides whether to participate or not), (b) bidding during the auction and (c) bidding at the end of the auction. Park et al. (2005) build an integrated model, based on bidders' willingness to pay at any given auction round, of bidding behavior incorporating three main factors: which bidder placed a bid, the timing of the bid and its amount. Similar, in spirit, to our model, Carare and Rothkopf (2005) assume that bidders incur a fixed cost of waiting and returning later to the auction. Unlike our research, however, they analyze a Dutch auction mechanism deriving, in a simple two-bidder, two-valuation framework with transaction costs, pure- and mixed-strategy Nash equilibria in bidder strategy in a Dutch auction with a linear price function.

In a series of papers, Roth and Ockenfels analyze last-minute bidding in online auctions. Roth and Ockenfels (2002) and Ockenfels and Roth (2002) provide empirical evidence of late bidding and give strategic and non-strategic hypotheses justifying the phenomenon while Ockenfels and Roth (2005) show that last-minute bidding can occur at equilibrium in fixed price auctions if very late bids have a positive probability of being rejected. In all their studies, they observe that auctions with a floating deadline (see §2.3 for definition) experience lesser last-minute bidding than

fixed deadline auctions. Taking a different approach, Bajari and Hortacısu (2003) rationalize late bidding by considering a model with common values where bidders have an incentive to hide their private information by bidding at the last-minute.

Other aspects of online auctions are addressed by Segev et al. (2001) who model an online auction as a two-dimension Markov chain (the two dimensions being the current price and the number of bidders) to estimate the final selling price of a product. While a significant number of online auctions offer multiple units, literature analyzing multi-unit online auctions is fairly limited. In two papers Bapna et al. (2000) and Bapna et al. (2001) study bidding strategies in multi-unit auctions classifying bidders as opportunists, participators and evaluators based on when and how often they bid. Pinker et al. (2001) formulate a dynamic program for solving the problem of allocating inventory across several multi-unit auctions. They extend their model to develop a framework where information from earlier auctions is utilized to update seller's beliefs about bidder valuations and consequently improve the lot-sizing decisions.

Publicly available bidding data on auction websites has led to a number of empirical studies on online auctions. Kaufmann and Wood (2004) investigate factors that make bidders pay more for exactly the same item and find that items sold on weekends, items with a picture and items sold by experienced sellers tend to sell at higher prices. A similar study by Lucking-Reiley et al. (2000) concludes that the auction selling price is higher when the seller has higher feedback ratings and the auction is longer. Several other papers including Houser and Wooders (2005), Melnik and Alm (2002), Ba and Pavlou (2002) and McDonald and Slawson (2002) study the effect on seller's feedback rating on auction outcome and conclude that seller ratings positively affect the selling price. Durham et al. (2004) study the "Buy-it-Now" option offered on *eBay* and find that seller reputation also has a significant impact on buyout price auctions – sellers with higher reputation are more likely to offer the buyout option and, the probability of option exercise increases with seller reputation. In another study, comparing auctions with online catalogs Vakrat and Seidmann (1999) find that, on an average, an item sells at significant discount (between 25% - 39%) when offered via an auction as opposed to a fixed-price mechanism.

1.1.2 Literature on Buyout Price auctions

While the literature on auction theory, online and otherwise, is large, existing research work on buyout prices is recent and relatively limited. Indeed, the comprehensive 1999 survey of the auction literature by Klemperer (1999) makes no mention of buyout prices, and while Lucking-Reiley (2000) observes the use of buyout prices in his 2000 survey of internet auction practices, he points out that he is “[...] not aware of any theoretical literature which examines the effect of such a buyout price in an auction.”

Most papers written since on buyout prices consider models where, in contrast with most actual online bidding interactions, the number of bidders is known in advance to all participants. Studying such a model with two risk averse bidders having two possible valuations for an item, Budish and Takeyama (2001) show that augmenting an English auction with a permanent buyout price can improve the seller’s profit. Kirkegaard and Overgaard (2003) investigate the impact of a permanent buyout option when two bidders with multi-unit demand face two sequential auctions of one item each. In the presence of two competing sellers they find that the one running the first auction benefits from using a buyout option. When a single seller runs both auctions, they show that his total revenue increases if bidders expect him to use a buyout option in the second auction. In contrast, we consider an auction for a single item run by a monopolistic seller, so that our model does not offer any insights on the issues of competition between sellers, sequential auctions and multi-unit demand.

Other papers still assume that the number of bidders is known to all, but allow that number to be arbitrarily high. In a model with n bidders, Reynolds and Wooders (2003) focus on the effect of bidder risk aversion on seller revenue in auctions with a buyout price (temporary or permanent). For either type of buyout option they find that with risk-averse bidders an auction with the optimal buyout price increases the seller’s revenue. Hidvégi et al. (2005) also find that, in the presence of risk aversion by either the bidders or the seller, such buyout price does increase the seller’s revenue. However in their model, where participants are not time-sensitive, bidders’ utility does not increase through the use of the buyout option. In a series of three papers

investigating variations of the same basic model, Mathews (2003b), Mathews (2004) and Mathews (2003a) focuses instead on the temporary buyout option. Specifically, he considers a fixed number of time-sensitive bidders with different arrival times, and explicitly captures the fact that early bidders may prevent later ones from exercising the option. He shows that a risk averse or time-sensitive seller facing either risk neutral or risk averse bidders will choose a buyout price ensuring that the buyout option is exercised with positive probability; he also finds that, depending on the valuation distribution, the buyout option either makes all bidders weakly better off, or low valuation bidders weakly better off and high valuation bidders strictly worse off. Note that we do not investigate bidder welfare in the present paper. By assuming a deterministic number of bidders, the papers mentioned above assume that every bidder knows with certainty upon his arrival how many competing bidders have already arrived and how many others are yet to come before the auction closes, which we believe to damage realism.

Furthermore, of all the papers analyzing buyout prices cited so far, the papers by Mathews are the only ones that capture, as we do in our model, the timing of bidder arrivals. That is, all others do not model the sequence in which bidders come to the auction site, and thus ignore the impact of each bidder's arrival time on his strategy. This is crucial as auction duration (and indeed the bidder arrival time) is an important factor affecting bidder participation strategy in buyout price auctions as illustrated by Wan et al. (2003) who, in a survey conducted by them, find that "38.7% respondents agree that the duration of an auction is a consideration in choosing the buyout option. 25% of the respondents replied that an auction with a long duration (7 to 10 days) encourages them to use the "Buy It Now" option". The importance of time sensitivity of auction participants on buyout exercise is also confirmed by *eBay* – its user guidelines state the reduction in waiting time as the very first reason why both sellers ("Sell your items fast.") and buyers ("Buy items instantly") would want to use their buy it now feature⁵) – and, more generally, by the discussion forums of experienced online auction users; below we list several examples:

⁵<http://pages.ebay.com/services/buyandsell/buyitnow.html>

“This guide is for people that are tired of losing auctions at the last second or for people who are just in a hurry to get their item. By using the Buy-It-Now option, you can quickly find the best price for the item you are searching for.” – Reviews and Guides, *eBay* (2005)

“One of the best features to come along in quite awhile is eBay’s “Buy It Now” feature. This allows bidders to buy it immediately for a price that you set. It’s great for buyers who don’t want to wait days until an auction ends to see if theyve won. Sellers also benefit. I have sold items within a half an hour by offering the “BIN” option.”⁶

“You can also sell in “Buy It Now” mode, where you establish a fixed price, and the “auction” ends as soon as someone agrees to pay that price. The fees are [t]he same as with standard auctions, but you could sell multiple copies of the same item in the time it would have taken you to run a single “standard” auction. [...] It’s also good when you don’t want the delays of standard auctions or the uncertainties of a variable auction price.” – Seltzer (2004)

“Neither you nor the buyer needs to wait for the end of an auction cycle – you get your payment sooner, and buyers get their merchandize faster. This is a great way to move more merchandize, especially if you have a product that’s in demand.”⁷

Moreover, by assuming that all bidders can potentially exercise the buyout option irrespective of when they arrive, these papers do not capture a key feature of buyout price auctions – a bidder arriving earlier can make the buyout option unavailable to subsequent bidders by either exercising it (for both temporary and permanent option) or placing a bid in the auction (for the temporary option only). This leads to a material difference as we exhibit robust equilibrium strategies where bidders arriving earlier in the auction do bid/buyout immediately on arrival.

⁶<http://auction.lifetips.com/subcat/69131/selling/special-features/index.html>

⁷<http://www.allbusiness.com/articles/BuyingSellingBusiness/3250-29-2806.html>

A model that ignores the arrival time of bidders also fails to capture the timing of bids placed in the auction. This is crucial as last-minute bidding and, in general, the timing of bids in online auctions has received considerable attention from both theorists – Roth and Ockenfels (2002) provide strategic and non-strategic hypotheses justifying late bidding – and practitioners – there are special softwares like *eSnipe* and *Last Minute Bidder*⁸ which allow bidders to place bids at the last minute. Also, auction sites frequently implement auction mechanisms that discourage late bids – for example, *Amazon* and *Yahoo* use a floating auction deadline that automatically extends if a bid is placed near the end of the auction; *Amazon* also offers a first bidder discount of 10% to promote early bidding. Despite the importance of bid timing, most of the literature analyzing bidder behavior in buyout price auctions either assumes exogenous bidding times (Caldentey and Vulcano (2004), cited below) or neglects the issue of bid times altogether (Budish and Takeyama (2001), Kirkegaard and Overgaard (2003), Reynolds and Wooders (2003), Hidvégi et al. (2005)). In contrast, the time when bidders act after their arrival is endogenous in our model, and we find in fact that with a permanent buyout option buyers submitting regular bids are likely to do so only at the very end of the auction.

While the papers cited above analyze models with a deterministic number of bidders, Caldentey and Vulcano (2004), who consider a multi-unit auction with a permanent buyout option, assume like we do that bidder arrivals follow a Poisson process whose future outcome is not known to participants. In addition, they also assume that auction participants are risk-neutral and time-sensitive, and use in fact the exact same utility functions we do. Unsurprisingly, they find, as in our research, that bidder strategies characterized by a threshold depending on arrival time and buyout price form an equilibrium. However, there are important differences between their work and the analysis we develop for the permanent buyout option: The model in Caldentey and Vulcano (2004) assumes that bidders are only informed about the initial number of units and not the number of units remaining. In the single-unit auction we investigate, this would correspond to bidders not knowing whether the

⁸See <http://www.esnipe.com/> and <http://www.lastminutebidder.com/>

item listed is still available or not. We assume instead that bidders have access to this information, which is a material difference since the specific threshold strategy we obtain as a result is different. In addition, as discussed above Caldentey and Vulcano (2004) assume bidders to act immediately (bid or buyout) upon their arrival. We do not however consider the multi-unit case here.

In summary, of all the papers cited above that analyze buyout prices, only Mathews (2004) models the arrival time of bidders and considers endogenous bidding times. It however assumes a deterministic number of bidders and only analyzes the temporary buyout option. Reynolds and Wooders (2003) – which as we pointed out ignores the impact of arrival and bid times – is the only paper we are aware of which attempts to study like we do both temporary and permanent buyout options in the same framework. Also, our model features more realistic information structure and strategy space than all others discussed, and we believe to be the first to provide an analysis of dynamic buyout prices.

Chapter 2

Market Environment and Auction Mechanism

In this chapter, we first describe our game-theoretic model, focusing on the market environment in §2.1 and the auction mechanism in §2.2. We then discuss its realism in §2.3.

2.1 Market Environment

We consider a monopolistic seller opening at time 0 a market for one item. From that point on, he faces an arrival stream of potential buyers (or bidders) which is non-observable per se, but is correctly believed by all participants to follow a Poisson process with a known, exogenous and constant rate λ . Bidders valuations (or the prices at which they are indifferent between purchasing the item and not participating in the market) are assumed to follow an independent private values model – see Klemperer (1999) for background. Specifically, each bidder has a privately known valuation, and all other participants initially share the correct belief that this valuation has been drawn independently from a distribution with cdf F and compact support $[\underline{v}, \bar{v}]$ (define $m = \bar{v} - \underline{v}$).

All participants are risk-neutral and time-sensitive. In particular, the utility of

the seller when earning revenue R at time τ is assumed to be

$$U^S(R, \tau) \triangleq e^{-\alpha\tau} R, \quad (2.1)$$

where $\alpha > 0$ denotes his time discounting factor.

Likewise, a bidder arriving at time $t > 0$ with valuation $v \in [\underline{v}, \bar{v}]$ who purchases the item at time $\tau \geq t$ for a payment of x gets utility

$$U(v, t, \tau) \triangleq e^{-\beta(\tau-t)}(v - x), \quad (2.2)$$

where $\beta > 0$ denotes his time discounting factor, assumed to be the same for all bidders. A losing bidder is assumed to derive zero utility from the market.

2.2 Auction Mechanism

The basic market mechanism we consider is a second-price auction with a time-limited bidding period $[0, T]$. That is, any bidder arriving at time $t \in [0, T]$ may submit a bid at any time in $[t, T]$, provided it is larger than any other he may already have submitted (i.e. bidders are not allowed to renege on their purchasing offers). At time T , the item is sold to the highest bidder who pays then a price equal to the second highest bid; if only one bidder has submitted a bid by T the item is sold to him for a price of \underline{v} , and if there are no bids the item is not sold. Note that the lower bound of the distribution support \underline{v} thus effectively corresponds to a publicly advertised minimum required bid (any bids lower than \underline{v} are ignored).

In addition to all the other information described previously, every bidder is assumed to know at every time τ subsequent to his arrival the value of I_τ , defined as the payment that would be made by the winning bidder if the auction were instead terminated at τ . That is, I_τ is equal to (i) the second highest bid submitted over $[0, \tau]$ if there are at least two such bids; (ii) \underline{v} if there is only one; and (iii) 0 if there is none. As is the case on all auction websites we are aware of, we assume that I_t must be truthfully revealed to any arriving bidder. For the ease of exposition, we however

allow bids below the current second highest bid to be placed in the auction unlike most auction sites. Notice that this does not affect the utility of either the bidders or the seller: such a bidder gets zero utility (neglecting his bidding cost) whether he bids or not while bids placed below the second highest bid clearly do not affect the seller's revenue.

The basic auction mechanism just defined (or a closely related version of it) is investigated for example in Vakrat and Seidmann (2001) and Gallien (2006). The critical extension that we study in the present paper is the upfront addition by the seller of a buyout price p , either temporary or permanent. Any bidder may exercise that buyout option at any time between his arrival and the end of the auction T , provided the option is still open then; this amounts to purchasing the item instantaneously at a price of p , effectively terminating the auction. A temporary buyout option remains open from the beginning until its exercise or the first time that a regular bid is submitted by any bidder, while a permanent buyout option remains open until its exercise or the end of the auction. In line with observed practice, we assume that all participants know at any point in time whether the buyout option is still open.

Notice that we assume that bids higher than the buyout price can be placed in the auction. While this is in line with practice – for instance *eBay* and *Yahoo* allow bids above the buyout price – these websites do recommend that bidders must bid lesser than the buyout price. For example, a help page on *Yahoo*¹ suggests bidders to “[...] make sure to place your maximum bid below the buy price amount.” Similarly, placing a bid higher than the buyout price on *eBay* leads to the following warning “Your maximum bid is above or equal to the Buy It Now price. We recommend you simply purchase the item via Buy It Now.” Notice that, even when the buyout option is available, a bidder with valuation higher than the buyout price may find it optimal to bid in the auction if he believes that he is likely to face very little competition from other bidders, and, as a consequence, he will be able to obtain the product at a price much lower than the buyout price (since the second highest bid is likely to be lower).

¹<http://help.yahoo.com/help/us/auct/abid/abid-15.html>

While we assume in Chapter 3 that the buyout price p remains constant throughout the auction, we study dynamic buyout prices in Chapter 4. In the dynamic extension we consider then, the seller commits upfront to a function of time $[p(t)]_{t \in [0, T]}$ describing the evolution of the buyout price (either temporary or permanent) over time, and that function is known to all bidders.

2.3 Model Discussion

Our model is motivated by the online auctions occurring on large auction sites such as *eBay* and *Yahoo*; in that spirit Figure 2-1 includes a screenshot made on October 20, 2004 of an actual ongoing auction, along with pointers to the quantities in our model representing some of its features. As can be seen in that example the buyout option is still open although eight regular bids have already been submitted, indicating that it is permanent as opposed to temporary.

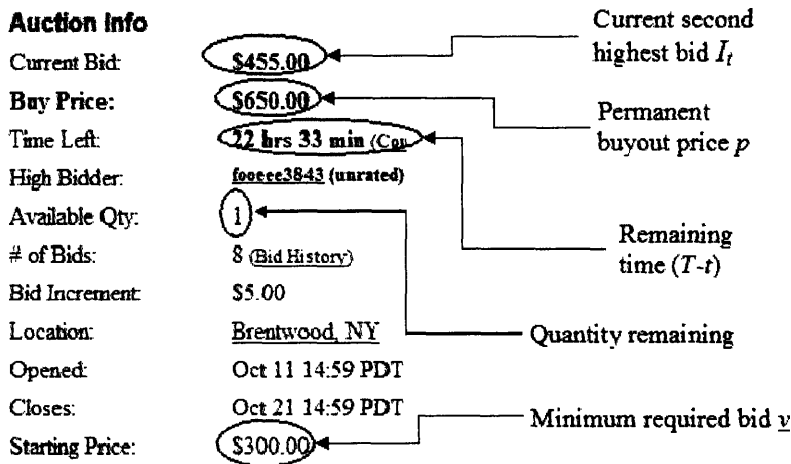


Figure 2-1: Snapshot of an online auction webpage

We first comment on our allocation mechanism. Online auction sites now typically feature “proxy bidding” systems, allowing bidders to enter the maximum amount they are willing to pay for the item. The system then submits bids on behalf of the bidder, increasing his outstanding bid whenever necessary and by as little as possible to maintain his position as the highest bidder, up until the maximum amount stated

is reached². As observed by Lucking-Reiley (2000), an online auction with a proxy bidding system effectively amounts to a second-price auction, the payment mechanism we assume.

For the closing rule, we assume a hard bidding expiration deadline similar to the one used on *eBay*, whereas some other sites such as *Amazon* use instead a floating deadline that automatically extends (within some limits) whenever a new bid close to the current deadline is submitted. As pointed out in Roth and Ockenfels (2002), this difference is material and *eBay*-like hard bidding deadlines account for a demonstrably higher concentration of bids near the end of the auction. In principle, our model allows to predict such surge of bids shortly before the end, because while we assume exogenous bidder arrival times, their bidding times are endogenous. In fact, our analysis in §3.2.1 confirms the intuition that last-minute bids seem more likely with a permanent buyout option than with a temporary one. However, our model does not capture some of the important reasons why last-minute bidding may occur: presence of inexperienced (irrational) bidders; possibility that late bids may not reach the auction site due to network transmission delays; informational value of bids when the item being sold has a common value component... while we refer the reader to Roth and Ockenfels (2002) for an excellent discussion and empirical study of this phenomenon, we argue that factors such as the loss of last minute bids due to network transmission capacity and the presence of inexperienced bidders may not remain as prevalent in the long run, partly justifying these modeling choices (otherwise primarily motivated by tractability considerations). As a result, truthful and immediate bidding is a weakly dominant strategy in the model we assume for an online auction without a buyout option.

Another feature of the market mechanism we consider is the possible presence of a publicly announced minimum required bid, denoted “Starting Price” in Figure 2-1, effectively captured in our model by the lower bound \underline{v} of the valuation distribution support. Note that this is distinct from what some auction sites (such as *eBay*) call a “reserve price”, which is likewise set by the seller as a minimum selling price for the

²See <http://pages.ebay.com/help/buy/proxy-bidding.html> for a detailed description.

item but, in contrast with the minimum required bid we use, is not publicly announced – when used by the seller, bidders are typically only informed that a reserve price has been set for the auction, and whether or not it has already been met by any of the existing bids. We assume that the seller does not use such concealed reserve price, in part because this would entail some inference of its value by the bidders, and may lead to further strategic interactions in the form of post auction negotiations between the winning bidder and the seller.

Several limitations of our analysis also stem from the market environment we consider. Our assumption that bidder arrivals follow a Poisson process seems more realistic than assuming that the number of bidders is known to all with certainty, and is partly justified by the classical Palm limit theorem on the superposition of counting processes. Nevertheless, the assumption that its arrival rate is constant and known to all participants (common to all other auction models assuming Poisson bidder arrivals that we are aware of) is still a strong one. In practice, the arrival rate of potential bidders to an auction could be variable; in particular, there may be a high concentration of bidder arrivals at the beginning of the auction – such an arrival process, for instance, can occur on auction websites, like *eBay*, that allow bidders to track newly introduced auctions. The high arrival rate of bidders at the start of the auction could be modeled by assuming that bidders arrive as a non-homogeneous Poisson process with a high arrival rate at the beginning of the auction; while this model is not analyzed in this thesis, our intuition suggests that most of the insights of our work will also be applicable to such a model. In practice, the arrival rate of bidders to a specific auction could also be endogenous, and depend for example on the bidding activity it has generated to date; it would also be influenced by factors such as advertising, the presence of a reserve price, the seller’s feedback ratings, the presence and quality of photographs describing the item, etc. Our assumption of a constant known arrival rate saliently implies that bidders, including those arriving early in the auction when only little bidding history is available, correctly synthesize the impact of these factors when estimating how many competing bidders they are likely to face.

In reality, the estimations of both the arrival rate of competing bidders and the distribution of their valuations may differ among participants. Intuition suggests however that the items for which sellers are likely to use a buyout price that will be exercised with some non-negligible probability should coincide with those for which relatively substantial historical transaction data is available – this is also supported by the results in Gallien (2006) showing the lower robustness of fixed prices relative to auctions in the presence of market uncertainty. Because large auction sites make the same extensive historical transaction data available to all participants, our assumption of common beliefs seems legitimate as a first approximation in our view. We also observe that the lower bound \underline{v} of the valuation distribution support may correspond in our model to a requested starting price, which (as can be seen on Figure 2-1, see also discussion of reserve price above) is announced to all participants.

The structure assumed here for the utility functions of the seller and the bidders is also used for example in Caldentey and Vulcano (2004) and Gallien (2006), and reflects a priori the proposed time sensitivity and risk neutrality of participants. While all the results in the paper have been derived for auctions with risk neutral participants, they can be easily generalized for risk averse auction participants – see §3.1.1 for a detailed discussion. The exponential time discounting that we assume for the seller applies to a monetary income, so that his utility can be interpreted as a straightforward net present value. As for the bidders, their exponential time discounting applies to the difference between their valuation for the item and their payment; this plausibly represents how a bidder may evaluate various actions (e.g. exercising the buyout option or submitting a regular bid) with different waiting time implications. Finally, our assumption that all bidders have the same time discounting factor is also a strong one, since bidders in online auctions are frequently end-consumers who are unlikely to share a single objective metric such as target ROI or reference interest rate when assessing their dislike of waiting.

In summary, while our model does capture some of the key features of an online auction, there are some others that it does not reproduce as faithfully. We point out that an actual online auction is inherently a complex and random process involving

multiple heterogeneous bidders with various incentives and rationality levels interacting in a dynamic manner. As such, any tractable analytical model designed to predict its outcome (including ours and every other one described in the literature) must necessarily rely on fairly restrictive assumptions. Given our primary research objective of understanding the differential impact of temporary and permanent buyout prices, we observe that several of these assumptions (e.g. common beliefs, bidder arrival process) may not specifically impact our model predictions when one type of buyout option is used as opposed to the other. From that perspective, we find it reassuring that our results rationalize some of the actual practices of auction sites using buyout options (see Chapter 6).

Chapter 3

Static Buyout Prices

In this chapter, we analyze auctions where the price of the buyout option remains fixed throughout the auction. We analyze a temporary buyout option in §3.1 and then, in §3.2, discuss a permanent buyout option.

3.1 Temporary Buyout Option

It is assumed for this section that the seller uses a fixed temporary buyout price p which disappears if a bid above the reserve price is placed in the auction. We characterize an equilibrium in bidder strategy for a temporary buyout price auction game (§3.1.1), analyze the robustness of the strategy (§3.1.2), and formulate the seller's optimization problem and discuss its solution in some asymptotic regimes (§3.1.3).

3.1.1 Outcome Prediction

For any bidder arriving at time t with valuation v , consider the following family $\mathcal{T}[\cdot]$ of threshold strategies:

$$\mathcal{T}[\nu](v, t) : \begin{cases} \text{Buyout at } p \text{ immediately} & \text{if buyout option available and } v > \nu(t) \\ \text{Bid } v \text{ immediately} & \text{if buyout option available and } v \leq \nu(t), \\ \text{Bid } v \text{ at any time in } [t, T] & \text{otherwise} \end{cases} \quad (3.1)$$

where $\nu : [0, T] \rightarrow [\underline{v}, \bar{v}]$ is a threshold valuation function. In the following we use the same notation for a strategy and the symmetric strategy profile obtained when every bidder plays that strategy, since no ambiguity arises from the present context.

Our main result in this section is the following which establishes the existence of a threshold function ν_{tmp} such that $\mathcal{T}[\nu_{tmp}]$ forms a Bayesian Nash equilibrium, and also provides a characterization of that function.

Theorem 1. *Define function ν_{tmp} as $\nu_{tmp}(t) = \min(\hat{v}(t), \bar{v})$ where $\hat{v}(t)$ is the unique solution on $[\underline{v}, +\infty)$ of the equation*

$$\hat{v}(t) - p = e^{-(\lambda+\beta)(T-t)} \int_{\underline{v}}^{\hat{v}(t)} e^{\lambda(T-t)F(x)} dx. \quad (3.2)$$

Then the symmetric strategy profile $\mathcal{T}[\nu_{tmp}]$ is a Bayesian Nash equilibrium for the online auction game with a temporary buyout price p .

In the equilibrium described in Theorem 1, the first incoming bidder compares upon his arrival the relative attractiveness of the buyout option and that of a regular bid, accounting for the likely competition resulting from the specific auction time remaining then; the dynamic threshold ν_{tmp} valuation characterized in (3.2) corresponds to the valuation of a bidder who at that time would be indifferent between the two options. Note that strategy $\mathcal{T}[\nu_{tmp}]$ and the associated equilibrium result just stated do not provide a prediction of when the second and subsequent bidders will submit their bid. That is, the timing of bid submissions for these bidders does not

have any strategic implication within the strict boundaries of our model definition. In practice however, it could be affected in various ways by features not captured by our model; for example a high cost of monitoring the auction could hasten bid submissions, while common value signaling could delay them – see §2.3 for a more complete discussion and related references.

The result in Theorem 1 is obtained by first deriving, for an arbitrary threshold function ν , a best response strategy to profile $\mathcal{T}[\nu]$, that is a strategy maximizing the utility of a bidder entering an auction where every other bidder uses strategy $\mathcal{T}[\nu]$. Specifically, denoting $\mathcal{R}(\mathcal{T}[\nu])$ the set of these best response strategies, we characterize a threshold function ν_{tmp} such that $\mathcal{T}[\nu_{tmp}] \in \mathcal{R}(\mathcal{T}[\nu])$. We further show that $\mathcal{T}[\nu_{tmp}] \in \mathcal{R}(\mathcal{T}[\nu_{tmp}])$, establishing that the profile $\mathcal{T}[\nu_{tmp}]$ constitutes indeed a Nash equilibrium.

Indeed consider a bidder A with type (v, t) in an auction where every other bidder uses strategy $\mathcal{T}[\nu]$, where ν is an arbitrary threshold function. If A is not the first bidder, the first bidder would have either placed a bid or exercised the option immediately on arrival (following strategy $\mathcal{T}[\nu]$), so that the buyout option is not available to bidder A . In that case, bidder A 's weakly dominant strategy is to bid his true valuation v , as shown in Vickrey (1961). His bid submission time in $[t, T]$ will not affect his utility in any way, so that bidding v at any time in $[t, T]$ constitutes then a best response.

Suppose now that A is the first bidder, so that the buyout option is available to him. We introduce the following notation for the three possible actions he may take at time t :

bid(t) : Bid in the auction at time t (in which case it is a dominant strategy for him to bid his valuation v);

buy(t) : Buyout at time t ;

wait(t, τ) : Wait for $\tau - t$ time units before deciding to bid (if the auction is still open) or buy out (if the option is still available).

We define the utility of bidder A with type (v, t) and taking action $a \in \{bid(t), buy(t), wait(t, \tau)\}$ as $U_a(v, t)$. If bidder A chooses $bid(t)$, i.e. bids immediately, the buyout option disappears. Following strategy $\mathcal{T}[\nu]$, all subsequent bidders will bid their true valuation. Denoting by $N(t, T)$ the random number of bidders arriving in interval $(t, T]$ and $N(t)$ the cumulative number of arrivals up to t , this implies:

$$\mathbf{E}[U_{bid(t)}(v, t) | N_t = 0, N(t, T)] = e^{-\beta(T-t)} \int_{\underline{v}}^v F(x)^{N(t, T)} dx, \quad (3.3)$$

and using the model assumption that $N(t, T)$ is Poisson with parameter $\lambda(T-t)$, we obtain the expected utility of the first bidders when bidding his valuation v upon his arrival at t :

$$\mathbf{E}[U_{bid(t)}(v, t) | N_t = 0] = e^{-(\lambda+\beta)(T-t)} \int_{\underline{v}}^v e^{\lambda(T-t)F(x)} dx \quad (3.4)$$

$$\triangleq B_1(v, t). \quad (3.5)$$

Conditional on A being the first bidder (i.e. $N_t = 0$), the utility from exercising the buyout option immediately is:

$$\mathbf{E}[U_{buy(t)}(v, t) | N_t = 0] = v - p \quad (3.6)$$

The key to deriving bidder A 's best response is the following Lemma, which establishes that bidder A 's expected utility from acting immediately upon his arrival (i.e. choosing either $bid(t)$ or $buy(t)$) is always as large as that obtained from waiting, i.e. $\mathbf{E}[U_{wait(t, \tau)}(v, t) | N_t = 0]$:

Lemma 1. $\mathbf{E}[U_{wait(t, \tau)}(v, t) | N_t = 0] \leq \max \{B_1(v, t), v - p\}$

Proof. Let $\mathcal{E} = \{N(t, \tau) = 0\}$ be the event that no bidder arrives in the interval (t, τ) . In this case bidder A remains the first bidder so that

$$\begin{aligned} \mathbf{E}[U_{bid(\tau)}(v, t) | N_t = 0, \mathcal{E}] &= e^{-\beta(\tau-t)} \mathbf{E}[U_{bid(\tau)}(v, \tau) | N_\tau = 0] \\ &= e^{-\beta(\tau-t)} B_1(v, \tau), \end{aligned} \quad (3.7)$$

and the buyout option is still available thus

$$\mathbf{E}[U_{wait(t,\tau)}(v,t)|N_t = 0, \mathcal{E}] = e^{-\beta(\tau-t)} \max \{B_1(v, \tau), v - p\}.$$

The complementary event $\bar{\mathcal{E}} = \{N(t, \tau) > 0\}$ corresponds to one or more arrivals occurring in the interval (t, τ) . In that case the buyout option is no longer available, so that

$$\begin{aligned} \mathbf{E}[U_{wait(t,\tau)}(v,t)|N_t = 0, \bar{\mathcal{E}}] &= \mathbf{E}[U_{bid(\tau)}(v,t)|N_t = 0, \bar{\mathcal{E}}] \\ &= e^{-\beta(\tau-t)} \mathbf{E}[U_{bid(\tau)}(v, \tau)|N_t = 0, \bar{\mathcal{E}}]. \end{aligned} \quad (3.8)$$

Note that the event $\bar{\mathcal{E}}$ includes the event that one of the bidders who arrived during (t, τ) exercised the buyout option, in which case bidder A 's utility is zero. The expected utility of the first bidder A if he waits up to time $\tau > t$ is thus

$$\begin{aligned} &\mathbf{E}[U_{wait(t,\tau)}(v,t)|N_t = 0] \\ &= e^{-\beta(\tau-t)} \left(\max \{B_1(v, \tau), v - p\} \cdot P(\mathcal{E}) + \mathbf{E}[U_{bid(\tau)}(v, \tau)|N_t = 0, \bar{\mathcal{E}}] \cdot P(\bar{\mathcal{E}}) \right) \end{aligned} \quad (3.9)$$

By the law of conditional expectation, we also have:

$$B_1(v, t) = \mathbf{E}[U_{bid(t)}(v, t)|N_t = 0, \mathcal{E}] \cdot P(\mathcal{E}) + \mathbf{E}[U_{bid(t)}(v, t)|N_t = 0, \bar{\mathcal{E}}] \cdot P(\bar{\mathcal{E}}) \quad (3.10)$$

Define \mathcal{G} as the event that the first bidder, say B , arriving in (t, τ) with type (v_B, t_B) (where $t_B \in (t, \tau)$) has valuation $\nu(t_B) < v_B \leq v$. Notice that $P(\mathcal{G}|\bar{\mathcal{E}}) \geq 0$; in particular, $P(\mathcal{G}|\bar{\mathcal{E}}) = 0$ if $v \leq \nu(t_B)$ for all $t_B \in (t, \tau)$. Then (3.10) can be rewritten as:

$$\begin{aligned} B_1(v, t) &= \mathbf{E}[U_{bid(t)}(v, t)|N_t = 0, \mathcal{E}] \cdot P(\mathcal{E}) + \mathbf{E}[U_{bid(t)}(v, t)|N_t = 0, \bar{\mathcal{E}}, \mathcal{G}] \cdot P(\mathcal{G}|\bar{\mathcal{E}}) \cdot P(\bar{\mathcal{E}}) \\ &\quad + \mathbf{E}[U_{bid(t)}(v, t)|N_t = 0, \bar{\mathcal{E}}, \bar{\mathcal{G}}] \cdot P(\bar{\mathcal{G}}|\bar{\mathcal{E}}) \cdot P(\bar{\mathcal{E}}) \end{aligned} \quad (3.11)$$

where $\bar{\mathcal{G}}$ is the complementary event.

Conditional on the event \mathcal{E} (i.e. $N(t, \tau) = 0$), the expected utility of bidding is same whether A bids at time t or τ , i.e.

$$\mathbf{E}[U_{bid(t)}(v, t)|N_t = 0, \mathcal{E}] = \mathbf{E}[U_{bid(\tau)}(v, t)|N_t = 0, \mathcal{E}] \quad (3.12)$$

Now consider the case when the event $\mathcal{G} \cap \bar{\mathcal{E}}$ occurs, i.e. the bidder B with type (v_B, t_B) has valuation $\nu(t_B) < v_B \leq v$. If bidder A bids in the auction at time t then the buyout option disappears and so B also bids in the auction; however if bidder A waits up to τ , then the buyout option is still present at time $t_B \in (t, \tau)$ and bidder B , following strategy $\mathcal{T}[\nu]$, exercises the buyout option. As a result, we have

$$\mathbf{E}[U_{bid(t)}(v, t)|N_t = 0, \bar{\mathcal{E}}, \mathcal{G}] > \mathbf{E}[U_{bid(\tau)}(v, t)|N_t = 0, \bar{\mathcal{E}}, \mathcal{G}] = 0 \quad (3.13)$$

where $\mathbf{E}[U_{bid(t)}(v, t)|N_t = 0, \bar{\mathcal{E}}, \mathcal{G}] > 0$ since $v_B \leq v$.

Additionally, conditional on the event $\bar{\mathcal{E}} \cap \bar{\mathcal{G}}$ we have

$$\mathbf{E}[U_{bid(t)}(v, t)|N_t = 0, \bar{\mathcal{E}}, \bar{\mathcal{G}}] = \mathbf{E}[U_{bid(\tau)}(v, t)|N_t = 0, \bar{\mathcal{E}}, \bar{\mathcal{G}}] \quad (3.14)$$

This can be explained as follows: the event $\bar{\mathcal{G}}$ implies that either

1. $v_B > v$ – In this case the expected utility from bidding is zero irrespective of when bidder A bids,
2. $v_B \leq \nu(t_B)$ – In this case bidder B , following strategy $\mathcal{T}[\nu]$, bids in the auction immediately if A waits up to τ and thus the buyout option disappears. If A bids at time t then also the buyout option disappears and thus, irrespective of when A bids, the buyout option is not exercised. Hence the expected utility from bidding for A is same from both actions $bid(t)$ and $bid(\tau)$.

Using (3.14), (3.12) and (3.13) in (3.11) we get

$$\begin{aligned}
B_1(v, t) &\geq \mathbf{E}[U_{bid(\tau)}(v, t)|N_t = 0, \mathcal{E}] \cdot P(\mathcal{E}) + \mathbf{E}[U_{bid(\tau)}(v, t)|N_t = 0, \bar{\mathcal{E}}, \mathcal{G}] \cdot P(\mathcal{G}|\bar{\mathcal{E}}) \cdot P(\bar{\mathcal{E}}) \\
&\quad + \mathbf{E}[U_{bid(\tau)}(v, t)|N_t = 0, \bar{\mathcal{E}}, \bar{\mathcal{G}}] \cdot P(\bar{\mathcal{G}}|\bar{\mathcal{E}}) \cdot P(\bar{\mathcal{E}}) \\
&= \mathbf{E}[U_{bid(\tau)}(v, t)|N_t = 0, \mathcal{E}] \cdot P(\mathcal{E}) + \mathbf{E}[U_{bid(\tau)}(v, t)|N_t = 0, \bar{\mathcal{E}}] \cdot P(\bar{\mathcal{E}}) \quad (3.15)
\end{aligned}$$

Furthermore

$$e^{-\beta(\tau-t)} B_1(v, \tau) \geq B_1(v, t), \quad (3.16)$$

because while both sides of the above inequality have the same time discounting, the right hand side is conditioned on $N_t = 0$ and the left hand side is conditioned on $N_\tau = 0$ (implying fewer competing bidders). Additionally, as indicated earlier in (3.7), we have $e^{-\beta(\tau-t)} B_1(v, \tau) = \mathbf{E}[U_{bid(\tau)}(v, t)|N_t = 0, \mathcal{E}]$. Equation (3.15) and inequality (3.16) thus imply together that

$$\mathbf{E}[U_{bid(\tau)}(v, t)|N_t = 0, \bar{\mathcal{E}}] \leq B_1(v, t). \quad (3.17)$$

Consider now the following two cases:

- **Case 1:** $v - p \leq B_1(v, \tau)$

Equation (3.9) becomes then

$$\begin{aligned}
&\mathbf{E}[U_{wait(t, \tau)}(v, t)|N_t = 0] \\
&= e^{-\beta(\tau-t)} B_1(v, \tau) \cdot P(\mathcal{E}) + e^{-\beta(\tau-t)} \mathbf{E}[U_{bid(\tau)}(v, \tau)|N_t = 0, \bar{\mathcal{E}}] \cdot P(\bar{\mathcal{E}}) \\
&= \mathbf{E}[U_{bid(\tau)}(v, t)|N_t = 0, \mathcal{E}] \cdot P(\mathcal{E}) + \mathbf{E}[U_{bid(\tau)}(v, t)|N_t = 0, \bar{\mathcal{E}}] \cdot P(\bar{\mathcal{E}}) \\
&\leq B_1(v, t) \leq \max \{ B_1(v, t), v - p \},
\end{aligned}$$

where the second equality follows from (3.7) and (3.8) and the first inequality follows from (3.15).

- **Case 2:** $v - p > B_1(v, \tau)$

In this case notice that

$$\begin{aligned}
e^{-\beta(\tau-t)}(v-p) &> e^{-\beta(\tau-t)}B_1(v, \tau) \\
&\geq B_1(v, t) \\
&\geq \mathbf{E}[U_{bid(\tau)}(v, t)|N_t = 0, \bar{\mathcal{E}}] \\
&= e^{-\beta(\tau-t)}\mathbf{E}[U_{bid(\tau)}(v, \tau)|N_t = 0, \bar{\mathcal{E}}], \tag{3.18}
\end{aligned}$$

where the second and third inequalities follow from (3.16) and (3.17) respectively, and the final equality follows from (3.8).

Equation (3.9) thus implies

$$\begin{aligned}
\mathbf{E}[U_{wait(t, \tau)}(v, t)|N_t = 0] &= e^{-\beta(\tau-t)}\left((v-p)P(\mathcal{E}) + \mathbf{E}[U_{bid(\tau)}(v, \tau)|N_t = 0, \bar{\mathcal{E}}]P(\bar{\mathcal{E}})\right) \\
&< e^{-\beta(\tau-t)}(v-p)P(\mathcal{E}) + e^{-\beta(\tau-t)}(v-p)P(\bar{\mathcal{E}}) \\
&= e^{-\beta(\tau-t)}(v-p) \\
&< (v-p) \leq \max\{B_1(v, t), v-p\},
\end{aligned}$$

where the first inequality follows from (3.18) and the second inequality from the law of total probability.

Because cases 1 and 2 above are exhaustive, the proof is complete. \square

We have thus established the best response for bidder A , if he sees the buyout option, is to act immediately upon his arrival. Defining now $\delta(v, t) \triangleq \mathbf{E}[U_{buy(t)}(v, t)|N_t = 0] - \mathbf{E}[U_{bid(t)}(v, t)|N_t = 0]$ as the expected utility difference from exercising the buyout option and placing a bid immediately for the first bidder, equations (3.3) and (3.6) imply

$$\delta(v, t) = v - p - e^{-(\lambda+\beta)(T-t)} \int_{\underline{v}}^v e^{\lambda(T-t)F(x)} dx. \tag{3.19}$$

Notice that $\delta(v, t)$ is continuous and differentiable on $[\underline{v}, +\infty) \times [0, T]$, and it is

increasing in v for all $t \in [0, T]$ since

$$\frac{\partial \delta(v, t)}{\partial v} = 1 - e^{(-\beta - \lambda(1-F(v)))(T-t)} > 0. \quad (3.20)$$

Assuming without loss of generality that $p \geq \underline{v}$ implies that $\delta(\underline{v}, t) \leq 0$ for all $t \in [0, T]$ which combined with (3.20) proves the existence of a unique $\hat{v}(t) \in [\underline{v}, +\infty)$ such that $\delta(\hat{v}(t), t) = 0$. Defining $\nu_{tmp}(t) \triangleq \min(\hat{v}(t), \bar{v})$ and denoting $\mathcal{R}(\mathcal{T}[\nu])$ the set of best response strategies to the symmetric profile $\mathcal{T}[\nu]$, we have thus proven that $\mathcal{T}[\nu_{tmp}] \in \mathcal{R}(\mathcal{T}[\nu])$. But because the characterization of ν_{tmp} provided by $\delta(\hat{v}(t), t) = 0$ does not depend on the choice of ν as can be seen from equation (3.19), we also have $\mathcal{T}[\nu_{tmp}] \in \mathcal{R}(\mathcal{T}[\nu_{tmp}])$, that proving that $\mathcal{T}[\nu_{tmp}]$ is a Bayesian Nash equilibrium of the temporary buyout price auction game. This completes the proof of Theorem 1.

The following proposition provides a closed-form expression for the equilibrium described in the statement of Theorem 1 for the special case of uniformly distributed valuations:

Proposition 1. *When bidder valuations are uniformly distributed on $[\underline{v}, \bar{v}]$, the threshold function ν_{tmp} characterizing the Bayesian Nash equilibrium described in Theorem 1 is*

$$\nu_{tmp}(t) = \min \left(p - \frac{m}{\lambda(T-t)} \left(W \left(-e^{-e^{-(\lambda+\beta)(T-t) + \frac{(p-\underline{v})\lambda(T-t)}{m} - (\lambda+\beta)(T-t)}} + e^{-(\lambda+\beta)(T-t)} \right), \bar{v} \right), \quad (3.21)$$

where W is Lambert's W or omega function, i.e. the inverse of $W \mapsto We^W$.

Proof. In Appendix. □

Before discussing the robustness of the equilibrium strategy derived above, we comment on the extension of the equilibrium results for the case when bidders are risk averse. The structure of the equilibrium strategy derived above remains the same if we assume, say, that bidders are risk averse with a CARA type utility function, i.e.

a bidder with valuation v who purchases the item at price x gets utility

$$U^R(v) \triangleq 1 - e^{-r(v-x)}$$

where $r > 0$ is the coefficient of risk aversion. Indeed under, certain technical conditions, Theorem 1 can be extended to show that for a temporary buyout price auction with risk averse bidders, a threshold strategy of the form $\mathcal{T}[\cdot]$ defines a Bayesian Nash equilibrium with a threshold function $\nu_{tmp}^{(r)}$. We prove the following result.

Theorem 2. *Let $\hat{v}(t)$ be the solution on $[\underline{v}, \bar{v}]$, if such a solution exists, of the equation*

$$\begin{aligned} & 1 - e^{-r(\hat{v}(t)-p)} \\ &= e^{-\lambda(T-t)} \left(e^{\lambda(T-t)F(\hat{v}(t))} - e^{-r(\hat{v}(t)-\underline{v})} - \lambda(T-t) \int_{\underline{v}}^{\hat{v}(t)} e^{-r(\hat{v}(t)-x)+\lambda(T-t)F(x)} f(x) dx \right) \end{aligned} \quad (3.22)$$

Define function $\nu_{tmp}^{(r)}$ as $\nu_{tmp}^{(r)}(t) = \hat{v}(t)$ if (3.22) has a solution on $[\underline{v}, \bar{v}]$; otherwise $\nu_{tmp}^{(r)}(t) = \bar{v}$. Then if p is such that

$$p > \frac{1}{r} \ln \left(e^{-r\bar{v}} - r \int_{\underline{v}}^{\bar{v}} e^{rx-\lambda T(1-F(x))} dx \right) \quad (3.23)$$

the symmetric strategy profile $\mathcal{T}[\nu_{tmp}^{(r)}]$ is a Bayesian Nash equilibrium for the online auction game with a temporary buyout price p .

The extra condition (3.23) on the buyout price is required to ensure that (3.22) has at most one solution on $[\underline{v}, \bar{v}]$. Notice that (3.23) is only a sufficient condition and indeed seems pretty strong – in all our numerical experiments even when this condition was violated, the equation (3.22) had at most one solution on $[\underline{v}, \bar{v}]$.

Proof of Theorem 2. Consider a bidder A with type (v, t) in an auction where every other bidder uses strategy $\mathcal{T}[\nu]$, where ν is an arbitrary threshold function. If A is not the first bidder, the first bidder would have either placed a bid or exercised the option immediately on arrival (following strategy $\mathcal{T}[\nu]$), so that the buyout option is

not available to bidder A . In that case, bidder A 's weakly dominant strategy is to bid his true valuation v . His bid submission time in $[t, T]$ will not affect his utility in any way, so that bidding v at any time in $[t, T]$ constitutes then a best response.

Suppose now that A is the first bidder, so that the buyout option is available to him. The result of Lemma 1 holds for risk averse bidders also thus proving that A 's expected utility from acting immediately upon arrival (i.e. choosing either $bid(t)$ or $buy(t)$) is always as large as that obtained from waiting. Thus the best response for bidder A , if he sees the buyout option, is to act immediately upon his arrival.

The expected utility from bidding for risk averse bidders $\mathbf{E}[U_{bid(t)}^R(v, t)|N_t = 0]$ can be shown to be

$$\begin{aligned} \mathbf{E}[U_{bid(t)}^R(v, t)|N_t = 0] \\ = e^{-\lambda(T-t)} \left(e^{\lambda(T-t)F(v)} - e^{-r(v-\underline{v})} - \lambda(T-t) \int_{\underline{v}}^v e^{-r(v-x)+\lambda(T-t)F(x)} f(x) dx \right) \end{aligned}$$

where $f(x) = \frac{\partial}{\partial x} F(x)$ is the probability density function corresponding to the cdf $F(\cdot)$. The utility from exercising the buyout option is given by $\mathbf{E}[U_{buy(t)}^R(v, t)|N_t = 0] = 1 - e^{-r(v-p)}$. We next derive a sufficient condition such that the equation

$$\mathbf{E}[U_{buy(t)}^R(v, t)|N_t = 0] = \mathbf{E}[U_{bid(t)}^R(v, t)|N_t = 0] \quad (3.24)$$

has at most one solution on $[\underline{v}, \bar{v}]$.

Lemma 2. *If*

$$p > \frac{1}{r} \log \left(e^{-r\bar{v}} - r \int_{\underline{v}}^{\bar{v}} e^{rx-\lambda T(1-F(x))} dx \right)$$

then (3.24) has at most one solution on $[\underline{v}, \bar{v}]$.

Proof. A sufficient condition for proving that (3.24) does not have multiple solutions on $[\underline{v}, \bar{v}]$ is that for any $t \in [0, T]$

$$\frac{\partial}{\partial v} \mathbf{E}[U_{buy(t)}^R(v, t)|N_t = 0] > \frac{\partial}{\partial v} \mathbf{E}[U_{bid(t)}^R(v, t)|N_t = 0], \quad \forall v \in [\underline{v}, \bar{v}]$$

Using the expressions for $\mathbf{E}[U_{buy(t)}^R(v, t)|N_t = 0]$ and $\mathbf{E}[U_{bid(t)}^R(v, t)|N_t = 0]$ this condition can be rewritten as

$$e^{rp} > \max_{v \in [\underline{v}, \bar{v}]} e^{-\lambda(T-t)} \left[e^{rv + \lambda(T-t)F(v)} - r \int_{\underline{v}}^v e^{rx + \lambda(T-t)F(x)} dx \right] \quad (3.25)$$

$$= e^{r\bar{v}} - r \int_{\underline{v}}^v e^{rx - \lambda(T-t)(1-F(x))} dx \quad (3.26)$$

where the equality follows since the expression on the right hand side of (3.25) is increasing in v . Since the right hand side of the inequality (3.26) is decreasing in t it is sufficient to impose the condition at $t = 0$ thus completing the proof of the lemma. \square

Define function $\nu_{tmp}^{(r)}$ as $\nu_{tmp}^{(r)}(t) = \hat{v}(t)$ where $\hat{v}(t)$ is the solution of (3.24) if a solution exists on $[\underline{v}, \bar{v}]$; otherwise $\nu_{tmp}^{(r)}(t) = \bar{v}$. Combined with the fact that for $p \in [\underline{v}, \bar{v}]$, $\mathbf{E}[U_{buy(t)}^R(\underline{v}, t)|N_t = 0] \leq \mathbf{E}[U_{bid(t)}^R(v, t)|N_t = 0]$ proves that $\mathcal{T}[\nu_{tmp}^{(r)}] \in \mathcal{R}(\mathcal{T}[\nu])$. But because the characterization of $\nu_{tmp}^{(r)}$ does not depend on the choice of ν as can be seen from equation (3.24), we also have $\mathcal{T}[\nu_{tmp}^{(r)}] \in \mathcal{R}(\mathcal{T}[\nu_{tmp}^{(r)}])$, thus proving that $\mathcal{T}[\nu_{tmp}^{(r)}]$ is a Bayesian Nash equilibrium of the temporary buyout price auction game. \square

Consider any t where the threshold valuation $\nu_{tmp}^{(r)}$ is such that

$$\mathbf{E}[U_{buy(t)}^R(\nu_{tmp}^{(r)}(t), t)|N_t = 0] = \mathbf{E}[U_{bid(t)}^R(\nu_{tmp}^{(r)}(t), t)|N_t = 0],$$

that is,

$$1 - e^{-r(\nu_{tmp}^{(r)}(t) - p)} = \mathbf{E}[U_{bid(t)}^R(\nu_{tmp}^{(r)}(t), t)|N_t = 0].$$

Then, since $U^R(\cdot)$ is concave, using Jensen's inequality, we have that

$$1 - e^{-r(\nu_{tmp}^{(r)}(t) - p)} \leq 1 - e^{-r\mathbf{E}[U_{bid(t)}^R(\nu_{tmp}^{(r)}(t), t)|N_t = 0]}$$

which implies that

$$\nu_{tmp}^{(r)}(t) - p \leq \mathbf{E}[U_{bid(t)}(\nu_{tmp}^{(r)}(t), t) | N_t = 0]$$

This combined with (3.19) shows that $\nu_{tmp}^{(r)}(t) \leq \nu_{tmp}(t)$ where $\nu_{tmp}(t)$ is the threshold valuation for risk-neutral bidders with $\beta = 0$. Similarly if $\nu_{tmp}^{(r)}(t) = \bar{v}$ then for all $v \in [\underline{v}, \bar{v}]$

$$\begin{aligned} 1 - e^{-r(v-p)} &\leq \mathbf{E}[U_{bid(t)}^R(v, t) | N_t = 0] \\ &\leq 1 - e^{-r\mathbf{E}[U_{bid(t)}(v, t) | N_t = 0]} \end{aligned}$$

where the second inequality follows again from Jensen's inequality. This implies that for all $v \in [\underline{v}, \bar{v}]$

$$v - p \leq \mathbf{E}[U_{bid(t)}(v, t) | N_t = 0]$$

which then means that $\nu_{tmp}(t) = \bar{v}$. Thus for all $t \in [0, T]$, $\nu_{tmp}^{(r)}(t) \leq \nu_{tmp}(t)$; intuitively, if bidders are risk averse then exercising the buyout option, which clearly involves no risk (as opposed to bidding where both the price and success of getting the product is uncertain), is much more attractive and hence the threshold valuation for exercising the option is lower. The results for the permanent buyout price case, derived in §3.2, can be similarly generalized for risk averse bidders.

3.1.2 Equilibrium Refinements

An important observation concerning Theorem 1 is that the equilibrium $\mathcal{T}[\nu]$ it specifies is not unique. Indeed, for any $w > 0$ one may choose a threshold function $\nu : [0, T] \rightarrow [\underline{v}, \bar{v}]$ such that the strategy $\mathcal{T}^{(w)}[\nu]$ defined as

$$\mathcal{T}^{(w)}[\nu](v, t) : \begin{cases} \text{Buyout at } p \text{ immediately} & \text{if buyout option available and } v > \nu(t) \\ \text{Bid } v \text{ after } \min(w, T - t) \text{ time units} & \text{if buyout option available and } v \leq \nu(t) \\ \text{Bid } v \text{ at any time in } [t, T] & \text{otherwise} \end{cases} \quad (3.27)$$

also constitutes an equilibrium. That is, in the equilibria $\mathcal{T}^{(w)}[\nu]$ with $w > 0$, a bidder finding the buyout option still available when he arrives may wait for some time before submitting a bid. We next argue that, in contrast to $\mathcal{T}[\nu]$, such an equilibrium does not survive two equilibrium refinement techniques, and therefore does not provide a robust outcome prediction.

While our model assumes all bidders are rational and have the same utility function, in practice a bidder in an online auction faces bidders with different preferences, bidding experience and levels of rationality, and he may be uncertain about the payoff function of the other bidders. To incorporate this uncertainty, we assume that there is some randomness associated with bidders' payoff functions and test which strategies still define an equilibrium of the buyout price auction game under this perturbation; see Harsanyi (1973) and van Damme (1987) for a discussion of games with perturbed payoffs. Another technique used to refine the set of equilibria in incomplete information games is the concept of trembling hand perfection which we also discuss in this section.

Payoff perturbations

Let G denote the online auction game with a temporary buyout option described in §2.2 and §2.1. We consider perturbations in the payoff function (see van Damme (1987)) and define $G^{(\epsilon)}$ as a game identical to G except that with a small probability $\epsilon > 0$ an arriving bidder is *desperate*, meaning that his utility from the auction with a type (v, t) is described instead by

$$U_D(v, t) = \begin{cases} +M & \text{if he obtains the item at } t; \\ -M & \text{if he bids in the auction;} \\ 0 & \text{otherwise,} \end{cases} \quad \text{where } M \gg 0. \quad (3.28)$$

In words, desperate bidders greatly value the item auctioned, have an outside alternative with negligible value, and cannot wait under any circumstances; the dominant strategy for a desperate bidder is obviously to exercise the buyout option if it is available and to not participate at all otherwise. This specific perturbation seems

appealing, because it may reveal the limiting impact of irrational bidders or bidders with different time sensitivities that our model otherwise assumes away (see §2.3).

We prove the following result:

Theorem 3. *The game $G^{(\epsilon)}$ does not have any Bayesian Nash equilibrium where a non-desperate bidder, who arrives when the buyout option is present, waits before bidding (e.g. plays $T^{(w)}[\cdot]$ with $w > 0$). In addition, there exists a threshold function $\nu_{tmp}^{(\epsilon)} : [0, T] \rightarrow [\underline{v}, \bar{v}]$ such that for non-desperate bidders the strategy profile $T[\nu_{tmp}^{(\epsilon)}]$ is a Bayesian Nash equilibrium of the game $G^{(\epsilon)}$, and $\lim_{\epsilon \rightarrow 0} \nu_{tmp}^{(\epsilon)} = \nu_{tmp}$ where ν_{tmp} is defined in Theorem 1.*

Proof. Suppose a bidder, say A , arrives in the auction when the buyout option is present. If A is *desperate* then his strictly dominant strategy is to exercise the buyout option immediately and thus any strategy where he waits cannot be an equilibrium of the game $G^{(\epsilon)}$.

Now suppose A is not *desperate* and is indeed of the type (v, t) . We next show that the utility from bidding immediately is strictly greater than the utility from waiting for w units of time (where $0 < w < T - t$) and then bidding, i.e. in the notation of Theorem 1, $\mathbf{E}[U_{bid(t)}(v, t)|\mathcal{B}] > \mathbf{E}[U_{bid(t+w)}(v, t)|\mathcal{B}]$ for all $w \in (0, T - t]$ where \mathcal{B} denotes the event that the buyout option is present when A arrives. The utility from bidding is calculated assuming the subsequent non-*desperate* bidders follow any arbitrary strategy while the *desperate* bidders follow their dominant strategy which is to exercise the buyout option immediately, if available, and to not participate in the auction otherwise. Notice that, as before, bidders are assumed to be rational and thus if, and when, they choose to bid in the auction they will bid their true valuation (which is their weakly dominant strategy since this is a second-price auction).

Similar to the proof of Theorem 1, using the law of conditional expectation, we have:

$$\mathbf{E}[U_{bid(t)}(v, t)|\mathcal{B}] = \mathbf{E}[U_{bid(t)}(v, t)|\mathcal{B}, \mathcal{D}] \cdot P(\mathcal{D}|\mathcal{B}) + \mathbf{E}[U_{bid(t)}(v, t)|\mathcal{B}, \bar{\mathcal{D}}] \cdot P(\bar{\mathcal{D}}|\mathcal{B}) \quad (3.29)$$

where \mathcal{D} denotes the event that the buyout option is exercised by a *desperate* bidder if bidder A waits up to time $t + w$. Notice that the event $\{\text{first bidder arriving in } (t, t + w) \text{ is } \textit{desperate}\} \subset \{\mathcal{D}|\mathcal{B}\}$ and thus we have

$$P(\mathcal{D}|\mathcal{B}) \geq P(\text{first bidder arriving in } (t, t + w) \text{ is } \textit{desperate}) > 0 \quad (3.30)$$

Since *desperate* bidders do not participate in the auction if the buyout option is not present, we have

$$\mathbf{E}[U_{bid(t)}(v, t)|\mathcal{B}, \mathcal{D}] > \mathbf{E}[U_{bid(t+w)}(v, t)|\mathcal{B}, \mathcal{D}] = 0 \quad (3.31)$$

Bidder A bids in the auction at $t + w$, if it is still open, and hence the buyout option surely disappears at $t + w$. Thus no *desperate* bidders arriving in the interval $(t + w, T]$ participate in the auction. In addition if the event \mathcal{D} does not occur then it implies that no *desperate* bidder arriving in the interval $(t, t + w)$ participates in the auction. Thus the presence of *desperate* bidders does not affect the expected utility of bidder A if the event $\bar{\mathcal{D}}$ occurs and hence the analysis used to obtain (3.15) can be essentially repeated, with minor modifications to incorporate the fact that subsequent bidders follow some arbitrary strategy, to get:

$$\mathbf{E}[U_{bid(t)}(v, t)|\mathcal{B}, \bar{\mathcal{D}}] \geq \mathbf{E}[U_{bid(t+w)}(v, t)|\mathcal{B}, \bar{\mathcal{D}}] \quad (3.32)$$

Using (3.32), (3.31) and (3.30) in (3.29), we obtain

$$\mathbf{E}[U_{bid(t)}(v, t)|\mathcal{B}] > \mathbf{E}[U_{bid(t+w)}(v, t)|\mathcal{B}, \mathcal{D}] \cdot P(\mathcal{D}|\mathcal{B}) + \mathbf{E}[U_{bid(t+w)}(v, t)|\mathcal{B}, \bar{\mathcal{D}}] \cdot P(\bar{\mathcal{D}}|\mathcal{B}) \quad (3.33)$$

$$= \mathbf{E}[U_{bid(t+w)}(v, t)|\mathcal{B}] \quad (3.34)$$

which proves that bidder A is strictly better off bidding immediately. Thus any strategy where a bidder, who arrives when the buyout option is present, waits for w units of time ($w > 0$) is not a Bayesian Nash equilibrium of $G^{(\epsilon)}$.

The second result of the theorem involves characterizing a threshold function ν such that the strategy where the non-*desperate* bidders play $\mathcal{T}[\nu]$ is a Bayesian Nash equilibrium of the game $G^{(\epsilon)}$.

If the first bidder arriving in the auction is *desperate* then his dominant strategy is to exercise the buyout option immediately and the auction ends. Otherwise if the first bidder, say A and of type (v, t) , is non-*desperate* then the analysis of Theorem 1 can be repeated exactly to show that the best response strategy of A is to bid immediately if his valuation $v \leq \nu_{tmp}^{(\epsilon)}(t)$ and to exercise the buyout option immediately otherwise. The threshold valuation $\nu_{tmp}^{(\epsilon)} = \min(\hat{v}(t), \bar{v})$ where $\hat{v}(t)$ is the unique solution in $[\underline{v}, +\infty)$ of

$$\hat{v}(t) - p = e^{-(\lambda(1-\epsilon)+\beta)(T-t)} \int_{\underline{v}}^{\hat{v}(t)} e^{\lambda(1-\epsilon)(T-t)F(x)} dx, \quad (3.35)$$

which is same as (3.1) except that λ is replaced by $\lambda(1-\epsilon)$ since the arrival rate of non-*desperate* bidders in the game $G^{(\epsilon)}$ is $\lambda(1-\epsilon)$. Thus the strategy $\mathcal{T}[\nu_{tmp}^{(\epsilon)}]$ is a Bayesian Nash equilibrium of $G^{(\epsilon)}$ and, in addition, as $\epsilon \rightarrow 0$, the right hand side of (3.35) converges to the right hand side of (3.1) and it follows that $\lim_{\epsilon \rightarrow 0} \nu_{tmp}^{(\epsilon)} \rightarrow \nu_{tmp}$. \square

The intuitive explanation for Theorem 3 is that when the first bidder decides to bid in the auction he is strictly better off bidding immediately and remove the buyout option then, because this prevents any subsequent desperate bidders from participating. The equilibrium $\mathcal{T}[\nu_{tmp}]$ characterized in Theorem 1 is thus the limit of a sequence of equilibria corresponding to perturbed versions of the original game.

Let \mathcal{S} denote the set of all strategies where, if a bidder decides to place a bid in the auction, he bids his true valuation. This is without loss of generality since \mathcal{S} only excludes strategies where there are bidders who start with a low bid and then increment their bid, in one or multiple steps, up to their true valuation. However, an auction with a temporary buyout price becomes a standard second price auction once a bid is placed, and under the assumption that bidders' valuation is private, a strategy where bidders bid in multiple steps (up to their true valuation) is equivalent to a strategy where they bid their true valuation in one step (which belongs to \mathcal{S}). Theorem 3 implies that in any equilibrium of $G^{(\epsilon)}$ a bidder, who arrives when the

buyout option is present, acts (bids/buyout) immediately and we have already shown that in such a case his choice is determined by the threshold ν_{tmp} . Once a bid has been placed in the auction, it becomes a standard second-price auction and for all subsequent bidders a weakly dominant strategy is to bid their true valuation. Hence this proves that, among the strategies in \mathcal{S} , the strategy $\mathcal{T}[\nu_{tmp}]$ is the unique equilibrium of a temporary buyout price auction that is robust to the payoff perturbation discussed above.

Another standard robustness test for outcome prediction is to consider the concept of trembling-hand perfect equilibrium (Fudenberg and Tirole 1991).

Trembling hand Perfection

In games of incomplete information, Fudenberg and Tirole (1991) argue that the concept of subgame perfection is not very useful “since the players do not know the others’ types, the start of a period does not form a well-defined subgame until the players’ posterior beliefs are specified, and so we cannot test whether the continuation strategies are a Nash equilibrium”. Two solution concepts often used for such games are the notions of *Perfect Bayesian equilibrium* and *sequential equilibrium* both of which explicitly consider players’ beliefs about what has transpired in the game before their move. We instead use the notion of a trembling-hand perfect equilibrium (“perfect equilibrium”) which is a related but stronger concept, see Mas-Colell et al. (1995), than both *Perfect Bayesian* and *sequential* equilibrium.

A perfect equilibrium requires that the strategies be the limit of totally mixed strategies and that, subject to the requirement that it must put at least a minimum weight (must tremble) on each pure strategy on the converging sequence, each player’s strategy is (constrained) optimal against his opponents’ (which includes trembles themselves) (see Fudenberg and Tirole (1991), Selten (1975)). In other words, it entails that strategies should be optimal even if there is a small probability that other players exhibit off-equilibrium path behavior (“tremble”). These trembles may arise due to players’ irrationality or inexperience or due to a mistake when playing the strategy.

Recall that a weakly dominated strategy is one that leads to at most as much payoff as the strategy that dominates it. An important property of a perfect equilibrium, as noted by Morrow (1994), is that a “perfect equilibrium eliminates [weakly dominated strategies] because there is a small chance that a tremble will lead [a] player to a node where the dominating strategy produces a better outcome for the player”.

Notice that the auction game in question is not finite because a bidder can potentially visit the auction site infinite number of times before bidding or exercising the buyout option. However since the utility of a bidder is a function only of his initial arrival time and the timing of his bid (or buyout exercise), his intermediate arrivals to the auction can be ignored. Thus the set of pure strategies of a bidder can be assumed to be $\{\text{exercise buyout option immediately, bid true valuation immediately, exercise buyout option at some later time (if available), bid true valuation at some later time}\}$ where if the bidder chooses to act (bid/buyout) at some later time then the exact time may be chosen either immediately on his first arrival or after repeated arrivals to the auction site.

Considering the normal (or strategic) form representation of the above game (with the modified strategy space), we argue heuristically that the strategy $\mathcal{T}^{(w)}[\cdot]$ for any $w > 0$ does not satisfy the perfectness concept while the strategy $\mathcal{T}[\nu_{tmp}]$ does. The intuitive justification of this observation is that if a subsequent bidder has a positive probability of exercising the buyout option even though he gains a negative utility from this action, then the first bidder is strictly better off bidding immediately and thus making the buyout option unavailable to future bidders. In other words, by bidding immediately the first bidder protects himself from the possibility that a subsequent bidder may exercise the buyout option by mistake or because he is irrational/inexperienced.

We use the following equivalent definition of a perfect equilibrium, due to Myerson (1978),:

Definition 1. *Strategy profile σ^ϵ of a strategic form is an ϵ -perfect equilibrium if it is completely mixed and for all players i and any strategy s_i , if there exists a strategy s'_i with utility $u_i(s_i, \sigma_{-i}^\epsilon) < u_i(s'_i, \sigma_{-i}^\epsilon)$, then $\sigma_i^\epsilon(s_i) < \epsilon$. A perfect equilibrium σ is*

any limit of σ -perfect strategy profiles σ^ϵ for some sequence ϵ of positive numbers that converge to 0.

We seek to characterize an ϵ -perfect equilibria of the auction game with the above-defined strategy space. Consider a bidder, say A , of type (v, t) who arrives when the buyout option is present. If bidder A chooses to bid his true valuation in the auction at some later time $\tau (> t)$ then his utility can be written as:

$$\mathbf{E}[U_{bid(\tau)}(v, t)|\mathcal{B}] = \mathbf{E}[U_{bid(\tau)}(v, t)|\mathcal{B}, \mathcal{E}]P(\mathcal{E}|\mathcal{B}) + \mathbf{E}[U_{bid(\tau)}(v, t)|\mathcal{B}, \bar{\mathcal{E}}]P(\bar{\mathcal{E}}|\mathcal{B}) \quad (3.36)$$

where \mathcal{E} is defined as the event that a bidder with valuation less than v exercises the buyout option in interval (t, τ) while \mathcal{B} is the event that the buyout option is present when bidder A first arrives (at time t).

As we are seeking an ϵ -perfect equilibrium, all players are assumed to play totally mixed strategies and under this assumption $P(\mathcal{E}|\mathcal{B}) > 0$ and consequently $P(\bar{\mathcal{E}}|\mathcal{B}) < 1$ for all $\tau > t$. Furthermore if \mathcal{E} occurs, the utility of bidder A is zero since the buyout option is exercised by another bidder and thus $\mathbf{E}[U_{bid(\tau)}(v, t)|\mathcal{B}, \mathcal{E}] = 0$. Using a sample path argument it can be easily shown that $\mathbf{E}[U_{bid(\tau)}(v, t)|\mathcal{B}, \bar{\mathcal{E}}]$ is equal to bidder A 's expected utility from bidding his true valuation in the auction immediately $\mathbf{E}[U_{bid(t)}(v, t)|\mathcal{B}]$.

We thus have that

$$\mathbf{E}_t[U_{bid(\tau)}(v, t)|\mathcal{B}] = \mathbf{E}_t[U_{bid(\tau)}(v, t)|\mathcal{B}, \bar{\mathcal{E}}]P(\bar{\mathcal{E}}|\mathcal{B}) < \mathbf{E}_t[U_{bid(t)}(v, t)|\mathcal{B}]$$

i.e bidding immediately leads to a strictly higher than bidding at some later time in the auction.

Similarly if the bidder chooses to exercise the buyout option at some later time $\tau (> t)$ (if available) his utility is

$$\mathbf{E}[U_{buy(\tau)}(v, t)|\mathcal{B}] = \mathbf{E}[U_{buy(\tau)}(v, t)|\mathcal{B}, \mathcal{E}]P(\mathcal{E}|\mathcal{B}) + \mathbf{E}[U_{buy(\tau)}(v, t)|\mathcal{B}, \bar{\mathcal{E}}]P(\bar{\mathcal{E}}|\mathcal{B}) \quad (3.37)$$

where we now define \mathcal{E} as the event that the buyout option is exercised in the interval

(t, τ) by another bidder; \mathcal{B} is as defined before. Clearly $\mathbf{E}[U_{buy(\tau)}(v, t)|\mathcal{B}, \mathcal{E}] = 0$ since the buyout option is exercised by another bidder while $\mathbf{E}[U_{buy(\tau)}(v, t)|\mathcal{B}, \bar{\mathcal{E}}] = e^{-\beta(\tau-t)}(v-p)$ since the bidder waits for $(\tau-t)$ units of time and then exercises the buyout option at price p . We thus have

$$\mathbf{E}[U_{buy(\tau)}(v, t)|\mathcal{B}] = e^{-\beta(\tau-t)}(v-p)P(\bar{\mathcal{E}}|\mathcal{B}) \quad (3.38)$$

$$< (v-p) = \mathbf{E}[U_{buy(t)}(v, t)|\mathcal{B}] \quad (3.39)$$

i.e. exercising the buyout option immediately leads to a strictly higher utility than exercising the buyout option at a later time.

Thus, by definition, any ϵ -constrained equilibrium σ^ϵ will have

$$\sigma_A^\epsilon(\textit{bid true valuation at some later time}) < \epsilon, \text{ and}$$

$$\sigma_A^\epsilon(\textit{exercise buyout option at some later time, if available}) < \epsilon.$$

Hence taking the limit as $\epsilon \rightarrow 0$ implies, in particular, that there is no perfect equilibrium of the auction game of the form $\mathcal{T}^{(w)}$ for any $w > 0$.

Analysis similar to the proof of Theorem 1 can be repeated to show that for any $\epsilon > 0$ there exists an ϵ -constrained equilibrium σ^ϵ of the form:

$$\sigma^\epsilon(v, t) = \begin{cases} (1 - 3\delta(\epsilon), \delta(\epsilon), \delta(\epsilon), \delta(\epsilon)) & \text{if buyout option present, } \mathbf{E}[U_{bid(t)}(v, t)|\mathcal{B}] > v - p \\ (\delta(\epsilon), 1 - 3\delta(\epsilon), \delta(\epsilon), \delta(\epsilon)) & \text{if buyout option present, } \mathbf{E}[U_{bid(t)}(v, t)|\mathcal{B}] \leq v - p \\ (0, y, 0, 1 - y) & \text{if buyout option not present} \end{cases}$$

where $0 < \delta(\epsilon) < \epsilon$ and $y \in (0, 1)$. The quadruplet vectors are the probabilities with which the strategies $\{\textit{exercise buyout option immediately, bid true valuation immediately, exercise buyout option at some later time (if available), bid true valuation at some later time}\}$ are mixed. When bidders play σ^ϵ , the expected utility from bidding

if the buyout option is present, can be bounded as following:

$$\begin{aligned} \mathbf{E}\left[e^{-\beta(T-t)} \int_{\underline{v}}^v F(x)^{N(t,T)+\delta(\epsilon)N(0,t)} dx\right] &\leq \mathbf{E}[U_{bid(t)}(v, t)|\mathcal{B}] \\ &\leq \mathbf{E}\left[e^{-\beta(T-t)} \int_{\underline{v}}^v F(x)^{N(t,T)} dx\right] \end{aligned}$$

where $N(0, t)$ (resp. $N(t, T)$) is the number of bidders arriving in the interval $(0, t)$ (resp. (t, T)). Hence in the limit as $\epsilon \rightarrow 0$,

$$\mathbf{E}[U_{bid(t)}(v, t)|\mathcal{B}] \rightarrow \mathbf{E}\left[e^{-\beta(T-t)} \int_{\underline{v}}^v F(x)^{N(t,T)} dx\right]$$

and thus $\sigma^\epsilon \rightarrow \mathcal{T}[\nu_{tmp}]$ which shows that $\mathcal{T}[\nu_{tmp}]$ is a perfect equilibrium.

Thus we have shown that in any perfect equilibrium a bidder who arrives when the buyout option is present acts (bids/buyout) immediately and indeed his choice is determined by the threshold ν_{tmp} . Once a bid has been placed in the auction, it becomes a standard second-price auction and for all subsequent bidders a weakly dominant strategy is to bid their true valuation. Hence this proves that the strategy $\mathcal{T}[\nu_{tmp}]$ is indeed the unique trembling-hand perfect equilibrium, among the strategies in \mathcal{S} , for an auction with a temporary buyout option.

These observations support in our view the use of equilibrium $\mathcal{T}[\nu_{tmp}]$ in the remainder of this analysis as a predictor for the outcome of an online auction with a temporary buyout price.

3.1.3 Seller's Optimization Problem

We now consider the revenue maximization problem faced by the seller, using the equilibrium characterized in Theorem 1 as a prediction of the game's outcome. Specifically, we seek to determine the temporary buyout price p maximizing the seller's expected discounted revenue $\mathbf{E}[U_{tmp}^S(p)]$ when all bidders follow strategy $\mathcal{T}[\nu_{tmp}]$ defined by (3.1) and (3.2). Note that p is the only decision variable we consider here (see Vakrat and Seidmann (2001) and Gallien (2006) for optimization studies focusing on the decision variables T and \bar{v}).

Making the dependence of ν_{tmp} on p explicit from now on and conditioning on both the arrival time and the action of the first bidder, the problem can be stated mathematically as

$$\begin{aligned} \max_p \mathbf{E}[U_{tmp}^S(p)] &= \int_0^T e^{-\alpha T} \mathbf{E}_t[\max(\underline{v}, v_{N(t,T)+1}^{(2)}) | v_1 \leq \nu_{tmp}(p, t)] F(\nu_{tmp}(p, t)) \lambda e^{-\lambda t} dt \\ &+ \int_0^T e^{-\alpha t} p (1 - F(\nu_{tmp}(p, t))) \lambda e^{-\lambda t} dt, \end{aligned} \quad (3.40)$$

where the expectation \mathbf{E}_t in the first integrand is with respect to the number $N(t, T)$ of arrivals in interval $(t, T]$ of a Poisson process with rate λ and the second highest value $v_{N(t,T)+1}^{(2)}$ among $N(t, T)+1$ independent draws $v_1, \dots, v_{N(t,T)+1}$ from the valuation distribution with cdf F , where by convention $v_1^{(2)} = 0$ – note that the first and second integrals in (3.40) correspond respectively to the seller’s expected revenue when the first bidder submits a regular bid upon his arrival and when he exercises the buyout option.

While solving analytically the optimization problem (3.40) in the general case appears to be particularly challenging, computing a numerical solution p_{tmp}^* to this problem through a line search over p is relatively straightforward: for each value of p , one may numerically solve (3.2) for $\nu_{tmp}(p, t)$, then estimate $\mathbf{E}[U_{tmp}^S(p)]$ through Monte-Carlo simulation by generating repeated random bidder arrival streams $\{(v_1, t_1), (v_2, t_2), \dots\}$. This is the method we implement to obtain the numerical results we report in §5.1.

In the remainder of this subsection, we discuss the solution of the optimization problem (3.40) in some limiting regimes of α, β and λ .

Let $g : [0, \infty) \rightarrow [0, \infty)$ be any function satisfying $\lim_{x \rightarrow \infty} g(x) \rightarrow 0$ and $f_i : [0, \infty) \rightarrow [0, \infty)$, $i \in \{1, 2\}$ be functions such that $\lim_{x \rightarrow 0} f_i(x) \rightarrow 0$. Define buyout price $\tilde{p} = \arg \max_p p(1 - F(p))$ and $\hat{p} = \arg \max_p p(1 - F(p)) + \underline{v}F(p)$. Here, and in the remainder of this subsection, it is assumed that the distribution function $F(\cdot)$ is strictly increasing on $[\underline{v}, \bar{v}]$ and is such that \tilde{p} and \hat{p} are unique. It can be easily shown that $\hat{p} \geq \tilde{p}$.

Now consider a market environment when the bidder arrival rate is high ($\lambda \rightarrow \infty$).

We first derive the asymptotic optimal buyout price for the case when seller sensitivity is low ($\alpha = g(\lambda)$). We will use the following lemma.

Lemma 3. *Consider a continuous function $h(p, \lambda) : X \times [0, \infty) \rightarrow [0, \infty)$ where $X = [\underline{v}, \bar{v}]$ and let $p^*(\lambda) = \arg \max_{p \in X} h(p, \lambda)$. Suppose $h(p, \lambda)$ is such that $\lim_{\lambda \rightarrow \infty} h(\lambda, p) \rightarrow h(p)$ (in the sup norm) where $h(p)$ is also continuous on the set X . Then if $h(\cdot)$ has a unique maximizer $p^* = \arg \max_{p \in X} h(p)$, $\lim_{\lambda \rightarrow \infty} p^*(\lambda) \rightarrow p^*$.*

Proof. We show that for any $\epsilon > 0$ there exists Λ, δ such that for all $\lambda > \Lambda$,

$$h(p^*, \lambda) - h(p, \lambda) > \delta \quad \forall p \notin (p^* - \epsilon, p^* + \epsilon)$$

hence proving that $|p^*(\lambda) - p^*| < \epsilon$ for all $\lambda > \Lambda$.

Consider an $\epsilon > 0$. We have

$$\begin{aligned} h(p^*, \lambda) - h(p, \lambda) &= h(p^*, \lambda) - h(p^*) + h(p) - h(p, \lambda) + h(p^*) - h(p) \\ &\geq -|h(p^*, \lambda) - h(p^*)| - |h(p, \lambda) - h(p)| + h(p^*) - h(p) \end{aligned} \quad (3.41)$$

Now since $h(p)$ is continuous and p^* is the unique maximizer, there exists $\delta_1 > 0$ such that

$$h(p^*) - h(p) > \delta_1 \quad \forall p \notin (p^* - \epsilon, p^* + \epsilon) \quad (3.42)$$

Also since $\lim_{\lambda \rightarrow \infty} h(\lambda, p) \rightarrow h(p)$ in the sup norm, \exists a $\Lambda > 0$ such that $\forall \lambda > \Lambda$

$$|h(p^*, \lambda) - h(p)| < \delta_1/3 \quad \forall p \in [\underline{v}, \bar{v}] \quad (3.43)$$

Substituting (3.42) and (3.43) in (3.41), we obtain that for any $p \notin (p^* - \epsilon, p^* + \epsilon)$

$$h(p^*, \lambda) - h(p, \lambda) > -\delta_1/3 - \delta_1/3 + \delta_1 = \delta_1/3$$

Setting $\delta = \delta_1/3$ completes the proof. \square

Now, notice that we have:

$$\begin{aligned}\lim_{\lambda \rightarrow \infty} \nu_{tmp}(p, t) &\rightarrow p \quad \forall t \\ \lim_{\lambda \rightarrow \infty} \mathbf{E}_t[\max(\underline{v}, v_{N(t,T)+1}^{(2)}) | v_1 \leq \nu_{tmp}(p, t)] &\rightarrow \bar{v}\end{aligned}$$

Using the above and the fact that $\alpha = g(\lambda)$, the seller's revenue can be written as

$$\begin{aligned}\lim_{\lambda \rightarrow \infty} \mathbf{E}[U_{tmp}^S(p)] &= \lim_{\lambda \rightarrow \infty} \left(\int_0^T e^{-g(\lambda)T} \bar{v} F(p) \lambda e^{-\lambda t} dt + \int_0^T e^{-g(\lambda)t} p(1 - F(p)) \lambda e^{-\lambda t} dt \right) \\ &= \bar{v} F(p) + p(1 - F(p))\end{aligned}$$

The function $r(p) = \bar{v} F(p) + p(1 - F(p))$ is uniquely maximized at \bar{v} and this implies, using Lemma 3, that in the case $\alpha = g(\lambda)$, $\lim_{\lambda \rightarrow \infty} p_{tmp}^* \rightarrow \bar{v}$.

A similar argument can be used to show that when the seller sensitivity is high ($\alpha = 1/g(\lambda)$) the optimal buyout price converges to \bar{p} in the limit as $\lambda \rightarrow \infty$.

We next analyze a market environment where the bidder arrival rate is small, bidders have high time-sensitivity ($\lambda \rightarrow 0, \beta = 1/f_2(\lambda)$) and the seller's sensitivity is low, i.e. $\alpha = f_1(\lambda)$. The analysis for the case when $\alpha = 1/f_1(\lambda)$ is similar.

Since $\beta \rightarrow \infty$ the threshold valuation $\nu_{tmp}(p, t) \rightarrow p$, and in addition since the bidder arrival rate $\lambda \rightarrow 0$, we have $\mathbf{E}_t[\max(\underline{v}, v_{N(t,T)+1}^{(2)}) | v_1 \leq \nu_{tmp}(p, t)] \rightarrow \underline{v}$. Now notice that in this case the seller's revenue approaches zero in the limit as $\lambda \rightarrow 0$ and so we instead consider the following ratio

$$\begin{aligned}\lim_{\lambda \rightarrow 0} \frac{\mathbf{E}[U_{tmp}^S(p)]}{\mathbf{E}[U_{tmp}^S(\hat{p})]} &= \lim_{\lambda \rightarrow 0} \frac{\int_0^T \left(e^{-f_1(\lambda)T} \underline{v} F(p) + e^{-f_1(\lambda)t} p(1 - F(p)) \right) \lambda e^{-\lambda t} dt}{\int_0^T \left(e^{-f_1(\lambda)T} \underline{v} F(\hat{p}) + e^{-f_1(\lambda)t} \hat{p}(1 - F(\hat{p})) \right) \lambda e^{-\lambda t} dt} \\ &= \frac{\underline{v} F(p) + p(1 - F(p))}{\underline{v} F(\hat{p}) + \hat{p}(1 - F(\hat{p}))}\end{aligned}$$

By definition \hat{p} uniquely maximizes the function $g_2(p) = p(1 - F(p)) + \underline{v} F(p)$ and hence it is also the unique maximizer of the function $\frac{g_2(p)}{g_2(\hat{p})}$. Thus, using Lemma 3, in the case $\beta = 1/f_2(\lambda)$ and $\alpha = f_1(\lambda)$, the optimal buyout price converges to \hat{p} in the limit $\lambda \rightarrow 0$.

Finally, we consider the regime when $(\alpha = 1/f_1(\lambda), \beta = f_2(\lambda), \lambda \rightarrow 0)$. It turns out that Lemma 3 is not applicable in this case and so we prove the convergence of the optimal solution using basic principles. Indeed we show that for any $\epsilon > 0$ there exists a $\Lambda > 0$ such that for all $\lambda < 1/\Lambda$, $|p_{tmp}^*(\lambda) - \underline{v}| < \epsilon$ thus proving that the optimal buyout price converges to \underline{v} in this regime.

Consider an $\epsilon > 0$. Recall that $\nu_{tmp}(p, t) = \min(\hat{v}(p, t), \bar{v})$ where $\hat{v}(p, t)$ is the solution of (3.2). Notice that when $\beta = f_2(\lambda)$ (recall that $\lim_{\lambda \rightarrow 0} f_2(\lambda) \rightarrow 0$), $\lim_{\lambda \rightarrow 0} \hat{v}(p, t) \rightarrow \infty, \forall t \in [0, T]$ if $p > \underline{v}$. Thus there exists a $\Lambda_1 > 0$ such that for all $\lambda < 1/\Lambda_1$ we have $\hat{v}(p, t) \geq \bar{v}, \forall t \in [0, T], p \geq \underline{v} + \epsilon$. This implies that $\nu_{tmp}(p, t) = \bar{v}, \forall t \in [0, T]$ and $p \geq \underline{v} + \epsilon$.

For $\lambda < 1/\Lambda_1$, and any $p \geq \underline{v} + \epsilon$, consider the ratio

$$\begin{aligned} \frac{\mathbf{E}_{tmp}^S(p)}{\mathbf{E}_{tmp}^S(\underline{v})} &= \frac{\int_0^T e^{-\alpha T} \mathbf{E}_t[\max(\underline{v}, v_{N(t,T)+1}^{(2)}) |v_1 \leq \nu_{tmp}(p, t)] \lambda e^{-\lambda t} dt}{\int_0^T e^{-\alpha t} \underline{v} \lambda e^{-\lambda t} dt} \\ &< \frac{\bar{v} \int_0^T e^{-\alpha T} e^{-\lambda t} dt}{\underline{v} \int_0^T e^{-\alpha t} e^{-\lambda t} dt} = \frac{\bar{v}}{\underline{v}} e^{-\alpha T} + \frac{\bar{v}}{\underline{v} \lambda} \alpha e^{-\alpha T} \end{aligned}$$

Now since $\alpha = 1/f_1(\lambda)$ and $\lim_{\lambda \rightarrow 0} f_1(\lambda) \rightarrow 0$, we obtain that $\alpha \rightarrow \infty$ as $\lambda \rightarrow 0$. Also notice that $\lim_{x \rightarrow \infty} x e^{-x} \rightarrow 0$ and thus $\exists \Lambda_2 > 0$ such that for all $\lambda < 1/\Lambda_2$

$$\frac{\bar{v}}{\underline{v}} e^{-\alpha T} < 1/2 \quad \text{and} \quad \frac{\bar{v}}{\underline{v} \lambda} \alpha e^{-\alpha T} < 1/2$$

Thus for all $\lambda < \min(\frac{1}{\Lambda_1}, \frac{1}{\Lambda_2})$ we obtain that

$$\frac{\mathbf{E}_{tmp}^S(p)}{\mathbf{E}_{tmp}^S(\underline{v})} < 1 \quad \forall p \geq \underline{v} + \epsilon$$

hence proving that $p_{tmp}^*(\lambda) \in [\underline{v}, \underline{v} + \epsilon)$. Setting $\Lambda = 1/\min(\frac{1}{\Lambda_1}, \frac{1}{\Lambda_2})$ completes the proof.

The results derived above can be summarized as following:

The case $p_{tmp}^* \rightarrow \underline{v}$ effectively amounts to using a fixed price mechanism, since no bidding activity will ever occur then; this is optimal for a highly time-sensitive

	$\lambda \rightarrow \infty$		$\lambda \rightarrow 0$
$\alpha = g(\lambda)$	$p_{tmp}^* \rightarrow \bar{v}$	$\alpha = f_1(\lambda), \beta = 1/f_2(\lambda)$	$p_{tmp}^* \rightarrow \hat{p}$
$\alpha = 1/g(\lambda)$	$p_{tmp}^* \rightarrow \tilde{p}$	$\alpha = 1/f_1(\lambda), \beta = 1/f_2(\lambda)$	$p_{tmp}^* \rightarrow \tilde{p}$
		$\alpha = 1/f_1(\lambda), \beta = f_2(\lambda)$	$p_{tmp}^* \rightarrow \underline{p}$

Table 3.1: Optimal buyout price in asymptotic regimes

seller facing a relatively small expected number of bidders (i.e. $\alpha \rightarrow \infty$, $\lambda \rightarrow 0$) all of whom have little time-sensitivity ($\beta \rightarrow 0$). Indeed, a seemingly large number of auction listings on eBay now feature only a "Buy It Now" option and no "Place Bid" option, providing anecdotal evidence for the relevance of this case in practice. On the other extreme, for low α and high λ , the optimal buyout price $p_{tmp}^* = \bar{v}$, which is equivalent to an auction without a buyout option since the buyout price is never exercised, i.e. a seller with relatively high market power and low time-sensitivity finds it beneficial to not use any buyout option at all and only rely on a traditional bidding mechanism – there are clearly many examples of such sellers on auction sites as well. These results are thus reminiscent of those obtained by Harris and Raviv (1981), who study a mechanism design model in which the seller should use an auction when demand exceeds supply but a posted price otherwise (see also Gallien (2006)). In our model, the relative values of the seller's and bidders' time sensitivity (α and β) and the expected number of bidders λ effectively capture the ratio between supply and demand and the seller's market power, and the hybrid mechanism relying on both bidding and posted price enabled by the buyout option makes for a continuous, smoother transition between those two mechanisms.

The case when both α and λ are high ($\alpha = 1/g(\lambda)$, $\lambda \rightarrow \infty$) corresponds to a seller who faces a high demand but is very time-sensitive. Such a seller could obtain almost \bar{v} for the product if he is willing to wait; however his high time-sensitivity means that he gets almost zero utility from selling the product at time T . He thus offers a buyout option with a price \tilde{p} which is chosen such that the expected revenue obtained from the event that the buyout option is exercised by the first bidder is maximized.

Next consider a market environment where demand for the auctioned item is low but the bidders are highly time-sensitive ($\beta = 1/f_2(\lambda)$, $\lambda \rightarrow 0$). In this case a bidder

could obtain the product at \underline{v} by bidding in the auction; however since β is high the bidder gains almost zero utility from getting the product at the end of the auction and thus is willing to exercise the buyout option if the buyout price is lesser than his valuation (i.e. $\nu_{tmp}(p, t) \rightarrow p$). A seller with a high time-sensitivity ($\alpha = 1/f_1(\lambda)$) gets zero utility from selling the product at time T and thus offers a buyout option with price \bar{p} which, as before, maximizes the expected revenue obtained from the event that the first bidder exercises the buyout option. If instead the seller has low time-sensitivity ($\alpha = f_1(\lambda)$) then he can wait up to the end of the auction to sell the product and hence finds it optimal to offer a higher buyout price $\hat{p} \geq \bar{p}$.

One regime not covered above is when $\alpha = f_1(\lambda), \beta = f_2(\lambda)$ and $\lambda \rightarrow 0$ which corresponds to a case when both the seller and the bidders have low time-sensitivity and the bidder arrival rate is small. For this case, it can be shown that, in the limit $\lambda \rightarrow 0$, the seller is indifferent between choosing any buyout price $p \in [\underline{v}, \bar{v}]$ since all prices lead to the same utility for the seller.

Approximate optimal buyout price

We next derive an approximate closed-form expression for the optimal temporary buyout price p_{tmp}^* when valuations are uniformly distributed and bidders are impatient, that is $\beta \rightarrow +\infty$. This limiting case for the bidders' time sensitivity is of special interest because it may also reflect a form of strong risk aversion, which may be realistic in some settings – see §2.3 for a related discussion. Concretely, it is characterized by bidders who will always exercise an open buyout option if their valuation is larger than the buyout price: for those bidders, the more distant and risky prospect of purchasing the item through the auction, even at a much lower price, is never more attractive than securing a purchase immediately. Formally, the equilibrium threshold function now specializes to $\nu_{tmp}(p, t) = \min(p, \bar{v})$, which we can directly substitute in (3.40). When valuations are uniformly distributed, an explicit expression for the term $\mathbf{E}_t[\max(\underline{v}, v_{N(t,T)+1}^{(2)}) | v_1 \leq p]$ can be derived (see appendix). As a result, (3.40) becomes then a concave maximization problem in one variable. As a second approximation, we now ignore the information that the first bidder's valuation is less than

the threshold when computing the expected auction price conditional on the buy-out option not being exercised in the first integral of (3.40). Formally, this can be stated as $\mathbf{E}_t[\max(\underline{v}, v_{N(t,T)+1}^{(2)}) | v_1 \leq \nu_{tmp}(p, t)] \simeq \mathbf{E}_t[\max(\underline{v}, v_{N(t,T)+1}^{(2)})]$, which seems intuitively a good approximation when the expected number of bidder arrivals λT is larger than five or six, at which point the relative impact on the expected second highest valuation that a single one of them is probabilistically smaller becomes negligible. Substitution in (3.40) and a straightforward calculation yield the following expression for the seller's approximate discounted revenue:

$$\mathbf{E}[\tilde{U}_{tmp}^S(p)] \triangleq p(1-F(p)) \frac{\lambda(1 - e^{-(\lambda+\alpha)T})}{\lambda + \alpha} + e^{-\alpha T} F(p) \left(\left(\bar{v} - \frac{2m}{\lambda T} \right) (1 - e^{-\lambda T}) + 2m\lambda T \right), \quad (3.44)$$

where F is the cdf of a uniform distribution on $[\underline{v}, \bar{v}]$ – the first term in (3.44) corresponds to the expected discounted revenue from the exercise of the option, while the second corresponds to the expected discounted revenue from the basic auction mechanism. Note that the function defined on $[0, +\infty)$ and obtained by substituting $F(p)$ with $\frac{p-\underline{v}}{m}$ in (3.44) is concave, coincides with $\mathbf{E}[\tilde{U}_{tmp}^S(\cdot)]$ on $[\underline{v}, \bar{v}]$ and achieves its unique maximum on $[0, +\infty)$ at

$$\hat{p} = \frac{\bar{v}}{2} + e^{-\alpha T} \frac{\bar{v}(1 - e^{-\lambda T}) - \frac{2m}{\lambda T} (1 - (1 + \lambda T)e^{-\lambda T})}{\frac{2\lambda}{\lambda + \alpha} (1 - e^{-(\lambda+\alpha)T})}. \quad (3.45)$$

Consequently, it is easy to show that the buyout price \tilde{p}_{tmp}^* in $[\underline{v}, \bar{v}]$ maximizing $\mathbf{E}[\tilde{U}_{tmp}^S(p)]$ on $[0, +\infty)$ is given by

$$\tilde{p}_{tmp}^* = \begin{cases} \underline{v} & \text{if } \hat{p} < \underline{v} \\ \hat{p} & \text{if } \underline{v} \leq \hat{p} \leq \bar{v} \end{cases}. \quad (3.46)$$

We have found that in a wide variety of environments the performance of the approximate optimal buyout price \tilde{p}_{tmp}^* characterized by (3.46) is close to that of the optimal buyout price p_{tmp}^* computed numerically through Monte-Carlo simulation – the numerical results testing the sub-optimality of the approximate optimal buyout

price \tilde{p}_{tmp}^* are provided in Chapter 5.

3.2 Permanent Buyout Option

We assume now that the seller uses a fixed permanent buyout price p . The equilibrium analysis is provided in §3.2.1, the robustness of the strategy is discussed in §3.2.2 while the seller's optimization is formulated in §3.2.3.

3.2.1 Outcome Prediction

For any bidder with valuation v arriving at time t and observing then a current second-highest bid I_t (see §2.2 for definition), consider the following family $\mathcal{P}[\cdot]$ of threshold strategies:

$$\mathcal{P}[\nu](v, t, I_t) : \begin{cases} \text{Buyout at } p \text{ immediately} & \text{if } v > \nu(t, I_t) \\ \text{Bid } v \text{ at time } T & \text{if } v \leq \nu(t, I_t) \end{cases}, \quad (3.47)$$

where $\nu : [0, T] \times [\underline{v}, \bar{v}] \cup \{0\} \rightarrow [\underline{v}, \bar{v}]$ is a continuous function. Note that the action of bidding at time T in the definition of $\mathcal{P}[\cdot]$ is clearly a theoretical limit, and would correspond in practice to submitting a bid as close as possible to the end of the auction with the goal of denying other bidders the opportunity to respond.

In this subsection, we prove the following result which establishes the existence of a threshold function ν_{prm} such that the symmetric strategy profile $\mathcal{P}[\nu_{prm}]$ constitutes a Bayesian Nash equilibrium of the permanent buyout price auction game, and also provides a characterization for ν_{prm} .

Theorem 4. *Consider a maximal solution $\tilde{v}(\cdot)$ of the following functional equation on $[0, T] \rightarrow [\underline{v}, +\infty)$:*

$$\tilde{v}(t) - p = \mathbf{E}_t \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\tilde{v}(t)} \frac{\prod_{i=1}^{N(t)} F(\min(\tilde{v}(t_i), x))}{\prod_{i=1}^{N(t)} F(\tilde{v}(t_i))} F(x)^{N(t,T)} dx \right) \right], \quad (3.48)$$

where the expectation \mathbf{E}_t is with respect to the number $N(t)$ and epochs $t_1, \dots, t_{N(t)}$

of arrivals in $[0, t)$ of a non-homogeneous Poisson process with rate $\lambda F(\tilde{v}(\tau))$ for $\tau \in [0, t)$, and number $N(t, T)$ of arrivals in $(t, T]$ of a Poisson process with rate λ . Let $\tilde{v}(t, I)$ be a continuous extension of $\tilde{v}(\cdot)$ to $[0, T] \times [\underline{v}, \bar{v}] \cup \{0\}$ such that $\tilde{v}(t, 0) = \tilde{v}(t)$ and $\tilde{v}(t, I)$ is non-increasing in I for all t , non-decreasing in t for all I , and define $\nu_{prm}(t, I) = \min(\tilde{v}(t, I), \bar{v})$. The symmetric strategy profile $\mathcal{P}[\nu_{prm}]$ is a Bayesian Nash equilibrium for the online auction game with a permanent buyout price p .

Denoting by $\mathcal{R}(\mathcal{P}[\nu])$ the set of best response strategies to the profile where every other player follows strategy $\mathcal{P}[\nu]$ we first establish that $\mathcal{P}[\nu_{prm}] \in \mathcal{R}(\mathcal{P}[\nu_{prm}])$ if and only if $\nu_{prm}(t, 0)$ is the solution of the following functional equation:

$$\tilde{v}(t) - p = \mathbf{E}_t \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\tilde{v}(t)} \frac{\prod_{i=1}^{N(t)} F(\min(\tilde{v}(t_i), x))}{\prod_{i=1}^{N(t)} F(\tilde{v}(t_i))} F(x)^{N(t, T)} dx \right) \right], \quad (3.49)$$

where the expectation \mathbf{E}_t is with respect to the number $N(t)$ and epochs $t_1, \dots, t_{N(t)}$ of arrivals in $[0, t)$ of a non-homogeneous Poisson process with rate $\lambda F(\tilde{v}(\tau))$ for $\tau \in [0, t)$, and number $N(t, T)$ of arrivals in $(t, T]$ of a Poisson process with rate λ . The most challenging part of the proof then consists of proving the existence of a solution to (3.49); to do so we establish that a generalization of Schauder's fixed point theorem applies to an appropriately defined functional space and continuous mapping on that space.

Consider a bidder A with type (v, t) and information I_t in an auction where all other bidders play strategy $\mathcal{P}[\nu]$, where $\nu : [0, T] \times [\underline{v}, \bar{v}] \cup \{0\} \rightarrow [\underline{v}, \bar{v}]$ is a continuous function such that $\nu(t, 0)$ is non-decreasing in t and $\nu(t, I)$ is non-increasing in I for any t . The requirement that $\nu(t, I)$ be non-decreasing for I fixed is intuitive: from the auction still running at time t it can be inferred that $v_i \leq \nu(t_i, I_{t_i})$ for all bidders i with type (v_i, t_i) observing a current second-highest bid $I_{t_i} \leq I$ upon their arrival in $(0, t)$. Consequently as t increases with I fixed, the expected final second highest valuation among all bidders decreases, thus increasing the expected utility from bidding in the auction relative to exercising the buyout option; this effect is compounded with the reduced relative discounting of the utility from bidding as t

increases. The requirement that $\nu(t, I)$ be non-increasing in I for every t is likewise easily interpreted: holding t fixed, a higher value of I implies that the expected second highest bid in the auction is higher, which lowers the expected utility from bidding relative to exercising the buyout option.

We first derive bidder A 's utility if he bids his true valuation at time T . Following strategy $\mathcal{P}[\nu]$ all other bidders also bid at T and thus $I_\tau = 0, \forall \tau \in [0, T]$. Now bidder A wins the auction if no bidder exercises the buyout option and if every bidder has a valuation less than bidder A 's valuation, i.e. the event $\{A \text{ wins}\} = \{v_\tau \leq \min(\nu(\tau, 0), v) \text{ for every bidder } (v_\tau, \tau, 0)\}$ where the notation v_τ indicates the valuation of a bidder arriving at time τ . Also, since the auction is open at t , it can be inferred that all bidders $(v_\tau, \tau, 0)$ with $\tau \in (0, t)$ have valuation $v_\tau \leq \nu(\tau, 0)$ and thus the arrival rate of bidders at any $\tau \in (0, t)$ is $\lambda F(\nu(\tau, 0))$. Then probability that A wins the auction is

$$Pr(A \text{ wins} | \{t_i\}_{i=1}^{N(t)}, \{\hat{t}_i\}_{i=1}^{N(t,T)}) = \prod_{i=1}^{N(t)} \frac{F(\min(\nu(t_i, 0), v))}{F(\nu(t_i, 0))} \prod_{i=1}^{N(t,T)} F(\min(\nu(\hat{t}_i, 0), v)), \quad (3.50)$$

where $N(t)$ is a counting process denoting the number of bidder arrivals in $(0, t)$ in a non-homogeneous Poisson process with arrival rate $\lambda(\tau) = \lambda F(\nu(\tau, 0)), \forall \tau \in (0, t)$; and $\{t_i\}_{i=1}^{N(t)}$ are the corresponding arrival epochs. $N(t, T)$ is a counting process denoting the number of arrivals in $(t, T]$ in a Poisson process with arrival rate λ and $\{\hat{t}_i\}_{i=1}^{N(t,T)}$ are the arrival epochs. The first term of the product in (3.50) is the probability that the event $\{v_{t_i} \leq \min(\nu(t_i, 0), v) | v_{t_i} \leq \nu(t_i, 0)\}$ occurs for all bidders arriving in $(0, t)$. The second term is the probability that the valuation $v_{\hat{t}_i}$ of every bidder arriving in $(t, T]$ satisfies $v_{\hat{t}_i} \leq \min(\nu(\hat{t}_i, 0), v)$. Here, and in the remainder of the paper, we assume that if $k = 0$ then $\prod_{i=1}^k (\cdot) = 1$.

Conditional on bidder A winning, the distribution of the highest bid, $v^{(1)}$, among the other $N(t) + N(t, T)$ bidders arriving at epochs $\{t_i\}_{i=1}^{N(t)}$ and $\{\hat{t}_i\}_{i=1}^{N(t,T)}$ is:

$$F_{v^{(1)} | A \text{ wins}, \{t_i\}_{i=1}^{N(t)}, \{\hat{t}_i\}_{i=1}^{N(t,T)}}(x) = \prod_{i=1}^{N(t)} \frac{F(\min(\nu(t_i, 0), x))}{F(\min(\nu(t_i, 0), v))} \prod_{j=1}^{N(t,T)} \frac{F(\min(\nu(\hat{t}_j, 0), x))}{F(\min(\nu(\hat{t}_j, 0), v))},$$

for all $x \in [0, v]$. Using the above distribution function, we have

$$\begin{aligned} \mathbf{E}[v^{(1)} | A \text{ wins}, \{t_i\}_{i=1}^{N(t)}, \{\hat{t}_i\}_{i=1}^{N(t,T)}] &= \int_0^v \left(1 - F_{v^{(1)} | A \text{ wins}, \{t_i\}_{i=1}^{N(t)}, \{\hat{t}_i\}_{i=1}^{N(t,T)}}(x)\right) dx \\ &= v - \int_{\underline{v}}^v F_{v^{(1)} | A \text{ wins}, \{t_i\}_{i=1}^{N(t)}, \{\hat{t}_i\}_{i=1}^{N(t,T)}}(x) dx, \end{aligned}$$

and thus the expected discounted utility from bidding at T for bidder A is

$$\begin{aligned} \mathbf{E}[U_{bid(T)}(v, t, 0)] &= \mathbf{E} \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^v \prod_{i=1}^{N(t)} \frac{F(\min(\nu(t_i, 0), x))}{F(\nu(t_i, 0))} \right. \right. \\ &\quad \left. \left. \times \prod_{j=1}^{N(t,T)} F(\min(\nu(\hat{t}_j, 0), x)) dx \right) \right] \quad (3.51) \end{aligned}$$

where the expectation on the right hand side is over $\{t_i\}_{i=1}^{N(t)}$ and $\{\hat{t}_i\}_{i=1}^{N(t,T)}$.

The rest of the derivation proceeds as follows: We first show using Lemma 4 that if bidder A bids in the auction he must do so at time T , and next prove that the bidder is weakly better off making a decision immediately in Lemma 5. Consequently we derive, in Lemma 6, the best response strategy $\mathcal{R}(\mathcal{P}[\nu])$ of a bidder when all other bidders play $\mathcal{P}[\nu]$ and then characterize a threshold function ν_{prm} such that $\mathcal{P}[\nu_{prm}] \in \mathcal{R}(\mathcal{P}[\nu_{prm}])$ thus establishing that $\mathcal{P}[\nu_{prm}]$ is an equilibrium strategy.

Lemma 4. $\mathbf{E}[U_{bid(\tau)}(v, t, 0)] \leq \mathbf{E}[U_{bid(T)}(v, t, 0)]$ for all $t \leq \tau \leq T$.

Proof. This result is a direct implication of the assumption that $\nu_{prm}(t, I_t)$ is non-increasing in I_t and admits the following justification: while bidding earlier does not increase the utility of a bidder it reveals information about his valuation to other bidders, who can use it this information their advantage. More formally suppose that the bidder A bids immediately, i.e. $\tau = t$ while all other bidders, following strategy $\mathcal{P}[\nu]$, bid at T and hence $I_\omega = \underline{v}$, $\forall \omega \in (t, T)$. Then the probability that the bidder A wins, from (3.50), is

$$Pr(A \text{ wins} | \{t_i\}_{i=1}^{N(t)}, \{\hat{t}_i\}_{i=1}^{N(t,T)}) = \prod_{i=1}^{N(t)} \frac{F(\min(\nu_{prm}(t_i, 0), v))}{F(\nu_{prm}(t_i, 0))} \prod_{i=1}^{N(t,T)} F(\min(\nu_{prm}(\hat{t}_i, \underline{v}), v)),$$

and the corresponding expected utility from bidding for bidder A is:

$$\mathbf{E}[U_{bid(t)}(v, t, 0)] = \mathbf{E} \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^v \prod_{i=1}^{N(t)} \frac{F(\min(\nu_{prm}(t_i, 0), x))}{F(\nu_{prm}(t_i, 0))} \times \prod_{j=1}^{N(t,T)} F(\min(\nu_{prm}(\hat{t}_j, \underline{v}), x)) dx \right) \right] \quad (3.52)$$

Since threshold $\nu(t, I)$ is non-increasing in I , we have

$$F(\min(\nu_{prm}(\hat{t}_j, \underline{v}), x)) \leq F(\min(\nu_{prm}(\hat{t}_j, 0), x)) \quad \forall j = 1, 2, \dots, N(t, T) \quad (3.53)$$

Comparing (3.52) with (3.51) using (3.53) we obtain

$$\mathbf{E}[U_{bid(t)}(v, t, 0)] \leq \mathbf{E}[U_{bid(T)}(v, t, 0)]$$

If the first bidder bids at some $t < \tau < T$ then $I_\omega = \underline{v}$, $\forall \omega \in (\tau, T)$ and then the above analysis can be repeated to obtain

$$\mathbf{E}[U_{bid(\tau)}(v, t, 0)] \leq \mathbf{E}[U_{bid(T)}(v, t, 0)]$$

□

More generally if bidder A places any bid in the auction (not necessarily his true valuation) at time τ ($\tau < T$) then $I_\tau > 0$ and the threshold valuation in (τ, T) is lower than if A bids at time T . Thus the probability that the buyout option is exercised by a bidder in (τ, T) is higher, and since bidder A gets zero utility in such an event he will not bid in the auction at any time $\tau < T$. Moreover if a bidder is bidding at time T then his weakly dominant strategy is to bid his true valuation.

We next establish that bidder A is weakly better off making a decision immediately upon his arrival, i.e. he instantaneously decides either to exercise the buyout option immediately or place a bid in the auction at time T .

Lemma 5. *When facing bidders who follow strategy \mathcal{PS} , bidder A is weakly better*

off making a decision immediately i.e.

$$\mathbf{E}[U_{wait(t,\tau)}(v, t, 0)] \leq \max \left\{ \mathbf{E}[U_{buy(t)}(v, t, 0)], \mathbf{E}[U_{bid(T)}(v, t, 0)] \right\}$$

Proof. Here we give an intuitive argument: a formal proof can be constructed on the lines of the proof of Lemma 1. We have already proven in Lemma 4 that if a bidder decides to bid in the auction he must do so at time T and thus his expected utility from bidding is independent of when he makes the decision to bid. Indeed if we let $\mathbf{E}[U_{bid(T)}^{(\tau)}(v, t, 0)]$ denote the utility of a bidder of type $(v, t, 0)$ who waits up to time τ ($\tau > t$) and then decides to place a bid in the auction at time T , then it can shown that

$$\mathbf{E}[U_{bid(T)}^{(\tau)}(v, t, 0)] = \mathbf{E}[U_{bid(T)}(v, t, 0)]$$

where recall that $\mathbf{E}[U_{bid(T)}(v, t, 0)]$ denotes the utility of a bidder $(v, t, 0)$ who decides immediately to place a bid at time T .

Additionally while the buyout price remains constant throughout the auction, waiting decreases the bidder's utility from exercising the buyout option because of his time-sensitivity. Thus, if a bidder waits before making a decision, his expected utility from bidding remains constant, but the utility from exercising the buying option decreases and so he is weakly better off making a decision immediately. \square

Thus we have shown, in Lemma 4 and Lemma 5, that the bidder A must choose, at time t , one of the two actions $\{bid(T), buy(t)\}$. We now show that the best response strategy $\mathcal{R}(\mathcal{P}[\nu])$ to $\mathcal{P}[\nu]$ is indeed a threshold strategy.

Lemma 6. *The best response strategy to $\mathcal{P}[\nu]$ is*

$$\mathcal{R}(\mathcal{P}[\nu])(v, t, 0) : \begin{cases} \text{Buyout at } p \text{ immediately} & \text{if } v > \min(\tilde{v}_{[\nu]}(t, 0), \bar{v}) \\ \text{Bid } v \text{ at time } T & \text{if } v \leq \min(\tilde{v}_{[\nu]}(t, 0), \bar{v}) \end{cases}$$

where $\tilde{v}_{[\nu]}(t, 0)$ is such that

$$\mathbf{E}[U_{buy(t)}(\tilde{v}_{[\nu]}(t, 0), t, 0)] = \mathbf{E}[U_{bid(T)}(\tilde{v}_{[\nu]}(t, 0), t, 0)] \quad (3.54)$$

Notice that Lemma 6 only specifies the equilibrium path behavior of bidders. The best response strategy is completely specified by choosing a continuous threshold function $\tilde{v}_{[\nu]}(t, I)$ which is non-increasing in I for all t and such that $\tilde{v}_{[\nu]}(t, 0)$ satisfies (3.54).

Proof. The derivative of the expected utility from bidding $\mathbf{E}[U_{bid(T)}(v, t, 0)]$ with respect to v is

$$\begin{aligned} \frac{\partial}{\partial v} \mathbf{E}[U_{bid(T)}(v, t, 0)] = \\ \mathbf{E} \left[e^{-\beta(T-t)} \left(\prod_{i=1}^{N(t)} \frac{F(\min(\nu(t_i, 0), v))}{F(\nu(t_i, 0))} \prod_{j=1}^{N(t,T)} F(\min(\nu(\hat{t}_j, 0), v)) \right) dx \right] \end{aligned} \quad (3.55)$$

For every realization of $N(t), N(t, T), \{t_i\}_{i=1}^{N(t)}$ and $\{\hat{t}_i\}_{i=1}^{N(t,T)}$, the term inside the expectation on the right hand side of (3.55) is non-negative and less than 1. Thus $0 \leq \frac{\partial}{\partial v} \mathbf{E}[U_{bid(T)}(v, t, 0)] < 1$ for all $v \in [\underline{v}, +\infty)$ and $t \in (0, T)$.

The utility from exercising the buyout option $\mathbf{E}[U_{buy(t)}(v, t, 0)] = v - p$ and thus

$$0 \leq \frac{\partial}{\partial v} \mathbf{E}[U_{bid(T)}(v, t, 0)] < 1 = \frac{\partial}{\partial v} \mathbf{E}[U_{buy(t)}(v, t, 0)]. \quad (3.56)$$

Assuming, without loss of generality, that $p \geq \underline{v}$ we get $\mathbf{E}[U_{buy(t)}(\underline{v}, t, 0)] = \underline{v} - p \leq 0 = \mathbf{E}[U_{bid(T)}(\underline{v}, t, 0)]$. This together with (3.56) implies that there exists a unique valuation $\tilde{v}_{[\nu]}(t, 0) \geq \underline{v}$ such that

$$\mathbf{E}[U_{buy(t)}(\tilde{v}_{[\nu]}(t, 0), t, 0)] = \mathbf{E}[U_{bid(T)}(\tilde{v}_{[\nu]}(t, 0), t, 0)], \quad (3.57)$$

where the notation $\tilde{v}_{[\nu]}(t, 0)$ indicates the dependence of this valuation on strategy $\mathcal{P}[\nu]$ and the fact that this corresponds to the case when $I = 0$.

Thus a bidder $(v, t, 0)$ with $v \leq \tilde{v}_{[\nu]}(t, 0)$ bids in the auction at time T since $\mathbf{E}[U_{buy(t)}(v, t, 0)] \leq \mathbf{E}[U_{bid(T)}(v, t, 0)]$. On the other hand if $v > \tilde{v}_{[\nu]}(t, 0)$ then

$$\mathbf{E}[U_{buy(t)}(v, t, 0)] > \mathbf{E}[U_{bid(T)}(v, t, 0)]$$

and thus it is profitable for bidder A to exercise the buyout option. \square

Next we characterize a continuous threshold function $\nu_{prm}(t, I)$ such that $\nu_{prm}(t, 0)$ is non-decreasing in t , $\nu_{prm}(t, I)$ is non-increasing in I for all t and is such that

$$\mathcal{P}[\nu_{prm}] \in \mathcal{R}(\mathcal{P}[\nu_{prm}]) \quad (3.58)$$

Now notice that $\mathcal{R}(\mathcal{P}[\nu_{prm}])$ is also a threshold strategy and indeed (3.58) holds, if $\nu_{prm}(t, I) = \min(\tilde{v}_{[\nu_{prm}]}(t, I), \bar{v})$ for all t, I .

For $I = 0$ substituting $\nu_{prm}(t, 0) = \min(\tilde{v}_{[\nu_{prm}]}(t, 0), \bar{v})$ in (3.54) implies that $\tilde{v}_{[\nu_{prm}]}(t, 0) = \tilde{v}(t)$ where $\tilde{v}(t)$ must satisfy

$$\tilde{v}(t) - p = \mathbf{E}_t \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\tilde{v}(t)} \prod_{i=1}^{N(t)} \frac{F(\min(\tilde{v}(t_i), x))}{F(\tilde{v}(t_i))} \prod_{j=1}^{N(t,T)} F(\min(\tilde{v}(\hat{t}_j), x)) dx \right) \right] \quad (3.59)$$

for all $t \in [0, T]$.

For $I > 0$, choosing any $\tilde{v}_{[\nu_{prm}]}(t, I)$, which is non-increasing in I for all t , suffices and so we set $\tilde{v}_{[\nu_{prm}]}(t, I) = \nu_{prm}(t, I)$.

We thus obtain that if a threshold function $\nu_{prm}(t, I)$ is non-increasing in I for all t and $\nu_{prm}(t, 0) = \min(\tilde{v}(t), \bar{v})$ is non-decreasing in t where $\tilde{v}(\cdot)$ is the solution of (3.59) then the corresponding strategy $\mathcal{P}[\nu_{prm}]$ defines an equilibrium. We next prove the existence of ν_{prm} .

Firstly consider the following equation obtained by substituting $F(\min(\tilde{v}(\hat{t}_j), x))$ with $F(x)$ in (3.59) for all bidders arriving in the interval $(t, T]$:

$$\tilde{v}(t) - p = \mathbf{E}_t \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\tilde{v}(t)} \prod_{i=1}^{N(t)} \frac{F(\min(\tilde{v}(t_i), x))}{F(\tilde{v}(t_i))} \prod_{j=1}^{N(t,T)} F(x) dx \right) \right] \quad (3.60)$$

Notice that the right hand side of (3.60) at any time t depends only on $[\tilde{v}(\tau)]_{\tau \in [0, t]}$ while the right hand side of (3.59) is the function of $[\tilde{v}(\tau)]_{\tau \in [0, T]}$. However if the solution $\tilde{v}(t)$ to (3.60) is non-decreasing in t then $\min(\tilde{v}(\hat{t}_j), x) = x$ for all $x \in [\underline{v}, \tilde{v}(t)]$, $\hat{t}_j \in (t, T]$ and thus $\tilde{v}(t)$ also solves (3.59). We next show that (3.59) has a solution.

Lemma 7. For any $\varepsilon > 0$, there exists a solution to (3.60) in the interval $[0, T - \varepsilon]$.

Proof. Define

$$G(\phi)(t) = \mathbf{E}_t \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\phi(t)} \prod_{i=1}^{N(t)} \frac{F(\min(\phi(t_i), x))}{F(\phi(t_i))} \prod_{j=1}^{N(t,T)} F(x) dx \right) \right] + p$$

Using this notation, (3.60) becomes

$$\phi(t) = G(\phi)(t), \tag{3.61}$$

which we seek to solve on the interval $[0, T - \varepsilon]$.

To prove the existence of a solution of the above equation, we use the following theorem (Theorem 4.1 of Smart (1974)) which is a slight generalization of Schauder's fixed point theorem, Schauder (1930).

Theorem 5 (Smart (1974)). Let \mathcal{M} be a non-empty convex subset of a normed space \mathcal{B} . Let T be a continuous mapping of \mathcal{M} into a compact set $\mathcal{K} \subset \mathcal{M}$. Then T has a fixed point.

Using the above theorem, we show that (3.61) has a solution on the interval $[0, T - \varepsilon]$ for any $\varepsilon > 0$. For $M > 0$ let \mathcal{F} be the set of continuous functions with domain $[0, T - \varepsilon]$ and range $[\underline{v}, q]$ ($q > \underline{v}$) that satisfy the following Lipschitz condition

$$|\phi(t') - \phi(t)| \leq M|t' - t|; \quad t', t \in [0, T - \varepsilon] \tag{3.62}$$

Let $\mathcal{K} = \{G(\phi) | \phi \in \mathcal{F}\}$, i.e. G maps the set \mathcal{F} to \mathcal{K} . We first prove the following lemma.

Lemma 8. If $q > \frac{p - e^{-\beta\varepsilon}\underline{v}}{1 - e^{-\beta\varepsilon}}$ and $M > \frac{e^{-\beta\varepsilon}(2\lambda + \beta)q}{1 - e^{-\beta\varepsilon}}$ then $\mathcal{K} \subset \mathcal{F}$.

Proof. In Appendix. □

Thus if we choose $q > \frac{p - e^{-\beta\varepsilon}\underline{v}}{1 - e^{-\beta\varepsilon}}$ and $M > \frac{e^{-\beta\varepsilon}(2\lambda + \beta)q}{1 - e^{-\beta\varepsilon}}$ the operator G maps \mathcal{F} to $\mathcal{K} \subset \mathcal{F}$. We next show that the set \mathcal{K} is compact in \mathbb{C} where \mathbb{C} is the space of continuous functions with domain $[0, T - \varepsilon]$.

Lemma 9. \mathcal{K} is compact in \mathbb{C} .

Proof. Firstly notice that since $q > \frac{p-e^{-\beta\varepsilon}v}{1-e^{-\beta\varepsilon}}$ it follows from Lemma 8 that $G(\phi)(t) \leq q$ for all $\phi \in \mathcal{F}$ and $t \in [0, T - \varepsilon]$.

It follows from (A.22) (in the proof of Lemma 8) that for any $\kappa > 0$ and all $|t - t'| < \kappa/M$

$$|G(\phi)(t') - G(\phi)(t)| \leq M|t' - t| < \kappa \quad \forall \phi \in \mathcal{F}; t, t' \in [0, T - \varepsilon], \quad (3.63)$$

proving that the set $\mathcal{K} = \{G(\phi) | \phi \in \mathcal{F}\}$ is equicontinuous on the interval $[0, T - \varepsilon]$ (§7.22 in Rudin (1976)). The compactness of set \mathcal{K} then follows from the Arzelà-Ascoli Theorem (Theorem 3 (3.I) in Kantorovich and Ailov (1964)). \square

We next show that the operator G is continuous on the set \mathcal{F} .

Lemma 10. G is continuous on the set \mathcal{F} .

Proof. In Appendix. \square

Thus G is a continuous mapping of \mathcal{F} (which is non-empty and convex) into a compact set $\mathcal{K} \subset \mathcal{F}$ and hence, by Theorem 5, G has a fixed point, i.e. there exists a solution to the equation (3.61) on the interval $[0, T - \varepsilon]$. \square

Since Lemma 7 holds for any $\varepsilon > 0$, it proves that a solution to (3.60) exists on the interval $[0, T]$. We next show that $\tilde{v}(t)$ satisfying (3.60) on the interval $[0, T]$, is non-decreasing in t . For that we first need the following result. Let $\mathbf{E}[H(v, t)]$ be the right hand side of (3.60), i.e.

$$\mathbf{E}[H(v, t)] = \mathbf{E}_t \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^v \prod_{i=1}^{N(t)} \frac{F(\min(\tilde{v}(t_i), x))}{F(\tilde{v}(t_i))} \prod_{j=1}^{N(t,T)} F(x) dx \right) \right] \quad (3.64)$$

Lemma 11. $\mathbf{E}_t[H(v, t')] > \mathbf{E}_t[H(v, t)]$ for $t' > t$; $t, t' \in [0, T]$.

Proof. Let $t' > t$ for some $t, t' \in [0, T]$. Suppose that there are k bidders in $(0, t)$ and l bidders in $(t, T]$ and they arrive at time $0 < t_1 < t_2 < \dots < t_k < t$ and

$t < \hat{t}_1 < \hat{t}_2 < \dots < \hat{t}_l < T$ respectively. Suppose also that j of the l bidders ($0 \leq j \leq l$) arrive in the interval (t, t') .

Then from (3.64), we have

$$\begin{aligned} \mathbf{E}[H(v, t') | k, l, j, \{t_i\}_{i=1}^k, \{\hat{t}_j\}_{i=1}^l] &= e^{-\beta(T-t')} \left(\int_{\underline{v}}^v \frac{\prod_{i=1}^k F(\min(\tilde{v}(t_i), x))}{\prod_{i=1}^k F(\tilde{v}(t_i))} \right. \\ &\quad \left. \times \frac{\prod_{i=1}^j F(\min(\tilde{v}(\hat{t}_i), x))}{\prod_{i=1}^j F(\tilde{v}(\hat{t}_i))} \prod_{i=j+1}^l F(x) dx \right) \end{aligned}$$

For this arrival stream of bidders, $H(v, t)$ is:

$$\mathbf{E}[H(v, t) | k, l, \{t_i\}_{i=1}^k, \{\hat{t}_j\}_{i=1}^l] = e^{-\beta(T-t)} \left(\int_{\underline{v}}^v \frac{\prod_{i=1}^k F(\min(\tilde{v}(t_i), x))}{\prod_{i=1}^k F(\tilde{v}(t_i))} \prod_{i=1}^l F(x) dx \right)$$

Since $\frac{F(\min(\tilde{v}(\hat{t}_i), x))}{F(\tilde{v}(\hat{t}_i))} \geq F(x)$ for $i = 1, \dots, j$ and $e^{-\beta(T-t')} > e^{-\beta(T-t)}$, we have

$$\mathbf{E}[H(v, t') | k, l, j, \{t_i\}_{i=1}^k, \{\hat{t}_j\}_{i=1}^l] > \mathbf{E}[H(v, t) | k, l, \{t_i\}_{i=1}^k, \{\hat{t}_j\}_{i=1}^l] \quad (3.65)$$

The inequality (3.65) is true for any realization of the random bidder arrival process and, hence, the inequality holds if we take the expectation over the arrival process. Thus, we have

$$\mathbf{E}[H(v, t')] > \mathbf{E}[H(v, t)]$$

□

We next use Lemma 11 to prove $\tilde{v}(t)$ is non-decreasing in t .

Lemma 12. *The solution $\tilde{v}(t)$ of (3.60) is non-decreasing in t .*

Proof. By definition of $\tilde{v}(t)$, we have

$$\tilde{v}(t) - p = \mathbf{E}[H(\tilde{v}(t), t)] \quad (3.66)$$

Assume for contradiction that $\tilde{v}(t) < \tilde{v}(t - dt)$ for some $t \in [0, T)$ and $dt > 0$ (and

such that $t - dt \in [0, T)$. Then there exists $dv > 0$, such that

$$\tilde{v}(t) = \tilde{v}(t - dt) - dv \quad (3.67)$$

Using Lemma 11, we have

$$\mathbf{E}[H(\tilde{v}(t), t)] > \mathbf{E}[H(\tilde{v}(t), t - dt)] \quad (3.68)$$

Substituting (3.67) in (3.66) and using (3.68), we get

$$\begin{aligned} \tilde{v}(t - dt) - dv - p &= \mathbf{E}[H(\tilde{v}(t - dt) - dv, t)] \\ &> \mathbf{E}[H(\tilde{v}(t - dt) - dv, t - dt)] \end{aligned} \quad (3.69)$$

By the definition of $\tilde{v}(t - dt)$, we have

$$\tilde{v}(t - dt) - p = \mathbf{E}[H(\tilde{v}(t - dt), t - dt)] \quad (3.70)$$

Using (3.70) in (3.69), we get

$$\mathbf{E}[H(\tilde{v}(t - dt), t - dt)] - \mathbf{E}[H(\tilde{v}(t - dt) - dv, t - dt)] > dv$$

However, by arguments similar to those in the proof of Lemma 6, it can be shown that $\frac{\partial}{\partial v} \mathbf{E}[H(v, t)] \leq 1$ for $t \in [0, T)$, which contradicts the above result. Hence $\tilde{v}(t)$ is non-decreasing in t .

□

Thus we have shown that $\tilde{v}(t)$, satisfying (3.60) on the interval $[0, T)$, is non-decreasing in t . Next, consider any function $\nu_{prm}(t, I)$ which is non-increasing in I and satisfies $\nu_{prm}(t, 0) = \min(\tilde{v}(t), \bar{v})$ where $\tilde{v}(t)$ is the solution of (3.59) (and (3.60)) on the interval $[0, T)$. Then Lemma 12 shows that $\nu_{prm}(t, 0)$ is non-decreasing in t , and by Lemma 6 the corresponding strategy $\mathcal{P}[\nu_{prm}]$ satisfies $\mathcal{P}[\nu_{prm}] \in \mathcal{R}(\mathcal{P}[\nu_{prm}])$, thus defining a Bayesian Nash equilibrium for an online auction game with a permanent

buyout price p . This proves Theorem 4.

Note that Theorem 4 only provides a stringent characterization of the equilibrium threshold function value $\nu_{prm}(t, I)$ for $I = 0$. This is because when all bidders follow strategy $\mathcal{P}[\nu]$ then on the equilibrium path $I_t = 0$ for all t in $[0, T)$, since all bidders not exercising the buyout option only bid then at time T . Indeed, equation (3.48) specifies quantitatively the valuation for which an incoming bidder should be indifferent between exercising the option and submitting a regular bid, accounting for the information about the valuations of potential competing bidders provided by the presence of an open buyout option. Other values of $\nu_{prm}(t, I)$ correspond to off-equilibrium path behavior, and are only required to satisfy the monotonicity properties discussed above. For the special case where valuations follow a uniform distribution, the following proposition shows that the characterization of the equilibrium threshold function stated in Theorem 4 specializes to a nonlinear first-order differential equation:

Proposition 2. *When bidder valuations follow a uniform distribution with cdf F on $[\underline{v}, \bar{v}]$, the threshold function ν_{prm} characterizing the Bayesian Nash equilibrium described in Theorem 4 satisfies $\nu_{prm}(t, 0) = \min(\tilde{v}(t), \bar{v})$ where $\tilde{v}(t)$ is the unique solution on $[0, T]$ of the differential equation*

$$\frac{d\tilde{v}(t)}{dt} = \frac{\left(\beta + \lambda(1 - F(\tilde{v}(t)))\right)(\tilde{v}(t) - p)}{1 - e^{-\left(\beta + \lambda(1 - F(\tilde{v}(t)))\right)(T-t)}} \quad (3.71)$$

with initial value $\nu_{tmp}(0)$ as defined in (3.21).

Proof. In Appendix. □

3.2.2 Equilibrium Refinements

We next discuss the robustness of the outcome prediction provided by the equilibrium $\mathcal{P}[\nu_{prm}]$ characterized by Theorem 4. An important observation is that for the permanent buyout price auction game there exist equilibrium strategies other than

the ones characterized by strategy $\mathcal{P}[\nu]$. Indeed for any threshold function ν which satisfies the conditions of Theorem 3, and is such that $\nu(t, I) \geq p, \forall t, I$, it can be shown that a strategy

$$\bar{\mathcal{P}}[\nu](v, t, I_t) : \begin{cases} \text{Buyout at } p \text{ immediately} & \text{if } v > \nu(t, I_t) \\ \text{Bid } v \text{ at time } T & \text{if } p < v \leq \nu(t, I_t), \\ \text{Bid } v \text{ at any time in } [t, T] & \text{if } v \leq p \end{cases}, \quad (3.72)$$

which is same as $\mathcal{P}[\nu]$ except that a bidder with valuation $v \leq p$ bids in the auction at any time subsequent to his arrival (as opposed to bidding at time T as imposed by $\mathcal{P}[\nu]$), also constitutes an equilibrium. Notice that a bidder of type (v, t, I) , with utility function as defined in §3.1, who has a valuation $v < p$ gets negative utility from exercising the buyout option for any t and I thus justifying the condition $\nu(t, I) \geq p, \forall t, I$.

As in the temporary case, we next argue that unlike $\mathcal{P}[\cdot]$ the equilibria specified by strategy $\bar{\mathcal{P}}[\cdot]$ are not robust to some specific payoff perturbations. Indeed suppose that with a small probability an arriving bidder is *cautious* meaning that his participation behavior is characterized by a threshold function $\nu^P(t, I)$ which is such that $\nu^P(t, 0) = \min(\tilde{v}(t), \bar{v})$ and $\nu^P(t, I) = \underline{v}, \forall I > 0, t$. Such a bidder behaves like a *normal* bidder (with utility function as in §3.1) when no bids are placed in the auction, i.e. when $I = 0$; however once there is any bidding activity he exercises the buyout option immediately irrespective of the buyout price. Analysis similar to the proof of Theorem 3 in the temporary case shows that in the presence of such bidders the permanent buyout price auction game does not have any Bayesian Nash equilibrium where *normal* bidders play $\bar{\mathcal{P}}[\cdot]$; indeed this payoff perturbation filters out any equilibrium strategy where a bid is placed before time T . However, a strategy where the *normal* bidders play $\mathcal{P}[\nu_{prm}]$ still constitutes an equilibrium of the perturbed game and thus the strategies characterized in Theorem 4 are the only equilibria that survive the above payoff perturbation.

Unlike the temporary analysis, the concept of trembling-hand perfection does not

filter out any equilibria in the permanent case and so that analysis is omitted.

3.2.3 Seller's Optimization Problem

As in §3.1.3 we now turn to the revenue maximization problem faced by the seller. Specifically, we seek to determine the permanent buyout price p maximizing the seller's expected discounted revenue $\mathbf{E}[U_{prm}^S(p)]$ when all bidders follow strategy $\mathcal{P}[\nu_{prm}]$ defined by (3.47) and Theorem 4.

We now make the dependence of ν_{prm} on p explicit and denote by $\nu_{prm}(p, t)$ the value of the threshold function on the equilibrium path (i.e. the variable $I_t = 0$ is omitted). In equilibrium, the arrivals of bidders who will exercise the buyout option follow a non-homogeneous Poisson process with instantaneous rate $\lambda(1 - F(\nu_{prm}(p, t)))$, and we denote its counting measure by N_{buy} . Likewise, the arrivals of bidders who will wait until the end of the auction to submit a bid follow a non-homogeneous Poisson process with instantaneous rate $\lambda F(\nu_{prm}(p, t))$, and we denote its counting measure by N_{bid} . As a result, the probability that the buyout option will not be exercised is $P(N_{buy}(T) = 0) = \exp(-\lambda \int_0^T (1 - F(\nu_{prm}(p, t))) dt)$, and the problem can be stated as

$$\begin{aligned} \max_p \mathbf{E}[U_{prm}^S(p)] &= \int_0^T e^{-\alpha t} p \lambda (1 - F(\nu_{prm}(p, t))) e^{-\lambda \int_0^t (1 - F(\nu_{prm}(p, \tau))) d\tau} dt \\ &+ e^{-\lambda \int_0^T (1 - F(\nu_{prm}(p, t))) dt} e^{-\alpha T} \mathbf{E}[\mathbf{1}_{\{N_{bid}(T) > 0\}} \max(\underline{v}, v_{N_{bid}(T)}^{(2)}) | v_i \leq \nu_{prm}(p, t_i) \forall i], \end{aligned} \quad (3.73)$$

where the expectation \mathbf{E} is with respect to the number $N_{bid}(T)$ and epochs $t_1, \dots, t_{N_{bid}(T)}$ of arrivals in $[0, T]$ of the second Poisson process defined above, and second highest value $v_{N_{bid}(T)}^{(2)}$ among $v_1, \dots, v_{N_{bid}(T)}$ (by convention $v_0^{(2)} = v_1^{(2)} = 0$), where the i -th valuation v_i follows a distribution with cdf $F_i(v) = F(v)/F(\nu_{prm}(p, t_i))$. The first term in (3.73) is equal to the seller's expected discounted revenue from the option, while the second term is the expected discounted revenue from regular bidding, which only occurs if the buyout option is not exercised.

The challenge of finding analytically the optimal permanent buyout price p_{prm}^*

solving (3.73) seems even greater than with a temporary option. In the special case where valuations are uniformly distributed however, the seller's expected discounted revenue $\mathbf{E}[U_{prm}^S(p)]$ corresponding to a given buyout price p can be easily estimated through Monte-Carlo simulation by solving the differential equation (3.71) characterizing ν_{prm} using standard numerical methods, then generating many random bidder arrival streams $\{(v_1, t_1), (v_2, t_2), \dots\}$. A line search can then be performed to estimate the value p_{prm}^* maximizing $\mathbf{E}[U_{prm}^S(p)]$, which is the method we follow in the numerical experiments described in §5.1.

Similar to the temporary case, we next analyze the solution of the optimization problem (3.73) in some limiting regimes of α, β and λ . It turns out that except the case when $\alpha = 1/f_1(\frac{1}{\lambda})$, $\lambda \rightarrow \infty$, in all other regimes the asymptotic optimal price of a permanent buyout option is same as the corresponding price of the temporary option (shown in Table 3.1) so that the interpretation provided for \tilde{p}_{tmp}^* in §3.1.3 also applies here. When $\alpha = 1/f_1(\frac{1}{\lambda})$, $\lambda \rightarrow \infty$ the optimal permanent buyout price depends on $\lim_{\lambda \rightarrow \infty} \alpha/\lambda$. Recall that the function f_1 is such that $\lim_{x \rightarrow 0} f_1(x) \rightarrow 0$; if in addition it satisfies the condition that $\lim_{\lambda \rightarrow \infty} \frac{\alpha}{\lambda} = \lim_{\lambda \rightarrow \infty} \frac{1/f_1(\frac{1}{\lambda})}{\lambda} = k$ then it can be shown that the optimal buyout price $p_{prm}^* \rightarrow p^*(k)$ where $p^*(k)$ is defined as the maximizer (assumed unique) of $\frac{p(1-F(p))}{k+1-F(p)}$ on the interval $[\underline{v}, \bar{v}]$. Notice that $\lim_{k \rightarrow \infty} p^*(k) \rightarrow \bar{p}$, i.e. in a market environment where the arrival rate λ approaches infinity but the seller's sensitivity α goes to infinity faster than λ , the seller finds it optimal to choose a buyout price which maximizes his expected revenue from the event that the first bidder exercises the buyout option (same as that for a temporary option). In this case, because the seller is highly time-sensitive he obtains negligible utility from waiting for a subsequent bidder, and so maximizes the revenue he can obtain from the first bidder. For a less time-sensitive seller where f_1 is such that $k \rightarrow 0$, the optimal buyout price $p^*(k) \rightarrow \bar{v}$. In this case because λ goes to infinity faster than α , the seller is willing to wait for a small period of time, and since a lot of bidders arrive in this interval the seller can charge a much higher buyout price.

For comparison purposes, we now consider the special case of problem (3.73) when valuations are uniformly distributed and participating bidders are impatient,

i.e. $\beta \rightarrow +\infty$. Their buyout threshold valuation $\nu_{prm}(p, t) = \min(p, \bar{v})$, and the seller's revenue maximization problem (3.73) can then be expressed as the following

$$\begin{aligned} \max_{p \in [\underline{v}, \bar{v}]} & \frac{p\lambda(1 - F(p))}{\alpha + \lambda(1 - F(p))} \left(1 - e^{-\alpha T - \lambda T(1 - F(p))}\right) \\ & + e^{-\alpha T - \lambda T(1 - F(p))} \left(p(1 - e^{-\lambda T}) - 2\frac{(p - \underline{v})}{\lambda T} (1 - e^{-\lambda T} - \lambda T e^{-\lambda T})\right) \end{aligned} \quad (3.74)$$

where $F(p) = \frac{p - \underline{v}}{m}$, $\forall p \in [\underline{v}, \bar{v}]$. The optimization problem (3.74) is a nonlinear program in one variable which is straightforward to solve numerically.

Chapter 4

Dynamic Buyout Prices

In this chapter, we study the mechanism obtained when the buyout price, both temporary (in §4.1 and permanent (in 4.2), is no longer constant but instead varies over the length of the auction according to a pre-announced trajectory $[p(t)]_{t \in [0, T]}$. While we are not aware of any actual auction site currently implementing such a feature, our goal is to develop a theoretical analysis providing some prediction for what the outcome of such mechanism is likely to be, and bound the maximum expected revenue achievable by the seller when setting this buyout price trajectory optimally.

4.1 Temporary Buyout Option

For a restricted set of buyout price trajectories we derive equilibrium strategies for a temporary buyout price auction game in §4.1.1 and discuss the associated seller's problem of maximizing expected discounted by optimally choosing the buyout price trajectory in §4.1.2.

4.1.1 Outcome Prediction

In an auction with a temporary buyout price following a dynamic trajectory $[p(t)]_{t \in [0, T]}$, consider the extension of strategy $\mathcal{T}[\nu]$ obtained for any function $\nu : [0, T] \rightarrow [\underline{\nu}, \bar{\nu}]$

by substituting $p(t)$ with p in the first line of (3.1):

$$\mathcal{T}[\nu](v, t) : \begin{cases} \text{Buyout at } p(t) \text{ immediately} & \text{if buyout option available and } v > \nu(t) \\ \text{Bid } v \text{ immediately} & \text{if buyout option available and } v \leq \nu(t) ; \\ \text{Bid } v \text{ at any time in } [t, T] & \text{otherwise} \end{cases} \quad (4.1)$$

for notational simplicity we will still refer to the resulting strategy as $\mathcal{T}[\nu]$. The following result establishes that any non-decreasing continuous threshold function ν can be supported by some price trajectory in equilibrium:

Theorem 6. *For any non-decreasing continuous function $\nu : [0, T] \rightarrow [\underline{v}, \bar{v}]$, define function $p : [0, T] \rightarrow [\underline{v}, \bar{v}]$ as*

$$p(t) = \nu(t) - e^{-(\lambda+\beta)(T-t)} \int_{\underline{v}}^{\nu(t)} e^{\lambda(T-t)F(x)} dx. \quad (4.2)$$

The symmetric strategy profile $\mathcal{T}[\nu]$ is then a Bayesian Nash equilibrium for the auction with temporary buyout price trajectory $[p(t)]_{t \in [0, T]}$.

Theorem 6 can be interpreted as following: any threshold function ν that is continuous and non-decreasing with time corresponds to a buyout price trajectory such that the strategy profile $\mathcal{T}[\nu]$ forms an equilibrium. In fact, the negative of the second term in the right-hand side of (4.2) represents the expected utility that a bidder arriving at time t and having a valuation equal to the threshold would obtain by submitting a regular bid (as opposed to exercising the buyout option). Therefore, (4.2) expresses that the buyout price $p(t)$ it defines is such that a bidder arriving at time t with a valuation equal to the threshold $\nu(t)$ would be indifferent between submitting a regular bid and exercising the buyout option (provided it is still open) at that price. However, setting the buyout price $p(t)$ according to (4.2) is only a necessary condition in general, and would not eliminate alone the possibility that a bidder could benefit from waiting beyond his arrival before choosing between these two options – this could occur for example if the buyout price is known to substantially decrease in the future, and would give rise to a competitive optimal stopping

situation in which strategy $\mathcal{T}[\nu]$ would not form an equilibrium. Theorem 6 actually establishes that in a temporary buyout price auction no rational bidder will ever find such wait to be more profitable a priori than acting immediately when the target valuation threshold is non-decreasing over time. Note that this does not imply that the buyout price itself is non-decreasing – in fact, for a constant valuation threshold $\nu(t) = \nu \in [\underline{\nu}, \bar{\nu}]$, which satisfies the conditions of Theorem 6, the price trajectory defined by (4.2) is decreasing. Only, in the incoming bidders' assessment it does not decrease fast enough for the possible utility increase derived from waiting to strictly overcome time discounting and the risk associated with the arrival of another bidder while the option is still open.

The result in Theorem 6 can be proven as following: Consider any bidder, say A , with type (v, t) . Assuming other bidders play $\mathcal{T}[\nu]$ with any continuous non-decreasing threshold function ν , we show that A 's best response strategy is to himself play $\mathcal{T}[\nu]$ if

$$\begin{aligned} p(t) &= \nu(t) - \mathbf{E}[U_{bid(t)}(\nu(t), t) | N_t = 0] \\ &= \nu(t) - e^{-(\lambda+\beta)(T-t)} \int_{\underline{\nu}}^{\nu(t)} e^{\lambda(T-t)F(x)} dx \end{aligned}$$

for all $t \in [0, T]$. This proves that $\mathcal{T}[\nu]$ is a Bayesian Nash equilibrium for the auction with temporary buyout price trajectory $[p(t)]_{t \in [0, T]}$ defined above.

If A is not the first bidder, the first bidder (following strategy $\mathcal{T}[\nu]$) would have either bid in the auction or exercised the buyout option immediately and hence the buyout option is not available to bidder A . In that case, the auction progresses as a standard second-price auction and A 's weakly dominant strategy is to bid his true valuation.

If A is the first bidder and the buyout option is available to him then he can either act immediately – exercise the buyout option or place a bid – or wait in the auction. The following lemma derives a condition that the threshold function ν must satisfy to ensure that the first bidder is weakly better off immediately.

Lemma 13. *When other bidders follow strategy $\mathcal{T}[\nu]$ in an auction with a temporary*

dynamic buyout price, the first bidder is weakly better off acting immediately i.e. the utility from acting immediately is at least as much as from waiting, if and only if the threshold valuation $\nu(t)$ is non-decreasing in t for all $t \in [0, T]$.

Proof. Suppose the first bidder is of type (v, t) . If the bidder waits up to time τ ($\tau > t$), his expected utility, using the notation in the proof of Theorem 1, is

$$\begin{aligned} \mathbf{E}[U_{wait(t,\tau)}(v, t)|N_t = 0] &= e^{-\beta(\tau-t)} \left(\max \{B_1(v, \tau), v - p(\tau)\} \cdot P(\mathcal{E}) \right. \\ &\quad \left. + \mathbf{E}[U_{bid(\tau)}(v, \tau)|N_t = 0, \bar{\mathcal{E}}] \cdot P(\bar{\mathcal{E}}) \right) \end{aligned} \quad (4.3)$$

Then using (3.15), we get

$$\begin{aligned} \mathbf{E}[U_{wait(t,\tau)}(v, t)|N_t = 0] \\ \leq e^{-\beta(\tau-t)} \left(\max \{v - p(\tau) - \mathbf{E}[U_{bid(\tau)}(v, \tau)|N_\tau = 0], 0\} \cdot P(\mathcal{E}) \right) + B_1(v, t) \end{aligned} \quad (4.4)$$

Now the first bidder of type (v, t) makes a decision immediately if he cannot gain by waiting, i.e. if

$$\max\{v - p(t), B_1(v, t)\} \geq \mathbf{E}[U_{wait(t,\tau)}(v, t)|N_t = 0] \quad \forall \tau > t \quad (4.5)$$

Indeed for the result of the lemma to hold in general, this condition must be true for all $v \in [\underline{v}, \bar{v}]$ and $t \in [0, T]$. Thus, we enforce the following constraint, for all $v \in [\underline{v}, \bar{v}]$; $t, \tau \in [0, T]$ and $\tau > t$

$$\begin{aligned} \max\{v - p(t), B_1(v, t)\} \\ \geq e^{-\beta(\tau-t)} \left(\max \{v - p(\tau) - \mathbf{E}[U_{bid(\tau)}(v, \tau)|N_\tau = 0], 0\} \cdot P(\mathcal{E}) \right) + B_1(v, t) \end{aligned}$$

which can be rewritten as

$$\begin{aligned} & \max \{v - P(t) - \mathbf{E}[U_{bid(t)}(v, t)|N_t = 0], 0\} \\ & \geq e^{-\beta(\tau-t)} \left(\max \{v - p(\tau) - \mathbf{E}[U_{bid(\tau)}(v, \tau)|N_\tau = 0], 0\} P(\mathcal{E}) \right) \end{aligned} \quad (4.6)$$

where by definition $B_1(v, t) = \mathbf{E}[U_{bid(t)}(v, t)|N_t = 0]$. By (4.4), the constraint (4.6) implies the condition (4.5) and thus if (4.6) holds then the first bidder is weakly better off acting immediately.

For an arbitrary t and $\tau(\tau > t)$, consider the following two cases:

1. $v \leq \nu(t)$: In this case bidding is more attractive to the first bidder at time t , i.e. we have $v - p(t) - \mathbf{E}[U_{bid(t)}(v, t)|N_t = 0] \leq 0$. Thus the constraint (4.6) becomes

$$\begin{aligned} 0 &= \max \{v - p(t) - \mathbf{E}[U_{bid(t)}(v, t)|N_t = 0], 0\} \\ &\geq e^{-\beta(\tau-t)} \left(\max \{v - p(\tau) - \mathbf{E}[U_{bid(\tau)}(v, \tau)|N_\tau = 0], 0\} P(\mathcal{E}) \right) \end{aligned}$$

which holds if

$$v - p(\tau) - \mathbf{E}[U_{bid(\tau)}(v, \tau)|N_\tau = 0] \leq 0 \quad \forall v \leq \nu(t) \quad (4.7)$$

Now notice that $v - p(\tau) - \mathbf{E}[U_{bid(\tau)}(v, \tau)|N_\tau = 0]$ is increasing in valuation v since

$$\frac{\partial}{\partial v} \left(v - p(\tau) - \mathbf{E}[U_{bid(\tau)}(v, \tau)|N_\tau = 0] \right) = 1 - e^{-(\lambda(1-F(v))+\beta)(T-\tau)} \geq 0$$

and thus it is sufficient to impose the condition (4.7) at $v = \nu(t)$. This gives

$$\nu(t) - p(\tau) - \mathbf{E}[U_{bid(\tau)}(\nu(t), \tau)|N_\tau = 0] \leq 0$$

Substituting $\nu(t) = p(t) + \mathbf{E}[U_{bid(t)}(\nu(t), t)|N_t = 0]$, the condition that the first bidder must be better off acting immediately than at time $\tau > t$ can be re-

written as

$$p(\tau) - p(t) \geq \mathbf{E}[U_{bid(t)}(\nu(t), t) | N_t = 0] - \mathbf{E}[U_{bid(\tau)}(\nu(t), \tau) | N_\tau = 0] \quad (4.8)$$

2. $v > \nu(t)$: In this case $v - p(t) - \mathbf{E}[U_{bid(t)}(v, t) | N_t = 0] > 0$. Thus (4.6) becomes

$$v - p(t) - \mathbf{E}[U_{bid(t)}(v, t) | N_t = 0] \geq e^{-\beta(T-t)} P(\mathcal{E}) \left(v - p(\tau) - \mathbf{E}[U_{bid(\tau)}(v, \tau) | N_\tau = 0] \right)$$

for all $v > \nu(t)$. Using the fact that $P(\mathcal{E})$, the probability of no arrival in the interval (t, τ) , is $e^{-\lambda(\tau-t)}$ the above condition can be expressed as, for all $v > \nu(t)$,

$$p(\tau) - e^{(\lambda+\beta)(\tau-t)} p(t) \geq e^{(\lambda+\beta)(\tau-t)} \mathbf{E}[U_{bid(t)}(v, t) | N_t = 0] - \mathbf{E}[U_{bid(\tau)}(v, \tau) | N_\tau = 0] + v \left(1 - e^{(\lambda+\beta)(\tau-t)} \right)$$

which is equivalent to

$$\begin{aligned} & p(\tau) - e^{(\lambda+\beta)(\tau-t)} p(t) \\ & \geq \sup_{v > \nu(t)} \left(e^{(\lambda+\beta)(\tau-t)} \mathbf{E}[U_{bid(t)}(v, t) | N_t = 0] - \mathbf{E}[U_{bid(\tau)}(v, \tau) | N_\tau = 0] \right. \\ & \quad \left. + v \left(1 - e^{(\lambda+\beta)(\tau-t)} \right) \right) \\ & = e^{(\lambda+\beta)(\tau-t)} \mathbf{E}[U_{Bid}(\nu(t), t)] - \mathbf{E}[U_{Bid}(\nu(t), \tau)] + \nu(t) \left(1 - e^{(\lambda+\beta)(\tau-t)} \right) \quad (4.9) \end{aligned}$$

where the equality follows from the fact that the supremum of the above ex-

pression occurs at $v = \nu(t)$. To see this notice that for all $v \in [\underline{v}, \bar{v}]$

$$\begin{aligned}
& \frac{\partial}{\partial v} \left(e^{(\lambda+\beta)(\tau-t)} \mathbf{E}[U_{bid(t)}(v, t) | N_t = 0] - \mathbf{E}[U_{bid(\tau)}(v, \tau) | N_\tau = 0] + v(1 - e^{(\lambda+\beta)(\tau-t)}) \right) \\
&= e^{-(\lambda+\beta)(T-\tau)} \left(e^{\lambda(T-t)F(v)} - e^{\lambda(T-\tau)F(v)} \right) + 1 - e^{(\lambda+\beta)(\tau-t)} \\
&= e^{-(\lambda+\beta)(T-\tau)} e^{\lambda(T-\tau)F(v)} \left(e^{\lambda(\tau-t)F(v)} - 1 \right) + 1 - e^{(\lambda+\beta)(\tau-t)} \\
&\leq \left(e^{(\lambda+\beta)(\tau-t)} - 1 \right) \left(e^{-\lambda(1-F(v))+\beta(T-\tau)} - 1 \right) \\
&\leq 0
\end{aligned}$$

where the first inequality follows since $\lambda F(v) \leq \lambda + \beta$.

Substituting $\nu(t) = p(t) + \mathbf{E}[U_{bid(t)}(\nu(t), t) | N_t = 0]$ in (4.9) we get

$$p(\tau) - p(t) \geq \mathbf{E}[U_{bid(t)}(\nu(t), t) | N_t = 0] - \mathbf{E}[U_{bid(\tau)}(\nu(t), \tau) | N_\tau = 0]$$

which is same as the condition (4.8) obtained in Case 1.

Thus first bidder is weakly better off acting immediately if for all $t, \tau \in [0, T], \tau > t$

$$p(\tau) - p(t) \geq \mathbf{E}[U_{bid(t)}(\nu(t), t) | N_t = 0] - \mathbf{E}[U_{bid(\tau)}(\nu(t), \tau) | N_\tau = 0] \quad (4.10)$$

Substituting $p(t) = \nu(t) - \mathbf{E}[U_{bid(t)}(\nu(t), t) | N_t = 0]$ in (4.10) gives the condition

$$\nu(\tau) - \nu(t) - \mathbf{E}[U_{bid(\tau)}(\nu(\tau), \tau) | N_\tau = 0] + \mathbf{E}[U_{bid(\tau)}(\nu(t), \tau) | N_\tau = 0] \geq 0 \quad (4.11)$$

for all $\tau, t \in [0, T], \tau > t$. By setting $\tau = t + \Delta t$ ($\Delta t > 0$) we get that (4.11) holds if and only if for all $t \in [0, T]$

$$\lim_{\Delta t \rightarrow 0} \left(\nu(t + \Delta t) - \nu(t) \right) \left(1 - \frac{\partial}{\partial v} \mathbf{E}[U_{bid(t+\Delta t)}(v, t + \Delta t) | N_{t+\Delta t} = 0] \Big|_{v=\nu(t)} \right) \geq 0. \quad (4.12)$$

It can be easily shown that $\frac{\partial}{\partial v} \mathbf{E}[U_{Bid}(v, t) | N_t = 0] \Big|_{v=\nu(t)} < 1, \forall v \in [\underline{v}, +\infty), t \in [0, T]$ and thus (4.12) holds if and only if $\nu(\cdot)$ is non-decreasing in t for all $t \in [0, T]$.

We have thus shown that if $\nu(t)$ is non-decreasing in t then the first bidder does

not gain by waiting provided the other bidders play the strategy $\mathcal{T}[\nu]$. We now prove the other direction, i.e. if the first bidder is weakly better off acting immediately then the threshold valuation $\nu(t)$ is non-decreasing in t .

Assume, for contradiction, that $\nu(t)$ is not non-decreasing in t and indeed there exists an interval $[t_1, t_2] \subset [0, T]$ such that

$$\nu(t) < \nu(t_1) \quad \forall t \in (t_1, t_2] \quad (4.13)$$

We now show that if (4.13) holds then there exists a case when the first bidder is strictly better off waiting in the auction. Indeed suppose that the first bidder, say A , with type $(\nu(t_1), t_1)$ waits up to time $\tau = t_1 + \epsilon$ (where $t_1 < \tau < t_2$). Then his utility from the auction, as derived in (3.9), is

$$\begin{aligned} & \mathbf{E}[U_{wait(t_1, \tau)}(\nu(t_1), t) | N_{t_1} = 0] \\ &= e^{-\beta(\tau - t_1)} \left(\max \{ B_1(\nu(t_1), \tau), \nu(t_1) - p(\tau) \} P(\mathcal{E}) + \mathbf{E}[U_{bid(\tau)}(\nu(t_1), \tau) | N_{t_1} = 0, \bar{\mathcal{E}}] P(\bar{\mathcal{E}}) \right) \end{aligned} \quad (4.14)$$

where, recall that the event $\mathcal{E} = \{N(t_1, \tau) = 0\}$ and $\bar{\mathcal{E}}$ denotes the complimentary event.

If the event \mathcal{E} occurs then the buyout option is still available at time τ . Furthermore since $\nu(t_1) > \nu(\tau)$, i.e. $\nu(t_1) - p(\tau) > B_1(\nu(t_1), \tau)$, bidder A will choose to exercise the buyout option at time τ .

Next, by defining \mathcal{G} as the event that the first bidder, say B , arriving in (t_1, τ) with type (v_B, t_B) (where $t_B \in (t_1, \tau)$) has valuation $\nu(t_B) < v_B \leq \nu(t_1)$, (4.14) can be rewritten as

$$\begin{aligned} & \mathbf{E}[U_{wait(t_1, \tau)}(\nu(t_1), t) | N_{t_1} = 0] \\ &= e^{-\beta(\tau - t_1)} \left((\nu(t_1) - p(\tau)) \cdot P(\mathcal{E}) + \mathbf{E}[U_{bid(\tau)}(\nu(t_1), \tau) | N_{t_1} = 0, \bar{\mathcal{E}}, \mathcal{G}] \cdot P(\mathcal{G} | \bar{\mathcal{E}}) \cdot P(\bar{\mathcal{E}}) \right. \\ & \quad \left. + \mathbf{E}[U_{bid(\tau)}(\nu(t_1), \tau) | N_{t_1} = 0, \bar{\mathcal{E}}, \bar{\mathcal{G}}] \cdot P(\bar{\mathcal{G}} | \bar{\mathcal{E}}) \cdot P(\bar{\mathcal{E}}) \right) \end{aligned} \quad (4.15)$$

where $\bar{\mathcal{G}}$ denotes the complimentary event.

Bidder A 's utility from the auction if he acts immediately is $B_1(\nu(t_1), t_1)$ which can be rewritten as:

$$\begin{aligned} B_1(\nu(t_1), t_1) &= \mathbf{E}[U_{bid(t_1)}(\nu(t_1), t_1)|N_{t_1} = 0, \mathcal{E}]P(\mathcal{E}) + \mathbf{E}[U_{bid(t_1)}(\nu(t_1), t_1)|N_{t_1} = 0, \bar{\mathcal{E}}, \mathcal{G}]P(\mathcal{G}|\bar{\mathcal{E}})P(\bar{\mathcal{E}}) \\ &+ \mathbf{E}[U_{bid(t_1)}(\nu(t_1), t_1)|N_{t_1} = 0, \bar{\mathcal{E}}, \bar{\mathcal{G}}]P(\bar{\mathcal{G}}|\bar{\mathcal{E}})P(\bar{\mathcal{E}}) \end{aligned} \quad (4.16)$$

Let Δ denote the difference in utility of bidder A if he waits up to time τ as opposed to acting immediately at t_1 , i.e. $\Delta = \mathbf{E}[U_{wait(t_1, \tau)}(\nu(t_1), t_1)|N_{t_1} = 0] - B_1(\nu(t_1), t_1)$, then subtracting (4.16) from (4.15) we get, using (3.13) and (3.14)

$$\begin{aligned} \Delta &= \left(e^{-\beta(\tau-t_1)}(\nu(t_1) - p(\tau)) - \mathbf{E}[U_{bid(t_1)}(\nu(t_1), t_1)|N_{t_1} = 0, \mathcal{E}] \right) \cdot P(\mathcal{E}) \\ &\quad - \mathbf{E}[U_{bid(t_1)}(\nu(t_1), t_1)|N_{t_1} = 0, \bar{\mathcal{E}}, \mathcal{G}] \cdot P(\mathcal{G}|\bar{\mathcal{E}}) \cdot P(\bar{\mathcal{E}}) \end{aligned} \quad (4.17)$$

Now substituting $p(\tau) = \nu(\tau) - \mathbf{E}[U_{bid(\tau)}(\nu(\tau), \tau)|N_\tau = 0]$, the first term of (4.17) becomes:

$$\begin{aligned} &e^{-\beta(\tau-t_1)}(\nu(t_1) - p(\tau)) - \mathbf{E}[U_{bid(t_1)}(\nu(t_1), t_1)|N_{t_1} = 0, \mathcal{E}] \\ &= e^{-\beta(\tau-t_1)}(\nu(t_1) - \nu(\tau)) + e^{-\beta(\tau-t_1)}\mathbf{E}[U_{bid(\tau)}(\nu(\tau), \tau)|N_\tau = 0] \\ &\quad - \mathbf{E}[U_{bid(t_1)}(\nu(t_1), t_1)|N_{t_1} = 0, \mathcal{E}] \\ &= e^{-\beta(\tau-t_1)}\left(\nu(t_1) - \nu(\tau) + \mathbf{E}[U_{bid(\tau)}(\nu(\tau), \tau)|N_\tau = 0] - \mathbf{E}[U_{bid(\tau)}(\nu(t_1), \tau)|N_\tau = 0]\right) \\ &> 0 \end{aligned} \quad (4.18)$$

where the second equality follows since (3.13) and (3.8) imply that

$$\begin{aligned} \mathbf{E}[U_{bid(t_1)}(\nu(t_1), t_1)|N_{t_1} = 0, \mathcal{E}] &= \mathbf{E}[U_{bid(\tau)}(\nu(t_1), t_1)|N_\tau = 0] \\ &= e^{-\beta(\tau-t_1)}\mathbf{E}[U_{bid(\tau)}(\nu(t_1), \tau)|N_\tau = 0] \end{aligned}$$

The inequality in (4.18) follows since $v - \mathbf{E}[U_{bid(\tau)}(v, \tau)|N_\tau = 0]$ is increasing in v

for all τ and $\nu(t_1) > \nu(\tau)$. Now since $P(\mathcal{E}) = 1 - \lambda\epsilon + o(\epsilon)$ and $P(\bar{\mathcal{E}}) = \lambda\epsilon + o(\epsilon)$, we obtain that there exists an $\epsilon > 0$ such that $\Delta > 0$. This implies that

$$\Delta = \mathbf{E}[U_{wait(t_1, \tau)}(\nu(t_1), t_1) | N_{t_1} = 0] - B_1(\nu(t_1), t_1) > 0$$

where $\tau = t + \epsilon$. Thus bidder A is strictly better off waiting for $\epsilon > 0$ units of time which is a contradiction, thus proving that if the first bidder is weakly better off acting immediately then the threshold valuation $\nu(t)$ is non-decreasing in t . \square

Since $\nu(t)$ is assumed to be non-decreasing in t , the above lemma implies that bidder A will either exercise the buyout option immediately or place a bid in the auction immediately. Now notice that the buyout price $p(t) = \nu(t) - \mathbf{E}[U_{bid(T)}(\nu(t), t) | N_t = 0]$ is such that

$$U_{buy(t)}(\nu(t), t) = \mathbf{E}[U_{bid(t)}(\nu(t), t) | N_t = 0] \quad (4.19)$$

Additionally, as in the static buyout price case, the excess utility function

$$\delta(v, p(t), t) = v - p(t) - \mathbf{E}[U_{bid(t)}(v, t) | N_t = 0]$$

is increasing in valuation v . Combining this with (4.19), bidder A 's best response strategy is to exercise the buyout option immediately if $v > \nu(t)$ and bid his true valuation immediately otherwise.

Hence, A 's best response strategy to $\mathcal{T}[\nu]$ is to himself play $\mathcal{T}[\nu]$ and since bidder A is arbitrarily chosen, it proves that $\mathcal{T}[\nu]$ is a Bayesian Nash equilibrium of an auction game with temporary buyout price $p(t) = \nu(t) - \mathbf{E}[U_{bid(T)}(\nu(t), t) | N_t = 0]$ for $t \in [0, T]$ thus proving Theorem 6.

Note that, as is the case with static buyout options, there may exist other equilibria for the temporary buyout price games besides those characterized here. In contrast with the static buyout case unfortunately, we have not been able to develop any formal robustness results rationalizing the use for outcome prediction of these specific equilibria among all possible ones. We do however make the observation that the following form of reciprocity holds for Theorem 6: for every continuous valuation

threshold curve ν that is strictly decreasing with time on some interval, there exist bidders whose best response to the symmetric profile $\mathcal{T}[\nu]$ (resp. $\mathcal{P}[\nu]$) will not be $\mathcal{T}[\nu]$ (resp. $\mathcal{P}[\nu]$). This suggests that any equilibrium we may be ignoring is likely to involve strategic and possibly risky waiting behavior relative to exercising the buyout option, which in practice may be unattractive to some bidders for reasons that our model does not capture (e.g. cost of auction monitoring efforts).

4.1.2 Seller's Optimization Problem

In this subsection we study the maximum expected discounted revenue achievable by the seller through the choice of a temporary or permanent buyout price trajectory $[p(t)]_{t \in [0, T]}$, using the equilibria characterized in Theorem 6 as a prediction of the game outcome.

An important implication of Theorem 6 is that, within the range of equilibria considered, finding an optimal price trajectory $[p(t)]_{t \in [0, T]}$ exactly corresponds to finding its associated continuous and non-decreasing threshold function $\nu : [0, T] \rightarrow [\underline{v}, \bar{v}]$ subject to (4.2). Denoting by \mathcal{C}^+ the set of all such functions, for $\nu \in \mathcal{C}^+$ and $[p(t)]_{t \in [0, T]}$ given by (4.2), the seller's expected discounted revenue conditional on the first bidder arriving at $t_1 = t$ when all bidders follow strategy $\mathcal{T}[\nu]$ is given by

$$\begin{aligned}
 u_{tmp}(\nu(t), t) &\triangleq \mathbf{E}[U_{tmp}^S(\nu)|t_1 = t] = e^{-\alpha t} \mathbf{E}_t[\max(\underline{v}, v_{N(t, T)+1}^{(2)})|v_1 \leq \nu(t)]F(\nu(t)) \\
 &\quad + e^{-\alpha t} \left(\nu(t) - e^{-(\lambda+\beta)(T-t)} \int_{\underline{v}}^{\nu(t)} e^{\lambda(T-t)F(x)} dx \right) (1 - F(\nu(t)))
 \end{aligned} \tag{4.20}$$

where the expectation \mathbf{E}_t in the first integrand is with respect to the number $N(t, T)$ of arrivals in interval $(t, T]$ of a Poisson process with rate λ and the second highest value $v_{N(t, T)+1}^{(2)}$ among $N(t, T)+1$ independent draws $v_1, \dots, v_{N(t, T)+1}$ from the valuation distribution with cdf F , where by convention $v_1^{(2)} = 0$ – note that the first and second integrals in (4.20) correspond respectively to the seller's expected revenue when the first bidder submits a regular bid upon his arrival and when he exercises the buyout

option. Note that the instantaneous buyout price $p(t)$ has been substituted with the right-hand side of (4.2), and that the notation $u_{tmp}(\nu(t), t)$ introduced shows explicitly that the right-hand side of (4.20) only depends on the value of ν at t . The seller's revenue maximization problem can thus be stated as

$$Z_{tmp}^* \triangleq \sup_{\nu \in \mathcal{C}^+} \mathbf{E}[U_{tmp}^S(\nu)] = \sup_{\nu \in \mathcal{C}^+} \int_0^T u_{tmp}(\nu(t), t) \lambda e^{-\lambda t} dt. \quad (4.21)$$

We next establish that a discretized version of problem (4.21) provides an upper bound for the seller's maximum expected discounted revenue Z_{tmp}^* just defined.

Indeed consider the following problem:

$$\hat{Z}_{tmp} \triangleq \sup_{\nu \in \mathcal{C}_0^+} \mathbf{E}[U_{tmp}^S(\nu)] = \sup_{\nu \in \mathcal{C}_0^+} \int_0^T u_{tmp}(\nu(t), t) \lambda e^{-\lambda t} dt \quad (4.22)$$

where \mathcal{C}_0^+ denotes the set of all non-decreasing functions $\nu : [0, T] \rightarrow [\underline{v}, \bar{v}]$.

Clearly $\mathcal{C}^+ \subset \mathcal{C}_0^+$ and thus $\hat{Z}_{tmp} \geq Z_{tmp}^*$. The compactness of set \mathcal{C}_0^+ follows from the Helly compactness theorem (§7.9 in Ewing (1985)). In addition it can be shown that the objective function of (4.22) is continuous over \mathcal{C}_0^+ and thus there exists a $\nu^* \in \mathcal{C}_0^+$ that achieves the optimal utility \hat{Z}_{tmp} . We first prove the following result.

Lemma 14. *The function $u_{tmp}(\nu^*(t), t)$ is decreasing in t for $t \in [0, T]$, where $\nu^*(t)$ is the solution of (4.22).*

Proof. We proceed by first proving that $u_{tmp}(v, t)$ is decreasing in t for $t \in [0, T]$. Indeed consider the partial derivative of $u_{tmp}(v, t)$ with respect to t

$$\begin{aligned} \frac{\partial}{\partial t} u_{tmp}(v, t) &= -\alpha \left(v - \mathbf{E}[U_{bid(t)}(v, t) | N_t = 0] \right) (1 - F(v)) \\ &\quad + e^{-\alpha t} \left(-\frac{\partial}{\partial t} \mathbf{E}[U_{bid(t)}(v, t) | N_t = 0] \right) (1 - F(v)) \\ &\quad + e^{-\alpha T} F(v) \frac{\partial}{\partial t} \mathbf{E}_t[\max(\underline{v}, v_{N(t, T)+1}^{(2)}) | v_1 \leq v] \end{aligned}$$

Now notice that $\mathbf{E}[U_{bid(t)}(v, t) | N_t = 0]$ is increasing in t while $\mathbf{E}_t[\max(\underline{v}, v_{N(t, T)+1}^{(2)}) | v_1 \leq v]$ is decreasing in t . Using this in the above expression yields that $\frac{\partial}{\partial t} u_{tmp}(v, t) < 0$ and thus $u_{tmp}(v, t)$ is decreasing in t .

Now assume, for contradiction, that $u_{tmp}(\nu^*(t), t)$ is not decreasing in t and indeed there exists an interval $[t_1, t_2] \subset [0, T]$ such that

$$u_{tmp}(\nu^*(t), t) \leq u_{tmp}(\nu^*(t_2), t_2) \quad \forall t \in [t_1, t_2]$$

and thus

$$\int_{t_1}^{t_2} u_{tmp}(\nu^*(t), t) dt \leq \int_{t_1}^{t_2} u_{tmp}(\nu^*(t_2), t_2) dt = u_{tmp}(\nu^*(t_2), t_2)(t_2 - t_1) \quad (4.23)$$

Consider the following valuation trajectory

$$\hat{\nu}(t) = \begin{cases} \nu^*(t_2) & \forall t \in [t_1, t_2] \\ \nu^*(t) & \text{otherwise} \end{cases}$$

Since $\nu^*(t_1) \geq \nu^*(t_2)$, $\hat{\nu} \in C_0^+$ and is thus feasible for the problem (4.22). Hence since ν^* is the optimal solution of the problem (4.22), the utility obtained from using threshold valuation $\hat{\nu}$ must be less than or equal to the optimal utility. Since $\hat{\nu}(t) = \nu^*(t)$ for all $t \notin [t_1, t_2]$, the optimality of ν^* implies

$$\begin{aligned} \int_{t_1}^{t_2} u_{tmp}(\nu^*(t), t) dt &\geq \int_{t_1}^{t_2} u_{tmp}(\hat{\nu}(t), t) dt \\ &= \int_{t_1}^{t_2} u_{tmp}(\nu^*(t_2), t) dt \\ &> \int_{t_1}^{t_2} u_{tmp}(\nu^*(t_2), t_2) dt = u_{tmp}(\nu^*(t_2), t_2)(t_2 - t_1) \end{aligned}$$

where the second inequality follows since $u_{tmp}(v, t)$ is decreasing in t . This contradicts (4.23). \square

For any partition $\tau \triangleq (\tau_j)_{j \in \{0, \dots, m\}}$ of $[0, T]$ into m subintervals such that $\tau_0 = 0 < \tau_1 < \dots < \tau_m = T$, let $\Delta\tau_j \triangleq \tau_{j+1} - \tau_j$ for $j \in \{0, \dots, m-1\}$ and $\Delta\tau \triangleq \max_j \Delta\tau_j$ be

the mesh size of τ . Define a discretized problem

$$\begin{aligned} \bar{Z}_{tmp}(\tau) \triangleq & \max_{(\nu_j)_{j \in \{0, \dots, m\}}} \sum_{j=0}^{m-1} u_{tmp}(\nu_j, \tau_j) \lambda e^{-\lambda \tau_j} \Delta \tau_j \\ & \text{subject to: } \underline{\nu} \leq \nu_{j-1} \leq \nu_j \leq \bar{\nu} \text{ for all } j \in \{1, \dots, m\} \end{aligned} \quad (4.24)$$

Then we have

$$\begin{aligned} \hat{Z}_{tmp} &= \int_0^T u_{tmp}(\nu^*(\tau), \tau) \lambda e^{-\lambda \tau} d\tau = \sum_{i=0}^{m-1} \int_{\tau_i}^{\tau_{i+1}} u_{tmp}(\nu^*(\tau), \tau) \lambda e^{-\lambda \tau} d\tau \\ &\leq \sum_{i=0}^{m-1} u_{tmp}(\nu^*(\tau_i), \tau_i) \lambda e^{-\lambda \tau_i} \Delta \tau_i \\ &\leq \bar{Z}_{tmp}(\tau), \end{aligned}$$

where the first equality follows by definition of ν^* . The first inequality follows from Lemma (14), while the second inequality follows since $\{\nu_i = \nu^*(\tau_i)\}_{i=0,1,\dots,m-1}$ is a feasible solution to the discretized problem (4.24). Thus $Z_{tmp}^* \leq \hat{Z}_{tmp} \leq \bar{Z}_{tmp}(\tau)$ and hence the solution to (4.24) provides an upper bound on the seller's revenue from a temporary dynamic buyout price auction.

From a practical standpoint, the discretized problem (4.24) provides a way to construct an upper bound for the seller's maximum expected discounted revenue by solving a nonlinear program. Note however that the function u_{tmp} appearing in the objective of (4.24) may not be always easy to express analytically, because of the expectation \mathbf{E}_t in (4.20). Also, we do not provide here any description of the relationship between the mesh size of a partition τ (or size of nonlinear program (4.24)) and the quality of upper bound $\bar{Z}_{tmp}(\tau)$. For our numerical experiments in §5.2, we focus on the special case of uniform valuations, for which a closed-form expression for u_{tmp} is readily derived.

4.2 Permanent Buyout Option

The equilibrium analysis for a permanent buyout price auction game with a dynamically priced buyout option is provided in §4.2.1 while we formulate and discuss the seller's optimization problem in §4.2.2.

4.2.1 Outcome Prediction

In an auction with a permanent buyout price following trajectory $[p(t)]_{t \in [0, T]}$, for any function $\nu : [0, T] \times [\underline{v}, \bar{v}] \cup \{0\} \rightarrow [\underline{v}, \bar{v}]$ we consider the extension of strategy $\mathcal{P}[\nu]$ obtained by substituting $p(t)$ with p in the first line of (3.47):

$$\mathcal{P}[\nu](v, t, I_t) : \begin{cases} \text{Buyout at } p \text{ immediately} & \text{if } v > \nu(t, I_t) \\ \text{Bid } v \text{ at time } T & \text{if } v \leq \nu(t, I_t) \end{cases}, \quad (4.25)$$

and keep using the same notation. In this subsection, we prove the following result which is the exact analogue of Theorem 6 for the case of a permanent buyout option:

Theorem 7. *For any continuous function $\nu : [0, T] \times [\underline{v}, \bar{v}] \cup \{0\} \rightarrow [\underline{v}, \bar{v}]$ such that $\nu(t, 0) \triangleq \nu(t)$ is non-decreasing in t and $\nu(t, I)$ is decreasing in I for all t , define function $p : [0, T] \rightarrow [\underline{v}, \bar{v}]$ as*

$$p(t) = \nu(t) - e^{-\beta(T-t)} \mathbf{E}_t \left[\int_{\underline{v}}^{\nu(t)} \frac{\prod_{i=1}^{N(t)} F(\min(\nu(t_i), x))}{\prod_{i=1}^{N(t)} F(\nu(t_i))} (F(x))^{N(t, T)} dx \right] \quad (4.26)$$

where the expectation \mathbf{E}_t is with respect to the number $N(t)$ and epochs $t_1, \dots, t_{N(t)}$ of arrivals in $[0, t)$ of a non-homogeneous Poisson process with rate $\lambda F(\nu(\tau))$ with $\tau \in [0, t)$, and number $N(t, T)$ of arrivals in $(t, T]$ of a Poisson process with rate λ . The symmetric strategy profile $\mathcal{P}[\nu]$ is then a Bayesian Nash equilibrium for the auction with permanent buyout price trajectory $[p(t)]_{t \in [0, T]}$.

As in the proof of Theorem 6, we prove that $\mathcal{P}[\nu]$ is a Nash equilibrium by showing that the best response strategy of an arbitrarily chosen bidder to $\mathcal{P}[\nu]$ (with a non-decreasing threshold function ν) is to himself play $\mathcal{P}[\nu]$ if the buyout price $p(t)$ is set

to be

$$p(t) = \nu(t) - e^{-\beta(T-t)} \mathbf{E}_t \left[\int_{\underline{v}}^{\nu(t)} \frac{\prod_{i=1}^{N(t)} F(\min(\nu(t_i), x))}{\prod_{i=1}^{N(t)} F(\nu(t_i))} (F(x))^{N(t,T)} dx \right]$$

for $t \in [0, T]$.

Indeed consider any bidder A with type (v, t) and information $I_t = 0$. Since the threshold function $\nu(t, I)$ is assumed to be decreasing in I for all t , Lemma 4 shows that bidder A is weakly better off bidding at T .

We next show that bidder A cannot increase his utility by waiting before making a decision.

Lemma 15. *When other bidders follow strategy $\mathcal{P}[\nu]$ in an auction with a permanent dynamic buyout price, a bidder is weakly better off acting immediately, i.e. utility from acting immediately is at least as much as from waiting, if and only if the threshold valuation trajectory $\nu(t)$ is non-decreasing in t for all $t \in [0, T]$.*

Proof. Consider the bidder A who has type $(v, t, 0)$. To ensure that he makes a decision immediately, we enforce the constraint that his utility from acting immediately must be at least as much as the utility he obtains from waiting in the auction. Recall that we have already shown, in Lemma 4, that if a bidder decides to bid in the auction he must place a bid at time T and thus bidder A 's utility from the auction if he makes a decision immediately is $\max \{U_{buy(t)}(v, t), \mathbf{E}[U_{bid(T)}(v, t, 0)]\}$.

Suppose A waits up to time τ ($\tau > t$) and define \mathcal{E} as the event that the buyout option is not exercised in (t, τ) , i.e. every bidder $(\hat{v}, \hat{t}, 0)$ arriving in the interval (t, τ) has valuation $\hat{v} \leq \nu(\hat{t})$. Then, if $\mathbf{E}[U_{bid(T)}^{(\tau)}(v, t, 0)]$ denotes the expected utility from bidding for a bidder who arrives at time t , waits up to time τ ($\tau > t$) and then decides to place a bid in the auction at time T , we have

$$\mathbf{E}[U_{bid(T)}^{(\tau)}(v, t, 0) | \mathcal{E}] = e^{-\beta(\tau-t)} \mathbf{E}[U_{bid(T)}(v, \tau, 0)] \quad (4.27)$$

since if the event \mathcal{E} occurs, the buyout option is still present at time τ . Furthermore since no bids are placed in the auction the information bidder A receives at time

τ , $I_\tau = 0$. Thus apart from the waiting cost incurred by bidder A , the situation is equivalent to a case where bidder A arrives to the auction at time τ . Another consequence of this argument is that

$$\mathbf{E}[U_{bid(T)}(v, t, 0)|\mathcal{E}] = e^{-\beta(\tau-t)}\mathbf{E}[U_{bid(T)}(v, \tau, 0)] \quad (4.28)$$

The complementary event $\bar{\mathcal{E}}$ corresponds to the arrival of a bidder $(\hat{v}, \hat{t}, 0)$ with $\hat{t} \in (t, \tau)$ and valuation $\hat{v} > \nu(\hat{t})$. Such a bidder, following strategy $\mathcal{P}[\nu]$, exercises the buyout option and so $\mathbf{E}[U_{bid(T)}^{(\tau)}(v, t, 0)|\bar{\mathcal{E}}] = 0$.

Thus the utility from waiting up to time τ is:

$$\mathbf{E}[U_{wait(t,\tau)}(v, t, 0)] = e^{-\beta(\tau-t)} \max \left\{ v - p(\tau), \mathbf{E}[U_{bid(T)}(v, \tau, 0)] \right\} \cdot P(\mathcal{E}) \quad (4.29)$$

Using the law of conditional expectation, we have

$$\mathbf{E}[U_{bid(T)}(v, t, 0)] = \mathbf{E}[U_{bid(T)}(v, t, 0)|\mathcal{E}] \cdot P(\mathcal{E}) \quad (4.30)$$

where again $\mathbf{E}[U_{bid(T)}(v, t, 0)|\bar{\mathcal{E}}] = 0$.

Using (4.30) and (4.28), the utility from waiting up to τ can be rewritten as

$$\begin{aligned} & \mathbf{E}[U_{wait(t,\tau)}(v, t, 0)] \\ &= e^{-\beta(\tau-t)} \max \left\{ v - p(\tau) - \mathbf{E}[U_{bid(T)}(v, \tau, 0)], 0 \right\} \cdot P(\mathcal{E}) + \mathbf{E}[U_{bid(T)}(v, t, 0)] \end{aligned}$$

Thus a bidder of type $(v, t, 0)$ makes a decision immediately if and only if

$$\max \left\{ U_{buy(t)}(v, t), \mathbf{E}[U_{bid(T)}(v, t, 0)] \right\} \geq \mathbf{E}[U_{wait(t,\tau)}(v, t, 0)] \quad \forall \tau > t,$$

which can be expressed as

$$\max \left\{ v - p(t) - \mathbf{E}[U_{bid(T)}(v, t, 0)], 0 \right\} \geq e^{-\beta(\tau-t)} \max \left\{ v - p(\tau) - \mathbf{E}[U_{bid(T)}(v, \tau, 0)], 0 \right\} P(\mathcal{E}) \quad (4.31)$$

Indeed no bidder in the auction has an incentive to wait, if and only if the condition

(4.31) holds for all $t, \tau \in [0, T]$, and $v \in [\underline{v}, \bar{v}]$.

For some $\tau > t$, consider the following two cases:

1. $v \leq \nu(t)$: In this case bidding is more attractive to the bidder at t , i.e. we have $v - p(t) - \mathbf{E}[U_{bid(T)}(v, t, 0)] \leq 0$. Thus the condition (4.31) becomes

$$\begin{aligned} 0 &= \max \left\{ v - p(t) - \mathbf{E}[U_{bid(T)}(v, t, 0)], 0 \right\} \\ &\geq e^{-\beta(\tau-t)} \max \left\{ v - p(\tau) - \mathbf{E}[U_{bid(T)}(v, \tau, 0)], 0 \right\} P(\mathcal{E}) \end{aligned}$$

which holds if and only if

$$v - p(\tau) - \mathbf{E}[U_{bid(T)}(v, \tau, 0)] \leq 0 \quad \forall v \leq \nu(t) \quad (4.32)$$

Now notice that

$$\begin{aligned} &\frac{\partial}{\partial v} \left(v - p(\tau) - \mathbf{E}[U_{bid(T)}(v, \tau, 0)] \right) \\ &= 1 - \mathbf{E} \left[e^{-\beta(T-t)} \frac{\prod_{i=1}^{N(t)} F(\min(\nu(t_i), v))}{\prod_{i=1}^k F(\nu(t_i))} \prod_{j=1}^{N(t,T)} F(\min(\nu(\hat{t}_j), v)) \right] \geq 0 \end{aligned}$$

i.e. $v - p(\tau) - \mathbf{E}[U_{bid(T)}(v, \tau, 0)]$ is non-decreasing in valuation v and thus it is sufficient to impose (4.32) at $v = \nu(t)$. This gives the condition

$$\nu(t) - p(\tau) - \mathbf{E}[U_{bid(T)}(\nu(t), \tau, 0)] \leq 0,$$

which, on substituting $\nu(t) = p(t) + \mathbf{E}[U_{bid(T)}(\nu(t), t, 0)]$, becomes

$$p(\tau) - p(t) \geq \mathbf{E}[U_{bid(T)}(\nu(t), t, 0)] - \mathbf{E}[U_{bid(T)}(\nu(t), \tau, 0)] \quad (4.33)$$

2. $v > \nu(t)$: In this case $v - p(t) - \mathbf{E}[U_{bid(T)}(v, t, 0)] > 0$. Thus we need

$$v - p(t) - \mathbf{E}[U_{bid(T)}(v, t, 0)] \geq e^{-\beta(T-t)} \left(v - p(\tau) - \mathbf{E}[U_{bid(T)}(v, \tau, 0)] \right) P(\mathcal{E}), \forall v > \nu(t)$$

Using (4.30) and (4.28), the above condition can be expressed as

$$v - p(t) \geq e^{-\beta(T-t)}(v - p(\tau))P(\mathcal{E}) \quad \forall v > \nu(t) \quad (4.34)$$

which can be re-written as

$$p(\tau) - \frac{1}{e^{-\beta(\tau-t)}P(\mathcal{E})}p(t) \geq v \left(1 - \frac{1}{e^{-\beta(\tau-t)}P(\mathcal{E})}\right) \quad \forall v > \nu(t)$$

Now since $e^{-\beta(\tau-t)}P(\mathcal{E}) < 1$ the right hand side decreases with v and hence it is sufficient to impose (4.34) at $v = \nu(t)$. This gives

$$p(\tau) - \frac{1}{e^{-\beta(\tau-t)}P(\mathcal{E})}p(t) \geq \nu(t) \left(1 - \frac{1}{e^{-\beta(\tau-t)}P(\mathcal{E})}\right)$$

Substituting $\nu(t) = p(t) + \mathbf{E}[U_{bid(T)}(\nu(t), t, 0)]$ and using (4.30) and (4.28) in the above condition we get

$$p(\tau) - p(t) \geq \mathbf{E}[U_{bid(T)}(\nu(t), t, 0)] - \mathbf{E}[U_{bid(T)}(\nu(t), \tau, 0)]$$

which is the same as the condition (4.33) obtained in Case 1.

Thus, no bidder waits before making a decision if and only if $\forall \tau, t \in [0, T], \tau > t$

$$p(\tau) - p(t) \geq \mathbf{E}[U_{bid(T)}(\nu(t), t, 0)] - \mathbf{E}[U_{bid(T)}(\nu(t), \tau, 0)] \quad (4.35)$$

Substituting $p(t) = \nu(t) - \mathbf{E}[U_{bid(T)}(\nu(t), t, 0)]$ in (4.35) gives the condition

$$\nu(\tau) - \nu(t) - \mathbf{E}[U_{bid(T)}(\nu(\tau), \tau, 0)] + \mathbf{E}[U_{bid(T)}(\nu(t), \tau, 0)] \geq 0 \quad (4.36)$$

for all $\tau, t \in [0, T], \tau > t$. By setting $\tau = t + \Delta t$ ($\Delta t > 0$) in the above condition we get that (4.36) holds if and only if for all $t \in [0, T]$

$$\lim_{\Delta t \rightarrow 0} \left(\nu(t + \Delta t) - \nu(t) \right) \left(1 - \frac{\partial}{\partial v} \mathbf{E}[U_{bid(T)}(v, t + \Delta t, 0)] \Big|_{v=\nu(t)} \right) \geq 0. \quad (4.37)$$

We have shown earlier that $\frac{\partial}{\partial v} \mathbf{E}[U_{bid(T)}(v, t, 0)] < 1, \forall v \in [\underline{v}, +\infty), t \in [0, T)$ and thus (4.37) (and hence (4.36)) holds if and only if $\nu(\cdot)$ is non-decreasing in t for all $t \in [0, T)$. \square

Therefore bidder A immediately decides whether to bid or exercise the buyout option. Now the buyout price is chosen such that

$$U_{buy(t)}(\nu(t), t) = \mathbf{E}[U_{bid(T)}(\nu(t), t, 0)] \quad (4.38)$$

This combined with the fact that the excess utility function

$$\delta(v, p(t), t) = U_{buy(t)}(v, t) - \mathbf{E}[U_{bid(T)}(v, t, 0)]$$

is increasing in valuation v implies that a bidder with valuation $v > \nu(t)$ exercises the buyout option immediately while a bidder with $v \leq \nu(t)$ will choose to bid his true valuation at time T .

Thus bidder A 's best response strategy to $\mathcal{P}[\nu]$ is to himself play $\mathcal{P}[\nu]$ and since the choice of the bidder was arbitrary this proves that for any non-decreasing threshold function ν , $\mathcal{P}[\nu]$ is a Bayesian Nash equilibrium of an auction game with permanent buyout price $p(t) = \nu(t) - \mathbf{E}[U_{bid(T)}(\nu(t), t, 0)]$. This concludes the proof of Theorem 7.

4.2.2 Seller's Optimization Problem

Similar to Theorem 6, Theorem 7 also implies that, within the range of equilibria considered, finding an optimal price trajectory $[p(t)]_{t \in [0, T]}$ exactly corresponds to finding its associated continuous and non-decreasing threshold function $\nu : [0, T] \rightarrow [\underline{v}, \bar{v}]$ subject to (4.26).

The seller's revenue maximization problem can be stated as

$$\begin{aligned} \sup_{\nu \in \mathcal{C}^+} \mathbf{E}[U_{prm}^S(\nu)] &= \int_0^T e^{-\alpha t} p(t) \lambda (1 - F(\nu(t))) e^{-\lambda \int_0^t (1 - F(\nu(\tau))) d\tau} dt & (4.39) \\ &+ e^{-\lambda \int_0^T (1 - F(\nu(t))) dt} e^{-\alpha T} \mathbf{E}[\mathbf{1}_{\{N_{bid}(T) > 0\}} \max(\underline{v}, v_{N_{bid}(T)}^{(2)}) | v_i \leq \nu(t_i) \forall i] \\ &\text{subject to (4.26),} & (4.40) \end{aligned}$$

where the definition of \mathbf{E} , $N_{bid}(T)$, $\{t_1, \dots, t_{N_{bid}(T)}\}$ and $v_{N_{bid}(T)}^{(2)}$ is the same as in (3.73).

We will denote by Z_{prm}^* the optimal value defined by (4.39)–(4.40). Similar to our analysis for the temporary case, we now develop an upper bound for Z_{prm}^* . Firstly, observe that the price trajectory $p(t)$ appearing in (4.39) and given by (4.26) satisfies

$$p(t) \leq \nu(t) - e^{-\beta(T-t)} \mathbf{E}_t \left[\frac{\int_{\underline{v}}^{\nu(t)} (F(x))^{N(T)} dx}{(F(\nu(t)))^{N(t)}} \right] \triangleq \bar{p}(\nu(t), t), \quad (4.41)$$

where the right-hand side of the first inequality is obtained by substituting $\nu(t)$ with $\nu(t_i)$ in (4.26). This is because the second term in (4.26) corresponds to the expected utility of a bidder submitting a regular bid equal to his valuation $\nu(t)$ upon his arrival at t when every competing bidder already arrived at time $t_i < t$ is known to have a valuation lower than $\nu(t_i)$. In contrast, the modified version in (4.41) corresponds to the same expected utility when competing bidders already arrived are only known to have a valuation lower than $\nu(t)$. Because $\nu \in \mathcal{C}^+$ so that $\nu(t_i) \leq \nu(t)$ for all $t_i \leq t$, the bidder considered faces more competition in the scenario underlying that modified version and his expected utility is therefore smaller, justifying that $\bar{p}(\nu(t), t)$ defined in (4.41) is indeed an upper bound. Note also that the notation $\bar{p}(\nu(t), t)$ introduced reflects that this quantity only depends now on the value of ν at t instead of the entire trajectory $(\nu(\tau))_{\tau \leq t}$ of ν up to t as in (4.26). The function defined as

$$h(\nu(t), t) \triangleq e^{-\alpha t} \left(\bar{p}(\nu(t), t) \lambda (1 - F(\nu(t))) e^{-\lambda (1 - F(\nu(t)))} \right)$$

provides thus an upper bound for the integrand in the first term of (4.39). In addition, the expected seller revenue from regular bidding when no buyout price is used

($\mathbf{E}[\mathbf{1}_{\{N(T)>0\}} \max(\underline{v}, v_{N(T)}^{(2)})]$) constitutes an upper bound for the corresponding quantity with a permanent buyout price in the second term of (4.39), which establishes the following bound:

$$Z_{prm}^* \leq \bar{Z}_{prm} \triangleq \sup_{\nu \in \mathcal{C}^+} \left(\int_0^T h(\nu(t), t) dt + e^{-\lambda \int_0^T (1-F(\nu(t))) dt} e^{-\alpha T} \mathbf{E}[\mathbf{1}_{\{N(T)>0\}} \max(\underline{v}, v_{N(T)}^{(2)})] \right). \quad (4.42)$$

While the bound \bar{Z}_{prm} just defined is the optimal value of a calculus of variations problem and is thus difficult to compute in the general case, the following proposition shows that a discretized version of (4.42) that is easier to solve still provides a valid upper bound for Z_{prm}^* .

Proposition 3. Consider any partition $\tau \triangleq (\tau_j)_{j \in \{0, \dots, m\}}$ of $[0, T]$ into m subintervals such that $\tau_0 = 0 < \tau_1 < \dots < \tau_m = T$, define $\Delta\tau_j \triangleq \tau_{j+1} - \tau_j$ for $j \in \{0, \dots, m-1\}$ and let $\Delta\tau \triangleq \max_j \Delta\tau_j$ be the mesh size of τ . Then $Z_{prm}^* \leq \bar{Z}_{prm} \leq \bar{Z}_{prm}(\tau)$ where

$$\begin{aligned} \bar{Z}_{prm}(\tau) \triangleq & \max_{(\nu_j)_{j \in \{0, \dots, m\}}} \sum_{j=0}^{m-1} h(\nu_j, \tau_j) \Delta\tau_j \\ & + e^{-\lambda \sum_{j=1}^m (1-F(\nu_j)) \Delta\tau_{j-1}} e^{-\alpha T} \mathbf{E}[\mathbf{1}_{\{N(T)>0\}} \max(\underline{v}, v_{N(T)}^{(2)})] \\ \text{subject to: } & \nu_{j-1} \leq \nu_j \text{ for all } j \in \{1, \dots, m\} \\ & \underline{v} \leq \nu_0, \nu_m \leq \bar{v}. \end{aligned} \quad (4.43)$$

Proof. Consider the following problem

$$\hat{Z}_{prm} \triangleq \sup_{\nu \in \mathcal{C}_0^+} \left(\int_0^T h(\nu(t), t) dt + e^{-\lambda \int_0^T (1-F(\nu(t))) dt} e^{-\alpha T} \mathbf{E}[\mathbf{1}_{\{N(T)>0\}} \max(\underline{v}, v_{N(T)}^{(2)})] \right), \quad (4.44)$$

which has the same objective function as (4.42) but the supremum is taken over the set \mathcal{C}_0^+ . Clearly $\mathcal{C}^+ \subset \mathcal{C}_0^+$ and thus $Z_{prm}^* \leq \bar{Z}_{prm} \leq \hat{Z}_{prm}$. The objective of (4.44) can be shown to be continuous and then since the set \mathcal{C}_0^+ is compact there exists a solution $\nu^* \in \mathcal{C}_0^+$ which attains the maximum utility \hat{Z}_{prm} .

Lemma 16. The function $h(\nu^*(t), t)$ is decreasing in t for $t \in [0, T]$ where $\nu^*(t)$ is the solution of (4.44).

Proof. It is easily seen that $h(v, t)$ is decreasing in t for $t \in [0, T]$ since $\bar{p}(v, t)$, defined in (4.41), is decreasing in t . Now, assume for contradiction that there exists an interval $[t_1, t_2] \subset [0, T]$ such that

$$h(\nu^*(t), t) \leq h(\nu^*(t_2), t_2) \quad \forall t \in [t_1, t_2]$$

and thus

$$\int_{t_1}^{t_2} h(\nu^*(t), t) dt \leq \int_{t_1}^{t_2} h(\nu^*(t_2), t_2) dt = h(\nu^*(t_2), t_2)(t_2 - t_1) \quad (4.45)$$

Consider the following valuation trajectory

$$\hat{\nu}(t) = \begin{cases} \nu^*(t_2) & \forall t \in [t_1, t_2] \\ \nu^*(t) & \text{otherwise} \end{cases}$$

Since $\nu^*(t_1) \geq \nu^*(t_2)$, $\hat{\nu} \in C_0^+$ and is thus feasible for the problem (4.44). Hence since ν^* is the optimal solution of the problem (4.44), the utility obtained from using threshold valuation $\hat{\nu}$ must be less than or equal to the optimal utility. Since $\hat{\nu}(t) = \nu^*(t)$ for all $t \notin [t_1, t_2]$, the optimality of ν^* implies

$$\begin{aligned} & \int_{t_1}^{t_2} h(\nu^*(t), t) dt + e^{-\lambda \int_0^T (1-F(\nu^*(t))) dt} e^{-\alpha T} \mathbf{E}[\mathbf{1}_{\{N(T)>0\}} \max(\underline{v}, v_{N(T)}^{(2)})] \\ & \geq \int_{t_1}^{t_2} h(\hat{\nu}(t), t) dt + e^{-\lambda \int_0^T (1-F(\hat{\nu}(t))) dt} e^{-\alpha T} \mathbf{E}[\mathbf{1}_{\{N(T)>0\}} \max(\underline{v}, v_{N(T)}^{(2)})] \end{aligned} \quad (4.46)$$

Additionally since, by definition, $\hat{\nu}(t) \geq \nu^*(t)$ it follows that

$$e^{-\lambda \int_0^T (1-F(\nu^*(t))) dt} \leq e^{-\lambda \int_0^T (1-F(\hat{\nu}(t))) dt}, \quad (4.47)$$

and thus (4.46) implies that

$$\begin{aligned}
\int_{t_1}^{t_2} h(\nu^*(t), t) dt &\geq \int_{t_1}^{t_2} h(\hat{\nu}(t), t) dt \\
&= \int_{t_1}^{t_2} h(\nu^*(t_2), t) dt \\
&> \int_{t_1}^{t_2} h(\nu^*(t_2), t_2) dt = h(\nu^*(t_2), t_2)(t_2 - t_1)
\end{aligned}$$

where the second inequality follows since $h(v, t)$ is decreasing in t . This contradicts (4.45). \square

Then for partition τ defined in the statement of the proposition, Lemma 16 implies that

$$\int_0^T h(\nu^*(\tau), \tau) d\tau \leq \sum_{i=0}^{m-1} h(\nu^*(\tau_i), \tau_i) \Delta\tau_i \quad (4.48)$$

Additionally since $\nu^*(t)$ is non-decreasing in t , we have for all $i = 1, \dots, m$

$$e^{-\lambda \int_{\tau_{i-1}}^{\tau_i} (1-F(\nu^*(\tau))) d\tau} \leq e^{-\lambda (1-F(\nu^*(\tau_i))) \Delta\tau_{i-1}} \quad (4.49)$$

Combining (4.48) and (4.49) we get

$$\begin{aligned}
\hat{Z}_{prm} &\leq \sum_{i=0}^{m-1} h(\nu^*(\tau_i), \tau_i) \Delta\tau_i + e^{-\lambda \sum_{i=1}^m (1-F(\nu^*(\tau_i))) \Delta\tau_{i-1}} e^{-\alpha T} \mathbf{E}[\mathbf{1}_{\{N(T)>0\}} \max(\underline{v}, v_{N(T)}^{(2)})] \\
&\leq \bar{Z}_{prm}(\tau)
\end{aligned}$$

where the second inequality follows since $\{\nu_i = \nu^*(\tau_i)\}_{i=0,1,\dots,m-1}$ is a feasible solution to the discretized problem (4.43). \square

The upper bounds \bar{Z}_{prm} or $\bar{Z}_{prm}(\tau)$ are not as good as their analogue $\bar{Z}_{tmp}(\tau)$ for the temporary case; this is because the substitutions leading to (4.41) and (4.42) above are relatively coarse. Consequently, the resulting bound proved too loose to support any assertive statement, as evidenced by the fact that the piecewise constant solution obtained by solving the discretized problem for the permanent case performed significantly worse in all our simulation experiments than all other policies tested,

including not using a buyout price at all. As a result, the experimental results we report for dynamic permanent buyout prices in Chapter 5 are not quite as conclusive as for dynamic temporary buyout prices.

Chapter 5

Empirical Analysis, Numerical Results and Comparative Discussion

In this chapter, we present results of our numerical analysis comparing the equilibrium threshold function, optimal buyout price and seller revenue in temporary and permanent static buyout price auctions (§5.1). The seller's utility in a dynamic buyout price, static buyout price and standard auction (without a buyout price) is compared in §5.2. Next we validate our model predictions by empirically testing two hypotheses, suggested by the outcome prediction in temporary and permanent static buyout price auctions, with bidding data from *eBay* and *Yahoo* (§5.3).

5.1 Static buyout prices

In this section we compare the equilibrium behavior, optimal buyout price and seller's revenue associated with the temporary and permanent buyout options, drawing on both numerical experiments and our theoretical insights from Chapter 3. In all experiments in this section we assume that valuations are uniformly distributed on $[50, 500]$ and the auction duration is $T = 16$.

5.1.1 Equilibrium threshold valuation functions

A first insightful exercise is to compare the bidders' equilibrium buyout threshold functions ν_{tmp} and ν_{prm} (see statements of Theorems 1 and 4) corresponding to the same buyout price and market environment. For illustration purposes, Figure 5-1 shows a plot of these two functions for the specific case $p_{tmp} = p_{prm} = 350$, $\lambda = 0.25$, $T = 16$, $\beta = 0.03$.

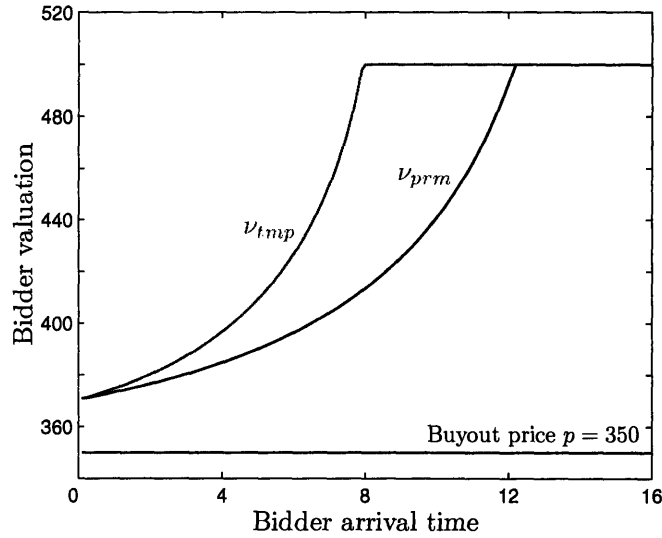


Figure 5-1: Equilibrium threshold valuation in temporary and permanent buyout price auction

A first observation is that both curves shown in Figure 5-1 are non-decreasing: either type of buyout option remaining open as time goes by indicates reduced competition among bidders participating in the auction and therefore progressively makes the buyout option less attractive relative to submitting a regular bid, so that fewer bidders will decide to exercise it. The temporary threshold function ν_{tmp} does lie above the permanent threshold function ν_{prm} however, suggesting that the effect just described is less pronounced with a permanent option than with a temporary option. Indeed, when participants follow the equilibrium strategy $\mathcal{T}[\nu_{tmp}]$ described by (3.1) and Theorem 1, the fact that a temporary option is still open when a bidder arrives indicates to him that he is the first bidder and that the only competition he is likely

to face should he submit a regular bid will come from bidders who are yet to arrive. On the other hand, under the strategy profile $\mathcal{P}[\nu_{prm}]$ described by (3.47) and Theorem 4, if a permanent option is still open when a bidder arrives he can only infer that all the bidders who have already arrived have valuations lower than the value of the threshold valuation at the time of their respective arrivals. Consequently, for such a bidder the decision to submit a regular bid appears less attractive relative to exercising the buyout option than it is for a bidder facing an open temporary option in circumstances that are otherwise the same. As a result, with identical buyout prices more bidders will tend to exercise a permanent option than a temporary one. Finally, note that the initial values $\nu_{tmp}(0)$ and $\nu_{prm}(0)$ shown in Figure 5-1 are identical, which is intuitive but can also be established analytically by calculating the right-hand sides of (3.2) and (3.49) for $t = 0$.

5.1.2 Approximate temporary buyout price

In an auction with a temporary buyout price with bidders having a finite time-sensitivity β , we propose using the approximate optimal price, \tilde{p}_{tmp}^* , as derived in equation (3.46), and assess its sub-optimality. Let $\mathbf{E}[U_{tmp}^S(\tilde{p}_{tmp}^*)]$ and $\mathbf{E}[U_{tmp}^S(p_{tmp}^*)]$ denote the seller's expected utility from an auction with temporary buyout price \tilde{p}_{tmp}^* and p_{tmp}^* respectively, where the optimal price p_{tmp}^* is obtained by performing a simulation-based line search as discussed in §3.1.3. The seller's expected utility from the basic auction mechanism without a buyout price the seller's expected utility, described in §2.2, is denoted by $\mathbf{E}[U_{nb}^S]$.

In Table 5.1, we compare $\Delta U_{tmp}^S[(p_{tmp}^*)] \left(= \frac{\mathbf{E}[U_{tmp}^S(p_{tmp}^*)] - \mathbf{E}[U_{nb}^S]}{\mathbf{E}[U_{nb}^S]} \times 100 \right)$ and $\Delta U^S[(\tilde{p}_{tmp}^*)] \left(= \frac{\mathbf{E}[U_{tmp}^S(\tilde{p}_{tmp}^*)] - \mathbf{E}[U_{nb}^S]}{\mathbf{E}[U_{nb}^S]} \times 100 \right)$, the percent increase in seller's utility (over an auction without a buyout price), achieved by introducing a buyout option with price p_{tmp}^* and \tilde{p}_{tmp}^* respectively, for different arrival rates λ and buyer time sensitivity β (with $\alpha = 0.03$).

From the table, it is evident that \tilde{p}_{tmp}^* performs well if the average number of bidders in the auction (λT) is high. However for low values of λT the increase in seller's utility achieved by using the approximate buyout price is significantly lower

λT	2			4			8		
β	0.01	0.03	0.05	0.01	0.03	0.05	0.01	0.03	0.05
$\Delta U_S[p_{tmp}^*]$	10.8%	16.0%	19.7%	11.2%	13.1%	14.6%	10.2%	10.4%	10.4%
$\Delta U_S[\tilde{p}_{tmp}^*]$	2.0%	10.5%	16.3%	9.2%	11.8%	13.8%	9.9%	10.2%	10.3%

Table 5.1: Percent utility increase achieved by temporary optimal and approximate buyout price

than the maximum achievable.

5.1.3 Temporary and permanent optimal buyout prices

Dependence of optimal buyout price on α , β and λ

We examine the variation of optimal static temporary buyout price with bidder arrival rate, and bidder and seller time-sensitivity. Similar to the last subsection, the optimal buyout price p_{tmp}^* is obtained by performing a simulation-based line search; here, and in the remainder of this section, for all values estimated by simulation, the true value is within 1% of the estimate with 95% confidence.

The temporary optimal buyout price p_{tmp}^* as a function of α and β is plotted in Figure 5-2 and 5-3 respectively. The two graphs confirm the intuition that the optimal buyout price increases with the bidder arrival rate. Figure 5-2 also suggests that the optimal buyout price decreases with seller time sensitivity (i.e. increasing α) – a more time-sensitive seller would prefer selling the product at a lower price early in the auction rather than waiting for the auction to end.

Furthermore, it can be observed from Figure 5-3 that the optimal buyout price increases with β ; a more time-sensitive bidder would be willing to pay a higher price for obtaining the product earlier.

The numerical results for the optimal permanent buyout price are similar and hence omitted.

Comparison of optimal permanent and temporary buyout prices

We next compare the optimal permanent and temporary buyout prices for the special case when participating bidders are impatient, i.e. $\beta \rightarrow \infty$. The optimal temporary

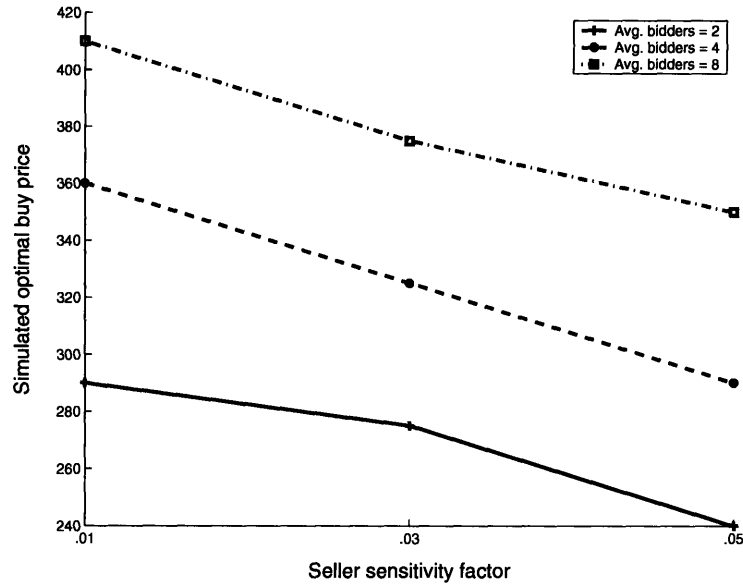


Figure 5-2: Optimal temporary buyout price ($\beta = 0.03$)

buyout price is obtained by solving numerically the concave problem obtained when substituting the impatient bidder condition in (3.40), while the optimal permanent buyout price is obtained by solving numerically the special case of (3.73) when valuations are uniformly distributed and bidders are impatient. It can be observed, from Figure 5-4, that the optimal price is higher for a permanent option than for a temporary one; our explanation follows from examining the individual terms of the equation for the seller's total expected discounted revenue

$$\mathbf{E}[pe^{-\alpha\tau_{buy}}|\text{buyout}]P(\text{buyout})+\mathbf{E}[e^{-\alpha T}\mathbf{1}_{\{N(T)>0\}}\max(\underline{v}, v_{N(T)}^{(2)})|\text{no buyout}]P(\text{no buyout}), \quad (5.1)$$

where the first term is the expected discounted revenue from the buyout auction (τ_{buy} denotes the conditional buyout exercise time), while the second is the expected discounted revenue from regular bidding. For a given buyout price p , the permanent buyout option is exercised with higher probability and, conditional on its exercise, later on average than the temporary option (it may be exercised by other bidders besides the first one). This suggests that the price maximizing the first term alone in (5.1), which is a unimodal function of the buyout price, will be larger with a

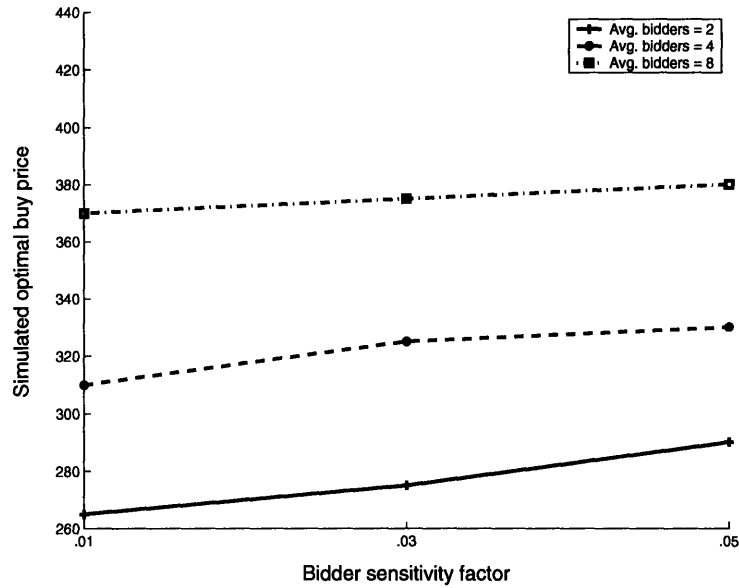


Figure 5-3: Optimal temporary buyout price ($\alpha = 0.03$)

permanent option than with a temporary one. Figure 5-1 also indicates that for any given buyout price both the expectation and the probability forming the expected revenue from bidding (second term in (5.1)), which is increasing in the buyout price, will be smaller with a permanent option than with a temporary one. The buyout price value at which the marginal decrease in expected buyout revenue equals the marginal increase in expected bidding revenue in (5.1) should thus be higher with a permanent option than with a temporary option. Finally, note that the higher the seller time-sensitivity α , the larger the difference between the conditional buyout revenues $\mathbf{E}[pe^{-\alpha\tau_{buy}}|\text{buyout}]$ for permanent and temporary options, explaining the larger difference between optimal permanent and temporary buyout prices observed in Figure 5-4.

Notice, that in the limiting regime where $\alpha, \lambda \rightarrow \infty$ with $k = \lim \frac{\alpha}{\lambda}$, we showed in §3.1.3 and §3.2.3 that the asymptotic optimal temporary and permanent buyout price is $\tilde{p} = \arg \max_p p(1 - F(p))$ and $p^*(k) = \arg \max_p \frac{p(1-F(p))}{k+1-F(p)}$ respectively. The

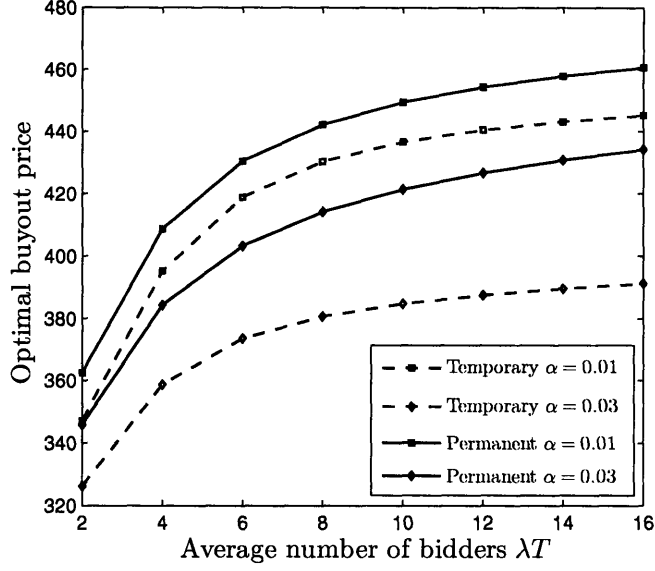


Figure 5-4: Optimal temporary and permanent buyout prices with impatient bidders

optimality of \tilde{p} and $p^*(k)$ imply that

$$\tilde{p}(1 - F(\tilde{p})) \geq p^*(k)(1 - F(p^*(k))) \text{ and } \frac{p^*(k)(1 - F(p^*(k)))}{k + 1 - F(p^*(k))} \geq \frac{\tilde{p}(1 - F(\tilde{p}))}{k + 1 - F(\tilde{p})}$$

which can be combined to show that $p^*(k) \geq \tilde{p}$ for all k . Thus, for the regime $\alpha, \lambda \rightarrow \infty$, this proves analytically the observation, from Figure 5-4, that the optimal buyout price is higher for a permanent option than a temporary option.

5.1.4 Gain in seller's utility enabled by a buyout price

Our last set of experiments focuses on the seller's relative gain in utility from an auction with temporary and permanent buyout options over an auction with no buyout price, that is $(\mathbf{E}[U_{tmp}^S(p_{tmp}^*)] - \mathbf{E}[U_{nb}^S]) / \mathbf{E}[U_{nb}^S]$ or $(\mathbf{E}[U_{prm}^S(p_{prm}^*)] - \mathbf{E}[U_{nb}^S]) / \mathbf{E}[U_{nb}^S]$, where $\mathbf{E}[U_{tmp}^S(p_{tmp}^*)]$ and $\mathbf{E}[U_{prm}^S(p_{prm}^*)]$ denote the seller's expected utility from an auction with optimal temporary and permanent buyout options respectively, and $\mathbf{E}[U_{nb}^S]$ the seller's expected utility from the basic auction mechanism without a buyout price. As described in §3.1.3 and §3.2.3 respectively, the optimal buyout prices

p_{tmp}^* and p_{prm}^* are obtained by performing a simulation-based line search.

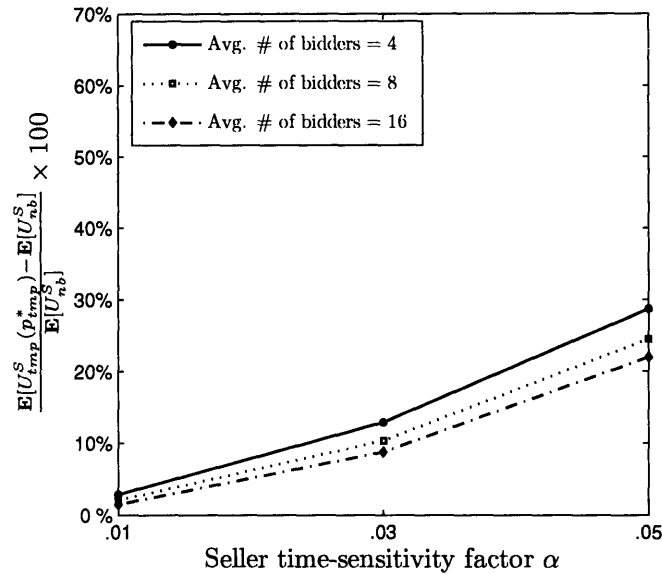


Figure 5-5: Relative increase in seller's utility from a temporary buyout option ($\beta = 0.03$)

The results from these experiments are plotted in Figure 5-5, 5-6, 5-7 and 5-8, which show the seller's relative utility increase for both option types in various environments. A first observation is that, as intuition suggests, the relative gain from both types of buyout option generally increases with both the seller's time sensitivity α and the bidders' time sensitivity β – the possibility of selling the item earlier is more valuable for a time-sensitive seller, and bidders with a high time-sensitivity are willing to pay more if they can get the product earlier. Figures 5-7 and 5-8 suggest however that the impact of the bidders' time sensitivity on the relative utility gain from a buyout option becomes insignificant when the expected number of bidders λT becomes moderately large, which partly justifies the approximation $\beta \rightarrow +\infty$ discussed in §3.1.3 and §3.2.3. On the other hand, the expected utility gain from a buyout option always seems to increase substantially with the seller's time sensitivity, independently of the expected number of bidders. Our interpretation is that while the seller's time sensitivity directly impacts his utility, the effect of the bidders' time

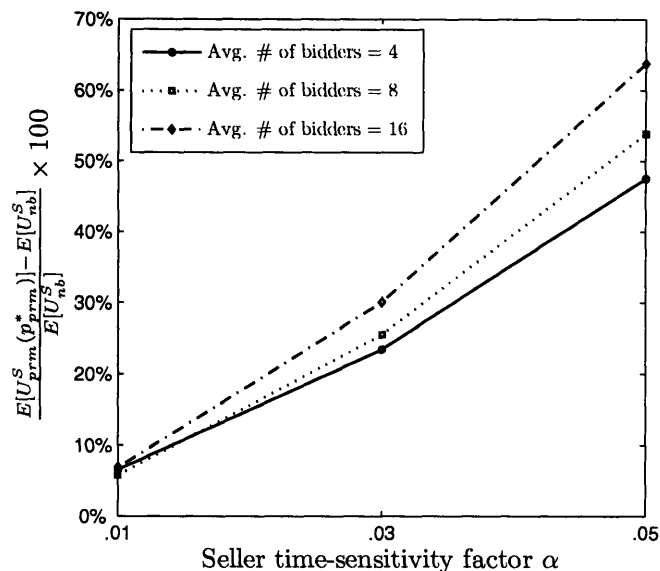


Figure 5-6: Relative increase in seller's utility from a permanent buyout option ($\beta = 0.03$)

sensitivity is more indirect in that it only affects the bidders' relative preference between the buyout option and the regular online auction, without otherwise affecting the seller's discounted revenue from either alternative. Moreover, when the number of bidders is large, affecting the probability that a single one of them will exercise the buyout option for a given time-sensitivity β becomes relatively easier.

Another important finding is that the optimal seller's utility derived from a permanent buyout option is always larger than that obtained with a temporary buyout option, as can be seen from comparing the vertical scales in Figures 5-5 and 5-7 with those in Figures 5-6 and 5-8; although unable to show this analytically, we have more generally observed this in all the experiments we have conducted besides the ones reported here. Within the strict boundaries of our model definition, a permanent buyout option seems like a more powerful instrument than a temporary one, because it allows to leverage the time-sensitivity of all participating bidders as opposed to only the first one. This interpretation ignores some of the features of actual online auctions that our model does not capture however, and we come back to this issue in Chapter 6.

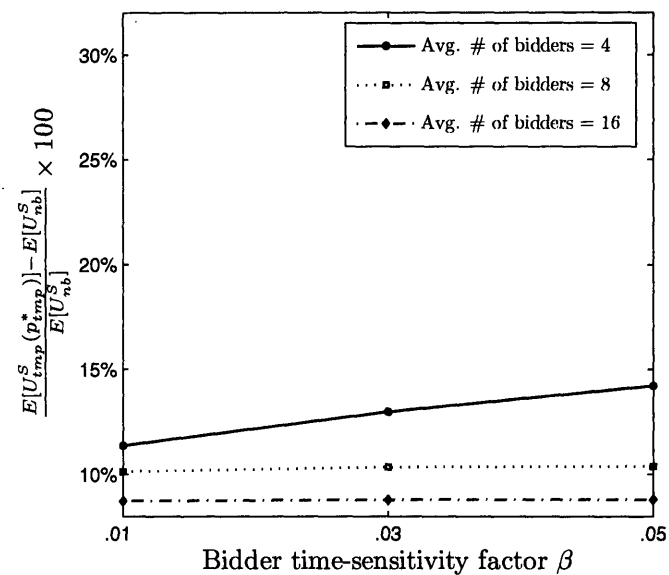


Figure 5-7: Relative increase in seller's utility from a temporary buyout option ($\alpha = 0.03$)

Finally, we observe that while the increase in seller's utility achieved by introducing a temporary buyout option (Figures 5-5 and 5-7) is decreasing in the bidder arrival rate, with a permanent buyout option the exact opposite occurs (Figures 5-6 and 5-8). Our interpretation is that since a temporary buyout option is only available to the first bidder, its relative impact diminishes in an environment with a high expected number of participants. On the other hand, a permanent option is potentially available to all arriving bidders and thus its relative impact does increase with the expected number of bidders.

5.2 Dynamic buyout prices

In this section we compare, under different market environments, the seller's utility from a dynamic buyout price auction with the utility achieved from an auction with a static buyout price. As before, for all numerical experiments, we assume that bidder valuations follow a uniform distribution with support $[50, 500]$ and the auction runs for $T = 16$ units.

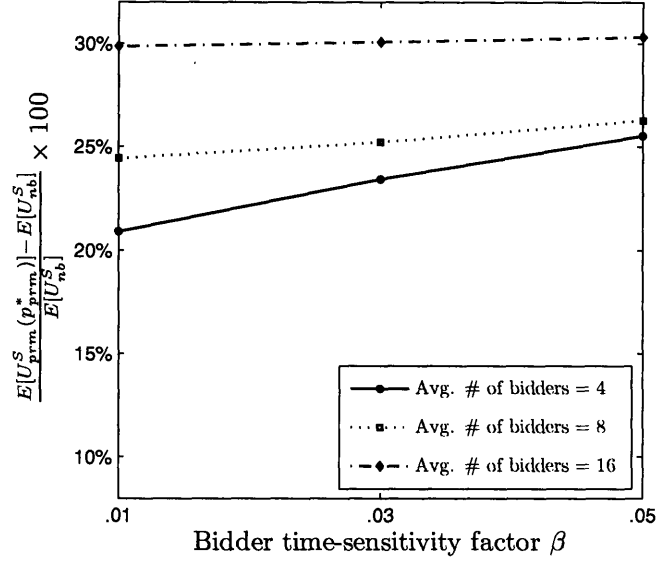


Figure 5-8: Relative increase in seller's utility from a permanent buyout option ($\alpha = 0.03$)

Let $\mathbf{E}[U_{tmp}^S(p_{tmp}^*)]$, $\mathbf{E}[U_{prm}^S(p_{prm}^*)]$ and $\mathbf{E}[U_{nb}^S]$ be as defined in §5.1. Like before these terms are estimated by simulation and are such that with 95% confidence the true values are within 1% of the estimate. For both temporary and permanent option, we also analyze a special case of the seller's revenue maximization problem where the seller's revenue is maximized over the set of fixed threshold valuation functions, i.e. $\nu(t) = v$ for all t . In this case the optimal fixed temporary (resp. permanent) threshold valuation v_{tmp}^* (resp. v_{prm}^*) is determined by numerically solving the concave maximization problem in one variable obtained from (4.21) (resp. (4.39)-(4.40)). In a slight abuse of notation, let $\mathbf{E}[U_{tmp}^S(v_{tmp}^*)]$ and $\mathbf{E}[U_{prm}^S(v_{prm}^*)]$ denote the corresponding expected utility of the seller in a temporary and permanent buyout option auction respectively.

For the case of a temporary buyout option we compare $\mathbf{E}[U_{tmp}^S(p_{tmp}^*)]$ and $\mathbf{E}[U_{tmp}^S(v_{tmp}^*)]$ with the upper bound $\bar{Z}_{tmp}(\tau)$ (derived in §4.1.2) where τ is a partition of $[0, T]$ such that $\Delta\tau_j = T/500$, $\forall j \in \{0, 1, \dots, 499\}$. As observed from Table 5.2, in case of a temporary buyout price the increase in utility obtained from both a fixed buyout price auction and a fixed threshold valuation auction is very close to the upper bound. Thus

α	0.01			0.03		
λT	4	8	16	4	8	16
$\frac{\mathbf{E}[U_{tmp}^S(p_{tmp}^*)] - \mathbf{E}[U_{nb}^S]}{\mathbf{E}[U_{nb}^S]}$	2.87%	2.07%	1.49%	12.89%	10.34%	8.79%
$\frac{\mathbf{E}[U_{tmp}^S(v_{tmp}^*)] - \mathbf{E}[U_{nb}^S]}{\mathbf{E}[U_{nb}^S]}$	3.65%	2.02%	1.40%	13.64%	10.31%	8.60%
$\frac{\bar{Z}_{tmp}(\tau) - \mathbf{E}[U_{nb}^S]}{\mathbf{E}[U_{nb}^S]}$	4.13%	2.87%	3.05%	13.91%	10.78%	10.38%

Table 5.2: Utility increase achieved by fixed and dynamic temporary buyout prices

in this case allowing for a dynamic buyout price leads to an insignificant increase in the seller's utility.

α	0.01			0.03		
λT	4	8	16	4	8	16
$\frac{\mathbf{E}[U_{prm}^S(p_{prm}^*)] - \mathbf{E}[U_{nb}^S]}{\mathbf{E}[U_{nb}^S]}$	6.55%	5.78%	6.88%	23.43%	25.22%	30.08%
$\frac{\mathbf{E}[U_{prm}^S(v_{prm}^*)] - \mathbf{E}[U_{nb}^S]}{\mathbf{E}[U_{nb}^S]}$	7.30%	6.68%	7.49%	24.57%	26.13%	30.47%
$\frac{\bar{Z}_{prm}(\tau) - \mathbf{E}[U_{nb}^S]}{\mathbf{E}[U_{nb}^S]}$	22.71%	25.06%	32.28%	42.84%	50.87%	69.84%
$\frac{\mathbf{E}[U_{prm}^S((\bar{v}_j)_{j \in \{0, \dots, m\}})] - \mathbf{E}[U_{nb}^S]}{\mathbf{E}[U_{nb}^S]}$	2.82%	-1.54%	-7.97%	20.73%	17.58%	14.54%

Table 5.3: Utility increase achieved by fixed and dynamic permanent buyout prices

For a permanent buyout option, we compare $\mathbf{E}[U_{prm}^S(p_{prm}^*)]$ and $\mathbf{E}[U_{prm}^S(v_{prm}^*)]$ with the upper bound $\bar{Z}_{prm}(\tau)$ (defined in Proposition 3) where τ is a partition of $[0, T]$ such that $\Delta\tau_j = T/100, \forall j \in \{0, 1, \dots, 99\}$. Let $(\bar{v}_j)_{j \in \{0, \dots, 100\}}$ be the threshold valuation achieving the upper bound $\bar{Z}_{prm}(\tau)$ in (4.43). Observe from Table 5.3 that for a permanent buyout option, the fixed threshold valuation auction leads to a slightly higher utility than a fixed buyout price auction but both are significantly lesser than the upper bound. However, as mentioned in §4.2.2, in this case the upper bound $\bar{Z}_{prm}(\tau)$ tends to be loose - a claim that is somewhat justified by the fact that in all cases $\mathbf{E}[U_{prm}^S((\bar{v}_j)_{j \in \{0, \dots, 100\}})]$, the seller's utility from using the valuation $(\bar{v}_j)_{j \in \{0, \dots, 100\}}$, is much lesser than $\mathbf{E}[U_{prm}^S(v_{prm}^*)]$.

5.3 Empirical Analysis of Bidding Data

Recall that we prove, in Theorem 1, that the following equilibrium strategy,

$$\mathcal{T}[\nu](v, t) : \begin{cases} \text{Buyout at } p \text{ immediately} & \text{if buyout option available and } v > \nu(t) \\ \text{Bid } v \text{ immediately} & \text{if buyout option available and } v \leq \nu(t), \\ \text{Bid } v \text{ at any time in } [t, T] & \text{otherwise} \end{cases} \quad (5.2)$$

defines a Bayesian Nash equilibrium of a temporary buyout price auction game for an appropriately chosen threshold function ν . Notice that the strategy $\mathcal{T}[\nu]$ suggests that the first bidder must act immediately in the auction while the remaining bidders can bid at any subsequent time after their arrival. In contrast, for an auction without a buyout price our model makes no predictions about the bid times – the weakly dominant strategy for bidders in this case is to bid their true valuation at any time subsequent to their arrival. This thus implies the following hypothesis which we test using bidding data from *eBay* auctions:

Hypothesis 1: The first activity (bid/buyout) in a temporary buyout price auction occurs earlier than in a standard auction (without a buyout price)

For a permanent buyout price auction we show, in Theorem 4, that a Bayesian Nash equilibrium is defined by

$$\mathcal{P}[\nu](v, t, I_t) : \begin{cases} \text{Buyout at } p \text{ immediately} & \text{if } v > \nu(t, I_t) \\ \text{Bid } v \text{ at time } T & \text{if } v \leq \nu(t, I_t) \end{cases}, \quad (5.3)$$

where ν is a suitably chosen threshold function. Notice that the strategy $\mathcal{P}[\nu]$ requires that all bids be placed just near the end of the auction. On the other hand, as mentioned earlier, our model makes no predictions about the bid times in an auction without a buyout price, thus suggesting the following hypothesis which we test using auction data from *Yahoo*:

Hypothesis 2: The average bid time in a permanent buyout price auction is higher than in a standard auction (without a buyout price)

In practice, bidding in online auctions is affected by several factors including other competing auctions, reserve price (Lucking-Reiley et al. (2000)), starting price (Wan et al. (2003), Lucking-Reiley et al. (2000)), seller's feedback ratings (Durham et al. (2004), and other papers studying seller reputation cited in §1.1), level of bidders' rationality and experience and their different incentives and, in all likelihood, the bidding data we collect depends on some or all of these factors. However, for the purpose of this analysis we assume that the difference in bid times (as observed below) is a consequence primarily of the presence of a buyout price.

For testing the hypotheses we collect bidding data on auctions belonging to the *Consumer Electronics* category which have the string "iPod"¹ in their title. This criterion gives data on auctions for iPods and its accessories (including headphones, iPod skins, chargers, cases); this market was chosen for several reasons:

1. The volume transacted is high as compared to other products
2. Most of these items are available through fixed price mechanisms and hence bidder valuations should have well defined upper and lower bounds
3. Furthermore, these items will most likely have little or no common value and so the independent private valuation assumption should hold for bidders participating in such auctions

A software program written in Perl, running on the Windows platform is used to collect data from auction websites. On *eBay*, which employs a temporary buyout price auction mechanism, once a bid is placed in the auction information about the presence of a buyout option and the buyout price disappears, and hence to obtain this information an auction needs to be tracked from the beginning. As a result, we use a Perl script which visits the *eBay* website every ten minutes and gathers information about all auctions, belonging to the market segment described above, which began in the last ten minutes and where no bids were placed. Once the auction is complete, the script revisits the auction site to gather the timings of all bids placed in the auction.

¹The search is not case-sensitive

Unlike *eBay*, collecting data by tracking newly introduced auctions would take a prohibitively long time on *Yahoo* since the number of open auctions in the market segment of our interest that satisfy certain conditions on the buyout price, starting price and winning bid (see §5.3.2) are very small. Instead we use a Perl script that visits the *Yahoo* auction website and collects bidding data on closed auctions that belong to the market segment specified above. However in the case of *Yahoo*, which employs a permanent buyout price auction mechanism, while information about the presence of a buyout option and its price is available after the auction has closed, only the actual auction ending time, and not the scheduled ending time (which is required for calculating the auction duration), is observable for a closed auction. For auctions where the buyout price is exercised, the auction ending time is the time of buyout price exercise and is usually different from the scheduled ending time, which is not observable, and hence we ignore data from such auctions. Ignoring auctions where the buyout price is exercised introduces a bias in the data which is discussed in §5.3.2.

For both *eBay* and *Yahoo* auctions, we collect the following data:

- Auction ID
- Starting time of the auction
- Scheduled auction ending time
- Auction duration (= *Starting time* - *Scheduled ending time*)
- Buyout price (if option is present) (*bp*)
- Time of buyout exercise (if option is present and exercised)
- Starting price (*sp*)
- Winning bid (*wb*)
- Bid times of all bids placed in the auction

Instead of recording the actual bid (or buyout) time, we calculate, for every bid, the following fraction:

$$f = \frac{\text{Bid Time} - \text{Starting time of the auction}}{\text{Auction Duration}}$$

which is the time elapsed in the auction (expressed as a fraction of the auction duration) before the bid is placed. This normalizes the bid times thus allowing for comparison across auctions with different durations.

5.3.1 *eBay* Data

A sample of 48,499 auctions starting and finishing between January 17, 2006 and January 31, 2006, that belonged to the market segment mentioned earlier was collected. There was bidding activity in 11,520 of these auctions, of which 6,561 were buyout price auctions while the remaining were standard second-price auctions without a buyout price. We exclude from this dataset all auctions where the buyout price (bp) is too close to the starting price (sp) ($\frac{bp}{sp} < 1.25$), since in such a case the auction effectively becomes a fixed price mechanism. Auctions where the buyout price is much higher than the winning bid wb , in particular $\frac{wb}{bp} < 0.8$ are also filtered out, since in such cases the buyout price may be set too high for any bidder to ever exercise it. Notice that, while the winning bid is a random variable depending on the bidder arrival process and bidder valuations, we assume that for auctions of the above category the variance of the winning bid is low enough so that it serves as a good indicator of the “actual price” of the auctioned item. We thus consider buyout price auctions where $\frac{bp}{sp} \geq 1.25$ and $\frac{wb}{bp} \geq 0.8$ which imply that $\frac{wb}{sp} \geq 1$ – this is the criterion we use for selecting auctions without a buyout price.

In Figure 5-9, the first activity time is plotted as a function of the winning bid for auctions without a buyout price. While it may not be obvious from the plot, it turns out that the first bid time is negatively correlated with the winning bid with a correlation factor of -0.6947 and an almost zero p-value (Matlab returns a 0 p-value). While we do not report the results here, the first activity time is negatively

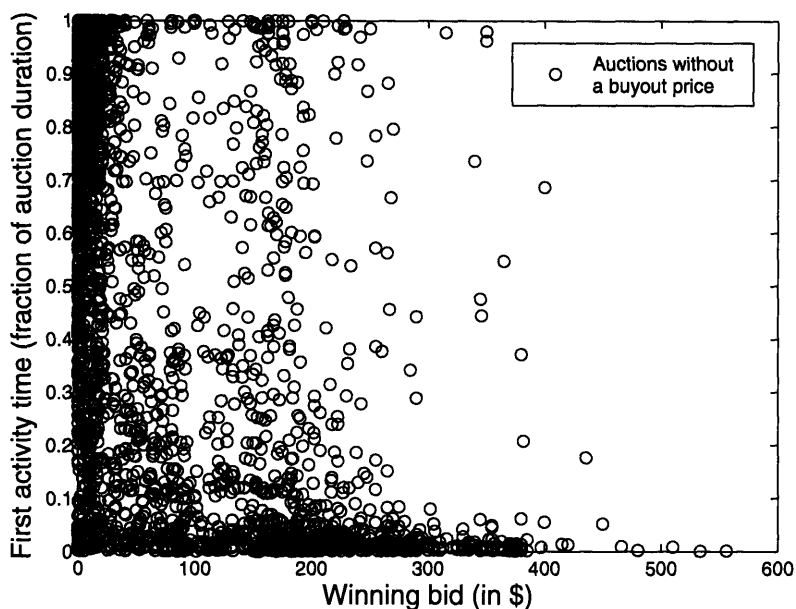


Figure 5-9: First activity time as a function of winning bid

correlated with the winning bid in buyout price auctions also. Furthermore, in our dataset almost 90% of the buyout price auctions have a winning bid of less than \$30 – these usually correspond to auctions for iPod accessories which, in most cases, are much cheaper than iPod music players. As a consequence, for the purpose of this analysis we only consider auctions where the winning bid is less than \$30. Table 5.4 summarizes the descriptive statistics of the data. The row $time_1$ represents the first bid time in auctions without a buyout price and the first activity time (bid/buyout) in buyout price auctions.

	Auctions without buyout price (3417 auctions)		Auctions with buyout price (1954 auctions)	
	Mean	Std. Dev.	Mean	Std. Dev.
Duration (days)	2.2721	0.0349	2.4390	0.0471
sp (\$)	2.7354	0.0704	2.3474	.7430
bp (\$)	NA	NA	4.0678	0.1059
wb (\$)	5.5482	0.1256	4.9275	0.1140
$time_1$	0.8694	0.0042	0.7811	0.0067

Table 5.4: Summary of auction data with $wb < \$30$, $\frac{wb}{sp} \geq 1.25$, and $\frac{wb}{bp} \geq 0.8$

Observe that the average time of the first activity in auctions without a buyout option is higher than in buyout price auctions. The two sample t-test gives a p-value of 2.75×10^{-27} implying that the hypothesis, that the average first activity time in buyout price auctions is lower than the first bid time in auctions without a buyout price, can be accepted. Furthermore, the average winning bid in buyout price auctions is lower and, since the first bid time tends to decrease as the winning bid increases, adjusting the data to ensure that the average winning bid is same for both cases would further lower the average first activity time in buyout price auctions (or alternately increase the average first bid time in auctions without a buyout price).

While we restrict the above analysis to auctions where the winning bid is less than \$30, we next study the effect of considering different cutoffs on the winning bid (it is still required that $\frac{bp}{sp} \geq 1.25$, and $\frac{wb}{bp} \geq 0.8$) – in particular, we calculate the mean first activity time and winning bid for auctions with a winning bid of less than \$10, \$20 and so on. The results are summarized in Table 5.5 and the corresponding plot is provided in Figure 5-10. In the table, we specify the mean values of $time_1$ (where, as before, $time_1$ is the first activity time) and wb (the winning bid); the standard deviation of the mean of $time_1$ is within 1% of the tabulated value in all cases. Observe, in Table 5.5, that in the last four rows (winning bid < \$200, \$300, \$400 and \$500) while the buyout price auctions have a higher mean first activity time as compared to standard auctions, the mean winning bid is much lower. Thus, since the first activity time is correlated with the mean winning bid, a comparison of first activity times is not meaningful in these cases.

Observe, from Figure 5-10, that for the same mean winning bid, the average first activity time is lower in buyout price auctions than in standard auctions, and that the average first activity time decreases with an increase in the mean winning bid.

In a separate analysis we find that in our dataset, the buyout option is exercised in about 53% of the auctions in which it is present with a mean exercise time, expressed as a fraction of the auction duration, of 0.7617 (standard deviation = 0.0064). The presence of a buyout option thus decreases the waiting time of the auction participants to about three-fourths of what they would have experienced if the buyout option was

Winning bid <	No buyout price			Buyout price		
	Number	$time_1$	$wb(\$)$	Number	$time_1$	$wb(\$)$
10	2696	0.892	2.284	1742	0.796	3.577
20	3183	0.877	4.172	1916	0.787	4.532
30	3413	0.866	5.568	1954	0.781	4.928
40	3537	0.856	6.570	1965	0.779	5.093
50	3646	0.848	7.766	1975	0.777	5.318
60	3714	0.839	8.621	1982	0.775	5.500
70	3764	0.832	9.362	1996	0.771	5.927
80	3821	0.824	10.332	2011	0.767	6.450
90	3882	0.815	11.500	2019	0.767	6.762
100	3918	0.810	12.257	2028	0.764	7.161
125	3968	0.803	13.548	2035	0.763	7.521
150	4065	0.791	16.559	2054	0.757	8.748
200	4461	0.740	30.659	2098	0.748	12.355
300	4833	0.690	46.807	2102	0.746	12.764
400	4935	0.678	53.153	2102	0.746	12.764
500	4948	0.676	54.130	2102	0.746	12.764

Table 5.5: Mean first activity time for different winning bid cutoffs

not present.

5.3.2 *Yahoo* Data

A sample of 1,475 closed *Yahoo* auctions belonging to the category “MP3 Players” and having the word “iPod” in their description was collected on January 18, 2006 – this gave us around six months of auction data with the earliest auction having been completed in July 2005. Recall that in this case we only collect data on closed auctions, and, among auctions with a buyout price, we restrict our analysis to those where the buyout price is not exercised. Notice that this biases the data since we are more likely to get auctions where the buyout price is higher than what bidders expect to pay for getting the product. We partially correct this bias by only considering, as in the earlier case, auctions where the ratio of the winning bid to the buyout price is high enough ($\frac{wb}{bp} \geq 0.8$); as before, we also ignore auctions where the buyout price is too close to the starting price by requiring that $\frac{bp}{sp} \geq 1.25$. Notice that this implies that among buyout price auctions we only consider those auctions where $\frac{wb}{sp} \geq 1 -$

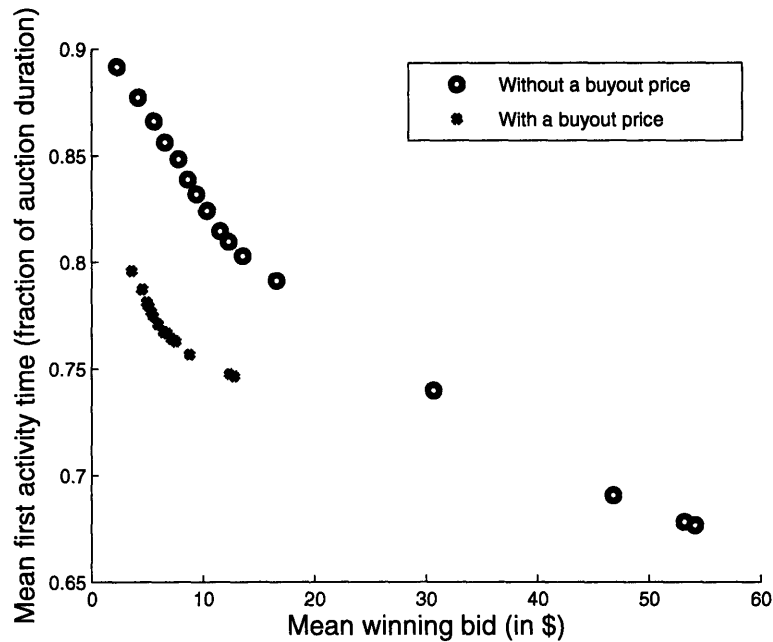


Figure 5-10: Plot of mean first activity time with mean winning bid in a temporary buyout price auction

this is also the criterion used for selecting auctions without a buyout price.

Recall that for permanent buyout price auctions, we test the following hypothesis:

Hypothesis 2: The average bid time in a permanent buyout price auction is higher than in a standard auction (without a buyout price).

Similar to the *eBay* analysis, in Figure 5-11 we plot the average bid time as a function of the winning bid for auctions without a buyout price. It can be observed from the plot that the average bid time decreases as the winning bid increases; indeed it turns out that average bid time is negatively correlated with the winning bid with a correlation factor of -0.3593 (p-value 1.1289×10^{-5}). Although, we do not include the result here, a similar negative correlation is also observed in buyout price auctions.

The median winning bid in the complete *Yahoo* data (including auctions where the buyout option is exercised) is \$152.51, as compared to \$5.95 in the *eBay* auction data, which presumably happens since the search for the string “iPod” on *Yahoo* leads to more auctions where iPod’s are sold as opposed to some of its accessories which are usually cheaper. Indeed it turns out that almost 95% of buyout price auctions

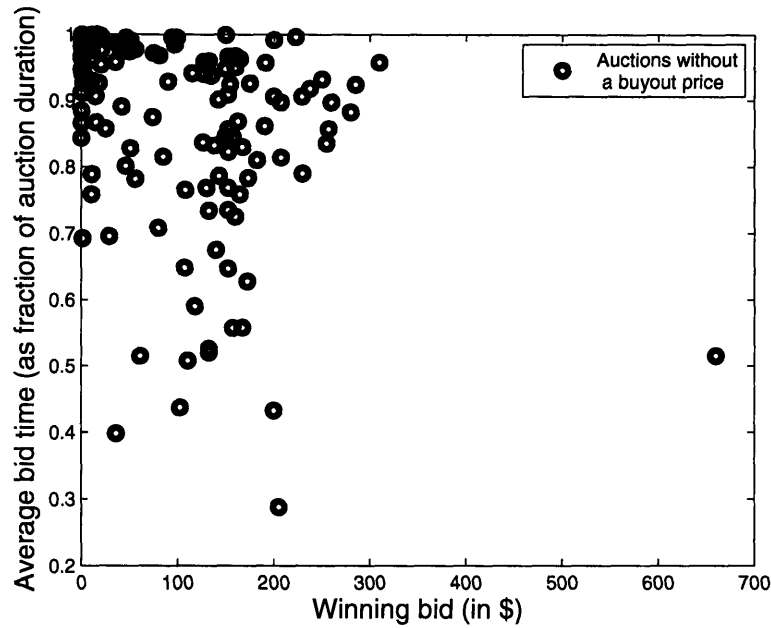


Figure 5-11: Average bid time as a function of winning bid

satisfying the above criterion (buyout option not exercised, $\frac{wb}{sp} \geq 0.8$, $\frac{bp}{sp} \geq 1.25$) have a winning bid greater than \$100 and hence we restrict our analysis to auctions where the winning bid is greater than \$100. The important statistics of the restricted dataset are summarized in Table 5.6; \overline{time} is the average bid time of all bids placed in an auction.

	Auctions without buyout price (184 auctions)		Auctions with buyout price (45 auctions)	
	Mean	Std. Dev.	Mean	Std. Dev.
Duration (days)	3.3082	0.1810	2.6728	0.1528
sp (\$)	108.5840	6.9045	183.5527	60.4369
bp (\$)	NA	NA	318.7093	75.1084
wb (\$)	180.3127	6.1154	268.9649	60.1669
\overline{time}	0.8072	0.0125	0.8301	0.0179

Table 5.6: Summary of auction data with $wb > \$100$, $\frac{bp}{sp} \geq 1.25$, and $\frac{wb}{sp} \geq 0.8$

Observe that the average bid time in buyout price auctions is higher than in auctions without a buyout price. However, a two sample t-test returns a p-value of 0.1493; thus, while the data suggests that bids arrive later in a permanent buyout

price auction as compared to an auction without a buyout price, the result is not very conclusive. However, observe that the average winning bid is higher in buyout price auctions, and hence the average bid time would further increase for buyout price auctions (or alternately, decrease for auctions without a buyout price) if the data were adjusted to ensure that the average winning bid is same for both cases. Due to the limited amount of data we have on *Yahoo* auctions, the analysis for different winning bid cutoffs is not meaningful in this case.

One of the primary reasons the result of the data analysis in the permanent case is not as conclusive as in the temporary case is that some *Yahoo* auctions use a slightly different auction mechanism than that assumed in this paper; in particular, while we assume a fixed auction end time, some *Yahoo* auctions have a floating deadline that extends if a bid is placed near the end of the auction. The presence of a floating deadline may alter bidding strategy – in particular notice that the strategy “bid just near the end of the auction” cannot be played because whenever a bid is placed in the auction, the deadline is automatically extended; see also Roth and Ockenfels (2002) who empirically test the effect of a floating deadline on bid times. The primary motivation of bidding late in an auction with a buyout price is to prevent bidders from utilizing the information provided by one’s bid; however, in an auction where the deadline extends when a bid is placed, other bidders always get some time to respond to a bid irrespective of when it is placed, thus decreasing the incentive of bidding late in the auction. Furthermore, as discussed before, our data is biased towards auctions with a buyout price which is higher than what bidders expect to pay for getting the product. As mentioned above, an important reason why bidders bid late in a permanent buyout price auction is that information about their valuation may cause other bidders to exercise the buyout option; however, a very highly priced buyout option has a small probability of exercise, and hence decreases the incentive of bidding late.

In summary, the empirical analysis suggests that, in a temporary buyout price auction, Hypothesis 1 can be accepted with a high degree of confidence; in the permanent case, while Hypothesis 2 seems to hold, the test is not as conclusive. As

mentioned before though, this simple empirical analysis does ignore other factors that impact bidder behavior. However, an extensive study of these factors is beyond the scope of this thesis.

Chapter 6

Conclusion

We have presented in this thesis a stylized game-theoretic model allowing to study the relative impact of temporary and permanent buyout options, two features increasingly widespread in large online auction sites. An important model prediction is that with both temporary and permanent options, bidders who exercise the buyout option will do so immediately upon (or shortly after) their arrival. Furthermore, with a temporary option the first bidder to submit a regular bid will also do so immediately upon arrival, but with a permanent option all regular bids should be submitted shortly before the end of the auction. Indeed econometric analysis of actual bidding data obtained from *eBay* and *Yahoo* indicates that on an average the first bid in a temporary buyout price auction arrives earlier than in a standard auction while the average bid time in a permanent buyout price auction is higher than in a standard auction. Note that our model does not provide any prediction for when regular bids from the second and subsequent bidders will be submitted in an auction with a temporary buyout option. In practice, the timing of bid submissions is also affected in various ways by features not captured here; for example a high cost of monitoring the auction could hasten bid submissions, while common value could delay them. However, our model does suggest that the marginal impact of a permanent buyout option relative to a temporary one is to delay the first bid (presumably a negative for the seller if bidding activity may be attracting more bidders), and concentrate bidding activity near the end of the auction. From that perspective, we find it remarkable

that *Amazon's* online auction site, one of the largest with a permanent buyout option, also features a rule whereby the first bidder is offered a 10% discount on the final selling price should he win the auction. This obvious incentive for early bidder involvement, which is not used on any site with a temporary buyout option we are aware of (most prominently *eBay*), lends support in our view to the robustness of our model predictions.

We also consider the seller's problem of finding the buyout price, either temporary or permanent, maximizing his expected discounted revenue. While this problem seems difficult to solve analytically in the general case, we present a method for efficiently computing its solution using simulation. For limiting values of bidder arrival rate, and bidder and seller time-sensitivity we derive asymptotic optimal buyout prices for both the temporary and permanent option. Besides being potentially useful in practice, these optimal buyout price expressions have mechanism design implications. Specifically, in our model where the relative values of the seller's and bidders' time sensitivity and the bidder arrival rate effectively capture market power and the ratio between supply and demand, a very time-sensitive seller facing bidders with little time-sensitivity, and a low bidder arrival rate should use a fixed posted price, while a time-insensitive seller facing a high bidder arrival rate should bypass the buyout option and only use a regular auction mechanism; the hybrid mechanism and smooth transition enabled by a buyout option is appropriate for a range of market environments between those two extremes.

Our numerical experiments confirm the intuition that the optimal permanent buyout price is higher than the temporary buyout price in a given market environment, and that both increase with seller and bidders time-sensitivities. Likewise, the relative increase in seller's utility from using a buyout option increases with both seller and bidders time-sensitivities, although its dependence on the bidders' time-sensitivity vanishes as the expected number of bidders grows; a distinguishing feature is that the relative attractiveness for the seller of a temporary buyout option decreases with the expected number of bidders, whereas it increases in the case of a permanent buyout option.

Finally, in all our experiments we found that the seller's expected discounted revenue derived from an optimal permanent buyout option was larger than that obtained with an optimal temporary option. Notice, however, that our numerical experiments assume the same bidder arrival rate for both auctions; while this assumption is reasonable in our model with a single isolated auction, it might be violated if bidders have multiple auctions to choose from and one auction type is more preferable than the other. For instance, since a permanent option is available at all times in the auction until exercised, a bidder in a permanent buyout price auction is constantly exposed to the possibility of losing the auction because the buyout option is exercised by another bidder. This may discourage bidders from participating in a permanent buyout price auction, who may instead prefer either a standard auction without a buyout price or a temporary buyout price auction where they can make the buyout option disappear by placing a bid. This would thus suggest that, everything else being equal, the bidder arrival rate in a permanent buyout price auction will be lower than in an auction with a temporary option. Furthermore, our equilibrium analysis suggests that all bidders must wait up to the end of the auction to place a bid in a permanent buyout price auction. In a market where the same item is sold via multiple auctions, some of these waiting bidders may balk on finding better offers for the same product elsewhere, thus also reducing the effective bidder arrival rate in a permanent option. Additionally, the higher incentives for late bidding associated with the permanent option may also negatively impact the seller's revenue. For these reasons, the numerical results just mentioned do not justify in our view an unambiguous recommendation to always use a permanent option over a temporary one, except perhaps for very time-sensitive sellers in environments with a high expected number of bidders, the conditions under which the predicted difference in expected discounted revenue was largest in our experiments. This nuanced interpretation also seems justified by the continued use by *eBay* (the largest and arguably most successful auction site currently operating) of a temporary buyout option.

Finally, our study also provides tentative answers to the question of how much a seller would stand to gain from using a buyout price varying dynamically according

to a pre-determined trajectory as the auction progresses. While our results are not quite as conclusive in the permanent case as in the temporary one, they still suggest that the potential revenue increase enabled by such dynamic buyout price is small, seemingly not justifying the associated implementation complexity and possible negative reactions from bidders; the fact that to the best of our knowledge no dynamic buyout price has ever been used in any actual auction site may also be corroborating our findings.

There are several interesting extensions of this work that are worthwhile considering. While focusing on the seller's perspective seemed justified in this first study because sellers typically choose auction sites and parameters, it would be valuable to explore the impact of buyout options on bidders' utilities. Another possible direction is to extend our analysis to the case of multi-item auctions, and also consider dynamic buyout prices that would not be pre-determined but rather modified according to actual bidding activity during the auction.

Appendix A

Appendix

A.1 Proof of Proposition 1

Proof. Let $\tilde{v}(t)$ be the solution of

$$\begin{aligned}\tilde{v}(t) - p &= e^{-(\lambda+\beta)(T-t)} \int_{\underline{v}}^{\tilde{v}(t)} e^{\lambda(T-t)\left(\frac{x-\underline{v}}{m}\right)} dx \\ &= \frac{me^{-(\lambda+\beta)(T-t)}}{\lambda(T-t)} \left(e^{\lambda(T-t)\left(\frac{\tilde{v}(t)-\underline{v}}{m}\right)} - 1 \right)\end{aligned}\tag{A.1}$$

which is the same equation as (3.2) except that $F(x)$ has been replaced by $\frac{x-\underline{v}}{m}$. The solution of (A.1) is given as:

$$\tilde{v}(t) = p - \frac{m}{\lambda(T-t)} \left(W \left(-e^{-e^{-(\lambda+\beta)(T-t) + \frac{(p-\underline{v})\lambda(T-t)}{m} - (\lambda+\beta)(T-t)}} \right) + e^{-(\lambda+\beta)(T-t)} \right).$$

If $\hat{v}(t) \leq \bar{v}$ for some t then (3.2) and (A.1) are equivalent since $F(x) = \frac{x-\underline{v}}{m}$, $\forall x \in [\underline{v}, \bar{v}]$ and $\hat{v}(t) = \tilde{v}(t)$. It thus follows that $\nu_{tmp}(t) = \hat{v}(t) = \min(\tilde{v}(t), \bar{v})$.

If $\hat{v}(t) > \bar{v}$ for some t then we have

$$\begin{aligned}\hat{v}(t) - e^{-(\lambda+\beta)(T-t)} \int_{\underline{v}}^{\hat{v}(t)} e^{\lambda(T-t)\left(\frac{x-\underline{v}}{m}\right)} dx &\leq \hat{v}(t) - e^{-(\lambda+\beta)(T-t)} \int_{\underline{v}}^{\hat{v}(t)} e^{\lambda(T-t)F(x)} dx \\ &= p,\end{aligned}$$

since $F(x) \leq \frac{x-\underline{v}}{m}$, $\forall x \in [\underline{v}, +\infty)$. Furthermore, notice that

$$v - e^{-(\lambda+\beta)(T-t)} \int_{\underline{v}}^v e^{\lambda(T-t)\left(\frac{x-\underline{v}}{m}\right)} dx$$

is increasing in v for all t and this combined with the fact that $\tilde{v}(t)$ is the solution of (A.1) implies that $\tilde{v}(t) \geq \hat{v}(t) > \bar{v}$. Thus $\nu_{tmp}(t) = \bar{v} = \min(\hat{v}(t), \bar{v}) = \min(\tilde{v}(t), \bar{v})$. \square

A.2 Expression for $\mathbf{E}_t[\max(\underline{v}, v_{N(t,T)+1}^{(2)}) | v_1 \leq p]$

We derive here an explicit expression for the expected revenue given that the first bidder arrives at time t and has valuation $v_1 \leq p$ (and thus bids in the auction). It is assumed that bidder valuations are uniformly distributed with support $[\underline{v}, \bar{v}]$.

Recall that $N(t, T)$ which is the number of bidders arriving in the auction in the interval $(t, T]$ is a Poisson random variable with parameter $\lambda(T - t)$. Now, let I be the number of bidders who have a valuation greater than p . Then the probability mass function of I is

$$p_I(i) = \binom{N(t, T)}{i} \left(\frac{\bar{v} - p}{m}\right)^i \left(\frac{p - \underline{v}}{m}\right)^{N(t, T) - i} \quad (\text{A.2})$$

where we have assumed without loss of generality that $p \in [\underline{v}, \bar{v}]$. Notice that $0 \leq I \leq N(t, T)$ (since the first bidder has a valuation less than p). For uniformly distributed valuations the expected revenue, as a function of $N(t, T)$ and I , is

$$\mathbf{E}_t[\max(\underline{v}, v_{N(t,T)+1}^{(2)}) | v_1 \leq p, N(t, T), I] = \begin{cases} p - \frac{2(p-\underline{v})}{N(t,T)+2} & I = 0 \\ p - \frac{(p-\underline{v})}{N(t,T)+1} & I = 1 \\ \bar{v} - \frac{2(\bar{v}-p)}{I+1} & 2 \leq I \leq N(t, T) \end{cases} \quad (\text{A.3})$$

Thus taking the expectation over I and $N(t, T)$ we obtain

$$\begin{aligned} \mathbf{E}_t[\max(\underline{v}, v_{N(t,T)+1}^{(2)})|v_1 \leq p] = \\ \sum_{n=0}^{\infty} \sum_{i=0}^n \mathbf{E}_t[\max(\underline{v}, v_{N(t,T)+1}^{(2)})|v_1 \leq p; N(t, T) = n, I = i] p_I(i) \frac{e^{-\lambda(T-t)} (\lambda(T-t))^n}{n!} \end{aligned} \quad (\text{A.4})$$

Evaluating the double summation, we get

$$\begin{aligned} \mathbf{E}_t[\max(\underline{v}, v_{N(t,T)+1}^{(2)})|v_1 \leq p] = e^{-\lambda(T-t)} \left(\bar{v} e^{\lambda(T-t)} - \frac{2m}{\lambda(T-t)} (e^{\lambda(T-t)} - 1) \right. \\ \left. + \frac{2m}{(\lambda(T-t))^2(1-f)} (e^{\lambda(T-t)(1-f)} - 1 - \lambda(T-t)(1-f)) \right. \\ \left. + \frac{\bar{v} - p}{\lambda(T-t)(1-f)} (e^{\lambda(T-t)(1-f)} - 1) \right) \end{aligned}$$

where $f = \frac{\bar{v}-p}{m}$.

A.3 Proof of Lemma 8

Proof. For any $\phi \in \mathcal{F}, t \in [0, T - \varepsilon]$ we have

$$\underline{v} \leq p \leq G(\phi)(t) \leq p + e^{-\beta\varepsilon}(q - \underline{v}) \quad (\text{A.5})$$

Clearly if $q > \frac{p - e^{-\beta\varepsilon}\underline{v}}{1 - e^{-\beta\varepsilon}}$ then $G(\phi)(t) \in [\underline{v}, q]$ for all $\phi \in \mathcal{F}, t \in [0, T - \varepsilon]$. Next we show that for any $\phi \in \mathcal{F}$

$$|G(\phi)(t) - G(\phi)(t')| \leq M|t - t'| \quad \forall t, t' \in [0, T - \varepsilon] \quad (\text{A.6})$$

Indeed for any $\phi \in \mathcal{F}$ and $t \in [0, T - \varepsilon]$ consider

$$G(\phi)(t) = \mathbf{E}_t \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\phi(t)} \prod_{i=1}^{N(t)} \frac{F(\min(\phi(t_i), x))}{F(\phi(t_i))} \prod_{j=1}^{N(t,T)} F(x) dx \right) \right] + p \quad (\text{A.7})$$

where recall that the expectation \mathbf{E}_t is with respect to the number $N(t)$ and epochs

$t_1, \dots, t_{N(t)}$ of arrivals in $[0, t)$ of a non-homogeneous Poisson process with rate $\lambda F(\tilde{v}(t))$, and number $N(t, T)$ of arrivals in $(t, T]$ of a Poisson process with rate λ .

Let

$$\mathbf{E}[H(\phi(t), t)] = G(\phi)(t) - p = \mathbf{E}_t \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\phi(t)} \prod_{i=1}^{N(t)} \frac{F(\min(\phi(t_i), x))}{F(\phi(t_i))} \prod_{j=1}^{N(t, T)} F(x) dx \right) \right].$$

Now to calculate $\mathbf{E}[H(\phi(t), t)]$, we condition on the number of arrivals in the interval $(t, t + \Delta t)$ where $\Delta t > 0$ and small and is such that $t + \Delta t \leq T - \varepsilon$.

For the sake of brevity, let

$$\Gamma(\{t_i\}_{i=1}^{N(t)}) = \prod_{i=1}^{N(t)} \frac{F(\min(\phi(t_i), x))}{F(\phi(t_i))} \quad (\text{A.8})$$

First suppose that there was arrival in $(t, t + \Delta t)$; an event which has probability $\lambda \Delta t$ since the arrival process is Poisson with rate λ . Then the conditional expectation is:

$$\begin{aligned} \mathbf{E}[H(\phi(t), t) | N(t, t + \Delta t) = 1] = \\ \sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\phi(t)} \Gamma(\{t_i\}_{i=1}^{N(t)}) \times F(x) \times F(x)^l dx \right) \right] P(N(t + \Delta t, T) = l) \quad (\text{A.9}) \end{aligned}$$

Note that here we first calculate the expected utility given $N(t + \Delta t, T) = l$ and then sum over all possible l . The expectation \mathbf{E}_t on the right hand side is over $N(t)$ and $\{t_i\}_{i=1}^{N(t)}$.

Now if there was no arrival in $(t, t + \Delta t)$, an event which has probability $(1 - \lambda \Delta t + o(\Delta t))$ where $o(\Delta t)$ indicates any function $f(\Delta t)$ such that $\lim_{\Delta t \rightarrow 0} \frac{f(\Delta t)}{\Delta t} = 0$, we get:

$$\begin{aligned} \mathbf{E}[H(\phi(t), t) | N(t, t + \Delta t) = 0] = \\ \sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\phi(t)} \Gamma(\{t_i\}_{i=1}^{N(t)}) \times F(x)^l dx \right) \right] P(N(t + \Delta t, T) = l) \quad (\text{A.10}) \end{aligned}$$

The probability of more than one arrival in an interval of length Δt is $o(\Delta t)$ and thus the unconditional expectation of $H(\phi(t), t)$ becomes:

$$\begin{aligned} \mathbf{E}[H(\phi(t), t)] &= \mathbf{E}\left[H(\phi(t), t)|N(t, t + \Delta t) = 1\right] \times (\lambda\Delta t) \\ &\quad + \mathbf{E}\left[H(\phi(t), t)|N(t, t + \Delta t) = 0\right] \times (1 - \lambda\Delta t) + o(\Delta t) \end{aligned} \quad (\text{A.11})$$

Substituting for the terms, we get

$$\begin{aligned} \mathbf{E}[H(\phi(t), t)] &= \\ &\left(\sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\phi(t)} \Gamma(\{t_i\}_{i=1}^{N(t)}) F(x) F(x)^l dx\right)\right] P(N(t + \Delta t, T) = l)\right) (\lambda\Delta t) \\ &+ \left(\sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\phi(t)} \Gamma(\{t_i\}_{i=1}^{N(t)}) F(x)^l dx\right)\right] P(N(t + \Delta t, T) = l)\right) (1 - \lambda\Delta t) + o(\Delta t) \end{aligned} \quad (\text{A.12})$$

Similar to the above analysis, we next condition on the number of arrivals in (t, t') to evaluate $\mathbf{E}[H(\phi(t'))]$, where for ease of exposition the notation $t' = t + \Delta t$ is used.

Now suppose that there is arrival in the interval (t, t') . For small Δt , the probability of this event is $\lambda F(\phi(t))\Delta t$, since the arrival process at t is non-homogeneous Poisson with rate $\lambda F(\phi(t))$. We then have:

$$\begin{aligned} \mathbf{E}[H(\phi(t'), t')|N(t, t') = 1] &= \\ &\sum_{l=0}^{\infty} \mathbf{E}\left[e^{-\beta(T-t')} \left(\int_{\underline{v}}^{\phi(t')} \Gamma(\{t_i\}_{i=1}^{N(t)}) \frac{F(\min(\phi(t), x))}{F(\phi(t))} F(x)^l dx\right)\right] P(N(t', T) = l) \end{aligned} \quad (\text{A.13})$$

Now if there was no arrival in (t, t') , an event with probability $(1 - \lambda F(\phi(t))\Delta t +$

$o(\Delta t)$), we get:

$$\begin{aligned} \mathbf{E}[H(\phi(t'), t') | N(t, t') = 0] = \\ \sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t')} \left(\int_{\underline{v}}^{\phi(t')} \Gamma(\{t_i\}_{i=1}^{N(t)}) \times F(x)^l dx \right) \right] P(N(t', T) = l) \end{aligned} \quad (\text{A.14})$$

The unconditional expectation of $H(\phi(t'), t')$ is:

$$\begin{aligned} \mathbf{E}[H(\phi(t'), t')] = \mathbf{E} \left[H(\phi(t'), t') | N(t, t') = 1 \right] (\lambda F(\phi(t)) \Delta t) \\ + \mathbf{E} \left[H(\phi(t'), t') | N(t, t') = 0 \right] \left(1 - \lambda F(\phi(t)) \Delta t \right) + o(\Delta t) \end{aligned} \quad (\text{A.15})$$

Substituting for the terms we get:

$$\begin{aligned} \mathbf{E}[H(\phi(t'), t')] = \\ \sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t')} \left(\int_{\underline{v}}^{\phi(t')} \Gamma(\{t_i\}_{i=1}^{N(t)}) \frac{F(\min(\phi(t), x))}{F(\phi(t))} F(x)^l dx \right) \right] P(N(t', T) = l) (\lambda F(\phi(t)) \Delta t) \\ + \sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t')} \left(\int_{\underline{v}}^{\phi(t')} \Gamma(\{t_i\}_{i=1}^{N(t)}) F(x)^l dx \right) \right] P(N(t', T) = l) \left(1 - \lambda F(\phi(t)) \Delta t \right) + o(\Delta t) \end{aligned} \quad (\text{A.16})$$

By the definition of the function G we have

$$G(\phi)(t') - G(\phi)(t) = \mathbf{E}[H(\phi(t'), t')] - \mathbf{E}[H(\phi(t), t)]$$

Substituting (A.12) and (A.16) in the above expression we obtain

$$\begin{aligned}
& G(\phi)(t') - G(\phi)(t) = \\
& \sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t')} \left(\int_{\underline{v}}^{\phi(t')} \Gamma(\{t_i\}_{i=1}^{N(t)}) \frac{F(\min(\phi(t), x))}{F(\phi(t))} F(x)^l dx \right) \right] P(N(t', T) = l) (\lambda F(\phi(t)) \Delta t) \\
& + \sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t')} \left(\int_{\underline{v}}^{\phi(t')} \Gamma(\{t_i\}_{i=1}^{N(t)}) F(x)^l dx \right) \right] P(N(t', T) = l) (1 - \lambda F(\phi(t)) \Delta t) + o(\Delta t) \\
& \quad - \left(\sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\phi(t)} \Gamma(\{t_i\}_{i=1}^{N(t)}) F(x) F(x)^l dx \right) \right] P(N(t', T) = l) \right) (\lambda \Delta t) \\
& - \left(\sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\phi(t)} \Gamma(\{t_i\}_{i=1}^{N(t)}) F(x)^l dx \right) \right] P(N(t', T) = l) \right) (1 - \lambda \Delta t) + o(\Delta t)
\end{aligned} \tag{A.17}$$

Simplifying the above expression we get the following bound

$$\begin{aligned}
& |G(\phi)(t') - G(\phi)(t)| \leq \\
& \left| \sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t')} \left(\int_{\underline{v}}^{\phi(t')} (\Gamma(\{t_i\}_{i=1}^{N(t)}) F(\min(\phi(t), x)) F(x)^l - F(x)^{l+1}) dx \right) \right] \times \right. \\
& \quad \left. \times P(N(t', T) = l) (\lambda \Delta t) \right| \\
& + \left| \sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t')} \left(\int_{\phi(t)}^{\phi(t')} \Gamma(\{t_i\}_{i=1}^{N(t)}) F(\min(\phi(\tau), x)) F(x)^l dx \right) \right] P(N(t', T) = l) (\lambda \Delta t) \right| \\
& + \left| \left(\sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t')} \left(\int_{\phi(t)}^{\phi(t')} \Gamma(\{t_i\}_{i=1}^{N(t)}) F(x)^l dx \right) \right] P(N(t', T) = l) \right) \right| \\
& + \left| \lambda \Delta t (1 - F(\phi(t))) \sum_0^{\infty} \mathbf{E} \left[e^{-\beta(T-t')} \left(\int_{\underline{v}}^{\phi(t)} \Gamma(\{t_i\}_{i=1}^{N(t)}) F(x)^l dx \right) \right] P(N(t', T) = l) \right| \\
& + \left| \lambda \Delta t \sum_0^{\infty} \mathbf{E} \left[e^{-\beta(T-t')} \left(\int_{\phi(t)}^{\phi(t')} \Gamma(\{t_i\}_{i=1}^{N(t)}) F(x)^l dx \right) \right] P(N(t', T) = l) \right| \\
& + \left| \left(e^{-\beta(T-t')} - e^{-\beta(T-t)} \right) \mathbf{E}[H(\phi(t), t)] \right| + o(\Delta t)
\end{aligned}$$

Now noting that $t, t' \leq (T - \varepsilon)$ and $H(\phi(t), t) \leq q \forall t \in [0, T - \varepsilon]$, the above bound

can be further simplified to obtain

$$|G(\phi)(t') - G(\phi)(t)| \leq e^{-\beta\varepsilon} \left(2q\lambda\Delta t + 2|\phi(t') - \phi(t)|\lambda\Delta t + |\phi(t') - \phi(t)| + (e^{\beta\Delta t} - 1)q \right) + o(\Delta t) \quad (\text{A.18})$$

Since $\phi \in \mathcal{F}$ we have $|\phi(t') - \phi(t)| \leq M|t' - t|$, $\forall t, t' \in [0, T - \varepsilon]$. Using this and rearranging terms to obtain

$$|G(\phi)(t') - G(\phi)(t)| \leq e^{-\beta\varepsilon} (q(2\lambda + \beta) + M)\Delta t + o(\Delta t) \quad (\text{A.19})$$

Thus if we choose $M > e^{-\beta\varepsilon} (q(2\lambda + \beta) + M)$, which can be rearranged to obtain $M > \frac{e^{-\beta\varepsilon}(2\lambda + \beta)q}{1 - e^{-\beta\varepsilon}}$ we get that

$$|G(\phi)(t') - G(\phi)(t)| \leq M(t' - t) \quad \forall 0 \leq t' - t \leq \delta; t, t' \in [0, T - \varepsilon] \quad (\text{A.20})$$

where $\delta = \sup \left\{ \delta > 0 \mid \frac{\alpha(\delta)}{\delta} < M - \frac{e^{-\beta\varepsilon}(2\lambda + \beta)q}{1 - e^{-\beta\varepsilon}} \right\}$. Notice that $\delta > 0$ since $\lim_{\delta \rightarrow 0} \frac{\alpha(\delta)}{\delta} \rightarrow 0$.

Since δ is independent of t , it follows from (A.20) that

$$|G(\phi)(t') - G(\phi)(t)| \leq M|t' - t| \quad \forall |t' - t| \leq \delta; t, t' \in [0, T - \varepsilon] \quad (\text{A.21})$$

where δ is as defined above.

Now for any $t, t' \in [0, T - \varepsilon]$, assuming without loss of generality that $t' > t$, we have

$$\begin{aligned} |G(\phi)(t') - G(\phi)(t)| &\leq |G(\phi)(t') - G(\phi)(t' - \delta)| + \dots + |G(\phi)(t' - n\delta) - G(\phi)(t)| \\ &\leq M|\delta| + \dots + M|t' - n\delta - t| = M|t' - t| \end{aligned}$$

where $n = \lfloor \frac{t' - t}{\delta} \rfloor$. The second inequality follows from (A.21).

Thus we have shown that for any $\phi \in \mathcal{F}$

$$|G(\phi)(t') - G(\phi)(t)| \leq M|t' - t| \quad \forall t, t' \in [0, T - \varepsilon] \quad (\text{A.22})$$

This combined with the fact that for all $\phi \in \mathcal{F}$

$$G(\phi)(t) \in [\underline{v}, q] \quad \forall t \in [0, T - \varepsilon] \quad (\text{A.23})$$

implies that the set $\mathcal{K} = \{G(\phi) | \phi \in \mathcal{F}\} \subset \mathcal{F}$. □

A.4 Proof of Lemma 10

Proof. Let ϱ denote the sup norm, i.e.

$$\varrho(\phi, \psi) = \max_{0 \leq t \leq T - \varepsilon} |\phi(t) - \psi(t)| \quad (\text{A.24})$$

Let $\varepsilon > 0$ be given. For any $\phi, \psi \in \mathcal{F}, t \in [0, T - \varepsilon]$ we have

$$\begin{aligned} |G(\phi)(t) - G(\psi)(t)| \leq & \left| \mathbf{E}_t \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\phi(t)} \left(\prod_{i=1}^{N(t)} \frac{F(\min(\phi(t_i), x))}{F(\phi(t_i))} \right. \right. \right. \right. \\ & \left. \left. \left. - \prod_{i=1}^{\tilde{N}(t)} \frac{F(\min(\psi(\tilde{t}_i), x))}{F(\psi(\tilde{t}_i))} \right) \prod_{j=1}^{N(t,T)} F(x) dx \right) \right] \right| \\ & + \left| \mathbf{E} \left[\int_{\phi(t)}^{\psi(t)} \prod_{i=1}^{\tilde{N}(t)} \frac{F(\min(\psi(\tilde{t}_i), x))}{F(\psi(\tilde{t}_i))} \prod_{j=1}^{N(t,T)} F(x) dx \right] \right| \quad (\text{A.25}) \end{aligned}$$

where $N(t)$ (resp. $\tilde{N}(t)$) is the number of arrivals in a non-homogeneous Poisson process with rate $\lambda F(\phi(\tau))$ (resp. $\lambda F(\psi(\tau))$) for $\tau \in (0, t)$, and the corresponding arrival times are given by $\{t_i\}_{i=1}^{N(t)}$ (resp. $\{\tilde{t}_i\}_{i=1}^{\tilde{N}(t)}$). $N(t, T)$ is the number of arrivals in the interval $(t, T]$ of a Poisson process with arrival rate λ .

If we assume that all bidders in $(t, T]$ bid in the auction, the first term on the right hand side of (A.25) can be interpreted as the difference in the expected utility for a bidder with valuation $\phi(t)$ if every bidder of type (v, τ) with $\tau \in (0, t)$ has valuation $v \leq \phi(\tau)$ as opposed to having valuation $v \leq \psi(\tau)$. This term is positive only if there is one or more arrival in the interval $(0, t)$ in a non-homogeneous Poisson process with rate $\lambda |F(\phi(\tau)) - F(\psi(\tau))|$. Now since F is a given continuous function for any $\kappa > 0$

there exists a $\delta(\kappa) > 0$ such that for all $\phi, \psi \in \mathcal{F}$, $\varrho(\phi, \psi) < \delta(\kappa)$

$$|F(\phi(\tau)) - F(\psi(\tau))| < \kappa \quad \forall \tau \in [0, T - \varepsilon]$$

Thus the rate $\lambda|F(\phi(\tau)) - F(\psi(\tau))|$ can be upper bounded by $\lambda\kappa$ for all $\tau \in [0, T - \varepsilon]$ if $\varrho(\phi, \psi) < \delta(\kappa)$. Now probability that there is one or more arrival in the interval $(0, t)$ in a Poisson process with rate $\lambda\kappa$ is $\lambda\kappa t + o(\kappa)$. Thus probability of at least one arrival in the interval $(0, t)$ in a non-homogeneous Poisson process with rate $\lambda|F(\phi(\tau)) - F(\psi(\tau))|$ can be upper bounded by $\lambda\kappa t + o(\kappa)$. In addition an arrival in the above mentioned process can lead to a difference in expected utility which is bounded above by q . Hence the first term in (A.25) can be upper bounded by $q(\lambda\kappa t + o(\kappa))$. In addition since $F(\min(\psi(\hat{t}_i), x)) \leq F(\psi(\hat{t}_i))$ and $F(\cdot) \leq 1$ the second term of (A.25) is bounded above by $|\psi(t) - \phi(t)|$. We thus get

$$\begin{aligned} |G(\phi)(t) - G(\psi)(t)| &\leq q(\lambda\kappa t + o(\kappa)) + |\phi(t) - \psi(t)| \\ &\leq q(\lambda\kappa T + o(\kappa)) + \varrho(\phi, \psi) \end{aligned}$$

Define $\kappa_1 = \frac{\varepsilon}{3q\lambda T}$ and $\kappa_2 = \sup\{\kappa | o(\kappa) < \varepsilon/3q\}$ ($\kappa_2 > 0$ since $\lim_{\kappa \rightarrow 0} o(\kappa)/\kappa \rightarrow 0$) and let $\kappa = \min(\kappa_1, \kappa_2)$. Thus for $\delta = \min(\delta(\kappa), \varepsilon/3)$, we have for all $\phi, \psi \in \mathcal{F}$ such that $\varrho(\phi, \psi) < \delta$

$$\begin{aligned} \varrho(G(\phi), G(\psi)) &= \sup_{t \in [0, T - \varepsilon]} |G(\phi)(t) - G(\psi)(t)| \leq q(\lambda\kappa T + o(\kappa)) + \varrho(\phi, \psi) \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

and hence the operator G is continuous on the set \mathcal{F} . □

A.5 Proof of Proposition 2

Proof. Let $\tilde{v}(t)$ be the solution of (3.60), i.e.

$$\tilde{v}(t) - p = \mathbf{E}_t \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\tilde{v}(t)} \prod_{i=1}^{N(t)} \frac{F(\min(\tilde{v}(t_i), x))}{F(\tilde{v}(t_i))} \prod_{j=1}^{N(t,T)} F(x) dx \right) \right] \quad (\text{A.26})$$

for all $t \in [0, T - \varepsilon]$ for some $\varepsilon > 0$ and small.

As discussed in the proof of Theorem 4 $\tilde{v}(t)$ is non-decreasing in t and thus it follows from (3.51) that the expected utility from bidding for a bidder with type $(v, t, 0)$, assuming other bidders follow strategy $\mathcal{P}[\tilde{v}]$, is

$$\mathbf{E}_t[U_{bid(T)}(v, t, 0)] = \mathbf{E} \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^v \prod_{i=1}^{N(t)} \frac{F(\min(\tilde{v}(t_i), x))}{F(\tilde{v}(t_i))} \prod_{j=1}^{N(t,T)} F(x) dx \right) \right]$$

Thus (A.26) can be restated as:

$$\tilde{v}(t) - p = \mathbf{E}_t[U_{bid(T)}(\tilde{v}(t), t, 0)] \quad (\text{A.27})$$

for all $t \in [0, T - \varepsilon]$.

For any $t, (t + \Delta t) \in [0, T - \varepsilon]$, $\tilde{v}(\cdot)$ must satisfy

$$\frac{\tilde{v}(t + \Delta t) - \tilde{v}(t)}{\Delta t} = \frac{\mathbf{E} \left[U_{bid(T)}(\tilde{v}(t + \Delta t), t + \Delta t, 0) \right] - \mathbf{E} \left[U_{bid(T)}(\tilde{v}(t), t, 0) \right]}{\Delta t} \quad (\text{A.28})$$

Consider a bidder A with type $(\tilde{v}(t), t)$ and information $I_t = 0$. Since the auction is running at time t , all bidders of type $(v_i, t_i, 0)$ arriving in the interval $(0, t)$ have valuation $v_i \leq \tilde{v}(t_i)$. Thus for bidder A the arrival process of other bidders is:

1. Non-homogeneous Poisson process in $(0, t)$ with arrival rate $\lambda(\tau) = \lambda F(v_{perm}(\tau))$ for $\tau \in (0, t)$
2. Homogeneous Poisson process in $(t, T]$ with arrival rate λ .

Now to calculate $\mathbf{E}[U_{bid(T)}(\tilde{v}(t), t, 0)]$, we condition on the number of arrivals in

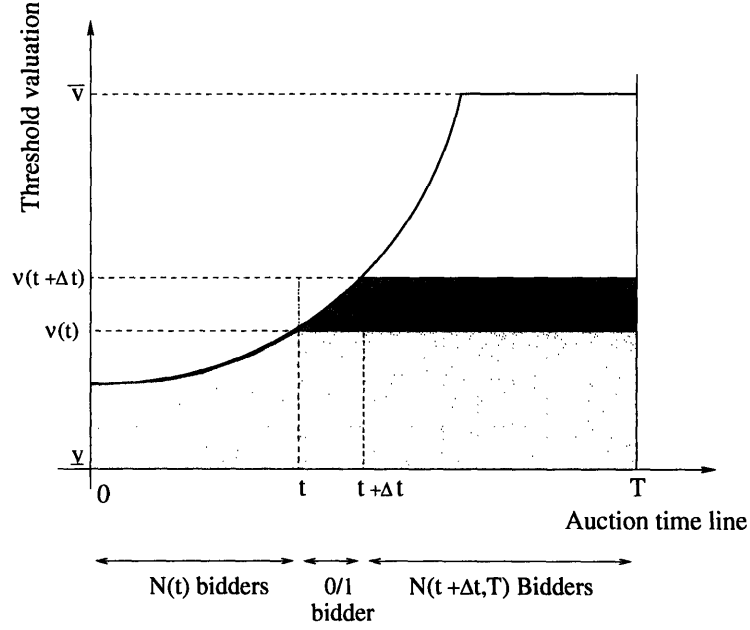


Figure A-1: Threshold valuation ν_{prm}

the interval $(t, t + \Delta t)$. To reduce notational complexity, let

$$\Gamma(\{t_i\}_{i=1}^{N(t)}) = \prod_{i=1}^{N(t)} \frac{F(\min(\tilde{v}(t_i), x))}{F(\tilde{v}(t_i))}. \quad (\text{A.29})$$

Suppose that there was arrival in $(t, t + \Delta t)$; this event has probability $\lambda\Delta t + o(\Delta t)$, where $o(\Delta t)$ indicates any function $f(\Delta t)$ such that $\lim_{\Delta t \rightarrow 0} \frac{f(\Delta t)}{\Delta t} = 0$, since the arrival process is Poisson with rate λ . Then the conditional expected utility is:

$$\begin{aligned} & \mathbf{E}[U_{bid(T)}(\tilde{v}(t), t, 0) | N(t, t + \Delta t) = 1] = \\ & \sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\tilde{v}(t)} \Gamma(\{t_i\}_{i=1}^{N(t)}) \times F(x) \times F(x)^l dx \right) \right] P(N(t + \Delta t, T) = l) \quad (\text{A.30}) \end{aligned}$$

Note that here we first calculate the expected utility given $N(t + \Delta t, T) = l$ and then sum over all possible l . The expectation \mathbf{E}_t on the right hand side is over $N(t)$ and $\{t_i\}_{i=1}^{N(t)}$.

Now if there was no arrival in $(t, t + \Delta t)$; an event which has probability $1 - \lambda\Delta t +$

$o(\Delta t)$, we get:

$$\begin{aligned} \mathbf{E}[U_{bid(T)}(\tilde{v}(t), t, 0) | N(t, t + \Delta t) = 0] = \\ \sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\tilde{v}(t)} \Gamma(\{t_i\}_{i=1}^{N(t)}) \times F(x)^l dx \right) \right] P(N(t + \Delta t, T) = l) \end{aligned} \quad (\text{A.31})$$

The probability of more than one arrival in an interval of length Δt is $o(\Delta t)$ and thus the unconditional expected utility from bidding becomes:

$$\begin{aligned} \mathbf{E}[U_{bid(T)}(\tilde{v}(t), t, 0)] &= \mathbf{E} \left[U_{bid(T)}(\tilde{v}(t), t, 0) | N(t, t + \Delta t) = 1 \right] \times (\lambda \Delta t) \\ &+ \mathbf{E} \left[U_{bid(T)}(\tilde{v}(t), t, 0) | N(t, t + \Delta t) = 0 \right] \times (1 - \lambda \Delta t) + o(\Delta t) \end{aligned} \quad (\text{A.32})$$

Substituting for the terms, we get

$$\begin{aligned} \mathbf{E}[U_{bid(T)}(\tilde{v}(t), t, 0)] &= \sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\tilde{v}(t)} \Gamma(\{t_i\}_{i=1}^{N(t)}) F(x)^{l+1} dx \right) \right] P(N(t + \Delta t, T) = l) (\lambda \Delta t) \\ &+ \sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t)} \left(\int_{\underline{v}}^{\tilde{v}(t)} \Gamma(\{t_i\}_{i=1}^{N(t)}) F(x)^l dx \right) \right] P(N(t + \Delta t, T) = l) (1 - \lambda \Delta t) + o(\Delta t) \end{aligned} \quad (\text{A.33})$$

Similar to the above analysis, we next condition on the number of arrivals in (t, t') to calculate $\mathbf{E}[U_{bid(T)}(\tilde{v}(t'), t', 0)]$, where to reduce notational complexity the notation $t' = t + \Delta t$ is used.

Consider a bidder B with type $(\tilde{v}(t'), t')$ and information $I_{t'} = 0$. Since the auction is running at time t' , all bidders of type $(v_i, t_i, 0)$ arriving in the interval $(0, t')$ have valuation $v_i \leq \tilde{v}(t_i)$. Thus for bidder B the arrival process of other bidders is:

1. Non-homogeneous Poisson process in $(0, t')$ with arrival rate $\lambda(\tau) = \lambda F(\tilde{v}(\tau))$ for $\tau \in (0, t)$
2. Homogeneous Poisson process in $(t', T]$ with arrival rate λ .

Now suppose that there is arrival in the interval (t, t') . For small Δt , the probability of this event is $\lambda F(\tilde{v}(t))\Delta t + o(\Delta t)$, since the arrival rate at t is $\lambda F(\tilde{v}(t))$. Therefore:

$$\begin{aligned} \mathbf{E}[U_{bid(T)}(\tilde{v}(t'), t', 0) | N(t, t') = 1] = \\ \sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t')} \left(\int_{\underline{v}}^{\tilde{v}(t')} \Gamma(\{t_i\}_{i=1}^{N(t)}) \frac{F(\min(\tilde{v}(t), x))}{F(\tilde{v}(t))} F(x)^l dx \right) \right] P(N(t', T) = l) \end{aligned} \quad (\text{A.34})$$

If there was no arrival in (t, t') , an event with probability $(1 - \lambda F(\tilde{v}))\Delta t + o(\Delta t)$, we get:

$$\begin{aligned} \mathbf{E}[U_{bid(T)}(\tilde{v}(t'), t', 0) | N(t, t') = 0] = \\ \sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t')} \left(\int_{\underline{v}}^{\tilde{v}(t')} \Gamma(\{t_i\}_{i=1}^{N(t)}) F(x)^l dx \right) \right] P(N(t', T) = l) \end{aligned} \quad (\text{A.35})$$

Since the probability of more than one arrival in an interval of length Δt is $o(\Delta t)$ the unconditional expected utility is:

$$\begin{aligned} \mathbf{E}[U_{bid(T)}(\tilde{v}(t'), t', 0)] = \mathbf{E} \left[U_{bid(T)}(\tilde{v}(t'), t', 0) | N(t, t') = 1 \right] (\lambda F(\tilde{v}(t))\Delta t) \\ + \mathbf{E} \left[U_{bid(T)}(\tilde{v}(t'), t', 0) | N(t, t') = 0 \right] (1 - \lambda F(\tilde{v}(t))\Delta t) + o(\Delta t) \end{aligned}$$

Substituting (A.34) and (A.35) in the above expression we get:

$$\begin{aligned} \mathbf{E}[U_{bid(T)}(\tilde{v}(t'), t', 0)] = \\ \sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t')} \left(\int_{\underline{v}}^{\tilde{v}(t')} \Gamma(\{t_i\}_{i=1}^{N(t)}) \frac{F(\min(\tilde{v}(t), x))}{F(\tilde{v}(t))} F(x)^l dx \right) \right] P(N(t', T) = l) (\lambda F(\tilde{v}(t))\Delta t) \\ + \sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t')} \left(\int_{\underline{v}}^{\tilde{v}(t')} \Gamma(\{t_i\}_{i=1}^{N(t)}) F(x)^l dx \right) \right] P(N(t', T) = l) (1 - \lambda F(\tilde{v}(t))\Delta t) + o(\Delta t) \end{aligned} \quad (\text{A.36})$$

Consider now the error associated with substituting $F(\min(\tilde{v}(\tau), x))$ with $F(x)$

in the first term of the right hand side of (A.36). The substitution effectively assumes that one bidder has valuation in the interval $[\underline{v}, \tilde{v}(t')]$ instead of $[\underline{v}, \tilde{v}(t)]$. When calculating the maximum valuation among the bidders, this assumption can lead to a difference $d(\tilde{v}(t), \tilde{v}(t'))$ which is bounded as follows:

$$0 \leq d(\tilde{v}(t), \tilde{v}(t')) \leq \tilde{v}(t') - \tilde{v}(t) \leq M(t' - t)$$

where we have used the fact that $\tilde{v}(\cdot)$ satisfies the Lipschitz condition for constant M . Thus if we let T_1 to be the first term in equation (A.36) and T_1^a be the corresponding approximate expression, then we have

$$\begin{aligned} 0 \leq T_1^a - T_1 &\leq \sum_{l=0}^{\infty} \left(\mathbf{E} \left[e^{-\beta(T-t')} (\tilde{v}(t') - \tilde{v}(t)) \right] \times Pr(N(t', T) = l) * \lambda F(v_{perm}(\tau)) \Delta t \right) \\ &\leq e^{-\beta(T-t')} M \lambda F(\tilde{v}(t)) (\Delta t)^2 \end{aligned}$$

and the error due to the approximation is thus $o(\Delta t)$.

Thus (A.36) can be rewritten as

$$\begin{aligned} \mathbf{E}[U_{bid}(\tilde{v}(t'), t')] &= \sum_{l=0}^{\infty} \mathbf{E} \left[e^{-\beta(T-t')} \left(\int_{\underline{v}}^{\tilde{v}(t')} \prod F(x)^{l+1} dx \right) \right] Pr(N(t', T) = l) (\lambda \Delta t) \\ &+ \sum_{l=0}^{\infty} \left(\mathbf{E} \left[e^{-\beta(T-t')} \left(\int_{\underline{v}}^{\tilde{v}(t')} \prod F(x)^l dx \right) \right] Pr(N(t', T) = l) \left(1 - \lambda F(\tilde{v}(\tau)) \Delta t \right) \right) + o(\Delta t) \end{aligned} \tag{A.37}$$

Subtracting equation (A.33) from equation (A.37), dividing by Δt and taking the limit $\Delta t \rightarrow 0$, we get, after simplification,

$$\frac{d\tilde{v}(t)}{dt} = \frac{\left(\beta + \lambda \left(1 - F(\tilde{v}(t)) \right) \right) (\tilde{v}(t) - P)}{1 - e^{-\left(\beta + \lambda \left(1 - F(\tilde{v}(t)) \right) \right) (T-t)}}$$

Recall that bidder valuations are assumed to be uniformly distributed and thus $F(x) = \frac{x-\underline{v}}{m}$ for $x \in [\underline{v}, \bar{v}]$.

Substituting $t = 0$ in (A.26), we get the initial value for the above differential

equation

$$\tilde{v}(0) = P - \frac{m}{\lambda T} \left(W \left(-e^{-(\beta+\lambda)T} - \frac{-(P - \underline{v})\lambda T + me^{-(\beta+\lambda)T}}{m} \right) + e^{-(\beta+\lambda)T} \right)$$

where as before W is Lambert's W function.

□

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