3.18 Unsteady Motion - Added Mass

D’Alembert: ideal, irrotational, unbounded, steady.

Example Force on a sphere accelerating \( (U = U(t), \text{unsteady}) \) in an unbounded fluid that is at rest at infinity.

\[
\phi = -U(t) \frac{a^3}{2r^2} \cos \theta
\]

K.B.C on sphere: \( \frac{\partial \phi}{\partial r} \bigg|_{r=a} = U(t) \cos \theta \)

Solution: Simply a 3D dipole (no stream)

\[
\phi = -U(t) \frac{a^3}{2r^2} \cos \theta
\]

Check: \( \frac{\partial \phi}{\partial r} \bigg|_{r=a} = U(t) \cos \theta \)
Hydrodynamic force:

\[ F_x = -\rho \int \int_B \left( \frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 \right) n_x dS \]

On \( r = a \),

\[
\frac{\partial \phi}{\partial t} \bigg|_{r=a} = -\dot{U} \frac{a^3}{2r^2} \cos \theta \bigg|_{r=a} = -\frac{1}{2} \dot{U} a \cos \theta
\]

\[
\nabla \phi \bigg|_{r=a} = \left( \frac{\partial \phi}{\partial r}, \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \right) = \left( U \cos \theta, \frac{1}{2} U \sin \theta, 0 \right)
\]

\[
|\nabla \phi|^2 \bigg|_{r=a} = U^2 \cos^2 \theta + \frac{1}{4} U^2 \sin^2 \theta; \hat{n} = -\hat{e}_r, n_x = -\cos \theta
\]

\[
\int \int_B dS = \int_0^\pi (a d\theta) (2\pi a \sin \theta)
\]
Finally,

\[
F_x = (-\rho)2\pi a^2 \int_0^\pi \sin \theta \left( -\cos \theta \right) \left( -\frac{1}{2} \dot{U} a \cos \theta + \frac{1}{2} \left( U^2 \cos^2 \theta + \frac{1}{4} U^2 \sin^2 \theta \right) \right)\]

\[
F_x = -\dot{U}(\rho a^3)\pi \int_0^{2/3} \sin \theta \cos^2 \theta \, d\theta + (\rho U^2)\pi a^2 \int_0^{2/3} \sin \theta \cos \theta \left( \cos^2 \theta + \frac{1}{4} \sin^2 \theta \right) \, d\theta
\]

\[
F_x = -\dot{U}(\rho a^3)\pi \int_0^{2/3} \sin \theta \cos^2 \theta \, d\theta + (\rho U^2)\pi a^2 \int_0^{2/3} \sin \theta \cos \theta \left( \cos^2 \theta + \frac{1}{4} \sin^2 \theta \right) \, d\theta = 0, \text{ D'Alembert revisited}
\]

Thus the **Hydrodynamic Force** on a sphere of diameter \( a \) moving with velocity \( U(t) \) in an unbounded fluid of density \( \rho \) is given by

\[
F_x = -\dot{U}(\rho a^3)\pi \int_0^{2/3} \sin \theta \cos^2 \theta \, d\theta + (\rho U^2)\pi a^2 \int_0^{2/3} \sin \theta \cos \theta \left( \cos^2 \theta + \frac{1}{4} \sin^2 \theta \right) \, d\theta
\]

Comments:

- If \( \dot{U} = 0 \rightarrow F_x = 0 \), i.e., steady translation \( \rightarrow \) no force (D'Alembert's Condition ok).
- \( F_x \propto \dot{U} \) with a \((-\) \) sign, i.e., the fluid tends to 'resist' the acceleration.
- \([\cdots]\) has the units of (fluid) mass \( \equiv m_a \)
- Equation of Motion for a body of mass \( M \) that moves with velocity \( U \):

\[
\underbrace{M}_{\text{Body mass}} \dot{U} = \underbrace{\Sigma F = F_H}_{\text{Hydrodynamic force}} + \underbrace{F_B}_{\text{All other forces on body}} = \left( -\dot{U} \underbrace{m_a}_{\text{Fluid mass}} \right) + F_B \iff
\]

\[
(M + m_a) \dot{U} = F_B
\]

i.e., the presence of fluid around the body acts as an **added** or **virtual** mass to the body.
3.19 General 6 Degrees of Freedom Motions

3.19.1 Notation Review

(3D) $U_1, U_2, U_3$: Translational velocities
$U_4 \equiv \Omega_1, U_5 \equiv \Omega_2, U_6 \equiv \Omega_3$: Rotational velocities

(2D) $U_1, U_2$: Translational velocities
$U_6 \equiv \Omega_3$: Rotational velocity
$U_3 = U_4 = U_5 = 0$

3.19.2 Added Mass Tensor (matrix)

$m_{ij} ; i, j = 1, 2, 3, 4, 5, 6$

$m_{ij}$: associated with force on body in $i$ direction due to unit acceleration in $j$ direction. For example, for a sphere:

$m_{11} = m_{22} = m_{33} = \frac{1}{2} \rho \mathcal{A} = (m_A)$ all other $m_{ij} = 0$
3.19.3 Added Masses of Simple 2D Geometries

- Circle

\[ m_{11} = m_{22} = \rho \pi a^2 \]

- Ellipse

\[ m_{11} = \rho \pi a^2, m_{22} = \rho \pi b^2 \]

- Plate

\[ m_{11} = \rho \pi a^2, m_{22} = 0 \]
- Square

\[ m_{11} = m_{22} \approx 4.754 \rho a^2 \]

A reasonable approximation to estimate the added mass of a 2D body is to use the displaced mass (\( \rho A \)) of an ‘equivalent cylinder’ of the same lateral dimension or one that ‘rounds off’ the body. For example, consider a square and approximate with an

(a) inscribed circle: \( m_A = \rho \pi a^2 = 3.14 \rho a^2 \).

(b) circumscribed circle: \( m_A = \rho \pi (\sqrt{2}a)^2 = 6.28 \rho a^2 \).

Arithmetic mean of (a) + (b) \( \approx 4.71 \rho a^2 \).
3.19.4 Generalized Forces and Moments

In this paragraph we are looking at the most general case where forces and moments are induced on rigid body moving with 6 DoF motions, in an unbounded fluid that is at rest at infinity.

**Body fixed reference frame**, i.e., $OX_1X_2X_3$ is **fixed** on the body.

\[ \ddot{U}(t) = (U_1, U_2, U_3), \text{ translational velocity} \]
\[ \ddot{\Omega}(t) = (\Omega_1, \Omega_2, \Omega_3) \equiv (U_4, U_5, U_6), \text{ rotational velocity with respect to } O \]

Consider a body with a 6 DoF motion $(\ddot{U}, \ddot{\Omega})$, and a fixed reference frame $OX_1X_2X_3$. Then the hydrodynamic forces and moments with respect to $O$ are given by the following relations (JNN §4.13)

- **Forces**

  \[ F_j = -\ddot{U}_i m_{ji} - E_{jkl} U_i \Omega_k m_{li} \quad \text{with} \quad i = 1, 2, 3, 4, 5, 6 \]

  and $j, k, l = 1, 2, 3$

- **Moments**

  \[ M_j = -\ddot{U}_i m_{j+3,i} - E_{jkl} U_i \Omega_k m_{i+3,l} - E_{jkl} U_k U_i m_{li} \quad \text{with} \quad i = 1, 2, 3, 4, 5, 6 \]

  and $j, k, l = 1, 2, 3$
Einstein’s Σ notation applies.

\[ E_{jkl} = \text{‘alternating tensor’} = \begin{cases} 
0 & \text{if any } j, k, l \text{ are equal} \\
1 & \text{if } j, k, l \text{ are in cyclic order, i.e.,} \\
& (1, 2, 3), (2, 3, 1), \text{ or } (3, 1, 2) \\
-1 & \text{if } j, k, l \text{ are not in cyclic order i.e.,} \\
& (1, 3, 2), (2, 1, 3), (3, 2, 1) 
\end{cases} \]

Note:

(a) if \( \Omega \equiv 0 \), \( F_j = -\dot{U}_i m_{ji} \) (as expected by definition of \( m_{ij} \)).
Also if \( \dot{U}_i \equiv 0 \), then \( F_j = 0 \) for any \( U_i \), no force in steady translation.

(b) \( B_t \sim U_i m_{li} \) ‘added momentum’ due to rotation of axes.
Then all the terms marked as 2. are proportional to \( \sim \tilde{\Omega} \times \tilde{B} \) where \( \tilde{B} \) is linear momentum (momentum from \( i \) coordinate into new \( x_j \) direction).

(c) If \( \Omega_k \equiv 0 : M_j = -\dot{U}_i m_{j+3,i} m_{ij} - \underbrace{E_{jkl} U_k U_l m_{ij}}_{\text{even with } U=0, M_j \neq 0 \text{ due to this term}} \)

**Moment** on a body due to pure steady translation – ‘Munk’ moment.
3.19.5 Example Generalized motions, forces and moments.

A certain body has non-zero added mass coefficients only on the diagonal, i.e. \( m_{ij} = \delta_{ij} \). For a body motion given by \( U_1 = t \), \( U_2 = -t \), and all other \( U_i, \Omega_i = 0 \), the forces and moments on the body in terms of \( m_i \) are:

\[
F_1 = \ldots, F_2 = \ldots, F_3 = \ldots, M_1 = \ldots, M_2 = \ldots, M_3 = \ldots
\]

Solution:

\[ m_{ij} = \delta_{ij} \]

\[
U_1 = t \quad U_2 = -t \quad U_i = 0 \quad i = 3, 4, 5, 6 \quad \Omega_k = 0 \quad k = 1, 2, 3
\]

\[
\dot{U}_1 = 1 \quad \dot{U}_2 = -1 \quad \dot{U}_i = 0 \quad i = 3, 4, 5, 6
\]

Use the relations from (JNN §4.13):

\[
F_j = -\dot{U}_i m_{ij} - E_{jkl} U_i \Omega_k m_{il} \xrightarrow{\Omega_k = 0} -\dot{U}_i m_{ij}
\]

\[
M_j = -\dot{U}_i m_{i(j+3)} - E_{jkl} U_i \Omega_k m_{i(l+3)} - E_{jkl} U_k U_i m_{li} \xrightarrow{\Omega_k = 0} -\dot{U}_i m_{i(j+3)} - E_{jkl} U_k U_i m_{li}
\]

where \( i = 1, 2, 3, 4, 5, 6 \) and \( j, k, l = 1, 2, 3 \)

For \( F_1, F_2, F_3 \) use the previous relationship for \( F_j \) with \( j = 1, 2, 3 \) respectively:

\[
F_1 = -\dot{U}_1 m_{11} - \dot{U}_2 m_{21} - \dot{U}_3 m_{31} - \dot{U}_4 m_{41} - \dot{U}_5 m_{51} - \dot{U}_6 m_{61} \xrightarrow{\dot{U}_1 = 1, \dot{U}_2 = 0, \dot{U}_3 = 0, \dot{U}_4 = 0, \dot{U}_5 = 0, \dot{U}_6 = 0} F_1 = -m_{11}
\]

\[
F_2 \xrightarrow{\text{Check}} -\dot{U}_2 m_{22} \rightarrow F_2 = m_{22}
\]

\[
F_3 \xrightarrow{\text{Check}} -\dot{U}_3 m_{33} \rightarrow F_3 = 0
\]
For $M_1, M_2, M_3$ use the previous relationship for $M_j$ with $j = 1, 2, 3$ respectively:

\[
M_1 = -\dot{U}_im_{i(1+3)} - E_{1kl}U_kU_i m_{li} \\
= -\dot{U}_im_{i4} - E_{1kl}U_kU_i m_{li} \\
= -\dot{U}_1 m_{14} - \dot{U}_2 m_{24} - \dot{U}_3 m_{34} - \dot{U}_4 m_{44} - \dot{U}_5 m_{54} - \dot{U}_6 m_{64} \\
- E_{123}U_2(U_{11} m_{13} + U_{22} m_{23} + U_{33} m_{33} + U_{44} m_{43} + U_{55} m_{53} + U_{66} m_{63}) \\
- E_{132} U_3(U_{11} m_{12} + U_{22} m_{22} + U_{33} m_{32} + U_{44} m_{42} + U_{55} m_{52} + U_{66} m_{62}) \\
\Rightarrow M_1 = 0
\]

\[
M_2 = -\dot{U}_im_{i5} - E_{2kl}U_kU_i m_{li} \\
= \dot{U}_5 m_{55} - E_{231} U_3U_i m_{1i} - E_{213} U_1U_i m_{3i} \\
= -E_{213} U_1 U_3 m_{33} \Rightarrow M_2 = 0
\]

\[
M_3 = -\dot{U}_im_{i6} - E_{3kl}U_kU_i m_{li} \\
= \dot{U}_6 m_{66} - E_{312} U_1U_i m_{2i} - E_{321} U_2U_i m_{1i} \\
= -U_1 \frac{m_{22}}{t} + U_2 \frac{m_{22}}{-t} - U_1 \frac{m_{11}}{t} \Rightarrow M_3 = t^2(m_{22} - m_{11})
\]
Example Munk Moment on a 2D submarine in steady translation

\[ U_1 = U \cos \theta \]
\[ U_2 = -U \sin \theta \]

Consider steady translation motion: \( \dot{U} = 0; \Omega_k = 0 \). Then

\[ M_3 = -E_{3kl}U_kU_im_i \]

For a 2D body, \( m_{3i} = m_{i3} = 0 \), also \( U_3 = 0, i, k, l = 1, 2 \). This implies that:

\[ M_3 = -E_{312}U_1 (U_1m_{21} + U_2m_{22}) - E_{321}U_2 (U_1m_{11} + U_2m_{12}) \]
\[ = -U_1U_2 (m_{22} - m_{11}) \]
\[ = U^2 \sin \theta \cos \theta \left( \frac{m_{22} - m_{11}}{>0} \right) \]

Therefore, \( M_3 > 0 \) for \( 0 < \theta < \pi/2 \) (‘Bow up’). Therefore, a submarine under forward motion is unstable in pitch (yaw). For example, a small bow-up tends to grow with time, and control surfaces are needed as shown in the following figure.
• Restoring moment \( \approx (\rho g\forall H)\sin\theta \).

• Critical speed \( U_{cr} \) given by:

\[
(\rho g\forall) H \sin \theta \geq U_{cr}^2 \sin \theta \cos \theta (m_{22} - m_{11})
\]

Usually \( m_{22} \gg m_{11}, m_{22} \approx \rho\forall \). For small \( \theta, \cos \theta \approx 1 \). So, \( U_{cr}^2 \leq gH \) or \( F_{cr} \equiv \frac{U_{cr}}{\sqrt{gH}} \leq 1 \). Otherwise, control fins are required.