1. Recall from Lecture 3 that the Kinematic Transport Theorem (KTT) is:

\[
\frac{d}{dt} \iiint_{V(t)} f(\vec{x}, t) dV = \iiint_{V(t)} \frac{\partial f}{\partial t} dV + \iint_{S(t)} f\vec{U}(\vec{x}, t) \cdot \hat{n} dS
\]

where \( V(t) \) is an arbitrary time-varying control volume, \( S(t) \) is its bounding surface, \( \vec{U}(\vec{x}, t) \) is the absolute velocity of the surface \( S \) with respect to a fixed frame, and \( \hat{n} \) is the normal to the surface \( S \) pointing out of the volume \( V \).

(a) In many applications, the control volume for a flow system can be chosen as fixed in space. Choose the property \( f \) to be the mass per unit volume, or density \( \rho(\vec{x}, t) \). Denote the fixed control volume as \( V_c \) and the corresponding control surface as \( S_c \). Write the simplified KTT for this case, leaving it in terms of general integrals as above:

(b) Now assume the flow is also steady. This further simplifies the KTT to the following:

(c) Now, start again with equation (1) but let the control volume be a material volume \( V_m(t) \) moving with the fluid and having a material surface \( S_m(t) \) whose points move with the fluid velocity \( \vec{V}(\vec{x}, t) \). Write equation (1) for unsteady flow, again leaving it in terms of general integrals:

(d) Why must the right and left-hand sides of the expression you derived in (c) be equal to zero?
(e) Simplify the version of the KTT obtained in (c) for the case of steady flow, remembering to set the expression equal to zero:

(f) The surface integral in (e) over the moving material surface \( S_m(t) \) is done at a specific instant of time. But this means we can then choose any control surface \( S \) in applying this version of the KTT, because any closed surface in the fluid at a given instant is a material surface. Rewrite the expression in (e) and denote the moving surface as \( S \):

IMPORTANT POINT: An equivalent interpretation of this expression is that the surface \( S \) that we instantaneously choose is actually fixed in space, and the velocity \( \vec{V} \) is then the velocity of the fluid relative to the fixed control surface. This often makes more sense from a problem-solving standpoint. We use it next.

(g) Now apply the steady flow version of the KTT obtained in (f) to a specific case where the fixed control volume and surface are shown below. The fluid enters normal to a surface \( S_1 \) with an area \( A_1 \), and the fluid exits normal to a surface \( S_2 \) with area \( A_2 \). No fluid passes through the top or bottom of the control volume. Assume the fluid velocity and density are uniform over the surfaces \( S_1 \) and \( S_2 \). Simplify all integrals to get an expression relating the inlet and outlet mass flow rates.
2. There is a steady flow of water through a horizontal pipe that has a bend in it as shown. The water enters normal to the inlet area $A_1$ and exits normal to the area $A_2$. Using a fixed control volume and the momentum theorem (see text SAH, Section 4.2), find the components $R_x$ and $R_y$ of the force reaction of the pipe bend ON the water in terms of
pressures $p_1$ and $p_2$, areas $A_1$ and $A_2$, velocity magnitudes $V_1$ and $V_2$, the constant density $\rho$, and the bend angle $\theta$. Pressures and velocities are uniform over $A_1$ and $A_2$. 

\[ 2\hat{n}_2, p_2, V_2 \]
\[ 1\hat{n}_1, p_1, V_1 \]

\[ \theta \]

\[ R_x \]
\[ R_y \]